Stability properties of a class of positive switched systems with rank one difference

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Abstract

Given a single-input continuous-time positive system, described by a pair $(A, \mathbf{b})$, with A a diagonal matrix, we investigate under what conditions there exists a state-feedback law $u(t) = \mathbf{c}^\top \mathbf{x}(t)$ that makes the resulting controlled system positive and asymptotically stable, by this meaning that $A + \mathbf{b}\mathbf{c}^\top$ is Metzler and Hurwitz. In the second part of this note we assume that the state-space model switches among different state-feedback laws ($\mathbf{c}_i^\top, i = 1, 2, \dots, p$) each of them ensuring the positivity, and show that the asymptotic stability of this type of switched system is equivalent to the asymptotic stability of all its subsystems, while its stabilizability is equivalent to the existence of an asymptotically stable subsystem.

Index Terms

Positive switched systems, asymptotic stability, stabilizability, Metzler Hurwitz matrices.

I. Introduction

Recent years have seen a growing interest in systems that are subject to a positivity constraint on their dynamical variables. There are several motivations for this interest, coming from different domains of science and technology. In fact, the positivity assumption is a natural one when describing physical, biological or economical processes whose variables represent quantities that are intrinsically nonnegative, such as pressures, concentrations, population levels, etc. [7].

By a continuous-time positive switched system (CPSS) we mean a dynamic system consisting of a family of continuous-time positive state-space models and a switching law, specifying

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when and how the switching takes place. CPSS have been fruitfully used in bioengineering and pharmacokinetics. For instance, the insulin-sugar metabolism is captured by two different compartmental models: one valid in steady-state and the other (of course, more complex) which is suitable to describe the evolution under perturbed conditions, following an oral consumption or an intravenous injection. The paper by Haddad, Chellaboina and Nersesov [13] provides a very interesting analysis of hybrid nonnegative systems and, in particular, of hybrid compartmental systems and their use in modeling physiological systems.

In intracellular systems biology, the continuous time dynamics of signaling pathways are often combined with the essentially logical machinery of gene expression. Together with transport delays in protein synthesis, this may lead to hybrid (in particular, switched) systems with time delays and positivity constraints on the describing variables [14]. Positive switched systems have also been used to design optimal drug treatments to cope with viral mutation [20].

CPSSs have been the object of an intense research activity, mainly focused on stability [5], [6], [8], [12], [16], [18], [19], [26] and stabilizability [1], [2], [25]. Special attention has been devoted to the class of CPSSs that switch among subsystems whose matrices differ by a rank one matrix [15], [18], [19], [21], [22]. The reason for the interest in these systems is twofold. On the one hand, they can be thought of as the possible configurations one obtains from a given single-input system, when applying different state-feedback laws that ensure the positivity of the resulting closed-loop system. For this reason, the subsystem matrices can be denoted by $A + \mathbf{b}\mathbf{c}_i^\top, i \in \{1, 2, \dots, p\}$. On the other hand, interesting connections have been highlighted [22] between the quadratic stability of CPSSs, switching between two subsystems of matrices A and $A + \mathbf{b}\mathbf{c}^\top$, and the SISO circle criterion for the transfer function $\mathbf{c}^\top(sI_n - A)^{-1}\mathbf{b}$.

In [19] it has been proved that, when a CPSS switches between $p = 2$ subsystems of dimension $n \leq 3$, the Hurwitz property of its subsystem matrices $A+\mathbf{b}\mathbf{c}_i^\top, i \in \{1, 2\}$, ensures the asymptotic stability of the associated CPSS. On the other hand, as one deduces by putting together the results of [12] and [9], this is also true when the CPSS has dimension $n = 2$ and consists of an arbitrary number of subsystems. At the present stage of research, it is not known whether the Metzler Hurwitz property of the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in \{1, 2, \dots, p\}$, of size $n > 2$ ensures that the associated CPSS is asymptotically stable. In this paper we prove that this is true under the additional assumption that the matrix A is a diagonal one.

CPSSs described by Metzler matrices $A+\mathbf{b}\mathbf{c}_i^\top, i \in \{1, 2, \dots, p\}$, with A diagonal, arise when investigating the behavior of non-homogeneous multi-agent systems, each of them described by a scalar system, evolving under the action of a unique input signal, that coordinates their behavior. If we assume that different state-feedback strategies may be employed to control the overall agent behavior, we naturally end up with this class of rank one CPSSs, having a diagonal system matrix. This kind of model arises also when dealing with compartmental models, with independent compartments, that are subject to different supervisory control strategies (e.g., tracers injections whose quantities depend on a weighted sum of the compartment concentrations, as it happens with some drug treatments).

In addition, the stability result derived in this paper is relevant also for non-positive switched systems whose subsystem matrices differ by a rank one matrix. Indeed, if we drop the positivity constraint, it is known that the Hurwitz property of the subsystem matrices alone does not ensure the asymptotic stability of the associated switched system, and additional conditions are required [23]. However, from the aforementioned result it follows that when A is diagonalizable, and the matrices $A+\mathbf{b}\mathbf{c}_i^\top$ leave invariant the polyhedral invariant cone generated by n (linearly independent, but otherwise arbitrarily chosen) eigenvectors of A, then the Hurwitz property of the matrices $A+\mathbf{b}\mathbf{c}_i^\top$ ensures the asymptotic stability of the switched system. It is conjectured that the existence of a proper polyhedral cone, left invariant by all the Hurwitz matrices $A+\mathbf{b}\mathbf{c}_i^\top$, may lead to obtain a complete characterization of the asymptotic stability property in the non-positive case.

In the second part of the paper, stabilizability of CPSSs with rank one difference, under the assumption that A is diagonal, is shown to be equivalent to the the asymptotic stability of at least one subsystem (i.e., existence of an index i such that $A+\mathbf{b}\mathbf{c}_i^\top$ is Hurwitz). While the sufficiency of this condition is obvious, its necessity is not, and essentially reveals that no smart switching strategy may overcome the drawback related to the fact that all subsystems are not asymptotically stable. Note that stabilizability of CPSSS that switch among subsystems whose matrices differ by a rank one matrix has not been addressed before in the literature, except in [9], where the main focus, however, is on convex combinations of the subsystem matrices.

In detail, the paper is organized as follows: in section II, we present some preliminary results and consider a continuous-time single-input state-space model with diagonal system matrix A. Conditions on the vectors $\mathbf{b}$ and $\mathbf{c}$ that ensure the positivity and the asymptotic stability of the resulting system $\dot{\mathbf{x}}(t) = (A+\mathbf{b}\mathbf{c}^\top)\mathbf{x}(t)$ are provided. Section III solves the stability problem of the class of rank one CPSSs, while stabilizability is the object of section IV. A preliminary version of part of the results appearing in this paper was recently presented at the ECC 2013 Conference [10].

Notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers and, for any pair of positive integers k, n, with $k \leq n$, $[k, n]$ is the set of integers $\{k, k+1, \dots, n\}$. The ith entry of a vector $\mathbf{v}$ is denoted by $[\mathbf{v}]_i$. We denote by $\mathbf{1}_n$ the n-dimensional vector with all unitary entries, and by $\mathbf{e}_i$ the ith canonical vector in $\mathbb{R}^n$ (n being clear from the context), with all zero entries except for the ith which is unitary. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called *nonnegative*, and if so we adopt the notation $A \geq 0$. If, in addition, A has at least one positive entry, the matrix is *positive* ($A > 0$), while if all its entries are positive, it is *strictly positive* ($A \gg 0$). A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. A square matrix A is *Hurwitz* if all its eigenvalues have negative real part.

II. Diagonal systems and positivity preserving stabilizing feedback laws

Consider a single-input state-space model

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}u(t), \qquad t \in \mathbb{R}_+, \tag{1}$$

where $\mathbf{x}(t)$ and $u(t)$ are the n-dimensional state variable and the scalar input, respectively, at time t. We assume that A is diagonal, namely $A = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}, \lambda_i \in \mathbb{R}$.
We consider a state feedback law $u(t) = \mathbf{c}^\top \mathbf{x}(t)$ that makes the resulting autonomous system positive, by this meaning that the matrix $A + \mathbf{b}\mathbf{c}^\top$ is Metzler. It is worth noticing that $A + \mathbf{b}\mathbf{c}^\top$ is Metzler if and only if $\mathbf{b}\mathbf{c}^\top$ is Metzler, and this introduces strong constraints on the sign of the nonzero entries of the vectors $\mathbf{b}$ and $\mathbf{c}$. In particular, if all the entries of the vectors $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ are nonzero, the product $\mathbf{b}\mathbf{c}^\top$ is Metzler if and only if one of the following applies:

- if $n = 1$, $\mathbf{b}$ and $\mathbf{c}$ can be arbitrary;
- if $n = 2$, either all entries of $\mathbf{b}$ and $\mathbf{c}$ have the same sign (in which case $\mathbf{b}\mathbf{c}^\top \gg 0$), or both $\mathbf{b}$ and $\mathbf{c}$ have two entries of opposite sign and $\mathbf{b}\mathbf{c}^\top$ has positive off-diagonal entries and negative diagonal entries;
- if $n > 2$, then all entries of $\mathbf{b}$ and $\mathbf{c}$ have the same sign (and hence, again $\mathbf{b}\mathbf{c}^\top \gg 0$).

We first explore the eigenvalue allocation problem, namely we investigate where the eigenvalues of the matrix $A + \mathbf{b}\mathbf{c}^\top$ can be located under the assumption that A is diagonal and $A + \mathbf{b}\mathbf{c}^\top$

is Metzler. To this end, it entails no loss of generality reordering the state components in such a way that $A, \mathbf{b}$ and $\mathbf{c}$ are block-partitioned (with corresponding blocks having the same size) as follows:

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} \mathbf{c}_1 \\ 0 \\ \mathbf{c}_3 \\ 0 \end{bmatrix}, \tag{2}$$

where all the entries of the blocks $\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}_1, \mathbf{c}_3$ are nonzero, and each block $A_i, i \in [1, 4]$, is diagonal of size n_i. This is a simple consequence of fact that the set $[1, n]$ can be partitioned into the four (possibly empty) disjoint sets:

$$\begin{aligned} I_1 &:= \{j \in [1, n] : [\mathbf{b}]_j \neq 0 \text{ and } [\mathbf{c}]_j \neq 0\}, \\ I_2 &:= \{j \in [1, n] : [\mathbf{b}]_j \neq 0 \text{ and } [\mathbf{c}]_j = 0\}, \\ I_3 &:= \{j \in [1, n] : [\mathbf{b}]_j = 0 \text{ and } [\mathbf{c}]_j \neq 0\}, \\ I_4 &:= \{j \in [1, n] : [\mathbf{b}]_j = 0 \text{ and } [\mathbf{c}]_j = 0\}. \end{aligned}$$

Moreover, we assume w.l.o.g. that $A_1 = \text{blockdiag}\{\tilde{\lambda}_1 I_{k_1},\ \tilde{\lambda}_2 I_{k_2},\ \ldots,\ \tilde{\lambda}_r I_{k_r}\}, \text{with } \tilde{\lambda}_1 > \tilde{\lambda}_2 > \ldots > \tilde{\lambda}_r$.

Proposition 1: Given a diagonal matrix $A \in \mathbb{R}^{n \times n}$, $n > 1$, and vectors $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, described as in (2), assume that the matrix $A + \mathbf{b}\mathbf{c}^\top$ is Metzler. Then

i) $\sigma(A + \mathbf{b}\mathbf{c}^\top) = \sigma(A_1 + \mathbf{b}_1\mathbf{c}_1^\top) \cup \sigma(A_2) \cup \sigma(A_3) \cup \sigma(A_4)$.

Moreover, the spectrum $(\mu_1, \mu_2, \ldots, \mu_{n_1})$ of $A_1 + \mathbf{b}_1\mathbf{c}_1^\top$ satisfies the following conditions:

ii) if $n_1 = 1$, then $\mu_1 = \tilde{\lambda}_1 + \mathbf{b}_1\mathbf{c}_1$;

iii) if $n_1 > 2$, then $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_r$ are eigenvalues of $A_1 + \mathbf{b}_1\mathbf{c}_1^\top$ of multiplicities $k_1 - 1, k_2 - 1, \ldots, k_r - 1$, while the remaining r eigenvalues of $A_1 + \mathbf{b}_1\mathbf{c}_1^\top$, say $\mu_1 > \mu_2 > \ldots > \mu_r$, satisfy

$$\tilde{\lambda}_r < \mu_r < \tilde{\lambda}_{r-1} < \mu_{r-1} < \ldots < \mu_2 < \tilde{\lambda}_1 < \mu_1; \tag{3}$$

iv) if $n_1 = 2$, then two cases possibly arise: (a) if the diagonal entries of $\mathbf{b}_1\mathbf{c}_1^\top$ are both positive, then the same conditions as in iii) hold; (b) if the diagonal entries of $\mathbf{b}_1\mathbf{c}_1^\top$ are both negative, then the 2 eigenvalues of $A_1 + \mathbf{b}_1\mathbf{c}_1^\top$, say $\mu_1 > \mu_2$, satisfy

$$\mu_2 < \tilde{\lambda}_2 < \mu_1 < \tilde{\lambda}_1, \tag{4}$$

if A_1 has two distinct eigenvalues, while the case $A_1 = \tilde{\lambda}_1 I_2$ leads to $\mu_1 = \tilde{\lambda}_1$ and $\mu_2 < \tilde{\lambda}_1$.

