Stability and stabilizability criteria for discrete-time positive switched systems

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***Abstract*—In this paper we consider the class of discrete-time switched systems switching between p autonomous positive subsystems. First, sufficient conditions for testing stability, based on the existence of special classes of common Lyapunov functions, are investigated, and these conditions are mutually related, thus proving that if a linear copositive common Lyapunov function can be found, then a quadratic positive definite common function can be found, too, and this latter, in turn, ensures the existence of a quadratic copositive common function. Secondly, stabilizability is introduced and characterized. It is shown that if these systems are stabilizable, they can be stabilized by means of a periodic switching sequence, which asymptotically drives to zero every positive initial state. Conditions for the existence of state-dependent stabilizing switching laws, based on the values of a copositive (linear/quadratic) Lyapunov function, are investigated and mutually related, too.**

Finally, some properties of the patterns of the stabilizing switching sequences are investigated, and the relationship between a sufficient condition for stabilizability (the existence of a Schur convex combination of the subsystem matrices) and an equivalent condition for stabilizability (the existence of a Schur matrix product of the subsystem matrices) is explored.

***Index Terms*—Switched system, positive linear system, asymptotic stability/stabilizability, linear/quadratic copositive Lyapunov function, positive definite Lyapunov function.**

I. INTRODUCTION

A Discrete-time positive switched system (DPSS) consists of a family of positive state-space models [12], [26] and a switching law, specifying when and how the switching among the various models takes place. This class of systems has some interesting practical applications. DPSS's have been adopted for describing networks employing TCP and other congestion control applications [41], for modeling consensus and synchronization problems [24], and, quite recently, for describing the viral mutation dynamics under drug treatment [21].

As for the broader classes of hybrid and switched systems, stability and stabilizability properties have been the two major issues to attract the researchers' attention. Clearly, all results so far obtained for general discrete-time switched systems hold true for DPSS's. In particular, the asymptotic stability of a DPSS switching into a finite set of matrices $\mathcal{A} := \{A_i, i \in [1,p]\}$, i.e. the convergence to zero of all infinite products of these matrices, is equivalent [10], [18], [25] to the fact that the joint spectral radius of $\mathcal{A}$, namely $\rho(\mathcal{A}) := \limsup_{k\to+\infty} \max\{\rho(A_{i_1}A_{i_2}\cdots A_{i_k})^{1/k} \ : \ A_{i_\ell} \in \mathcal{A}\}$, is smaller than 1. The *finiteness conjecture* [10], [28], assuming that for an asymptotically stable switched system an index $k \in \mathbb{N}$ and a product $A_{i_1}A_{i_2}\cdots A_{i_k}$ of matrices in $\mathcal{A}$ could always be found such that $\rho(\mathcal{A}) = \rho(A_{i_1}A_{i_2}\cdots A_{i_k})^{1/k}$, turned out to be false [6], [7]. This implies, in particular, that the convergence to zero of all state trajectories along periodic switching sequences does not ensure, in general, asymptotic stability. So, even if a number of algorithms was proposed to evaluate the joint spectral radius of a set of matrices in quite general conditions (branch-and-bound methods, the simple convex combinations method, geometric methods, and Lyapunov methods) [25], research efforts about stability and henceforth about stabilizability have also taken alternative directions and focused on different approaches. The variational approach to stability (see [32] for a complete survey in the continuous-time case) is based on the rather intuitive idea [39] that if one is able to characterize the most critical switching sequence, and such a sequence proves to be stabilizing, then all the other sequences are. This approach, which provides in turn necessary and sufficient conditions for stability, has rather significant advantages: most of all, it allows to use powerful tools from optimal control theory. Moreover, by investigating the system behavior under the worst possible switching path, it reveals the mechanisms that lead to instability.

The most popular approach to the investigation of stability and stabilizability, however, is undoubtedly the one based on common Lyapunov functions or multiple Lyapunov functions (see [5], [9], [43], [31], to quote just a few contributions). It is worthwhile to mention the work of Lee and Dullerud [30], [29] that provides quite interesting results regarding the stability and the stabilizability of discrete-time switched systems under the assumption that the path of each switching sequence is constrained by the graph of an irreducible matrix. In addition to a characterization of these properties in terms of LMIs, the Authors propose the concept of finite-path-dependent Lyapunov function, which allows to extend the stabilization techniques based on common Lyapunov functions and on multiple Lyapunov functions.

Also in the context of positive switched systems, stability and stabilizabilty properties have been investigated by resorting to Lyapunov functions techniques. Most of the results obtained so far, however, have been derived in the continuous-time case [14], [19], [27], [35], [36], [37], [44]. While conditions based on linear copositive functions find a straightforward extension to the discrete-time case, this is not true when dealing with quadratic stability and stabilizability, and at our knowledge the only contribution on this subject is [34]. Some recent work on the stabilization of discrete-time positive switched systems by Benzaouia and coauthors [2], [3] focuses on the different issue of state and output feedback

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stabilization, and provides stabilizability conditions based on the solution of certain LMIs.

In this paper we concentrate our attention on discrete-time positive switched systems, and investigate in detail stability and stabilizability properties for them. In section II several sufficient conditions for testing stability, based on the existence of special classes of common Lyapunov functions, are mutually related, thus proving that if a linear copositive common Lyapunov function can be found, then a quadratic positive definite common Lyapunov function can be found, too, and this latter, in turn, ensures the existence of a quadratic copositive common Lyapunov function.

