Asymptotic stability and stabilizability of special classes of discrete-time positive switched systems

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Index Terms

Switched system, positive linear system, (global uniform) asymptotic stability, stabilizability, monomial matrix, circulant matrix.

Abstract

In this paper we consider discrete-time positive switched systems, switching among autonomous subsystems, characterized either by monomial matrices or by circulant matrices. Necessary and sufficient conditions are provided guaranteeing either (global uniform) asymptotic stability or stabilizability (i.e. the possibility of driving to zero the state trajectory corresponding to any initial state by resorting to some switching sequence). Such conditions lead to simple algorithms that allow to easily detect, under suitable conditions, whether a given positive switched system is not stabilizable.

I. INTRODUCTION

A discrete-time positive switched system (DPSS) [24] consists of a family of discrete-time positive state-space models [9] and a switching law, specifying when and how the switching among the various subsystems takes place. This class of systems has interesting practical applications. They have been adopted for describing networks employing TCP and other congestion control applications [33], for modeling consensus and synchronization problems [18], and, quite recently, to describe the viral mutation dynamics under drug treatment [15].

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Asymptotic stability and stabilizability have been investigated in depth for continuous-time positive switched systems, by resorting to linear copositive and to quadratic Lyapunov functions [14], [20], [26], [27], [28]. However, no computationally effective necessary and sufficient condition for assessing either property is available, yet.

On the other hand, research efforts on asymptotic stability and stabilizability for the specific class of DPSS's have been rather limited [11], [25]. As a matter of fact, there is a long stream of research on the stability analysis of general (i.e. not necessarily positive) discrete-time switched systems. Interesting results have been obtained, basing on a variety of mathematical methods: Lyapunov-Metzler inequalities [12], piecewise quadratic control-Lyapunov functions [34], the maximum principle and the variational approach [29], [30], $\mathcal{H}^\infty$ control and ℓ^2 gain minimization [23], [22], ergodic measure theory [17]. The joint spectral radius, in particular, provides a powerful theoretical tool in assessing asymptotic stability. In fact, if a discrete time system switches among a finite number of subsystems, and we denote by $\mathcal{A} = \{A_1, A_2, \dots, A_p\}$ the set of matrices associated with these subsystems, asymptotic stability is equivalent [6], [13], [19] to the fact that the *joint spectral radius* of $\mathcal{A}$,

$$\rho(\mathcal{A}) := \lim_{k\to+\infty} \{\max \|(A_{i_1} \cdots A_{i_k}\|^{1/k} : A_i \in \mathcal{A}\} = \limsup_{k\to+\infty} (\max\{\rho(A_{i_1} \cdots A_{i_k})^{1/k} : A_i \in \mathcal{A}\})$$

is smaller than 1. It was conjectured (*finiteness conjecture*) [6], [21] that $q \in \mathbb{N}$ and a product $A_{i_1}A_{i_2}, \cdots A_{i_q}$ of q matrices of $\mathcal{A}$ could always be found such that $\rho(\mathcal{A})$ coincides with $\rho(A_{i_1}A_{i_2} \cdots A_{i_q})^{\frac{1}{q}}$. If this were the case, the convergence to zero of all state trajectories corresponding to periodic switching signals would ensure the convergence to zero of the state trajectories corresponding to any switching signal. This conjecture has been disproved in [3], [4]. For the classes of DPSS whose matrices $A_i, i \in [1,p]$, satisfy the *finiteness property*, asymptotic stability is always algorithmically decidable [19]. Families of matrices endowed with this property are, for instance, families of symmetric matrices or matrices whose associated Lie-algebra is solvable. In this paper we will show that, when restricting our attention to positive matrices, there are other families of matrices that satisfy the finiteness conjecture and hence for which asymptotic stability is equivalent to the convergence to zero of all the trajectories corresponding to periodic switching: families of monomial matrices or of (left/right) positive circulant matrices. As a further result, we will also show that, for DPSS characterized by these classes of matrices, asymptotic stability is equivalent to the existence of special classes of Lyapunov functions which are common to all matrices.

Stabilizability property, namely the possibility of driving to zero the state trajectory corresponding to any initial state, by resorting to a suitable switching sequence, is a much weaker requirement on the dynamical behavior of a DPSS. However, it is extremely important from the point of view of the system control. As we shall see, it admits quite interesting characterizations both for cyclic monomial and circulant matrices, that provide also, as a byproduct, some interesting conditions, that allow to detect when a general DPSS is not stabilizable.

In detail, the paper is organized as follows. Sections II, III and IV consider DPSS whose state transition matrices are monomial. Specifically, in section II necessary and sufficient conditions for asymptotic stability of these systems are provided, and it is shown that for these systems the finiteness property holds. As a consequence, asymptotic stability proves to be equivalent to global uniform asymptotic stability and all these properties can be checked by means of special classes of Lyapunov equations. In section III, stabilizability of DPSS described by monomial matrices having the same nonzero pattern is fully characterized. Further results are obtained for the special class of systems switching among diagonal matrices. It turns out that, for these classes of systems, stabilizability is equivalent to the existence of a Schur matrix product that involves a number of distinct matrices not greater than the system dimension. An extension of this result for two-dimensional and three-dimensional DPSS switching among arbitrary monomial matrices is provided in section IV. We conjecture that the extension is true for an arbitrary dimension.

Section V deals with asymptotic stability and stabilizability of DPSS whose state transition matrices are (either left or right) circulant. Finally, section VI provides a number of sufficient conditions, based on the previous results, that allow to check whether a generic DPSS cannot be stabilized.

Notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called *nonnegative* ($A \geq 0$). If, in addition,there is at least one positive entry, A is *positive* ($A > 0$), while if all its entries are positive, A is *strictly positive* ($A \gg 0$). The (ℓ, j)th entry of a matrix A is denoted by $[A]_{\ell j}$, while the ℓth entry of a vector $\mathbf{v}$ is $[\mathbf{v}]_\ell$. The ith column of a matrix A is $\mathrm{col}_i(A)$.

A vector $\mathbf{v} \in \mathbb{R}^n_+$ is a *monomial vector* if it exhibits a single positive entry. If such a positive entry is 1, then $\mathbf{v}$ is a canonical vector. The sum of the canonical vectors of $\mathbb{R}^n_+$, i.e. the n-

dimensional vector with all entries equal to 1, is denoted by $\mathbf{1}_n$. A *monomial (permutation) matrix* is a nonsingular square positive matrix whose columns are monomial (canonical) vectors. A monomial matrix can always be expressed as the product of a diagonal matrix, with positive diagonal entries, and of a permutation matrix. In particular, a monomial matrix described as

$$A = \begin{bmatrix} 0 & a_{12} & 0 & \dots & 0 \\ 0 & 0 & a_{23} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1,n} \\ a_{n1} & 0 & 0 & \dots & 0 \end{bmatrix}, \quad a_{12}, a_{23}, \dots a_{n1} > 0,$$

is an $n \times n$ *cyclic monomial matrix*. Notice that when A is cyclic monomial, A^k is diagonal if and only if k is a mulitple of n.

Two positive matrices $A_1, A_2 \in \mathbb{R}_+^{n\times n}$ are said to be *cogredient* if there exists a permutation matrix $P \in \mathbb{R}_+^{n\times n}$ such that $A_2 = P^{-1}A_1P = P^\top A_1 P$.

Given a matrix $A \in \mathbb{R}_+^{n\times n}$, we associate with it [5] a *digraph* $\mathcal{D}(A)$, with vertices $1, \dots, n$. There is an arc (j, ℓ) from j to ℓ if and only if $[A]_{\ell j} > 0$. If so, $[A]_{\ell j}$ represents the *weight* of the arc. A sequence $j_1 \to j_2 \to \dots \to j_k \to j_{k+1}$ is a *path* of length k from j_1 to j_{k+1} provided that $(j_1, j_2), \dots, (j_k, j_{k+1})$ are arcs of $\mathcal{D}(A)$. A closed path is called a *cycle*. In particular, a cycle γ with no repeated vertices is called *elementary,* and its length $|\gamma|$ coincides with the number of (distinct) vertices appearing in it. Note that the digraph of a cyclic monomial matrix consists of one elementary cycle with length n.

A square symmetric matrix P is *positive definite* ($\succ 0$) if for every nonzero vector $\mathbf{x}$, of compatible dimension, $\mathbf{x}^\top P\mathbf{x} > 0$, and negative definite ($\prec 0$) if $-P$ is positive definite.

A real square matrix A is *Metzler* if its off-diagonal entries are nonnegative, *Schur* if all its eigenvalues lie in the open unit disk (equivalently, its spectral radius, $\rho(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}$, is smaller than one), and *Hurwitz* if they all lie in the open left halfplane.