Proof: i) Follows trivially from the structure of $A + \mathbf{b}\mathbf{c}^\top$:

$$A + \mathbf{b}\mathbf{c}^\top = \begin{bmatrix} A_1 + \mathbf{b}_1\mathbf{c}_1^\top & 0 & \mathbf{b}_1\mathbf{c}_3^\top & 0 \\ \mathbf{b}_2\mathbf{c}_1^\top & A_2 & \mathbf{b}_2\mathbf{c}_3^\top & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix}.$$

ii) is obvious.

iii) We first note that if $n_1 > 2$, then, $A_1 + \mathbf{b}_1\mathbf{c}_1^\top$ can be Metzler if and only if $\mathbf{b}_1\mathbf{c}_1^\top$ is a strictly positive matrix. We observe that

$$\det(sI_n - A_1 - \mathbf{b}_1\mathbf{c}_1^\top) = d(s) - n(s),$$

where

$$d(s) := \det(sI_n - A_1), \qquad n(s) := \mathbf{c}_1^\top \mathrm{adj}(sI_n - A_1)\mathbf{b}_1.$$

Moreover, we can easily see that

$$\begin{aligned} d(s) \ &= \ \prod_{i=1}^{r}(s-\tilde{\lambda}_i)^{k_i} = \prod_{i=1}^{r}(s-\tilde{\lambda}_i)^{k_i-1} \cdot \prod_{i=1}^{r}(s-\tilde{\lambda}_i), \\ n(s) \ &= \ \prod_{i=1}^{r}(s-\tilde{\lambda}_i)^{k_i-1}\left[\sum_{i=1}^{r}\gamma_i \prod_{j\in[1,r],j\neq i}(s-\tilde{\lambda}_j)\right], \end{aligned}$$

where

$$\gamma_i := \sum_{k=k_1+k_2+\ldots+k_{i-1}+1}^{k_1+k_2+\ldots+k_{i-1}+k_i} [\mathbf{b}_1]_k[\mathbf{c}_1]_k, \qquad \forall\ i \in [1,r], \qquad (k_0 := 0),$$

are all positive coefficients. Consequently,

$$\det(sI_n - A_1 - \mathbf{b}_1\mathbf{c}_1^\top) \ = \ \prod_{i=1}^{r}(s-\tilde{\lambda}_i)^{k_i-1}\left[\prod_{i=1}^{r}(s-\tilde{\lambda}_i) - \sum_{i=1}^{r}\gamma_i \prod_{j\in[1,r],j\neq i}(s-\tilde{\lambda}_j)\right].$$

This immediately proves that $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_r$ are eigenvalues of $A_1 + \mathbf{b}_1\mathbf{c}_1^\top$ of multiplicities (at least) $k_1 - 1, k_2 - 1, \ldots, k_r - 1$. Set

$$\psi(s) \ := \ \prod_{i=1}^{r}(s-\tilde{\lambda}_i) - \sum_{i=1}^{r}\gamma_i \prod_{j\in[1,r],j\neq i}(s-\tilde{\lambda}_j).$$

We note that, by the positivity of the γ_i's and the ordering of the $\tilde{\lambda}_i$'s,

$$\begin{aligned}
\psi(s)|_{s=\tilde{\lambda}_1} &= 0 - \sum_{i=1}^{n} \gamma_i \prod_{j\in[1,n],j\neq i} (\tilde{\lambda}_1 - \tilde{\lambda}_j) = -\gamma_1 \prod_{j\in[1,n],j\neq 1} (\tilde{\lambda}_1 - \tilde{\lambda}_j) < 0, \\
\psi(s)|_{s=\tilde{\lambda}_2} &= 0 - \sum_{i=1}^{n} \gamma_i \prod_{j\in[1,n],j\neq i} (\tilde{\lambda}_2 - \tilde{\lambda}_j) = -\gamma_2 \prod_{j\in[1,n],j\neq 2} (\tilde{\lambda}_2 - \tilde{\lambda}_j) > 0, \\
\psi(s)|_{s=\tilde{\lambda}_3} &= 0 - \sum_{i=1}^{n} \gamma_i \prod_{j\in[1,n],j\neq i} (\tilde{\lambda}_3 - \tilde{\lambda}_j) = -\gamma_3 \prod_{j\in[1,n],j\neq 1} (\tilde{\lambda}_3 - \tilde{\lambda}_j) < 0, \\
&\vdots
\end{aligned}$$

By the change of signs of the polynomial $\psi(s)$ on the real line, we can deduce that it always has (independently of the specific values of the positive γ_i's) $r-1$ real zeros, ordinately located in the intervals $(\tilde{\lambda}_i, \tilde{\lambda}_{i-1}), i \in [2, r]$. On the other hand, as the leading coefficient of $\psi(s)$ is positive, and hence this characteristic polynomial eventually takes positive values on the positive real axis, it follows that $\mu_1 \in (\tilde{\lambda}_1, +\infty)$. This proves statement iii).

iv) If $n_1 = 2$, then either $\mathbf{b}_1\mathbf{c}_1^\top$ is strictly positive or it has negative diagonal entries and positive off-diagonal entries. The first case reduces to the one addressed in part iii). In the second case, the nonzero pattern of the matrix $\mathbf{b}_1\mathbf{c}_1^\top$ and the fact that it has rank 1 allow to express it as

$$\mathbf{b}_1\mathbf{c}_1^\top = \begin{bmatrix} 1 \\ -\beta \end{bmatrix} \begin{bmatrix} -\alpha K & K \end{bmatrix},$$

for suitable positive numbers α, β, K. Suppose, first, that A_1 has two distinct eigenvalues. Then

$$\begin{aligned}
\det(sI_2 - A_1 - \mathbf{b}_1\mathbf{c}_1^\top) &= (s-\tilde{\lambda}_1)(s-\tilde{\lambda}_2) - \begin{bmatrix} -\alpha K & K \end{bmatrix} \begin{bmatrix} s-\tilde{\lambda}_2 & 0 \\ 0 & s-\tilde{\lambda}_1 \end{bmatrix} \begin{bmatrix} 1 \\ -\beta \end{bmatrix} \\
&= (s-\tilde{\lambda}_1)(s-\tilde{\lambda}_2) + K(\alpha+\beta)\left[s - \frac{\beta\tilde{\lambda}_1 + \alpha\tilde{\lambda}_2}{\alpha+\beta}\right].
\end{aligned}$$

By considering the positive root locus parametrized by $\tilde{K} := K(\alpha+\beta)$, we easily see that for every positive value of $\tilde{K}$ the two roots of $\det(sI_2 - A_1 - \mathbf{b}_1\mathbf{c}_1^\top)$ satisfy (4). On the other hand, if A_1 has one single eigenvalue of multiplicity 2, then $\det(sI_2 - A_1 - \mathbf{b}_1\mathbf{c}_1^\top) = (s-\tilde{\lambda}_1)(s-\tilde{\lambda}_1 + K(\alpha+\beta))$, and hence $\mu_1 = \tilde{\lambda}_1$ is still an eigenvalue, meanwhile the other eigenvalue satisfies $\mu_2 < \tilde{\lambda}_1$. ■

In the light of Proposition 1, regarding the eigenvalue allocation problem, we can now investigate under what conditions the state-feedback law makes the resulting system not only positive but also asymptotically stable, which means that $A+\mathbf{b}\mathbf{c}^\top$ is both Metzler and Hurwitz.