In section III stabilizability is introduced and characterized. It is shown that if a DPSS is stabilizable, it can be stabilized by means of a periodic switching sequence, which asymptotically drives to zero every positive initial state. Conditions for the existence of state-dependent stabilizing switching laws, based on the values of a copositive (linear/quadratic) Lyapunov function, are investigated and related to each other in section IV. Interestingly enough, the mutual relationship between the various conditions for the existence of these special Lyapunov functions are very close to the analogous ones obtained for the stability characterization. In showing that the existence of copositive Lyapunov functions allows to define suitable switching strategies, we extend to the class of DPSS's a technique first explored in [43].

Finally, section V explores some patterns of the stabilizing switching sequences. In particular, it is shown that when a Schur convex combination of the matrices $A_i, i \in [1,p]$, can be found, and hence stabilizability is ensured, the combination coefficients can be related to the relative frequencies of the matrices A_i in a Schur matrix product and, consequently, in a convergent periodic switching sequence.

A preliminary version of the paper, regarding the stabilizability property only, has appeared in the Proceedings of the 49th IEEE Conference on Decision and Control [15].

Before proceeding, we introduce some notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is *nonnegative*, and if so we adopt the notation $A \geq 0$. If, in addition, it has at least one positive entry, A is *positive* ($A > 0$), while if all its entries are positive it is *strictly positive* ($A \gg 0$). Given two matrices A and B, of the same size, $A \geq B$, $A > B$ and $A \gg B$ are synonymous of $A - B \geq 0$, $A - B > 0$ and $A - B \gg 0$, respectively. In a similar way can be defined the symbols $\leq, <$ and $\ll$.

A vector $\mathbf{v} \in \mathbb{R}^n_+$ is a *monomial vector* if it all its entries are zero, except for a single positive one. If the value of the positive entry is 1, $\mathbf{v}$ is a *canonical vector*. A *monomial (permutation) matrix* is a nonsingular square positive matrix whose columns are monomial (canonical) vectors. $\mathbf{1}_n$ is the n-dimensional vector with all entries equal to 1. An $n \times n$ ($n > 1$) positive matrix A is *reducible* if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}_+$ such that

$$P^\top A P = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. If this is not the case, A is called *irreducible*.

A real square matrix A is *Metzler* if its off-diagonal entries are nonnegative, *Schur* if all its eigenvalues lie in the open unit disk (equivalently, its spectral radius, $\rho(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}$, is smaller than 1), and *Hurwitz* if they all lie in the open left complex halfplane.

A square symmetric matrix P is *positive definite* ($\succ 0$) if for every nonzero vector $\mathbf{x}$, of compatible dimension, $\mathbf{x}^\top P \mathbf{x} > 0$, and *positive semi definite* ($\succeq 0$) if for every vector $\mathbf{x}$, of compatible dimension, $\mathbf{x}^\top P \mathbf{x} \geq 0$. P is negative (semi)definite ($\prec 0$ or $\preceq 0$) if $-P$ is positive (semi)definite.

Given a family of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ in $\mathbb{R}^n$, the *convex hull* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ is the set of vectors $\left\{\sum_{i=1}^s \alpha_i \mathbf{v}_i : \alpha_i \geq 0, \sum_{i=1}^s \alpha_i = 1\right\}$.

Finally, we need some definitions borrowed from the algebra of non-commutative polynomials [40]. Given an *alphabet* $\Xi = \{\xi_1, \xi_2, \ldots, \xi_p\}$, we denote by Ξ^* the set of all *words* $w = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}, k \in \mathbb{N}, \ \xi_{i_h} \in \Xi$. The *length* k of w is denoted by $|w|$, while $|w|_i$ represents the number of occurrences of ξ_i in w. The product of words in Ξ^* is defined by concatenation, and ε, the empty word, is the unit element. $\mathbb{R}\langle\xi_1, \xi_2, \ldots, \xi_p\rangle$ is the algebra of polynomials in the noncommuting indeterminates $\xi_1, \xi_2, \ldots, \xi_p$. For every family $\{A_1, A_2, \ldots, A_p\}$ of p matrices in $\mathbb{R}^{n\times n}$, the map ψ defined by the assignments $\psi(\varepsilon) = I_n$ and $\psi(\xi_i) = A_i, i = 1, 2, \ldots, p$, uniquely extends to an algebra morphism of $\mathbb{R}\langle\xi_1, \xi_2, \ldots, \xi_p\rangle$ into $\mathbb{R}^{n\times n}$ (as an example, $\psi(\xi_1\xi_2) = A_1A_2 \in \mathbb{R}^{n\times n}$). If w is a word in Ξ^* (i.e. a monic monomial in $\mathbb{R}\langle\xi_1, \xi_2, \ldots, \xi_p\rangle$), the ψ-image of w is denoted by $w(A_1, A_2, \ldots, A_p)$.

II. Stability of discrete-time positive switched systems

A *discrete-time positive switched system* (DPSS) is described by the following equation

$$\mathbf{x}(t+1) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n_+$ denotes the value of the n-dimensional state variable at time t, σ is an arbitrary switching sequence, taking values in the set $[1,p] := \{1, 2, \ldots, p\}$, and for each $i \in [1,p]$ the matrix A_i is an $n \times n$ positive matrix.