Given a family of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s$ in $\mathbb{R}^n$, the *convex hull* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s$ is the set of vectors $\{\sum_{i=1}^s \alpha_i \mathbf{v}_i : \alpha_i \geq 0, \sum_{i=1}^s \alpha_i = 1\}$.

Finally, we need some definitions borrowed from the algebra of non-commutative polynomials [32]. Given the *alphabet* $\Xi = \{\xi_1, \xi_2, \dots, \xi_p\}$, the free monoid Ξ^* with base Ξ is the set of all *words* $w = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}, k \in \mathbb{N}, \xi_{i_h} \in \Xi$. The integer k is called the *length* of w and is denoted by $|w|$, while $|w|_i$ represents the number of occurencies of ξ_i in w. If $\tilde{w} = \xi_{j_1}\xi_{j_2}\cdots\xi_{j_p}$ is another element of Ξ^*, the product is defined by concatenation $w\tilde{w} = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}\xi_{j_1}\xi_{j_2}\cdots\xi_{j_p}$. This produces a monoid with $\varepsilon = \emptyset$, the empty word, as unit element. Clearly, $|w\tilde{w}| = |w| + |\tilde{w}|$ and $|\varepsilon| = 0$. $\mathbb{R}\langle\xi_1, \xi_2, \ldots, \xi_p\rangle$ is the algebra of polynomials in the noncommuting indeterminates $\xi_1, \xi_2, \ldots, \xi_p$. For every family $\{A_1, A_2, \ldots, A_p\}$ of p matrices in $\mathbb{R}^{n\times n}$, the map ψ defined on $\{\varepsilon, \xi_1, \xi_2, \ldots, \xi_p\}$ by the assignments $\psi(\varepsilon) = I_n$ and $\psi(\xi_i) = A_i$, $i = 1, 2, \ldots, p$, uniquely extends to an algebra morphism of $\mathbb{R}\langle\xi_1, \xi_2, \ldots, \xi_p\rangle$ into $\mathbb{R}^{n\times n}$ (as an example, $\psi(\xi_1\xi_2) = A_1A_2 \in \mathbb{R}^{n\times n}$). If w is a word in Ξ^* (i.e. a monic monomial in $\mathbb{R}\langle\xi_1, \xi_2, \ldots, \xi_p\rangle$), the ψ-image of w is denoted by $w(A_1, A_2, \ldots, A_p)$.

II. Asymptotic stability of switched systems with monomial matrices

A *discrete-time positive switched system* (DPSS) is described by the following equation

$$\mathbf{x}(t+1) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{1}$$

where $\mathbf{x}(t)$ denotes the n-dimensional state variable at time t, σ is an arbitrary switching sequence, taking values in $[1,p] := \{1, 2, \ldots, p\}$, and for each $i \in [1,p]$ A_i is the state transition matrix of a discrete-time positive system, which means that A_i is an $n \times n$ positive matrix.

Definition 1: [19] The DPSS (1) is *asymptotically stable* if, for every positive initial state $\mathbf{x}(0)$ and every switching signal σ, the state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, asymptotically converges to zero.

Linearity and positivity of the DPSS's allow to say that, as soon as there exists a single strictly positive initial state $\mathbf{x}(0)$ for which the state trajectories of (1) corresponding to any switching sequence σ converge to zero, asymptotic stability is guaranteed. A different, yet strictly related, concept is that of global uniform asymptotic stability.

Definition 2: [24] The DPSS (1) is *globally uniformly asymptotically stable* (GUAS, for short) if there exist a class $\mathcal{KL}$ function[1] β such that, for every positive initial state $\mathbf{x}(0)$ and every switching signal σ, the state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, satisfies the inequality

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(0)\|, t), \qquad \forall\ t \geq 0.$$

[1]A function $\beta : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ belongs to the $\mathcal{KL}$-class if, for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+} \beta(s,t) = 0$, and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s,t) = 0$.

For general (i.e. non necessarily positive) discrete time linear switched systems, asymptotic stability and GUAS are equivalent properties, and all of them depend on the spectral radius of the matrix set $\mathcal{A}$, as a system is asymptotically stable if and only if $\rho(\mathcal{A}) < 1$. This result, extremely important from a theoretic point of view, does not suggest, however, a finite procedure for deciding whether a system is asymptotically stable. For specific classes of DPSS, however, this is possible, and our contribution deals with two of them. In this section we focus on DPSS's whose matrices $A_i, i \in [1,p]$, are monomial and hence can be described as

$$A_i = D_i P_i, \tag{2}$$

where $D_i = \mathrm{diag}\{d_1^{(i)}, d_2^{(i)}, \ldots, d_n^{(i)}\}$ is a diagonal matrix with positive diagonal entries, and P_i is an $n \times n$ permutation matrix. The digraph $\mathcal{D}(A_i)$ consists on n vertices, and n arcs which form a number of disjoint elementary cycles, each vertex belonging to exactly one cycle. Consequently A_i is cogredient to a matrix of the following type

$$\begin{bmatrix} Z_1^{(i)} & & & \\ & Z_2^{(i)} & & \\ & & \ddots & \\ & & & Z_{r_i}^{(i)} \end{bmatrix},$$

with $Z_h^{(i)}$ a cyclic monomial matrix corresponding to some elementary cycle $\gamma_h^{(i)}$, and r_i the number of distinct cycles in $\mathcal{D}(A_i)$. The characteristic polynomial of A_i can be expressed as

$$\Delta_{A_i}(z) := \det(zI_n - A_i) = \prod_{h=1}^{r_i} \Delta_{Z_h^{(i)}}(z),$$

where $\Delta_{Z_h^{(i)}}(z) = z^{|\gamma_h^{(i)}|} - \prod_{v \in \gamma_h^{(i)}} d_v^{(i)}$, and $d_v^{(i)} := [D_i]_{vv} > 0$. Clearly, A_i is a Schur matrix if and only if all blocks $Z_h^{(i)}$'s are Schur, namely

$$\prod_{v \in \gamma_h^{(i)}} d_v^{(i)} < 1, \qquad \forall\ \gamma_h^{(i)} \in \mathcal{D}(A_i), \tag{3}$$

and A_i^k is a diagonal matrix if and only if k is a common multiple of the $|\gamma_h^{(i)}|$, $h = 1, 2, \ldots, r_i$. For every $w = \xi_{i_1}\xi_{i_2}\ldots\xi_{i_p} \in \Xi^*$, the matrix $w(A_1, A_2, \ldots, A_p) = A_{i_1}A_{i_2}\ldots A_{i_p}$ is always monomial, but its digraph may generally differ from the graph of any of the matrices A_{i_k}. Also, its (ℓ, j)th entry, if not zero, can be uniquely expressed as

$$[A_{i_1}A_{i_2}\ldots A_{i_k}]_{\ell j} = [A_{i_1}]_{\ell v_2}[A_{i_2}]_{v_2 v_3}\ldots[A_{i_k}]_{v_k j},$$

for suitable $v_2, v_3, \dots, v_k \in [1,n]$. We now provide a complete characterization of the asymptotic stability of system (1), under the assumption that all matrices $A_i, i \in [1,p]$, are monomial.

Proposition 1: Given a DPSS (1), with monomial matrices $A_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, the following facts are equivalent:

(i) the system is asymptotically stable;

(ii) for every $w \in \Xi^*$, $|w| \le n$, the matrix product $w(A_1, A_2, \dots, A_p)$ is a positive Schur matrix;

(iii) for every $w \in \Xi^*$, each diagonal entry of $w(A_1, A_2, \dots,\ A_p)$ is smaller than 1;

(iv) for each map $\pi : [1,n] \to [1,p]$, the matrix $A_\pi := [\,\mathrm{col}_1(A_{\pi(1)})\ \mathrm{col}_2(A_{\pi(2)})\ \ \dots\ \mathrm{col}_n(A_{\pi(n)})\,]$ is Schur;

(v) the A_i's admit a *common linear copositive function* [10], [11], [27], namely there exists $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top A_i \ll \mathbf{v}^\top$, for every $i \in [1,p]$;

(vi) the A_i's admit a *common diagonal Lyapunov function*, namely there exists $\Delta = \mathrm{diag}\{\delta_1, \delta_2, \dots, \delta_n\}$, $\delta_j > 0, j \in [1,n]$, such that $A_i^\top \Delta A_i - \Delta \prec 0$, for every $i \in [1,p]$;

(vii) the A_i's admit a *common quadratic copositive function of rank* 1 [10], [11], namely there exists $P = P^\top$ of rank 1 such that for every $\mathbf{x} > 0$ one finds $\mathbf{x}^\top P \mathbf{x} > 0$ and $\mathbf{x}^\top [A_i^\top P A_i - P]\mathbf{x} < 0$, for every $i \in [1,p]$.