Proposition 2: Given a diagonal matrix $A \in \mathbb{R}^{n\times n}$, $n > 1$, and vectors $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, described as in (2), assume that the matrix $A + \mathbf{b}\mathbf{c}^\top$ is Metzler. Then $A + \mathbf{b}\mathbf{c}^\top$ is Hurwitz if and only if A_2, A_3 and A_4 have negative diagonal entries, and one of the following conditions hold:

i) if $n_1 = 1$, then $\tilde{\lambda}_1 + \mathbf{b}_1\mathbf{c}_1 < 0$;

ii) if either $n_1 > 2$ or $n_1 = 2$ and $\mathbf{b}_1\mathbf{c}_1^\top \gg 0$, then A_1 has negative diagonal entries and

$$\det(sI_{n_1} - A_1 - \mathbf{b}_1\mathbf{c}_1^\top)\Big|_{s=0} > 0; \tag{5}$$

iii) if $n_1 = 2$, and the diagonal entries of the Metzler matrix $\mathbf{b}_1\mathbf{c}_1$ are both negative, then A_1 has at least one negative diagonal entry and (5) holds.

Proof: The proof reduces to analyze the Hurwitz property of the block $A_1 + \mathbf{b}_1\mathbf{c}_1^\top$. i) is obvious. As far as point ii) is concerned, from (3) it is clear that $A_1 + \mathbf{b}_1\mathbf{c}_1^\top$ is Hurwitz if and only if $\mu_1 < 0$. By the proof of Proposition 1, this happens if and only if all the $\tilde{\lambda}_i$'s are negative and the dominant zero of $\psi(s)$ is located in $(\tilde{\lambda}_1, 0)$. Since $\psi(\tilde{\lambda}_1) < 0$, and all the other zeros of $\psi(s)$ are smaller then $\tilde{\lambda}_1$, this latter condition is equivalent to the fact that $\psi(0) > 0$. On the other hand, the negativity of the $\tilde{\lambda}_i$'s implies that $\psi(0) > 0$ if and only if (5) holds.

iii) Also, in this case, by Proposition 1, $A_1 + \mathbf{b}_1\mathbf{c}_1^\top$ is Hurwitz if and only if $\mu_1 < 0$. This is the case if and only if either the diagonal entries of A_1 are both negative (in which case (5) is surely verified) or A_1 has a negative diagonal entry and $\mu_1 < 0$. As in the proof of the previous part, this is possible if and only if (5) holds. ■

III. Stability analysis of positive switched systems

In the rest of the paper we consider continuous-time positive switched systems (CPSSs) described by the following equation

$$\dot{\mathbf{x}}(t) = (A + \mathbf{b}\mathbf{c}_{\sigma(t)}^\top)\ \mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{6}$$

where $\mathbf{x}(t)$ is the n-dimensional state variable, $\sigma(t)$ the switching sequence at time t, taking values in the set $[1, p]$, $A \in \mathbb{R}^{n\times n}$, while $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n$, for every $i \in [1, p]$. We assume that for every index $i \in [1, p]$, the matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler. This latter condition ensures that the switched system (6) is positive, by this meaning that if the initial state $\mathbf{x}(0)$ is positive the state trajectory remains in the positive orthant $\mathbb{R}^n_+$ for every choice of the switching sequence.

Definition 1: The CPSS (6) is *asymptotically stable* if for every initial state $\mathbf{x}(0) > 0$ and every switching sequence $\sigma(t), t \in \mathbb{R}_+$, the state trajectory $\mathbf{x}(t), t \in \mathbb{R}_+$, converges to zero.

If the CPSS is asymptotically stable, all the system matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, are (Metzler and) Hurwitz. In this section we want to prove that, when A is a diagonal matrix, the fact that all matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, are Hurwitz is also sufficient for asymptotic stability. To this end, we first notice that we can always assume that $A, \mathbf{b}, \mathbf{c}_i, i \in [1, p]$, are block partitioned as

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{c}_i = \begin{bmatrix} \mathbf{c}_{i,1} \\ 0 \\ \mathbf{c}_{i,3} \\ 0 \end{bmatrix}, \tag{7}$$

where each block $A_k, k \in [1, 4]$, is diagonal of size n_k, all the entries of the blocks $\mathbf{b}_1$ and $\mathbf{b}_2$ are nonzero, while the blocks $\mathbf{c}_{i,1}$ and $\mathbf{c}_{i,3}$ are such that there is no index j such that $[\mathbf{c}_{i,1}]_j = 0, \forall\ i \in [1, p]$, or $[\mathbf{c}_{i,3}]_j = 0, \forall\ i \in [1, p]$. We accordingly partition the state-vector as:

$$\mathbf{x}(t) = [\, \mathbf{x}_1(t)^\top \quad \mathbf{x}_2(t)^\top \quad \mathbf{x}_3(t)^\top \quad \mathbf{x}_4(t)^\top \,]^\top,$$

and the common structure of all the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, easily shows that the CPSS (6) is asymptotically stable if and only if A_2, A_3, A_4 are Hurwitz and the switched system

$$\dot{\mathbf{x}}_1(t) = (A_1 + \mathbf{b}_1 \mathbf{c}_{\sigma(t),1}^\top)\ \mathbf{x}_1(t), \qquad t \in \mathbb{R}_+, \tag{8}$$

is asymptotically stable. So, from now on, we will focus on the switched system (8), or, equivalently, we will assume that the matrices $A + \mathbf{b}\mathbf{c}_i^\top$ of the switched system (6) satisfy these two constraints: $\mathbf{b}$ is devoid of zero entries and for every $k \in [1, n]$ there exists $i \in [1, p]$ such that $[\mathbf{c}_i]_k \neq 0$. Clearly, the case $n = 1$ is trivial, and hence we will always assume $n > 1$.

We first address the case when the vector $\mathbf{b}$ has entries of different signs. When so, upon a suitable permutation $\mathbf{b}$ can always be expressed as follows:

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_+ \\ \mathbf{b}_- \end{bmatrix}, \qquad \mathbf{b}_+ \gg 0, \quad \mathbf{b}_- \ll 0. \tag{9}$$

We set $n_+ := \dim\ \mathbf{b}_+$, and $n_- := \dim\ \mathbf{b}_-$. We have the following result.

Proposition 3: Given a diagonal matrix $A \in \mathbb{R}^{n\times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, assume that $\mathbf{b}$ is described as in (9). If $n_+ \geq 1$, $n_- \geq 1$, and the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are all Metzler and Hurwitz, then the CPSS (6) is asymptotically stable.

Proof: We distinguish the following subcases:

(a) $[n_+ > 1$ and $n_- > 1]$. If so, $\mathbf{b}\mathbf{c}_i^\top$ is Metzler if and only if $\mathbf{c}_i = 0$. Consequently, each matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler Hurwitz if and only if A is Hurwitz and $\mathbf{c}_i = 0$. Therefore, in this case, the Metzler Hurwitz property of the matrices guarantees the asymptotic stability of the CPSS (6).