Definition 1: A function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is *copositive* if $V(\mathbf{x}) > 0$ for every $\mathbf{x} > 0$, and $V(\mathbf{0}) = 0$. A copositive function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is a *common Lyapunov function* for the positive matrices $A_i, i \in [1,p]$, (or for the DPSS (1)) if

$$\forall \mathbf{x} > 0, \forall\ i \in [1,p] \qquad \Delta V_i(\mathbf{x}) := V(A_i\mathbf{x}) - V(\mathbf{x}) < 0,$$

or, equivalently,

$$\forall\ \mathbf{x} > 0 \qquad \max_{i\in[1,p]} \Delta V_i(\mathbf{x}) < 0. \tag{2}$$

In this paper we will consider three classes of copositive functions:

- *linear copositive* functions: $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, with $\mathbf{v} \in \mathbb{R}^n$ (necessarily) strictly positive;
- *quadratic copositive* functions: $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, with $P = P^\top \in \mathbb{R}^{n\times n}$ such that $\mathbf{x}^\top P \mathbf{x} > 0$ for every $\mathbf{x} > 0$;

- *quadratic positive definite* functions: $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$, with $P = P^\top \succ 0$.

A linear copositive function $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, with $\mathbf{v} \gg 0$, is a common Lyapunov function (CLF) for the matrices $A_i, i \in [1,p]$, if and only if $\mathbf{v}^\top A_i \mathbf{x} < \mathbf{v}^\top \mathbf{x}$ for every $i \in [1,p]$ and every $\mathbf{x} > 0$, which amounts to saying that

$$\mathbf{v}^\top (A_i - I_n) \ll 0, \qquad \forall\, i \in [1,p].$$

Similarly, a quadratic copositive function (and, in particular, a quadratic positive definite function) $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a CLF for the matrices $A_i, i \in [1,p]$, if and only if

$$\mathbf{x}^\top (A_i^\top P A_i - P)\mathbf{x} < 0 \qquad \forall\, i \in [1,p] \qquad \text{and} \qquad \forall\, \mathbf{x} > 0.$$

It is well known [14], [27], [34], [36], [37], that the existence of CLFs belonging to any of the previous three classes represents a sufficient condition for (uniform exponential, and hence uniform asymptotic) system stability. Also, in [14], [27], equivalent conditions for the existence of a linear copositive CLF have been provided. Finally, in [34] necessary or sufficient conditions for the existence of a quadratic positive definite CLF are given in terms of certain matrix pencils. In particular, for two-dimensional systems (i.e., when $n = 2$), a complete characterization of the existence of a quadratic positive definite CLF for two matrices, A_1 and A_2, is provided.

In this section we want to investigate how the conditions for the existence of these CLFs are mutually related.

Theorem 1: Let $A_1, A_2, \dots, A_p$ be $n \times n$ positive matrices. The following facts are equivalent[1]:

c1) $\exists\ \mathbf{v} \gg 0$ such that $\mathbf{v}^\top \sum_{i=1}^p \alpha_i (A_i - I_n) \ll 0,\ \forall\ \alpha_1, \alpha_2, \dots, \alpha_p \geq 0$ with $\sum_{i=1}^p \alpha_i = 1$;

c2) $\exists\ \mathbf{v} \gg 0$ such that $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ is a linear copositive CLF for $A_i, i \in [1,p]$;

c3) $\exists\ P = P^\top$ of rank 1 such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a quadratic copositive CLF for $A_i, i \in [1,p]$;

c4) for each map $\pi : [1,n] \to [1,p]$, the matrix

$$A_\pi := [\,\mathrm{col}_1(A_{\pi(1)})\ \mathrm{col}_2(A_{\pi(2)})\ \dots\ \mathrm{col}_n(A_{\pi(n)})\,]$$

is Schur;

c5) the convex hull of the columns of

$$W := [\,A_1 - I_n \quad A_2 - I_n \quad \dots \quad A_p - I_n\,] \in \mathbb{R}^{n \times np}$$

does not intersect the positive orthant of $\mathbb{R}^n$.

If c1)$-$c5) hold, then each of the following two equivalent conditions holds:

d1) $\exists\ \tilde{P} = \tilde{P}^\top \succ 0$ such that $V(\mathbf{x}) = \mathbf{x}^\top \tilde{P}\mathbf{x}$ is a quadratic positive definite CLF for $A_i, i \in [1,p]$;

d2) $\exists\ \tilde{P} = \tilde{P}^\top \succeq 0$ such that $V(\mathbf{x}) = \mathbf{x}^\top \tilde{P}\mathbf{x}$ is a quadratic copositive CLF for $A_i, i \in [1,p]$.

If d1)$-$d2) hold, then

e) $\exists\ P = P^\top$ such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is quadratic copositive CLF for $A_i, i \in [1,p]$.

Condition e), in turn, implies

f) the DPSS (1) is asymptotically stable,

which implies

g) $\sum_{i=1}^p \alpha_i A_i$ is Schur, $\forall\ \alpha_1, \alpha_2, \dots, \alpha_p \geq 0$, with $\sum_{i=1}^p \alpha_i = 1$.