Proof: (i) $\Rightarrow$ (ii) If (1) is asymptotically stable, all periodic switching sequences ensure convergence. Consequently, given any $w \in \Xi^*$, $|w| \le n$, for every $\mathbf{x}(0) > 0$ the trajectory $\mathbf{x}(k|w|) = w(A_1, A_2, \dots, A_p)^k \mathbf{x}(0)$ converges to zero as $k \to +\infty$. Thus $w(A_1, A_2, \dots, A_p)$ is Schur.

(ii) $\Rightarrow$ (iii) We preliminary notice that if $w(A_1, A_2, \dots, A_p)$ is a positive Schur matrix, $I_n - w(A_1, A_2, \dots, A_p)$ is an M-matrix [16]. Therefore all its principal minors and, in particular, its diagonal entries $1 - [w(A_1, A_2, \dots, A_p)]_{jj}$, $j \in [1,n]$, are positive. So, for every $w \in \Xi^*$, with $|w| \le n$, the matrix $w(A_1, A_2, \dots,\ A_p)$ has all diagonal entries smaller than 1.

Consider now $w \in \Xi^*$ with $|w| = k > n$, and assume $w(A_1, A_2, \dots, A_p) = A_{i_1} A_{i_2} \dots A_{i_k}$. As the digraph of this monomial matrix includes only elementary cycles, the diagonal element $[A_{i_1} A_{i_2} \dots A_{i_k}]_{v_1 v_1}$, $v_1 \in [1,n]$, is positive if and only if there exists a (unique) choice of vertices $v_2, v_3, \dots, v_k \in [1,n]$ such that $[A_{i_1} A_{i_2} \dots A_{i_k}]_{v_1 v_1} = [A_{i_1}]_{v_1 v_2} [A_{i_2}]_{v_2 v_3} \dots [A_{i_k}]_{v_k v_1}$.
As $k > n$, in the sequence $v_1, v_2, \dots, v_k$ some element appears (at least) twice and we can extract a subsequence, $v_h, v_{h+1}, \dots, v_{h+t}$, such that $v_h = v_{h+t} = v$ and all vertices $v_{h+s},\ s = 1, 2, \dots t-1$,

are both distinct and different from v. This implies $t \leq n$ and, by the first part of the proof,

$$[A_{i_h}]_{vv_{h+1}}[A_{i_{h+1}}]_{v_{h+1}v_{h+2}} \dots [A_{i_{h+t-1}}]_{v_{h+t-1}v} = [A_{i_h} A_{i_{h+1}} A_{i_{h+t-1}}]_{vv} < 1.$$

We therefore have $[A_{i_1}]_{v_1v_2}[A_{i_2}]_{v_2v_3} \dots [A_{i_k}]_{v_kv_1} < [A_{i_1}]_{v_1v_2} \dots [A_{i_{h-1}}]_{v_{h-1}v}[A_{i_{h+t}}]_{vv_{h+t+1}} \dots [A_{i_k}]_{v_kv_1}$ and, by iteratively proceeding in this way, we end up with an inequality of the following type:

$$[A_{i_1}]_{v_1v_2}[A_{i_2}]_{v_2v_3} \dots [A_{i_k}]_{v_kv_1} < [\tilde{w}(A_1, A_2, \dots, A_p)]_{v_1v_1},$$

where $\tilde{w} \in \Xi^*$ satisfies $|\tilde{w}| \leq n$. By the first part of the proof, $[\tilde{w}(A_1, A_2, \dots, A_p)]_{v_1v_1} < 1$, thus proving the result.

(iii) $\Rightarrow$ (iv) As all columns (but not necessarily all rows) of A_π are monomial, for every vertex $v \in [1, n]$ in $\mathcal{D}(A_\pi)$ there is one and only one outgoing arc. Therefore, in $\mathcal{D}(A_\pi)$

- there exists at least one elementary cycle;
- two distinct elementary cycles are disjoint;
- if a vertex v does not belong to an elementary cycle, there is a unique path from v to a unique elementary cycle.

Upon relabeling the vertices of $\mathcal{D}(A_\pi)$ one gets

$$P^\top A_\pi P = \left[\begin{array}{cccc|c} Z_1 & & & & \\ & Z_2 & & & \\ & & \ddots & & A_{12} \\ & & & Z_s & \\ \hline & & 0 & & A_{22} \end{array}\right], \tag{4}$$

where P is a permutation matrix, $Z_1, Z_2, \dots, Z_s$ are cyclic monomial matrices and A_{22} is nilpotent. So there exists $k \in \mathbb{N}$ such that

$$(P^\top A_\pi P)^k = \left[\begin{array}{c|c} D & * \\ \hline 0 & 0 \end{array}\right]$$

where D is a diagonal matrix with positive diagonal entries, and $*$ is nonnegative. To prove that A_π is Schur it is enough to show that the diagonal entries of D are smaller than 1. As the entries of the cyclic monomial blocks Z_h in (4) are entries of the matrices A_i's, each diagonal entry of D can be expressed as $[A_{i_1}]_{v_1v_2}[A_{i_2}]_{v_2v_3} \dots [A_{i_k}]_{v_kv_1}$ for a suitable choice of the indices $i_1, i_2, \dots, i_k \in [1, p]$ and $v_1, v_2, \dots, v_k \in [1, n]$, and assumption (iii) ensures that all diagonal entries of D are smaller than 1.

(iv) $\Rightarrow$ (v) A_π is positive Schur if and only if $A_\pi - I_n$ is Metzler Hurwitz. As it has been shown in [10], [20], if $A_\pi - I_n$ is a (Metzler) Hurwitz matrix for every choice of π, then there exists $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top(A_i - I_n) \ll 0, \forall\ i \in [1,p]$, and hence condition (v) holds.

(v) $\Rightarrow$ (i) If the matrices A_i have a common linear copositive Lyapunov function, the switched system (1) is asymptotically stable [27].

(v) $\Rightarrow$ (vi) $\Rightarrow$ (i) Assume that the matrices A_i's admit a common linear copositive Lyapunov function, associated with the vector $\mathbf{v} \gg 0$, and hence (by the previous part of the proof) system (1) is asymptotically stable. This, in turn, implies that the switched system

$$\mathbf{x}(t+1) = A^\top_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{5}$$

with $A^\top_{\sigma(t)} \in \{A^\top_1, A^\top_2, \dots, A^\top_p\}$, is asymptotically stable. As the matrices $A^\top_i$ are monomial, the asymptotic stability of (5) implies (again by (v) $\Leftrightarrow$ (i)) that there exists a vector $\mathbf{y} \gg 0$ such that $\mathbf{y}^\top A^\top_i \ll \mathbf{y}^\top, \forall\ i \in [1,p]$. As proved in [2], the diagonal matrix

$$\Delta = \mathrm{diag}\left\{\frac{y_1}{v_1}, \frac{y_2}{v_2}, \dots, \frac{y_n}{v_n}\right\}$$

satisfies condition (vi). As the matrices A_i have a common diagonal Lyapunov function, the switched system (1) is asymptotically stable.

(v) $\Leftrightarrow$ (vii) If $\mathbf{v} \gg 0$ satisfies condition (v), then we have $\mathbf{v}^\top A_i\mathbf{x} < \mathbf{v}^\top\mathbf{x}, \forall i \in [1,p],\ \forall \mathbf{x} > \mathbf{0}$, which in turn implies $\mathbf{x}^\top A_i^\top \mathbf{v}\mathbf{v}^\top A_i\mathbf{x} = |\mathbf{v}^\top A_i\mathbf{x}|^2 < |\mathbf{v}^\top\mathbf{x}|^2 = \mathbf{x}^\top\mathbf{v}\mathbf{v}^\top\mathbf{x},\ \forall i \in [1,p],\ \forall \mathbf{x} > \mathbf{0}$. So, (vii) is satisfied for $P := \mathbf{v}\mathbf{v}^\top$. Viceversa, il $P = P^\top$ satisfies (vii) and has rank 1, it can be expressed as $P = \mathbf{v}\mathbf{v}^\top$, for some vector $\mathbf{v}$. Moreover, $\mathbf{x}^\top P\mathbf{x} > 0,\ \forall \mathbf{x} > 0$, implies that all entries of $\mathbf{v}$ are nonzero and of the same sign. So, it entails no loss of generality assuming $\mathbf{v} \gg \mathbf{0}$. We therefore have $\mathbf{x}^\top[A_i^\top P A_i - P]\mathbf{x} = |\mathbf{v}^\top A_i\mathbf{x}|^2 - |\mathbf{v}^\top\mathbf{x}|^2 < 0, \quad \forall i \in [1,p],\ \forall \mathbf{x} > \mathbf{0}$ and by the nonnegativity of both $\mathbf{v}^\top A_i\mathbf{x}$ and $\mathbf{v}^\top\mathbf{x}$, we have also $\mathbf{v}^\top A_i\mathbf{x} < \mathbf{v}^\top\mathbf{x}$. This proves that condition (v) holds. ∎

Remark 1: Condition (ii) is equivalent to the "finiteness property", however, from a computational viewpoint, condition (v) is the easiest one to check to ascertain the asymptotic stability of the switched system (1), with monomial matrices.