(b) $[n_+ > 1$ and $n_- = 1]$ or $[n_+ = 1$ and $n_- > 1]$. Consider, first, the case when $n_+ > 1$ and $n_- = 1$. It is easily seen that $\mathbf{b}\mathbf{c}_i^\top$ is Metzler if and only if $\mathbf{c}_i = \alpha_i \mathbf{e}_n$, where $\alpha_i \geq 0$ and $\mathbf{e}_n$ is the nth canonical vector. When so, the Metzler matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is upper triangular. But a CPSS whose matrices are all Hurwitz and in the upper triangular form is necessarily asymptotically stable [17]. The case $n_+ = 1$ and $n_- > 1$ follows the same lines, but it deals with lower triangular Metzler Hurwitz matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$.

(c) $[n_+ = 1$ and $n_- = 1]$. In this case, we first observe that if the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are all Metzler Hurwitz, then (see Proposition 3 in [9]) all their convex combinations are Metzler Hurwitz, in turn. On the other hand, since we are dealing with a two-dimensional CPSS, the Metzler Hurwitz property of all the convex combinations of the system matrices ensures [12] that the CPSS (6) is asymptotically stable. ■

We now address the case when all the entries of the vector $\mathbf{b}$ have the same sign.

Proposition 4: Given a diagonal matrix $A \in \mathbb{R}^{n\times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, with $\mathbf{b}$ either strictly positive or strictly negative, if the matrices $A+\mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are all Metzler and Hurwitz, then the CPSS (6) is asymptotically stable.

Proof: Consider, first, the case when $\mathbf{b} \gg 0$. Then $\mathbf{b}\mathbf{c}_i^\top$ is Metzler if and only if $\mathbf{c}_i \geq 0$. Moreover, by suitable adjusting the result given in case ii) of Proposition 2, $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler Hurwitz if and only if A is Hurwitz and $\det(sI_n - A - \mathbf{b}\mathbf{c}_i^\top)\big|_{s=0} > 0$. We first note that, by the Hurwitz property of A it follows that $-A^{-1}$ is a positive matrix and $\det(-A) > 0$. On the other hand, as $\det(sI_n - A - \mathbf{b}\mathbf{c}_i^\top)\big|_{s=0} = \det(-A)[1 + \mathbf{c}_i^\top A^{-1}\mathbf{b}]$, it is clear that under the Hurwitz assumption on A, $\det(sI_n - A - \mathbf{b}\mathbf{c}_i^\top)\big|_{s=0} > 0$ holds if and only if $1 + \mathbf{c}_i^\top A^{-1}\mathbf{b} > 0$. Set $\mathbf{w} := -A^{-1}\mathbf{b}$. It is easy to see that $\mathbf{w} \gg 0$ and that $(A+\mathbf{b}\mathbf{c}_i^\top)\mathbf{w} = -(1+\mathbf{c}_i^\top A^{-1}\mathbf{b})\mathbf{b} \ll 0, \forall\, i \in [1,p]$. This ensures that there exists a common linear copositive Lyapunov function [8], [16] for the matrices $(A + \mathbf{b}\mathbf{c}_i^\top)^\top, i \in [1,p]$, and hence the positive switched system

$$\dot{\mathbf{z}}(t) = A_{\sigma(t)}^\top \mathbf{z}(t), \qquad A_\sigma(t) \in \{A + \mathbf{b}\mathbf{c}_1^\top, \dots, A + \mathbf{b}\mathbf{c}_p^\top\},$$

is asymptotically stable. But then, for each choice of the switching sequence, the product of the matrix exponentials converges to the zero matrix and so does its transpose, thus ensuring that also the positive switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)} \mathbf{x}(t), \qquad A_\sigma(t) \in \{A + \mathbf{b}\mathbf{c}_1^\top, \dots, A + \mathbf{b}\mathbf{c}_p^\top\},$$

is asymptotically stable. The case $\mathbf{b} \ll 0$ can be addressed along the same lines. ■

Propositions 3 and 4 together prove that, when the vector $\mathbf{b}$ is devoid of zero entries, the CPSS (6) is asymptotically stable if and only if the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are Metzler Hurwitz. On the other hand, we have shown that when $\mathbf{b}$ has also zero entries, and the matrices $A, \mathbf{b}$ and $\mathbf{c}_i, i \in [1,p]$, are described as in (7), the asymptotic stability of the CPSS (6) is equivalent to the Hurwitz property of the blocks A_2, A_3 and A_4 together with the asymptotic stability of the CPSS (8). So, by putting together all these results, we get the following theorem.

Theorem 1: Let $A \in \mathbb{R}^{n \times n}$ be a diagonal matrix, and let $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, be vectors such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler for every index $i \in [1,p]$. The following facts are equivalent:

i) $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz for every index $i \in [1,p]$;

ii) the CPSS (6) is asymptotically stable.

Remark 2: It is worthwhile to compare Theorem 1 with the results about absolute stability of positive systems derived in [3], [4]. To this end, it is convenient to reduce the CPSS (6) to the same form adopted in [3]. We first notice that every switching sequence $\sigma(t)$ induces a partition of the time set $\mathbb{R}_+$ into p disjoint sets

$$\Omega_i := \{t \in \mathbb{R}_+ : \sigma(t) = i\}, \qquad i \in [1,p].$$

Next, after introducing the output function

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1^\top \\ \mathbf{c}_2^\top \\ \vdots \\ \mathbf{c}_p^\top \end{bmatrix} \mathbf{x}(t) =: C\mathbf{x}(t),$$

we define the nonlinear function

$$\varphi(\mathbf{y}(t),t) = \begin{bmatrix} \varphi_1(y_1(t),t) \\ \varphi_2(y_2(t),t) \\ \vdots \\ \varphi_p(y_p(t),t) \end{bmatrix},$$

where

$$\varphi_i(y_i(t),t) = \begin{cases} y_i(t) = \mathbf{c}_i^\top \mathbf{x}(t), & \text{if } t \in \Omega_i; \\ 0, & \text{otherwise.} \end{cases}$$

So, by making use of this notation, we can equivalently represent the CPSS (6) as

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{10}$$

$$\mathbf{y}(t) = C\mathbf{x}(t), \tag{11}$$

$$\mathbf{u}(t) = \varphi(\mathbf{y}(t),t), \tag{12}$$

where $B = \mathbf{b}\mathbf{1}_p^\top$. As the nonlinear function φ, satisfies the *sector condition*:

$$\mu_i \le \varphi_i(y_i(t),t)/y_i(t) \le \nu_i, \qquad i \in [1,p], \tag{13}$$

for $\mu_i = 0$ and $\nu_i = 1$, upon setting

$$M = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_m \end{bmatrix} = 0_{p\times p} \qquad N = \begin{bmatrix} \nu_1 & & & \\ & \nu_2 & & \\ & & \ddots & \\ & & & \nu_m \end{bmatrix} = I_p,$$

we are in a position to discuss the possibility of applying Theorem 1 in [3]. This theorem states that, if B and C are nonnegative matrices, then system (10)-(11) is positively absolutely stable in the class of nonlinearities (12) satisfying (13) if and only if $A+BMC$ is Metzler and $A+BNC$ is Hurwitz (in the specific case we are considering, if and only if A is Metzler and $A+BC$ is Hurwitz). Theorem 6 in [4] provides a similar result, under the assumption that $M = -N$.