Proof: c1) $\Leftrightarrow$ c2) Condition c2) is obtained from c1) for special values of the $p$tuples $(\alpha_1, \alpha_2, \dots, \alpha_p)$. The reverse implication is obvious.

c2) $\Rightarrow$ c3) Suppose that for some $\mathbf{v} \gg \mathbf{0}$ condition $\mathbf{v}^\top A_i \mathbf{x} < \mathbf{v}^\top \mathbf{x}$ holds, $\forall i \in [1,p]$ and $\forall \mathbf{x} > 0$.
As all quantities involved are nonnegative, $\mathbf{x}^\top A_i^\top \mathbf{v}\mathbf{v}^\top A_i \mathbf{x} = (\mathbf{v}^\top A_i \mathbf{x})^2 < (\mathbf{v}^\top \mathbf{x})^2 = \mathbf{x}^\top \mathbf{v}\mathbf{v}^\top \mathbf{x}$ holds, $\forall i \in [1,p]$ and $\forall \mathbf{x} > 0$. So, c3) is satisfied for $P := \mathbf{v}\mathbf{v}^\top$.

c3) $\Rightarrow$ c2) If rank $P = 1$ and $P = P^\top$, then P can be expressed as $P = \mathbf{v}\mathbf{v}^\top$, for some vector $\mathbf{v}$. Moreover, as $\mathbf{x}^\top P \mathbf{x} = (\mathbf{v}^\top \mathbf{x})^2 > 0$, $\forall \mathbf{x} > 0$, all entries of $\mathbf{v}$ are nonzero and of the same sign, and it entails no loss of generality assuming that they are all positive. On the other hand, $\forall \mathbf{x} > 0$ and $\forall i \in [1,p]$, condition

$$\mathbf{x}^\top [A_i^\top P A_i - P]\mathbf{x} = \mathbf{x}^\top A_i^\top \mathbf{v}\mathbf{v}^\top A_i \mathbf{x} - \mathbf{x}^\top \mathbf{v}\mathbf{v}^\top \mathbf{x} < 0,$$

can be rewritten as $(\mathbf{v}^\top A_i \mathbf{x})^2 < (\mathbf{v}^\top \mathbf{x})^2$, and from the nonnegativity of both $\mathbf{v}^\top A_i \mathbf{x}$ and $\mathbf{v}^\top \mathbf{x}$, one gets condition c2), namely:

$$\mathbf{v}^\top A_i \mathbf{x} < \mathbf{v}^\top \mathbf{x}, \qquad \forall \mathbf{x} > 0,\ \forall i \in [1,p].$$

c2) $\Leftrightarrow$ c4) Condition c4) holds if and only if $A_\pi - I_n$ is a Metzler Hurwitz matrix for all π, which is equivalent [14], [27] to assuming that there exists $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top (A_i - I_n) \ll 0,\ \forall\ i \in [1,p]$, which is just c2).

c2) $\Leftrightarrow$ c5) By Lemma 2, in the Appendix, one and only one of the following alternatives holds:

$$\begin{aligned} &\text{either } \exists\ \mathbf{v} > \mathbf{0} \quad \text{such that} \quad \mathbf{v}^\top W \ll \mathbf{0}^\top && (3)\\ &\text{or } \exists\ \mathbf{z} > \mathbf{0} \quad \text{such that} \quad W\mathbf{z} \geq \mathbf{0}, && (4) \end{aligned}$$

and in (4) the vector $\mathbf{z}$ can be assumed w.l.o.g. stochastic (i.e., $\sum_{i=1}^{np} [\mathbf{z}]_i = 1$). If c2) (and hence (3)) holds true, (4) cannot be verified, and consequently no convex combination of the columns of W intersects the positive orthant of $\mathbb{R}^n$. Viceversa, if c5) holds, (4) does not, and hence (3) admits a positive solution $\mathbf{v} \in \mathbb{R}^n_+$. We want to prove that $\mathbf{v} \gg 0$. Suppose it is not. Then it entails no loss of generality assuming that $\mathbf{v}^\top = [\,0 \quad \mathbf{v}_2^\top\,]$, with $\mathbf{v}_2 \gg 0$. Indeed, we can always reduce ourselves to this situation by means of a suitable relabeling, which amounts to applying a suitable permutation. Partition the matrices A_i's accordingly as

$$A_i = \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ A_{21}^{(i)} & A_{22}^{(i)} \end{bmatrix},$$

with $A_{11}^{(i)}$ and $A_{22}^{(i)}$ square matrices. So, $\mathbf{v}^\top W \ll 0^\top$ implies

$$[\,0 \quad \mathbf{v}_2^\top\,] \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ A_{21}^{(i)} & A_{22}^{(i)} \end{bmatrix} \ll [\,0 \quad \mathbf{v}_2^\top\,], \qquad i \in [1,p],$$

which is clearly inconsistent.

c3) $\Rightarrow$ d2) If P is a symmetric matrix of rank 1 such that $\mathbf{x}^\top P\mathbf{x} > 0$ in every point of the positive orthant, except for

[1]The choice of labeling the theorem conditions starting from "c" is motivated by the fact that a set of analogous conditions will be derived later on for stabilizability, and in that case conditions "C" will be implied by sufficient conditions, labelled by "A" and "B".

the origin, then, as shown in c3) $\Rightarrow$ c2), $P = \mathbf{v}\mathbf{v}^\top$ for some $\mathbf{v} \gg 0$. This implies that P is also positive semidefinite.

d2) $\Rightarrow$ d1) Assume that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a quadratic copositive CLF for $A_i, i \in [1,p]$. Set $\tilde{P} := P + \varepsilon I_n$, with $\varepsilon > 0$. Clearly, $\tilde{P} \succ 0$. The two functions

$$\begin{aligned} f(\mathbf{x}) &:= \max_{i\in[1,p]} |\mathbf{x}^\top[A_i^\top A_i - I_n]\mathbf{x}|, \\ g(\mathbf{x}) &:= \max_{i\in[1,p]} \mathbf{x}^\top[A_i^\top P A_i - P]\mathbf{x}, \end{aligned}$$

are continuous in the compact set $\mathcal{S} := \mathbb{R}^n_+ \cap \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}$. So, by Weierstrass' theorem and assumption d2), we have

$$\begin{aligned} \max_{\mathbf{x}\in\mathcal{S}} f(\mathbf{x}) &= \max_{\mathbf{x}\in\mathcal{S}} \max_{i\in[1,p]} |\mathbf{x}^\top[A_i^\top A_i - I_n]\mathbf{x}| = M \geq 0, \\ \max_{\mathbf{x}\in\mathcal{S}} g(\mathbf{x}) &= \max_{\mathbf{x}\in\mathcal{S}} \max_{i\in[1,p]} \mathbf{x}^\top[A_i^\top P A_i - P]\mathbf{x} = -\delta < 0. \end{aligned}$$