Note that the equivalences (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vii) hold for any $p-$tuple of nonnegative matrices $A_i,\ i \in [1,p]$, irrespective of their nonzero patterns (see [10] for the continuous-time case).

It is immediate that each convex combination of the A_is is Schur if anyone of (iv), (v) and (vii) holds, irrespective of their nonzero patterns. Even more, as a consequence of the joint spectral radius theorem, this holds for every asymptotically stable switched system. Notice, however, that all convex combinations $\alpha A_1 + (1-\alpha)A_2$, $\alpha \in [0,1]$, of the pair of monomial matrices

$$A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

are Schur, but $A_1 A_2$ is not Schur. Consequently, the asymptotic stability of all convex combinations of the matrices A_i does not guarantee the asymptotic stability of a DPSS (1), even in the very particular case of (cyclic) monomial matrices.

The results of Proposition 1, stated under the assumption that we are dealing with monomial matrices, namely that the diagonal entries of D_i in (2) are positive for every $i \in [1,p]$, immediately extend to the case when some of these diagonal entries are zero.

III. Stabilizability of DPSS with monomial matrices having the same nonzero pattern

Definition 3: The DPSS (1) is *stabilizable* if for every positive initial state $\mathbf{x}(0)$ there exists a switching sequence (possibly depending on $\mathbf{x}(0)$) σ such that the state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, asymptotically converges to zero.

As shown in [11], if a DPSS (1) is stabilizable, then it can be stabilized by means of a periodic switching sequence. This property does not require that all periodic switching sequences, with a common bound on the period length, are stabilizing, but just asserts that a single periodic switching sequence (without any a priori bound on the period length) converges to zero or, equivalently, that some word $w \in \Xi^*$ can be found, such that $w(A_1, A_2, \ldots, A_p)$ is a Schur matrix. So, to characterize stabilizability, we will resort to this result.

Clearly, if one of the matrices, say A_ℓ, is Schur, then the system is stabilizable by means of the constant switching sequence $\sigma(t) = \ell$. For the class of DPSS's (1) that switch among cyclic monomial matrices this is the only case when stabilizability is possible.

Proposition 2: A DPSS (1), with cyclic monomial matrices $A_i \in \mathbb{R}_+^{n\times n},\ i \in [1,p]$, is stabilizable if and only if at least one of the matrices $A_i, i \in [1,p]$, is Schur.

Proof: According to the previous comments, if (1) is stabilizable there exists $w \in \Xi^*$ such that $w(A_1, A_2, \ldots, A_p)$ is a Schur matrix. As $w(A_1, A_2, \ldots, A_p)^n$ is a Schur diagonal matrix, its diagonal entries are smaller than one, and so is the product of its diagonal entries

$$\det w(A_1, A_2, \ldots, A_p)^n = \prod_{i=1}^{p} (a_{12}^{(i)} a_{23}^{(i)} \ldots a_{n1}^{(i)})^{n \cdot |w|_i} < 1.$$

This implies $a_{12}^{(i)} a_{23}^{(i)} \ldots a_{n1}^{(i)} < 1$ for at least one index $i \in [1, p]$, thus proving that one of the A_i's is Schur. The converse is obvious. ■

Remark 2: The stabilizability criterion of Proposition 2 trivially extends to the case when some of the entries $a_{j,j+1}^{(i)}, j \in [1, n-1]$, and $a_{n1}^{(i)}$ are zero. In fact, zeroing anyone of such entries in a cyclic monomial matrix A_i produces a nilpotent matrix.

When one of the matrices A_i's is Schur, a natural way to ensure asymptotic convergence of a state trajectory is by steadily remaining set on the asymptotically stable subsystem, which amounts to choosing a constant stabilizing switching sequence. It may be of interest, however, to know which periodic sequences ensure stabilizability or, equivalently, which matrix products are Hurwitz matrices.

Proposition 3: A DPSS (1), with cyclic monomial matrices $A_i \in \mathbb{R}_+^{n \times n}, i \in [1, p]$, is stabilizable if and only if there exist $k_1, k_2, \ldots, k_p \in \mathbb{Z}_+$ such that the following condition holds:

$$\begin{bmatrix} k_1 & k_2 & \ldots & k_p \end{bmatrix} \begin{bmatrix} \log \rho(A_1) \\ \log \rho(A_2) \\ \vdots \\ \log \rho(A_p) \end{bmatrix} < 0. \tag{6}$$

If this is the case, a word $w \in \Xi^*$ corresponds to a Schur matrix product $w(A_1, A_2, \ldots, A_p)$ if and only if (6) is satisfied for $k_i := |w|_i$.

Proof: Notice, first, that $a_{12}^{(i)} a_{23}^{(i)} \ldots a_{n1}^{(i)} = \rho(A_i)^n = \rho(A_i^n), \forall\ i \in [1, p]$. As in the previous proof, if $w(A_1, A_2, \ldots, A_p)$ is Schur, then also $w(A_1, A_2, \ldots, A_p)^n$ is Schur, thus implying

$$\prod_{i=1}^{p} (a_{12}^{(i)} a_{23}^{(i)} \ldots a_{n1}^{(i)})^{n \cdot |w|_i} = \prod_{i=1}^{p} \rho(A_i)^{n^2 \cdot |w|_i} < 1.$$

By applying the logarithm, we get $n^2 \cdot \sum_{i=1}^{p} |w|_i \cdot \log \rho(A_i) < 0$, that immediately proves the necessity (for $k_i := |w|_i$), as well as the final statement.

Viceversa, if (6) holds, then

$$B := A_1^{nk_1} A_2^{nk_2} \cdots A_p^{nk_p} = \left(\rho(A_1)^{nk_1} I_n\right) \left(\rho(A_2)^{nk_2} I_n\right) \cdots \left(\rho(A_p)^{nk_p} I_n\right)$$

is a scalar diagonal matrix, and the logarithm of (any of) its diagonal entries satisfies

$$\log\left[\rho(A_1)^{nk_1} \rho(A_2)^{nk_2} \cdots \rho(A_p)^{nk_p}\right] = n \sum_{i=1}^{p} k_i \log \rho(A_i) < 0.$$

Therefore B is a Schur matrix. ■

We now focus on the stabilizability of DPSS's (1) whose matrices $A_i, i \in [1,p]$, are monomial matrices with the same nonzero patterns, i.e. $A_i = D_i P$, where $D_i,\ i \in [1,p]$, are diagonal matrices with positive diagonal entries and P is a common $n \times n$ permutation matrix. So, all digraphs $\mathcal{D}(A_i), i \in [1,p]$, have the same structure, consisting of r disjoint elementary cycles, but have different weights for the various arcs. We assume w.l.o.g. that each A_i is expressed as

$$A_i = \begin{bmatrix} Z_1^{(i)} & & & \\ & Z_2^{(i)} & & \\ & & \ddots & \\ & & & Z_r^{(i)} \end{bmatrix} = D_i \begin{bmatrix} \Pi_1 & & & \\ & \Pi_2 & & \\ & & \ddots & \\ & & & \Pi_r \end{bmatrix}, \ D_i = \mathrm{diag}\{d_1^{(i)}, \dots, d_n^{(i)}\} \qquad (7)$$

where $Z_h^{(i)}$ and Π_h are cyclic monomial and cyclic permutation matrices, respectively, corresponding to some elementary cycle γ_h, and r is the number of distinct cycles in $\mathcal{D}(A_1) = \mathcal{D}(A_2) = \ldots = \mathcal{D}(A_p)$.

Obviously, A_i is a Schur matrix if and only if all $Z_h^{(i)}$'s are, which amounts to requiring that $\prod_{v\in\gamma_h} d_v^{(i)} = \rho\left(Z_h^{(i)}\right)^{|\gamma_h|} < 1$, for each elementary cycle $\gamma_h \in \mathcal{D}(A_i)$. Any matrix product $A_{i_1} A_{i_2} \ldots A_{i_k},\ i_1, i_2, \ldots, i_k \in [1,p]$, is still monomial with block diagonal structure:

$$A_{i_1} A_{i_2} \ldots A_{i_k} = \begin{bmatrix} Z_1^{(i_1)} Z_1^{(i_2)} \ldots Z_1^{(i_k)} & & & \\ & Z_2^{(i_1)} Z_2^{(i_2)} \ldots Z_2^{(i_k)} & & \\ & & \ddots & \\ & & & Z_r^{(i_1)} Z_r^{(i_2)} \ldots Z_r^{(i_k)} \end{bmatrix}.$$

(but its diagonal blocks are, in general, not cyclic). As a consequence, it will be Schur if and only if the blocks $Z_h^{(i_1)} Z_h^{(i_2)} \ldots Z_h^{(i_k)}$ are Schur for every $h \in [1,r]$.