As a matter of fact, Theorem 1 in our paper does not follow from Theorem 1 in [3], not even in the special case when $\mathbf{b}$ and $\mathbf{c}_i, i \in [1,p]$, are nonnegative vectors (which represents an additional assumption with respect to those adopted in our set-up). For instance, consider

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 7/8 \\ 0 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 0 \\ 7/8 \end{bmatrix},$$

and hence

$$B = [\,\mathbf{b} \quad \mathbf{b}\,] = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} \mathbf{c}_1^\top \\ \mathbf{c}_2^\top \end{bmatrix} = \begin{bmatrix} 7/8 & 0 \\ 0 & 7/8 \end{bmatrix}.$$

The matrices $A + \mathbf{b}\mathbf{c}_i^\top$, $i \in [1, 2]$, are Metzler Hurwitz, but $A + BC = A + \mathbf{b}\mathbf{c}_1^T + \mathbf{b}\mathbf{c}_2^T$ is not Hurwitz. So, the CPSS (6) is asymptotically stable, but the corresponding system (10)-(11) is not positively absolutely stable. This shows that Theorem 1 in [3] provides a condition stronger than asymptotic stability for the class of CPSSs (6) with nonnegative vectors $\mathbf{b}$ and $\mathbf{c}_i, i \in [1, p]$, and it cannot even be applied to this class of systems when any of these vectors have at least one positive entry.

IV. Stabilizability of positive switched systems

Definition 2: The CPSS (6) is *stabilizable* if for every positive initial state $\mathbf{x}(0)$ there exists a switching sequence $\sigma(t), t \in \mathbb{R}_+$, such that the state trajectory $\mathbf{x}(t), t \in \mathbb{R}_+$, converges to zero.

In the general case, a sufficient condition for stabilizability is that at least one of the system matrices is Hurwitz. More generally, if there exists a convex combination of the system matrices that is (Metzler and) Hurwitz, then the system is stabilizable [24]. For Metzler matrices that differ by a rank one matrix, these two sufficient conditions for stabilizability are in fact equivalent (independently of the fact that A is diagonal or not). To prove this, we first need this technical lemma that shows this result for a pair of matrices whose difference is a rank one matrix.

Lemma 1: Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix, and assume that $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ are column vectors such that $A + \mathbf{b}\mathbf{c}^\top$ is Metzler, in turn. There exists $\alpha \in [0, 1]$ such that $A(\alpha) := (1-\alpha)A + \alpha(A + \mathbf{b}\mathbf{c}^\top) = A + \alpha\mathbf{b}\mathbf{c}^\top$ is Hurwitz if and only if either A or $A + \mathbf{b}\mathbf{c}^\top$ is Hurwitz.

Proof: Sufficiency is trivial, so we focus only on the necessity. Clearly, if the Hurwitz convex combination $A(\alpha)$ corresponds either to $\alpha = 0$ or to $\alpha = 1$ there is nothing to prove. So we assume that $A(\alpha)$ is Hurwitz for some $\alpha \in (0, 1)$. Set $d(s) := \det(sI_n - A)$ and $n(s) = \mathbf{c}^\top \mathrm{adj}(sI_n - A)\mathbf{b}$, and note that $\det(sI_n - A(\alpha)) = d(s) - \alpha n(s)$.

We first assume that A and $A + \mathbf{b}\mathbf{c}^\top$ are both irreducible Metzler matrices. This ensures [7] that both $d(s)$ and $d(s) - n(s) = \det(sI_n - A - \mathbf{b}\mathbf{c}^\top)$ have a simple strictly dominant real zero. We let $p = \lambda_{\max}(A)$ denote the strictly dominant real zero of $d(s)$ and $z = \lambda_{\max}(A + \mathbf{b}\mathbf{c}^\top)$ denote the strictly dominant real zero of $d(s) - n(s)$. Upon assuming $\alpha = \frac{\gamma}{\gamma+1}$, we notice that there exists $\bar{\alpha} \in (0,1)$ such that $A(\bar{\alpha})$ is Hurwitz if and only if there exists $\bar{\gamma} > 0$ such that

$$\Delta_{\bar{\gamma}}(s) := d(s) + \bar{\gamma}[d(s) - n(s)]$$

is Hurwitz. So, we now consider the (positive) root locus corresponding to $\Delta_\gamma(s), \gamma > 0$. We make the following remarks:

i) as $\deg n(s) < \deg d(s)$, the two polynomials $d(s)$ and $d(s) - n(s)$ have the same degree, which implies that there are no branches going to infinity;

ii) for every $\gamma > 0$, $\Delta_\gamma(s)$ is (up to a rescaling factor) the characteristic polynomial of an irreducible Metzler matrix, and hence it has a simple strictly dominant real zero.

We consider the case when $p \geq z$; the case $p < z$ can be treated in a similar way. By the definition of z, in the strip $\mathcal{S} := \{s \in \mathbb{C} : z \leq \mathrm{Re}(s) \leq p\}$ there cannot be zeros of $d(s) - n(s)$, only zeros of $d(s)$. Also, none of the points of the real semiaxis $\{s \in \mathbb{C} : \mathrm{Re}(s) > p\}$ belongs to the root locus. We distinguish two cases:

a) in $\mathcal{S}$ there is no real zero of $d(s)$;

b) in $\mathcal{S}$ there are real zeros of $d(s)$. If so, we denote by p_m the maximum real root of $d(s)$ satisfying $p_m < p$.

In case a), by the rule that to the root locus of $\Delta_\gamma(s)$ belong all the points of the real axis that have at their right an odd number of zeros of $d(s)$ or $d(s) - n(s)$ (counted with their multiplicities), it follows that the line segment $[z, p]$ belongs to the root locus. This implies that, as γ varies between 0 and $+\infty$, the real dominant root of $\Delta_\gamma(s)$ must belong to that segment. So, the assumption that $\Delta_{\bar{\gamma}}(s)$ is Hurwitz implies that some point of that segment is negative, and hence $z < 0$. This implies that $A + \mathbf{b}\mathbf{c}^\top$ is Hurwitz.

In case b), by the same rule we previously mentioned, the line segment $[p_m, p]$ belongs to the root locus. But since p and p_m are two roots of $d(s)$, there must be two branches of the root locus starting from some point of the segment $[p_m, p]$ and leaving the real axis. But this means that there exist values of $\gamma > 0$ for which the zeros with maximum real part are not real, and this contradicts the fact that for every $\gamma > 0$, $\Delta_\gamma(s)$ has a simple strictly dominant real zero. So, case b) is not possible and the result is proved (under the assumption that A and $A + \mathbf{b}\mathbf{c}^\top$ are both irreducible).