Let ε be any positive number such that $\varepsilon M < \delta$. Then, for every $\mathbf{x} \in \mathcal{S}$,

$$\begin{aligned} & \max_{i\in[1,p]} \mathbf{x}^\top(A_i^\top \tilde{P} A_i - \tilde{P})\mathbf{x} \\ = & \max_{i\in[1,p]} \left[\mathbf{x}^\top(A_i^\top P A_i - P)\mathbf{x} + \varepsilon\left(\mathbf{x}^\top(A_i^\top A_i - I_n)\mathbf{x}\right)\right] \\ \leq & \max_{\mathbf{x}\in\mathcal{S}} \max_{i\in[1,p]} \left[\mathbf{x}^\top(A_i^\top P A_i - P)\mathbf{x}\right] \\ + & \varepsilon \cdot \max_{\mathbf{x}\in\mathcal{S}} \max_{i\in[1,p]} \left[|(\mathbf{x}^\top(A_i^\top A_i - I_n)\mathbf{x}|\right] = -\delta + \varepsilon M < 0. \end{aligned}$$

By the homogeneity of $V(\mathbf{x})$, the result holds for every $\mathbf{x} > 0$.

d1) $\Rightarrow$ d2) is obvious.

d2) $\Rightarrow$ e) is obvious, and the fact that e) implies f) (i.e. uniform asymptotic stability) is well-known in the literature. Also, f) $\Rightarrow$ g) follows from the fact that if the DPSS (1) is asymptotically stable, then [25] so is the DPSS switching among the convex combinations of the matrices $A_i, i \in [1,p]$. But this implies that all convex combinations $\sum_{i=1}^p \alpha_i A_i$ are Schur. ∎

Remark 1: The copositivity of $V(\mathbf{x})$ introduced in statement d2) ensures that $V(\mathbf{x}) = \mathbf{x}^\top \tilde{P}\mathbf{x}$ does not annihilate at any point of $\mathbb{R}^n_+$, even if it is only positive semidefinite. One may wonder whether dropping the copositivity assumption could lead to a further condition (apparently weaker than d2)), integrating the general pattern presented in Theorem 1, namely:

$\exists\, \tilde{P} = \tilde{P}^\top \succeq 0$ *such that* $V(\mathbf{x}) = \mathbf{x}^\top \tilde{P}\mathbf{x}$ *satisfies condition* $\mathbf{x}^\top(A_i^\top \tilde{P} A_i - \tilde{P})\mathbf{x} < 0$, *for every* $\mathbf{x} > 0$ *and every* $i \in [1,p]$.

As a matter of fact, this is just equivalent to d2). Indeed, assuming $\Delta V_i(\mathbf{x}) := \mathbf{x}^\top(A_i^\top \tilde{P} A_i - \tilde{P})\mathbf{x} < 0$, for every $\mathbf{x} > 0$ and every $i \in [1,p]$, rules out the possibility of having $V(\tilde{\mathbf{x}}) = 0$ for some $\tilde{\mathbf{x}} > 0$. Indeed, this would imply $V(A_i\tilde{\mathbf{x}}) < 0$, thus contradicting the positive semi-definiteness of $V(\mathbf{x})$. Consequently, positive semi-definiteness of $V(\mathbf{x})$ and the negativity assumption on the ΔV_i's imply the copositivity of $V(\mathbf{x})$.

Remark 2: While conditions c1)-c5) imply d1)-d2), the converse is not true. Consider the pair of positive Schur matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2/3 & 1/30 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1/2 & 1 \\ 0 & 1/3 \end{bmatrix}.$$

It is easy to see that the matrix

$$[\,\mathrm{col}_1(A_1) \quad \mathrm{col}_2(A_2)\,] = \begin{bmatrix} 0 & 1 \\ 2/3 & 1/3 \end{bmatrix}$$

is row stochastic and hence its spectral radius is 1. So, it is not a Schur matrix and condition c4) is not verified. However, it is a matter of simple calculation to show that the matrix

$$\tilde{P} = \begin{bmatrix} 1 & 4/5 \\ 4/5 & 2 \end{bmatrix} = \tilde{P}^\top \succ 0$$

makes $V(\mathbf{x}) = \mathbf{x}^\top \tilde{P}\mathbf{x}$ a quadratic positive definite CLF for A_1 and A_2, and hence d1) holds.

Remark 3: Condition g) does not ensure asymptotic stability of the DPSS. If one considers the two matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

it is clear that

$$A_1A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is not Schur, and hence the state trajectory corresponding to the periodic switching sequence

$$\sigma(t) = \begin{cases} 2, & \text{if } t \text{ is even;} \\ 1, & \text{if } t \text{ is odd,} \end{cases}$$

does not converge to zero corresponding to every positive $\mathbf{x}(0)$. However, for every $\alpha \in [0,1]$

$$\alpha A_1 + (1-\alpha)A_2 = \begin{bmatrix} 0 & \alpha \\ 1-\alpha & 0 \end{bmatrix}$$

has characteristic polynomial $z^2 - \alpha(1-\alpha)$ and hence it is Schur. Note that, when dealing with continuous-time positive switched systems of dimension $n = 2$, it is true that asymptotic stability is equivalent to the fact that all the convex combinations of the subsystem matrices are Hurwitz. It was initially conjectured [33] that the result could be extended to systems of arbitrary size n. However, this results was proved to be wrong [19], [11].