Proposition 4: Given a DPSS (1), with monomial matrices $A_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, having the same nonzero pattern, and hence described as in (7), the following facts are equivalent:

(i) the system is stabilizable;

(ii) there exist nonnegative integers $k_1, k_2, \dots, k_p$ such that

$$\left(\prod_{v\in\gamma_h} d_v^{(1)}\right)^{k_1} \left(\prod_{v\in\gamma_h} d_v^{(2)}\right)^{k_2} \dots \left(\prod_{v\in\gamma_h} d_v^{(p)}\right)^{k_p} < 1, \quad \forall\ h \in [1,r]; \tag{8}$$

(iii) there exist nonnegative integers $\bar{k}_1, \bar{k}_2, \dots, \bar{k}_p$ such that $A_1^{\bar{k}_1} A_2^{\bar{k}_2} \dots A_p^{\bar{k}_p}$ is Schur.

Proof: (i) $\Rightarrow$ (ii) Assume that $\tilde{w}(A_1, A_2, \dots, A_p),\ \tilde{w} \in \Xi^*$, is a Schur matrix, and let w be a power of $\tilde{w}$, such that $w(A_1, A_2, \dots, A_p)$ is a diagonal matrix. As $w(Z_h^{(1)}, Z_h^{(2)}, \dots, Z_h^{(p)})$, is a diagonal Schur matrix for every $h \in [1,r]$, all its diagonal entries are smaller than 1, i.e.

$$\Big(\prod_{v\in\gamma_h} d_v^{(1)}\Big)^{|w|_1} \Big(\prod_{v\in\gamma_h} d_v^{(2)}\Big)^{|w|_2} \dots \Big(\prod_{v\in\gamma_h} d_v^{(p)}\Big)^{|w|_p} < 1, \qquad \forall\ h \in [1,r].$$

So, condition (ii) holds for $k_i = |w|_i$.

(ii) $\Rightarrow$ (iii) Let $m = \text{l.c.m.}\{|\gamma_h| : h \in [1,r]\}$, and set $m_h := \frac{m}{|\gamma_h|}$. By assumption (ii),

$$\Big(\prod_{v\in\gamma_h} d_v^{(1)}\Big)^{m_h k_1} \Big(\prod_{v\in\gamma_h} d_v^{(2)}\Big)^{m_h k_2} \dots \Big(\prod_{v\in\gamma_h} d_v^{(p)}\Big)^{m_h k_p} < 1$$

holds for every $h \in [1,r]$. We want to prove that, $B := A_1^{mk_1} A_2^{mk_2} \dots A_p^{mk_p}$ is Schur. Indeed, as $A_i^{mk_i},\ i \in [1,p]$, are diagonal matrices, B is diagonal too and it can be expressed as

$$\begin{bmatrix} (Z_1^{(1)})^{m_1|\gamma_1|k_1} \dots (Z_1^{(p)})^{m_1|\gamma_1|k_p} & & & \\ & (Z_2^{(1)})^{m_2|\gamma_2|k_1} \dots (Z_2^{(p)})^{m_2|\gamma_2|k_p} & & \\ & & \ddots & \\ & & & (Z_r^{(1)})^{m_r|\gamma_r|k_1} \dots (Z_1^{(p)})^{m_r|\gamma_r|k_p} \end{bmatrix}.$$

For every $h \in [1,r]$ and $i \in [1,p]$, we have

$$(Z_h^{(i)})^{m_h|\gamma_h|k_i} = (\det Z_h^{(i)})^{m_h k_i} I_{|\gamma_h|} = \Big(\prod_{v\in\gamma_h} d_v^{(i)}\Big)^{m_h k_i} I_{|\gamma_h|}$$

and therefore the hth diagonal block of B is the scalar diagonal matrix

$$(Z_h^{(1)})^{m_h|\gamma_h|k_1} \dots (Z_h^{(p)})^{m_h|\gamma_h|k_p} = \Big(\prod_{v\in\gamma_h} d_v^{(1)}\Big)^{m_h k_1} \Big(\prod_{v\in\gamma_h} d_v^{(2)}\Big)^{m_h k_2} \dots \Big(\prod_{v\in\gamma_h} d_v^{(p)}\Big)^{m_h k_p} I_{|\gamma_h|},$$

whose entries are smaller than 1 by the assumption (ii).

(iii) $\Rightarrow$ (i) Obvious. ■

Upon assuming that all cycles γ_h have unit length, the results of Proposition (4) particularize to any DPSS (1) that switches among diagonal matrices $A_i = D_i = \text{diag}\{d_1^{(i)}, d_2^{(i)}, \dots, d_n^{(i)}\}$. Further characterizations can be derived, based on the $n \times p$ matrix

$$W := \begin{bmatrix} \log d_1^{(1)} & \log d_1^{(2)} & \dots & \log d_1^{(p)} \\ \log d_2^{(1)} & \log d_2^{(2)} & \dots & \log d_2^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \log d_n^{(1)} & \log d_n^{(2)} & \dots & \log d_n^{(p)} \end{bmatrix}. \tag{9}$$

Finally, it will be shown that, for this class of systems, stabilizability is equivalent to the existence of a Schur matrix product involving at most n factors $A_i^{k_i}$. Even if this does not constrain the length of the matrix product, namely how many times each matrix A_i appears in the product, however this is interesting from a system viewpoint, as it tells us that n is the maximum number of subsystems we have to switch among.

Corollary 1: Given a DPSS (1), with diagonal matrices $A_i \in \mathbb{R}_+^{n\times n}, i \in [1, p]$, the following facts are equivalent:

(i) the system is stabilizable;

(ii) there exist $k_1, k_2, \dots, k_p \in \mathbb{Z}_+$ such that $(d_h^{(1)})^{k_1}(d_h^{(2)})^{k_2}\dots(d_h^{(p)})^{k_p} < 1, \forall\ h \in [1, n]$;

(iii) there exist $k_1, k_2, \dots, k_p \in \mathbb{Z}_+$ such that $A_1^{k_1}A_2^{k_2}\dots A_p^{k_p}$ is Schur;

(iv) there exist a nonzero vector $\mathbf{k} \in \mathbb{Z}_+^p$ such that $W\mathbf{k} \ll 0$;

(v) the convex hull of the rows of W does not intersect the positive orthant of $\mathbb{R}^p$;

(vi) there exist $s \leq n$ indices $j_1, j_2, \dots, j_s \in [1, p]$, and $\bar{k}_1, \bar{k}_2, \dots, \bar{k}_s \in \mathbb{Z}_+$, such that $A_{j_1}^{\bar{k}_1}A_{j_2}^{\bar{k}_2}\cdots A_{j_s}^{\bar{k}_s}$ is Schur.

Proof: The equivalence of conditions (i) $\div$ (iii) follows from Proposition 4. (ii) $\Rightarrow$ (iv) can be proved along the same lines as in the proof of Proposition 3. (vi) $\Rightarrow$ (i) is obvious. So, to conclude the proof, we show that (iv) and (v) are equivalent, and that (iv) implies (vi).

(iv) $\Leftrightarrow$ (v) Note that one and only one of the following alternatives holds ([1], Corollary 3.49):

$$\text{either} \quad \exists\ \mathbf{v} > \mathbf{0} \text{ such that } W\mathbf{v} \ll \mathbf{0}, \tag{10}$$

$$\text{or} \quad \exists\ \mathbf{y} > \mathbf{0} \text{ such that } \mathbf{y}^\top W \geq \mathbf{0}^\top, \tag{11}$$

and in (11) the vector $\mathbf{y}$ can be assumed w.l.o.g. stochastic (i.e., $\sum_{i=1}^n [\mathbf{y}]_i = 1$). If (iv), and hence (10), hold true, (11) cannot be verified, and consequently no convex combination of the rows of

W intersects the positive orthant of $\mathbb{R}^p$. Viceversa, if (v) holds, (10) admits a nonzero solution $\mathbf{v} \in \mathbb{R}^p_+$, hence a nonzero solution $\mathbf{r} \in \mathbb{Q}^p_+$ and, consequently, a nonzero solution $\mathbf{k} \in \mathbb{Z}^p_+$.