Consider now the case when either A or $A + \mathbf{b}\mathbf{c}^\top$ is reducible, and there exists $\alpha \in [0, 1]$ such that $A(\alpha)$ is Hurwitz. Then there exists $\epsilon > 0$ such that $A(\alpha) + \epsilon \mathbf{1}_n \mathbf{1}_n^\top$ is both Hurwitz and irreducible. Moreover,

$$A(\alpha) + \epsilon \mathbf{1}_n \mathbf{1}_n^\top = (1-\alpha)(A + \epsilon \mathbf{1}_n \mathbf{1}_n^\top) + \alpha(A + \epsilon \mathbf{1}_n \mathbf{1}_n^\top + \mathbf{b}\mathbf{c}^\top)$$

and the two Metzler matrices $A + \epsilon \mathbf{1}_n \mathbf{1}_n^\top$ and $A + \epsilon \mathbf{1}_n \mathbf{1}_n^\top + \mathbf{b}\mathbf{c}^\top$ are both irreducible. So, by the previous part of the proof we can claim that either one of the matrices is Hurwitz. This implies that either A or $A + \mathbf{b}\mathbf{c}^\top$ is Hurwitz. ■

Proposition 5: Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix, and assume that $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, are column vectors such that the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are Metzler. There exist $\alpha_i, i \in [1,p], 0 \leq \alpha_i \leq 1$, with $\sum_{i=1}^p \alpha_i = 1$, such that $\sum_{i=1}^p \alpha_i (A + \mathbf{b}\mathbf{c}_i^\top)$ is Hurwitz if and only if there exists $i \in [1,p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

Proof: Sufficiency is obvious, so, again, we only deal with necessity. We prove necessity by induction on p. We have already shown, in Lemma 1, that the result is true for $p = 2$. Suppose it is true for $p - 1 \geq 2$ matrices. We want to show that the result is true for p matrices. Assume that $\sum_{i=1}^p \alpha_i (A + \mathbf{b}\mathbf{c}_i^\top)$ is Hurwitz. It entails no loss of generality assuming that $\alpha_p \neq 0$. Set $\tilde{A} := A + \mathbf{b}\mathbf{c}_p^\top$ and $\tilde{\mathbf{c}}_i := \mathbf{c}_i - \mathbf{c}_p$. Accordingly, for every $i \in [1, p-1]$, $A + \mathbf{b}\mathbf{c}_i^\top = \tilde{A} + \mathbf{b}\tilde{\mathbf{c}}_i^\top$ and

$$\begin{aligned}
\sum_{i=1}^{p} \alpha_i (A + \mathbf{b}\mathbf{c}_i^\top) &= \sum_{i=1}^{p-1} \alpha_i (\tilde{A} + \mathbf{b}\tilde{\mathbf{c}}_i^\top) + \alpha_p \tilde{A} \\
&= (1-\alpha_p) \sum_{i=1}^{p-1} \frac{\alpha_i}{1-\alpha_p} (\tilde{A} + \mathbf{b}\tilde{\mathbf{c}}_i^\top) + \alpha_p \tilde{A} \\
&= (1-\alpha_p) \left[\tilde{A} + \mathbf{b} \left(\sum_{i=1}^{p-1} \frac{\alpha_i}{1-\alpha_p} \tilde{\mathbf{c}}_i^\top \right) \right] + \alpha_p \tilde{A}.
\end{aligned}$$

Since the convex combination of the two Metzler matrices $\tilde{A}$ and $\tilde{A} + \mathbf{b}\left(\sum_{i=1}^{p-1} \frac{\alpha_i}{1-\alpha_p} \tilde{\mathbf{c}}_i^\top\right)$ is Hurwitz, then, by Lemma 1, either $\tilde{A}$ or $\tilde{A} + \mathbf{b}\left(\sum_{i=1}^{p-1} \frac{\alpha_i}{1-\alpha_p} \tilde{\mathbf{c}}_i^\top\right)$ is Hurwitz. If $\tilde{A}$ is Hurwitz, we are done. On the other hand, the matrix $\tilde{A} + \mathbf{b}\left(\sum_{i=1}^{p-1} \frac{\alpha_i}{1-\alpha_p} \tilde{\mathbf{c}}_i^\top\right)$ is a convex combination of the $p - 1$ matrices $A + \mathbf{b}\mathbf{c}_i^\top = \tilde{A} + \mathbf{b}\tilde{\mathbf{c}}_i^\top, i \in [1, p-1]$. So, by the inductive assumption, at least one of them is Hurwitz. ■

By making use of the previous result, we can provide an important characterization of the stabilizability property for the class of CPSSs described as in (6), under the additional assumption that A is a diagonal matrix. As in the previous section, we assume that $A, \mathbf{b}$ and each $\mathbf{c}_i, i \in [1,p]$, are described as in (7), where $A_k, k \in [1,4]$, are diagonal blocks of size n_k, all the entries of the blocks $\mathbf{b}_1$ and $\mathbf{b}_2$ are nonzero, while the blocks $\mathbf{c}_{i,1}$ and $\mathbf{c}_{i,3}$ are such that there is no index j such that $[\mathbf{c}_{i,1}]_j = 0, \forall\ i \in [1,p]$, or $[\mathbf{c}_{i,3}]_j = 0, \forall\ i \in [1,p]$. Also in this case it is easily seen that the CPSS (6) is stabilizable if and only if A_2, A_3 and A_4 are Hurwitz and the CPSS (8) is stabilizable. So, in the sequel we consider the stabilizability problem for CPSSs described as in (6), with A diagonal and $\mathbf{b}$ devoid of zero entries. Again, we address separately the case when $\mathbf{b}$ has entries of opposite signs and the case when the nonzero entries have all the same sign.

Proposition 6: Given a diagonal matrix $A \in \mathbb{R}^{n\times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, assume that $\mathbf{b}$ is described as in (9). If $n_+ \geq 1$, $n_- \geq 1$, and the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are all Metzler, then the CPSS (6) is stabilizable if and only if there exists an index $i \in [1,p]$ such that the matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

Proof: Sufficiency is obvious, so we only prove necessity. As in the proof of Proposition 3, we proceed by considering all possible cases:

(a) $[n_+ > 1$ and $n_- > 1]$. If so, $A+\mathbf{b}\mathbf{c}_i^\top$ is Metzler if and only if $\mathbf{c}_i = 0$, and hence all matrices $A+\mathbf{b}\mathbf{c}_i^\top$ coincide with A. So, stabilizability requires that all matrices $A+\mathbf{b}\mathbf{c}_i^\top = A$ are Metzler Hurwitz.

(b) $[n_+ > 1$ and $n_- = 1]$ or $[n_+ = 1$ and $n_- > 1]$. In the first case, all vectors $\mathbf{c}_i$ take the form $\mathbf{c}_i = \alpha_i \mathbf{e}_n$ for some $\alpha_i > 0$, and hence all matrices $A + \mathbf{b}\mathbf{c}_i^\top$ take the form:

$$A + \mathbf{b}\mathbf{c}_i^\top = \left[\begin{array}{cccc|c} \lambda_1 & & & & \\ & \lambda_2 & & & \alpha_i \mathbf{b}_+ \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ \hline & & & & \lambda_n + \alpha_i b_- \end{array}\right].$$

So, it is clear that the system is stabilizable only if $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are negative and there exists at least one index $i \in [1,p]$ such that $\lambda_n + \alpha_i b_- < 0$. But this means that there exists an index $i \in [1,p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz. The second case is symmetric.