The results of Theorem 1 are summarized in Figure 1. We ignore whether the implications d1)$-$d2) $\Rightarrow$ e) and e) $\Rightarrow$ f) can be reversed.

III. STABILIZATION

We introduce the concept of stabilizability for DPSS, also known in the literature on (general) switched systems [42] as pointwise asymptotic stabilizablility.

Definition 2: The DPSS (1) is *stabilizable* if for every positive initial state $\mathbf{x}(0)$ there exists a switching sequence $\sigma : \mathbb{Z}_+ \to [1,p]$ such that the state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.

Clearly, the stabilization problem is a non-trivial one only if all matrices A_i's are not Schur. So, in the following, we will steadily make this assumption. As remarked in the previous definition, the choice of the switching sequence σ may depend on the initial state $\mathbf{x}(0)$. A stronger definition of stabilizability

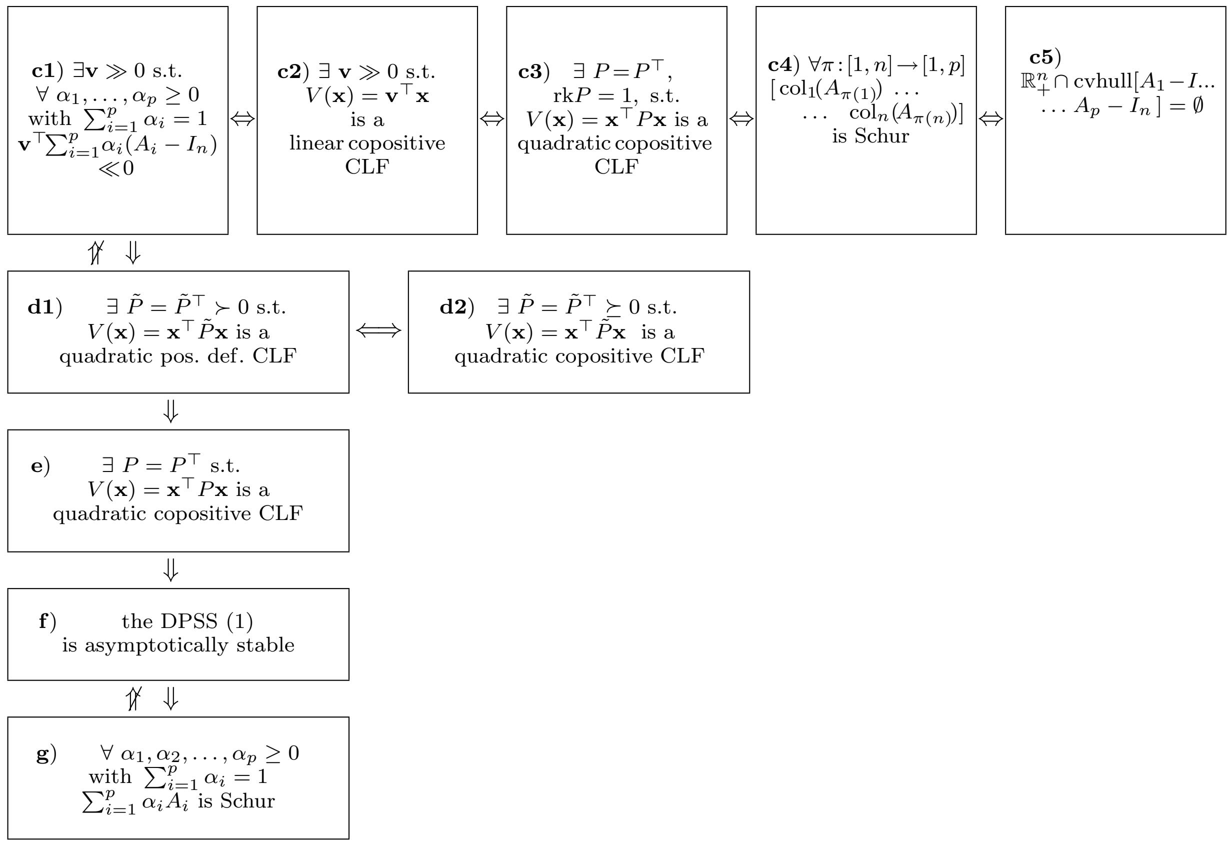

Figure 1: Stability conditions based on the existence of certain CLFs for $A_i, i \in [1,p]$.

requires that the stabilizing sequence does not depend on the initial state [42].

Definition 3: The DPSS (1) is *consistently stabilizable* if there exists a switching sequence $\sigma : \mathbb{Z}_+ \to [1,p]$ such that, for every positive initial state $\mathbf{x}(0)$, the corresponding state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.

It is clear that consistent stabilizability implies stabilizability. The natural question arises whether the converse is true.

In the general case, i.e. when there is no positivity assumption, discrete-time switched systems can be found (see the example at pages 112-113 in [42]) that are stabilizable, but not consistently stabilizable. However, it has been recently proven [23] that if a switching sequence exists that drives to zero any initial state, then there is an uncountable number of such switching sequences. In [42] (see Theorem 3.5.4) it is also shown that for discrete-time switched systems, without positivity constraints, consistent stabilizability is equivalent to the existence of a periodic switching sequence that asymptotically drives to zero the state evolution starting from every $\mathbf{x}(0) \in \mathbb{R}^n$. As we will see, when dealing with positive switched systems (1), consistent stabilizability and stabilizability are equivalent properties, and they are both equivalent to the possibility of stabilizing the system by means of a periodic switching sequence, independently of the positive initial state.