(iv) $\Rightarrow$ (vi) Set

$$\mathbf{w}^{(i)} := \begin{bmatrix} \log d_1^{(i)} \\ \vdots \\ \log d_n^{(i)} \end{bmatrix} = \mathrm{col}_i(W), \qquad i \in [1,p],$$

and assume that (iv) holds. This implies that there is a point $\mathbf{x}$, interior to the negative orthant and belonging to the cone generated by the vectors $\mathbf{w}^{(i)}, i \in [1,p]$. By Caratheodory's theorem [8], $\mathbf{x}$ belongs to the cone generated by some independent subset of the columns of W. So, there exist $s \le n$, indices $j_1,\ j_2, \dots, j_s \in [1,p]$, and $\beta_h \ge 0$ such that

$$\mathbf{x} := \sum_{h=1}^{s} \mathbf{w}^{(j_h)} \beta_h \ll \mathbf{0}$$

and, by continuity, an interior point of the negative orthant can be obtained also by combining the $\mathbf{w}^{(j_h)}$'s with suitable nonnegative rational numbers, and hence with nonnegative integers $\bar{k}_1, \bar{k}_2, \dots, \bar{k}_s$, i.e.

$$\left[\, \mathbf{w}^{(j_1)} \quad \mathbf{w}^{(j_2)} \quad \dots \quad \mathbf{w}^{(j_s)} \,\right] \begin{bmatrix} \bar{k}_1 \\ \bar{k}_2 \\ \vdots \\ \bar{k}_s \end{bmatrix} = \begin{bmatrix} \log d_1^{(j_1)} & \log d_1^{(j_2)} & \dots & \log d_1^{(j_s)} \\ \log d_2^{(j_1)} & \log d_2^{(j_2)} & \dots & \log d_2^{(j_s)} \\ \vdots & \vdots & \ddots & \vdots \\ \log d_n^{(j_1)} & \log d_n^{(j_2)} & \dots & \log d_n^{(j_s)} \end{bmatrix} \begin{bmatrix} \bar{k}_1 \\ \bar{k}_2 \\ \vdots \\ \bar{k}_s \end{bmatrix} \ll \mathbf{0},$$

Therefore the diagonal matrix $A_{j_1}^{\bar{k}_1} A_{j_2}^{\bar{k}_2} \dots A_{j_s}^{\bar{k}_s}$, having $(d_h^{(1)})^{\bar{k}_1} (d_h^{(2)})^{\bar{k}_2} \dots (d_h^{(p)})^{\bar{k}_p} < 1$ as hth diagonal entry, $h \in [1,n]$, is Schur. ∎

Corollary 1 allows to complete the stabilizability characterization provided in Proposition 4. Indeed, also for the class of positive switched systems (1), switching among monomial matrices having the same structure, stabilizability turns out to be equivalent to the existence a Schur matrix product that involves no more than n distinct matrices A_i's.

Proposition 5: A DPSS (1) with monomial matrices $A_i \in \mathbb{R}^{n\times n}_+$ given in (7), and hence having the same nonzero pattern, is stabilizable if and only if there exist $s \le r$, indices $j_1, j_2, \dots, j_s \in [1,p]$ and $h_1, h_2, \dots, h_s \in \mathbb{Z}_+$, such that $A_{j_1}^{h_1} A_{j_2}^{h_2} \dots A_{j_s}^{h_s}$ is Schur.

Proof: If $A_{j_1}^{h_1} A_{j_2}^{h_2} \dots A_{j_s}^{h_s}$ is Schur, clearly the DPSS is stabilizable. Viceversa, if the DPSS with matrices given in (7) is stabilizable, by Proposition 4 there exist $k_1, k_2, \dots, k_p \in \mathbb{Z}_+$ such that

$$\left(\prod_{v\in\gamma_h} d_v^{(1)}\right)^{k_1}\left(\prod_{v\in\gamma_h} d_v^{(2)}\right)^{k_2}\dots\left(\prod_{v\in\gamma_h} d_v^{(p)}\right)^{k_p} < 1, \qquad \forall\ h\in[1,r].$$

Hence the r-dimensional DPSS associated with the diagonal matrices

$$\tilde{A}_i = \begin{bmatrix} \prod_{v\in\gamma_1} d_v^{(i)} & & & \\ & \prod_{v\in\gamma_2} d_v^{(i)} & & \\ & & \ddots & \\ & & & \prod_{v\in\gamma_r} d_v^{(i)} \end{bmatrix} \in \mathbb{R}_+^{r\times r}, \quad i\in[1,p],$$

is stabilizable, since $\tilde{A}_1^{k_1}\tilde{A}_2^{k_2}\dots\tilde{A}_p^{k_p}$ is Schur. By Corollary 1, there exist $s\le r$ indices $j_1,j_2,\dots,j_s$ and nonnegative integers $\bar{k}_1,\bar{k}_2,\dots,\bar{k}_s$ such that $\tilde{A}_{j_1}^{\bar{k}_1}\tilde{A}_{j_2}^{\bar{k}_2}\dots\tilde{A}_{j_s}^{\bar{k}_s}$ is Schur. This implies that

$$\left(\prod_{v\in\gamma_h} d_v^{(j_1)}\right)^{\bar{k}_1}\left(\prod_{v\in\gamma_h} d_v^{(j_2)}\right)^{\bar{k}_2}\dots\left(\prod_{v\in\gamma_h} d_v^{(j_s)}\right)^{\bar{k}_s} < 1, \qquad \forall\ h\in[1,r].$$

So, by Proposition 4, we can claim that there are $s\le r$ indices $j_1,j_2,\dots,j_s\in[1,p]$ and nonnegative integers $h_1,h_2,\dots,h_s$ such that $A_{j_1}^{h_1}A_{j_2}^{h_2}\dots A_{j_s}^{h_s}$ is Schur. ∎

IV. Further results on the stabilizability of DPSS with monomial matrices

Proposition 5 shows that when dealing with n-dimensional DPSS's (1), switching among monomial matrices with the same nonzero pattern, stabilizability can always be achieved by switching among at most n subsystems. In this section we prove that, when $n=2$ or $n=3$, this result is true also for DPSS (1) switching among monomial matrices A_i with distinct patterns.

To address the two-dimensional case, we first observe that all matrices A_i take either one of the following two forms (diagonal and antidiagonal, respectively):

$$A_i = \begin{bmatrix} d_1^{(i)} & 0 \\ 0 & d_2^{(i)} \end{bmatrix} \qquad \text{or} \qquad A_i = \begin{bmatrix} 0 & d_1^{(i)} \\ d_2^{(i)} & 0 \end{bmatrix}. \tag{12}$$

Proposition 6: A DPSS (1), with monomial matrices $A_i\in\mathbb{R}_+^{2\times 2}, i\in[1,p]$, is stabilizable if and only if there exist $j_1,j_2\in[1,p]$ and $k_1,k_2\in\mathbb{Z}_+$ such that $A_{j_1}^{k_1}A_{j_2}^{k_2}$ is Schur.

Proof: Clearly, if $A_{j_1}^{k_1}A_{j_2}^{k_2}$ is Schur, the DPSS (1) is stabilizable. Viceversa, assume that there exists $w=\xi_{i_1}\xi_{i_2}\dots\xi_{i_k}\in\Xi^*$ such that $w(A_1,A_2,\dots,A_p)$ is a Schur matrix. As $w(A_1,A_2,\dots,A_p)$ is either a diagonal or an antidiagonal matrix, in both cases the Schur property ensures that

$$1>|\det w(A_1,A_2,\dots,A_p)| = |\det A_{i_1}||\det A_{i_2}|\cdots|\det A_{i_k}| = |\det A_1|^{k_1}|\det A_2|^{k_2}\cdots|\det A_p|^{k_p}$$

where $k_i := |w|_i$. So, for at least one index $j_1 \in [1,p]$, $|\det A_{j_1}| < 1$. If A_{j_1} is antidiagonal, then $\det(zI_2 - A_{j_1}) = z^2 - d_1^{(j_1)} d_2^{(j_1)}$, with $d_1^{(j_1)} d_2^{(j_2)} < 1$. Thus A_{j_1} is Schur and the result holds for $k_1 = 1$, j_2 arbitrary and $k_2 = 0$. If A_{j_1} is diagonal and there exists at least one matrix A_{j_2} which is antidiagonal, for $k_1 \in \mathbb{Z}_+$ large enough $(\det A_{j_1})^{k_1} \det A_{j_2} < 1$ and $A_{j_1}^{k_1} A_{j_2}$ is a Schur antidiagonal matrix. If all matrices are diagonal, the result follows from Corollary 1. ■

Proposition 7: A DPSS (1), with monomial matrices $A_i \in \mathbb{R}_+^{3\times 3}, i \in [1,p]$, is stabilizable if and only if there exist indices $j_1, j_2, j_3 \in [1,p]$ and $k_1, k_2, k_3 \in \mathbb{Z}_+$ such that $A_{j_1}^{k_1} A_{j_2}^{k_2} A_{j_3}^{k_3}$ is Schur.