(c) $[n_+ = 1$ and $n_- = 1]$. In this case, $A + \mathbf{b}\mathbf{c}_i^\top$ is a 2×2 Metzler matrix, and for a two-dimensional CPSS (6) stabilizability is equivalent [1], [2] to the existence of a Hurwitz convex combination of the matrices. By Proposition 5, this implies that at least one matrix $A+\mathbf{b}\mathbf{c}_i^\top$ is Hurwitz. ■

To prove the result in the case when all entries of $\mathbf{b}$ have the same sign, we need the following technical lemmas.

Lemma 2: [1], [2], [11] Consider the CPSS

$$\dot{\mathbf{x}}(t) = M_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{14}$$

with $\sigma(t)$ taking values in $[1,p]$, and $M_1, M_2, \dots, M_p \in \mathbb{R}^{n\times n}$ Metzler matrices. The CPSS (14) is stabilizable if and only if there exist $r \in \mathbb{Z}_+$, $i_1, i_2, \dots, i_r \in [1,p], \tau_1, \tau_2, \dots, \tau_r \in \mathbb{R}_+$, such that the matrix $e^{M_{i_1}\tau_1}e^{M_{i_2}\tau_2}\dots e^{M_{i_r}\tau_r}$ is Schur.

Lemma 3: Consider the CPSS (14), with $\sigma(t)$ taking values in $[1,p]$, and $M_1, M_2, \dots, M_p \in \mathbb{R}^{n\times n}$ Metzler matrices. If there exists $\mathbf{w} \gg 0$ such that $M_i\mathbf{w} \geq 0, \forall\ i \in [1,p]$, then the CPSS (14) is not stabilizable.

Proof: Introduce the CPSS

$$\dot{\mathbf{z}}(t) = M_{\sigma(t)}^\top\mathbf{z}(t), \qquad t \in \mathbb{R}_+. \tag{15}$$

Consider the Lyapunov function $V(\mathbf{z}) := \mathbf{w}^\top\mathbf{z}$ and its derivatives along the various subsystems of (15),

$$\dot{V}_i(\mathbf{z}) = \mathbf{w}^\top M_i^\top\mathbf{z}, \qquad i \in [1,p].$$

Clearly, for every choice of the switching sequence and every initial condition $\mathbf{z}(0) > 0$, $V(\mathbf{z}(t)) \geq V(\mathbf{z}(0)) = \mathbf{w}^\top\mathbf{z}(0) > 0$, and hence the CPSS (15) is not stabilizable. By Lemma 2, this implies that for every choice of $r \in \mathbb{Z}_+$, $i_1, i_2, \dots, i_r \in [1,p], \tau_1, \tau_2, \dots, \tau_r \in \mathbb{R}_+$, the matrix $Z := e^{M_{i_1}^\top\tau_1}e^{M_{i_2}^\top\tau_2}\dots e^{M_{i_r}^\top\tau_r}$ is not Schur. So, neither $Z^\top = e^{M_{i_r}\tau_r}\dots e^{M_{i_2}\tau_2}\dots e^{M_{i_1}\tau_1}$ is Schur, and this prevents, by Lemma 2, the stabilizability of the CPSS (14). ∎

Proposition 7: Let $A \in \mathbb{R}^{n\times n}, n > 1$, be a diagonal matrix and consider vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1,p]$, with $\mathbf{b}$ either strictly positive or strictly negative, such that the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1,p]$, are all Metzler. The CPSS (6) is stabilizable if and only if there exists an index $i \in [1,p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

Proof: Again, we only need to prove the necessity. Consider, first, the case when $\mathbf{b} \gg 0$. We preliminary notice that, as $\mathbf{b} \gg 0$, $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler only if $\mathbf{c}_i \geq 0$, and this ensures that $A + \mathbf{b}\mathbf{c}_i^\top \geq A$. If the CPSS (6) is stabilizable then, by Lemma 2, there exist $r \in \mathbb{Z}_+$, $i_1, i_2, \dots, i_r \in [1,p], \tau_1, \tau_2, \dots, \tau_r \in \mathbb{R}_+$, such that $Z := e^{(A+\mathbf{b}\mathbf{c}_{i_1}^\top)\tau_1}e^{(A+\mathbf{b}\mathbf{c}_{i_2}^\top)\tau_2}\dots e^{(A+\mathbf{b}\mathbf{c}_{i_r}^\top)\tau_r}$ is

Schur. But since $Z \geq e^{A\tau_1} e^{A\tau_2} \dots e^{A\tau_r} > 0$, this latter matrix must be Schur, too, and hence the diagonal matrix A must be Hurwitz. Since A is Hurwitz and $\mathbf{b} \gg 0$, if each $A + \mathbf{b}\mathbf{c}_i^\top$ were not Hurwitz then, by Proposition 2, it should be $\det(sI_n - A - \mathbf{b}\mathbf{c}_i^\top)\big|_{s=0} \leq 0$ for every $i \in [1, p]$. By proceeding as in the proof of Proposition 4, we can claim that this is equivalent to assuming that $1 + \mathbf{c}_i^\top A^{-1}\mathbf{b} \leq 0, \forall\, i \in [1, p]$. Set $\mathbf{w} := -A^{-1}\mathbf{b} \gg 0$ and note, again, that

$$(A + \mathbf{b}\mathbf{c}_i^\top)\mathbf{w} = -(1 + \mathbf{c}_i^\top A^{-1}\mathbf{b})\mathbf{b}.$$

So, if none of the system matrices were Hurwitz, there would be a vector $\mathbf{w} \gg 0$ such that

$$(A + \mathbf{b}\mathbf{c}_i^\top)\mathbf{w} \geq 0, \qquad \forall\, i \in [1, p],$$

thus preventing, by Lemma 3, stabilizability. So, $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz for some $i \in [1, p]$. ∎

By putting together Propositions 6 and 7, we finally get the following result.

Theorem 3: Let $A \in \mathbb{R}^{n \times n}$ be a diagonal matrix, and let $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n, i \in [1, p]$, be vectors such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler for every index $i \in [1, p]$. The following facts are equivalent:

i) there exists $i \in [1, p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz;

ii) the CPSS (6) is stabilizable.

V. Conclusions

In this paper we have investigated the class of CPSSs whose subsystems are described by Metzler matrices taking the form $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, n]$, where A is a diagonal matrix. For these systems, stability is equivalent to the seemingly weaker condition that all the subsystem matrices are Hurwitz, while stabilizability is equivalent to the seemingly stronger condition that one of the matrices $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz. As in general checking stability and stabilizability of CPSSs is a difficult task, these characterizations are very useful, and it would be of extreme interest to investigate to what classes of CPSSs the previous results, about stability or stabilizability, can be extended. In addition, for these systems, it would be important to characterize the set of of all stabilizing switching sequences, in addition to the trivial constant one.

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