Proposition 1: Given a DPSS (1), the following facts are equivalent:

i) the system is stabilizable;
ii) the system is consistently stabilizable;
iii) there exist $N > 0$ and indices $i_0, i_1, \dots, i_{N-1} \in [1,p]$, such that the matrix product $A_{i_{N-1}} A_{i_{N-2}} \cdots A_{i_1} A_{i_0}$ is a positive Schur matrix;
iv) there exists a periodic switching sequence that leads to zero every positive initial state.

Proof: i) $\Rightarrow$ ii) If a switching sequence σ asymptotically drives to zero the initial state $\hat{\mathbf{x}}(0) = \mathbf{1}_n$, it drives to zero every other positive state $\mathbf{x}(0)$. Indeed, let $\hat{\mathbf{x}}(t)$ and $\mathbf{x}(t), t \in \mathbb{Z}_+$, be the state evolutions originated from $\hat{\mathbf{x}}(0)$ and $\mathbf{x}(0)$, respectively, corresponding to the switching sequence σ. A positive number M can be found such that $0 < \mathbf{x}(0) \leq M\mathbf{1}_n$, and the positivity assumption on the matrices A_i's implies that, at each time $t \in \mathbb{Z}_+$, $0 \leq \mathbf{x}(t) \leq M\hat{\mathbf{x}}(t)$, thus ensuring that $\mathbf{x}(t)$ goes to zero as $t \to +\infty$. So, the system is consistently stabilizable.

ii) $\Rightarrow$ iii) Let σ be the switching sequence that makes the state evolution go to zero, independently of the initial state. Set $\mathbf{x}(0) = \mathbf{1}_n$ and $\varepsilon \in (0,1)$. Then a positive integer N can be found such that

$$\mathbf{x}(N) = A_{\sigma(N-1)} \cdots A_{\sigma(1)} A_{\sigma(0)} \mathbf{1}_n < \varepsilon \mathbf{1}_n.$$

This ensures (see Theorem 1.1, Chapter II, in [38]) that the spectral radius of the positive matrix $A_{\sigma(N-1)}\cdots A_{\sigma(1)}A_{\sigma(0)}$ is smaller than $\varepsilon < 1$ and hence the matrix is Schur. So, iii) holds for $i_k = \sigma(k), k \in [0, N-1]$.

iii) $\Rightarrow$ iv) If $A := A_{i_{N-1}}A_{i_{N-2}}\cdots A_{i_1}A_{i_0}$ is a positive Schur matrix, then A^k converges to zero as k goes to infinity. Consequently, the switching sequence $\sigma(t) = i_{(t \bmod N)}$ drives to zero the state evolution corresponding to every positive initial state.

iv) $\Rightarrow$ i) is obvious. ∎

Remark 4: If a DPSS is consistently stabilizable, it is so when one removes the positivity constraint on the initial condition. Indeed, if a switching sequence σ drives to zero the state evolution corresponding to every positive initial state, it does the same for all states $\mathbf{x}(0) \in \mathbb{R}^n$, as $\mathbf{x}(0)$ can be expressed as $\mathbf{x}(0) = \mathbf{x}_+(0) - \mathbf{x}_-(0)$, with $\mathbf{x}_+(0), \mathbf{x}_-(0) \in \mathbb{R}^n_+$, and the sequence σ ensures that both $\mathbf{x}_+(t)$ and $\mathbf{x}_-(t)$, the state trajectories corresponding to $\mathbf{x}_+(0)$ and $\mathbf{x}_-(0)$, converge to zero. Consequently, the equivalence of ii) and iv) in Proposition 1 could have been proved using Theorem 3.5.4 of [42].

IV. State-feedback stabilization

In the previous section we introduced the general stabilization problem for the class of DPSS's (1). According to Definition 2, the stabilizing switching sequence σ is a function of time (and of the initial state), and can be thought of as an open-loop control action that we apply to the system in order to ensure that the state converges to zero. An alternative solution can be that of searching for a stabilizing switching sequence whose value at time t depends on the specific value of the state $\mathbf{x}(t)$, thus representing a state-feedback stabilizing switching sequence. This strategy, which has been explored in [43], is also known as *variable structure control*. Indeed, in [43] it is shown that, given a continuous-time switched system (without any positivity constraint)

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{R}_+,$$

if there exists a quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ whose derivative in every point $\mathbf{x} \neq 0$ is negative along at least one of the subsystems, by this meaning that for every $\mathbf{x} \neq 0$ there exists $i \in [1,p]$ such that

$$\dot{V}_i(\mathbf{x}) := \mathbf{x}^\top(A_i^\top P + PA_i)\mathbf{x} < 0,$$

then it is possible to define a state-feedback switching strategy that makes the state evolution converge to zero[2].