Proof: Obviously, if $A_{j_1}^{k_1} A_{j_2}^{k_2} A_{j_3}^{k_3}$ is Schur, the DPSS is stabilizable. Viceversa, suppose there exists $w = \xi_{i_1}\xi_{i_2}\dots\xi_{i_k} \in \Xi^*$ such that $w(A_1, A_2, \dots, A_p)$ is a Schur matrix. By the same reasoning we previously adopted, for at least one matrix, say A_{j_1}, we have $|\det A_{j_1}| < 1$. It entails no loss of generality assuming that $|w|_i > 0$ for every $i \in [1,p]$. If not, we can simply reduce the number p of the subsystems and hence discard the corresponding monomial matrices. If the matrices $A_i, i \in [1,p]$, have all the same nonzero patterns, namely they are all (cogredient to, by means to the same permutation matrix P) either diagonal matrices, or cyclic matrices, or matrices having the structure

$$\begin{bmatrix} 0 & a_i & 0 \\ b_i & 0 & 0 \\ 0 & 0 & c_i \end{bmatrix}$$

for suitable positive a_i, b_i and c_i, then by making use of Proposition 5 we can obtain the result. So, we suppose, now, that, in order to obtain such a Schur matrix product $w(A_1, A_2, \dots, A_p)$, we need matrices with at least two distinct nonzero patterns. We distinguish the following cases:

(a) There exists $j_2 \in [1,p]$ (possibly coinciding with the index j_1 previously mentioned) such that A_{j_2} is (cogredient to) a cyclic monomial matrix. Let $k_1 \in \mathbb{Z}_+$ be such that $A_{j_1}^{k_1}$ is a diagonal matrix and $|\det A_{j_1}|^{k_1} \cdot |\det A_{j_2}| < 1$. Then $A_{j_1}^{k_1} A_{j_2}$ is (cogredient to) a cyclic matrix whose determinant has modulus smaller than 1, and hence it is Schur.

(b) There is no cyclic matrix, but there exist two indices $j_2, j_3 \in [1,p]$ (possibly coinciding with the index j_1 previously mentioned) such that A_{j_2} and A_{j_3} can be reduced, by means of the same permutation matrix P, to the forms

$$P^\top A_{j_2} P = \begin{bmatrix} 0 & a_{j_2} & 0 \\ b_{j_2} & 0 & 0 \\ 0 & 0 & c_{j_2} \end{bmatrix} \qquad P^\top A_{j_3} P = \begin{bmatrix} 0 & 0 & c_{j_3} \\ 0 & b_{j_3} & 0 \\ a_{j_3} & 0 & 0 \end{bmatrix}.$$

It entails no loss of generality assuming that $P = I_3$, namely that this permutation matrix has been applied to all matrices $A_i, i \in [1, p]$. But then

$$A_{j_2}A_{j_3} = \begin{bmatrix} 0 & a_{j_2}b_{j_3} & 0 \\ 0 & 0 & b_{j_2}c_{j_3} \\ c_{j_2}a_{j_3} & 0 & 0 \end{bmatrix}$$

is a cyclic matrix. So, by the same reasoning we applied in case (a), we can claim that there exists $k_1 \in \mathbb{Z}_+$ such that $A_{j_1}^{k_1}A_{j_2}A_{j_3}$ is a cyclic matrix with determinant of modulus smaller than 1, and hence Schur.

(c) If neither of the previous two cases applies, then all matrices involved in $w(A_1, A_2, \dots, A_p)$ are either diagonal or (upon a suitable common cogredience transformation) take the form

$$\begin{bmatrix} 0 & a_i & 0 \\ b_i & 0 & 0 \\ 0 & 0 & c_i \end{bmatrix}.$$

The matrices of these two types can both be described as

$$A_i = \begin{bmatrix} \Delta_i & 0 \\ 0 & c_i \end{bmatrix}, \qquad i \in [1, p],$$

where Δ_i takes one of the two alternative forms described in (12). So,

$$w(A_1, A_2, \dots, A_p) = \begin{bmatrix} w(\Delta_1, \Delta_2, \dots, \Delta_p) & 0 \\ 0 & w(c_1, c_2, \dots, c_p) \end{bmatrix}$$

is Schur if and only if $w(\Delta_1, \Delta_2, \dots, \Delta_p)$ is Schur and $w(c_1, c_2, \dots, c_p)$ is a (positive) scalar number smaller than 1. By recalling that $w = \xi_{i_1}\xi_{i_2}\dots\xi_{i_k}$, this implies that

$$|\det \Delta_{i_1}||\det \Delta_{i_2}|\dots|\det \Delta_{i_k}| < 1, \qquad c_{i_1}c_{i_2}\dots c_{i_k} < 1,$$

namely that there exist indices $h_1, h_2, \dots, h_p \in \mathbb{Z}_+$ (with $h_i := |w|_i$) such that

$$|\det \Delta_1|^{h_1}|\det \Delta_2|^{h_2}\dots|\det \Delta_p|^{h_p} < 1, \qquad c_1^{h_1}c_2^{h_2}\dots c_p^{h_p} < 1.$$

So, by applying the same reasoning we resorted to within the proof of Corollary 1, we can claim that there exist indices $j_1, j_2 \in [1, p]$ and nonnegative integers $k_1, k_2 \in \mathbb{Z}_+$, such that

$$|\det \Delta_{j_1}|^{k_1}|\det \Delta_{j_2}|^{k_2} < 1, \qquad c_{j_1}^{k_1}c_{j_2}^{k_2} < 1.$$

If at least one of the matrices Δ_{j_1} and Δ_{j_2} is antidiagonal, say Δ_{j_1}, we can slightly perturb k_1 and k_2 so that k_1 is odd ($\Delta_{j_1}^{k_1}$ is antidiagonal) and k_2 is even ($\Delta_{j_2}^{k_2}$ is diagonal). If so,

$A_{j_1}^{k_1} A_{j_2}^{k_2}$ will be a Schur matrix. If both Δ_{j_1} and Δ_{j_2} are diagonal (i.e., A_{j_1} and A_{j_2} are both diagonal), then, by the assumptions in case (c), there exists $j_3 \in [1,p]$ such that Δ_{j_3} is antidiagonal. So, if $m \in \mathbb{Z}_+$ is large enough to ensure that

$$(|\det \Delta_{j_1}|^{k_1} |\det \Delta_{j_2}|^{k_2})^m \cdot |\det \Delta_{j_3}| < 1, \qquad\qquad (c_{j_1}^{k_1} c_{j_2}^{k_2})^m \cdot c_{j_3} < 1,$$

then $A_{j_1}^{k_1 m} A_{j_2}^{k_2 m} A_{j_3}$ will be a Schur matrix.

∎

V. Asymptotic stability and stabilizability: the case of positive circulant matrices

In this section we focus on DPSS's described by either left or right positive circulant matrices. A right circulant matrix, simply known as a *circulant matrix*, is endowed with the following structure:

$$C = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ddots & a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \ddots & a_1 \\ a_1 & a_2 & a_3 & \dots & a_0 \end{bmatrix}. \tag{13}$$

Once we set

$$p(s) := a_0 + a_1 s + a_2 s^2 + \ldots + a_{n-1} s^{n-1},$$

the eigenvalues of the circulant matrix C_i are [7]

$$p(1),\ p(\varepsilon_n),\ p(\varepsilon_n^2),\ \ldots,\ p(\varepsilon_n^{n-1}),$$

where $\varepsilon_n = e^{j\frac{2\pi}{n}}$ is a primitive nth root of 1. If the circulant matrix is nonnegative, i.e. $a_\ell \geq 0, \forall\ \ell \in [0, n-1]$, the eigenvalue of maximal modulus is the first one, $p(1) = a_0 + a_1 + a_2 + \ldots + a_{n-1} = \rho(C_i)$. Hence C is Schur if and only if

$$\mathbf{1}_n^\top C = \mathbf{1}_n^\top \rho(C) \ll \mathbf{1}_n^\top, \tag{14}$$

i.e. the circulant matrix is strictly (column) sub-stochastic.

As the product of two positive circulant matrices, C_i and C_j, is positive circulant, we get

$$\rho(C_iC_j) = \sum_{k=1}^{n}[C_iC_j]_{k1} = \mathbf{1}_n^\top C_i \begin{bmatrix} [C_j]_{11} \\ [C_j]_{21} \\ \vdots \\ [C_j]_{n1} \end{bmatrix} = \mathbf{1}_n^\top \rho(C_i) \begin{bmatrix} [C_j]_{11} \\ [C_j]_{21} \\ \vdots \\ [C_j]_{n1} \end{bmatrix} = \rho(C_i) \cdot \rho(C_j).$$

A *left circulant matrix*

$$C^{(L)} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_{n-1} & a_0 \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ a_{n-2} & a_{n-1} & \iddots & a_{n-4} & a_{n-3} \\ a_{n-1} & a_0 & \dots & a_{n-3} & a_{n-2} \end{bmatrix}, \tag{15}$$

and a (right) circulant matrix C, having the same entries in the first row, can be simultaneously reduced [31], by a similarity transformation T which only depends on n, to the following forms:

$$T^{-1}CT = \begin{bmatrix} p(1) & & & & \\ & p(\varepsilon_n) & & & \\ & & p(\varepsilon_n^2) & & \\ & & & \ddots & \\ & & & & p(\varepsilon_n^{n-1}) \end{bmatrix}$$

$$T^{-1}C^{(L)}T = \begin{bmatrix} p(1) & & & & \\ & & & & p(\varepsilon_n) \\ & & & p(\varepsilon_n^2) & \\ & & \iddots & & \\ & p(\varepsilon_n^{n-1}) & & & \end{bmatrix}.$$

Consequently $\rho(C^{(L)}) = \rho(C) = p(1) = \sum_{k=0}^{n-1} a_k$. So, $C^{(L)}$ is Schur if and only if

$$\mathbf{1}_n^\top C^{(L)} = \mathbf{1}_n^\top \rho(C^{(L)}) \ll \mathbf{1}_n^\top, \tag{16}$$

or, equivalently, if and only if it is strictly (column) sub-stochastic. Given a family $\{A_1, A_2, \dots, A_p\}$ including both left and right circulant matrices, for every $w \in \Xi^*$, the matrix product $w(A_1, A_2, \dots, A_p)$ is a circulant matrix, left if the number of left circulant matrices involved in the matrix product is odd and right if it is even. In both cases,

$$\rho\left(w(A_1, A_2, \dots, A_p)\right) = \rho(A_1)^{|w|_1} \rho(A_2)^{|w|_2} \cdots \rho(A_p)^{|w|_p}. \tag{17}$$

As a consequence of the previous remarks, characterizing asymptotic stability and stabilizability of a DPSS with circulant matrices is rather easy.

Proposition 8: Given a DPSS (1), with positive (either left or right) circulant matrices $A_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, the following facts are equivalent:

(i) the system is asymptotically stable;

(ii) for every $w \in \Xi^*$, the matrix product $w(A_1, A_2, \dots, A_p)$ is a positive Schur matrix;

(iii) all matrices $A_1, A_2, \dots, A_p$ are Schur;

(iv) the matrices A_i's admit a common linear copositive function;

(v) for every choice of $\alpha_i \geq 0$, with $\sum_{i=1}^p \alpha_i = 1$, $\sum_{i=1}^p \alpha_i A_i$ is a positive (not necessarily circulant) Schur matrix;

Proof: Notice that if A_i is Schur, it satisfies either (14) or (16). Consequently (iii) $\Rightarrow$ (iv) is immediate, and (iii) $\Rightarrow$ (v) follows from $\mathbf{1}_n^\top \left(\sum_{i=1}^p \alpha_i A_i\right) \ll \mathbf{1}_n^\top \sum_{i=1}^p \alpha_i = \mathbf{1}_n^\top$. The remaining implications in the loops (i) $\Rightarrow$(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) and (iii) $\Leftrightarrow$ (v) are obvious. ■

Exactly like DPSS's with cyclic monomial matrices, stabilizability of DPSS's with circulant matrices requires the asymptotic stability of at least one subsystem.

Proposition 9: A DPSS (1) with positive (either left or right) circulant matrices $A_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, is stabilizable if and only if at least one of the matrices $A_i, i \in [1,p]$, is Schur.

Proof: If the DPSS is stabilizable, there exists $w \in \Xi^*$ such that $w(A_1, A_2, \dots, A_p)$ is a Schur matrix, and $1 > \rho\left(w(A_1, A_2, \dots, A_p)\right) = \rho(A_1)^{|w|_1} \cdot \rho(A_2)^{|w|_2} \cdots \rho(A_p)^{|w|_p}$ implies that $\rho(A_i) < 1$ for at least one index $i \in [1,p]$. The converse is obvious. ■

Finally, property (17) and the above discussion on DPSS with circulant matrices allow to refine the result of Proposition 9, thus providing a complete characterization of the composition of the matrix products $w(A_1, A_2, \dots, A_p)$ that are Schur.

Corollary 2: If a DPSS (1) with (either left or right) circulant matrices $A_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, is stabilizable, then a matrix product $w(A_1, A_2, \dots, A_p)$, corresponding to a word w in Ξ^*, is Schur if and only if condition (6) holds with $|w|_i = k_i$.

VI. Stabilizability criteria for general DPSS

The stabilizability results of the previous sections provide a set of simple sufficient conditions for checking the lack of stabilizability of a generic positive switched system (1).

Proposition 10: Consider a DPSS (1), with $A_i \in \mathbb{R}_+^{n\times n}$, $i \in [1,p]$, arbitrary positive matrices. If there exist matrices $\tilde{A}_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, that are either all cyclic monomial or all positive circulant, and each of them satisfies the following conditions:

$$\text{(a) } A_i \ge \tilde{A}_i, \qquad \text{and} \qquad \text{(b) } \tilde{A}_i \ \text{ is not Schur,}$$

then the DPSS (1) is not stabilizable.

Proof: For every $w \in \Xi^*$, $w(A_1, A_2, \dots, A_p) \ge w(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_p)$ implies

$$\rho\left(w(A_1, A_2, \dots, A_p)\right) \ge \rho\left(w(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_p)\right).$$

If all matrices $\tilde{A}_i$ are either cyclic monomial or positive circulant, then under assumption (b), $\rho\left(w(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_p)\right) \ge 1$ for every $w \in \Xi^*$. So, the DPSS (1) cannot be stabilizable. ∎

The above result can be easily extended, as shown in the following corollaries. We first consider the case when the submatrices are cyclic monomial.

Corollary 3: Consider a DPSS (1), with $A_i, i \in [1,p]$, arbitrary positive matrices. Suppose that there exists an elementary cycle γ, of length $k \le n$, say $j_1 \to j_2 \to j_3 \to \dots \to j_k \to j_1$, appearing in every digraph $\mathcal{D}(A_i)$, and the product of the weights of its edges is greater than or equal to 1 in every digraph $\mathcal{D}(A_i)$. Then the DPSS is not stabilizable.

Proof: It entails no loss of generality assuming that γ is the elementary cycle: $1 \to k \to k-1 \to \dots \to 2 \to 1$. If we retain only the entries of the matrices A_i that represent the weights of these arcs, the matrices:

$$\tilde{A}_i = \left[\begin{array}{ccccc|c} 0 & [A_i]_{12} & 0 & \dots & 0 & \\ 0 & 0 & [A_i]_{23} & \ddots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & [A_i]_{k-1,k} & \\ [A_i]_{k1} & 0 & 0 & \dots & 0 & \\ \hline & & & & & 0_{(n-k)\times(n-k)} \end{array}\right], \quad i \in [1,p],$$

are not Schur, and $w(A_1, A_2, \ldots, A_p) \geq w(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_p)$ for every $w \in \Xi^*$. This implies

$$\rho\left(w(A_1, A_2, \ldots, A_p)\right) \geq \rho\left(w(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_p)\right) \geq 1.$$

So none of matrix products $w(A_1, A_2, \ldots, A_p)$ is Schur and the DPSS is not stabilizable. ■

Example 1: Consider a DPSS (1), with $p = 2$ and

$$A_1 = \begin{bmatrix} 0.1 & 0.2 & 2 \\ 0.5 & 0.2 & 0.5 \\ 1 & 0.5 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0.01 & 0.1 & 1 \\ 0 & 0 & 0.1 \\ 3 & 0 & 0.2 \end{bmatrix}.$$

$\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$ have in common an elementary cycle including vertices 1 and 3, and

$$A_1 > \tilde{A}_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_2 > \tilde{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

As neither $\tilde{A}_1$ nor $\tilde{A}_2$ is Schur, the switched system is not stabilizable.

An extension of Proposition 10 to the case of circulant submatrices is provided below. The proof follows the same lines as the previous corollary.

Corollary 4: Consider a DPSS (1), with $A_i, i \in [1, p]$, arbitrary positive matrices. Suppose that there exist a permutation matrix P, a positive integer $k \leq n$ and positive (left or right) circulant matrices $\tilde{A}_i \in \mathbb{R}_+^{k \times k}, i \in [1, p]$, such that, for every index i,

$$\text{(a) } P^\top A_i P \geq \begin{bmatrix} \tilde{A}_i & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad \text{(b) } \tilde{A}_i \text{ is not Schur.}$$

Then the positive switched system is not stabilizable.

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