In this section we want to investigate, and mutually relate, the conditions for the existence of a copositive function $V(\mathbf{x})$ such that

$$\forall\ \mathbf{x} > 0, \exists\ i \in [1,p] \qquad \text{such that} \qquad \Delta V_i(\mathbf{x}) < 0,$$

[2]As a matter of fact, the switching strategy does not simply consist of setting $\sigma(t)$ equal to the value of the index i (or possibly, one of the indices $i \in [1,p]$) for which $\dot{V}_i(\mathbf{x})$ takes the minimum value, as this strategy would possibly lead to chattering (see [43] for the details). When dealing with discrete-time systems, however, this problem cannot arise.

or, equivalently,

$$\forall\ \mathbf{x} > 0 \qquad \min_{i\in[1,p]} \Delta V_i(\mathbf{x}) < 0, \tag{5}$$

where, as usual,

$$\Delta V_i(\mathbf{x}) := V(A_i\mathbf{x}) - V(\mathbf{x}), \tag{6}$$

in the various cases when $V(\mathbf{x})$ is quadratic positive (semi)definite, linear copositive or quadratic copositive. So, in a sense, we search for the counterpart for stabilizability of the characterization obtained in Theorem 1 for stability. As we will see, with respect to stability, we can provide a more detailed picture, especially if we restrict our attention to DPSS's switching between two subsystems.

Subsequently, we will prove that, when any such function $V(\mathbf{x})$ is available, a suitable switching law, based on the values taken by the various $\Delta V_i(\mathbf{x})$'s, can be found that proves to be stabilizing.

Theorem 2: Let $A_1, A_2, \dots, A_p$ be $n\times n$ positive matrices. Condition

B) $\exists\ P = P^\top \succ 0$ and $\alpha_1, \alpha_2, \dots, \alpha_p \geq 0$, with $\sum_{i=1}^p \alpha_i = 1$, such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ satisfies for every $\mathbf{x} > 0$

$$\sum_{i=1}^p \alpha_i \Delta V_i(\mathbf{x}) = \sum_{i=1}^p \alpha_i \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < 0,$$

implies any of the following equivalent facts:

C0) $\exists\ \alpha_1, \alpha_2, \dots, \alpha_p \geq 0$, with $\sum_{i=1}^p \alpha_i = 1$, such that $\sum_{i=1}^p \alpha_i A_i$ is Schur;

C1) $\exists\ \mathbf{v} \gg 0$ and $\exists\ \alpha_1, \alpha_2, \dots, \alpha_p \geq 0$, with $\sum_{i=1}^p \alpha_i = 1$, such that $\mathbf{v}^\top \sum_{i=1}^p \alpha_i(A_i - I_n) \ll 0$;

C2) $\exists\ \mathbf{v} \gg 0$ such that $V(\mathbf{x}) = \mathbf{v}^\top\mathbf{x}$ satisfies, for every $\mathbf{x} > 0$, $\min_{i\in[1,p]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,p]} \mathbf{v}^\top\ (A_i - I_n)\mathbf{x} < 0$;

C3) $\exists\ P = P^\top$ of rank 1 such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a quadratic copositive function that satisfies, for every $\mathbf{x} > 0$, $\min_{i\in[1,p]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,p]} \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < 0$.

If C0)−C3) hold, then any of the following two equivalent conditions holds:

D1) $\exists\ \tilde{P} = \tilde{P}^\top \succ 0$ such that $V(\mathbf{x}) = \mathbf{x}^\top\tilde{P}\mathbf{x}$ is a quadratic positive definite function that satisfies, for every $\mathbf{x} > 0$, $\min_{i\in[1,p]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,p]} \mathbf{x}^\top(A_i^\top\tilde{P}A_i - \tilde{P})\mathbf{x} < 0$;

D2) $\exists\ \tilde{P} = \tilde{P}^\top \succeq 0$ such that $V(\mathbf{x}) = \mathbf{x}^\top\tilde{P}\mathbf{x}$ is a quadratic copositive function that satisfies, for every $\mathbf{x} > 0$, $\min_{i\in[1,p]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,p]} \mathbf{x}^\top(A_i^\top\tilde{P}A_i - \tilde{P})\mathbf{x} < 0$.

If D1)−D2) holds, then

E) $\exists\ P = P^\top$ such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a quadratic copositive function that satisfies, for every $\mathbf{x} > 0$, $\min_{i\in[1,p]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,p]} \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < 0$.

Proof: B) $\Rightarrow$ C0) The proof follows a reasoning very similar to the one employed in [17] (see page 725). If $P \succ 0$, then

$$\begin{bmatrix} A_i^\top PA_i & A_i^\top P \\ PA_i & P \end{bmatrix} = \begin{bmatrix} A_i^\top \\ I_n \end{bmatrix} P \begin{bmatrix} A_i & I_n \end{bmatrix} \succeq 0, \quad \forall\ i \in [1,p].$$

Consequently

$$\begin{aligned}&\sum_{i=1}^p \alpha_i \begin{bmatrix} A_i^\top P A_i & A_i^\top P \\ PA_i & P \end{bmatrix} \\ = & \begin{bmatrix} \left(\sum_{i=1}^p \alpha_i A_i^\top P A_i\right) & \left(\sum_{i=1}^p \alpha_i A_i^\top\right) P \\ P\left(\sum_{i=1}^p \alpha_i A_i\right) & P \end{bmatrix} \succeq 0.\end{aligned}$$

By the Schur complement's formula, this implies that for every $\mathbf{x}$, and hence, in particular, for every $\mathbf{x} > 0$,

$$\mathbf{x}^\top \left[\left(\sum_{i=1}^p \alpha_i A_i^\top P A_i\right) - \left(\sum_{i=1}^p \alpha_i A_i^\top\right) P \left(\sum_{i=1}^p \alpha_i A_i\right)\right] \mathbf{x} \ge 0,$$

namely

$$\mathbf{x}^\top\left[\sum_{i=1}^p \alpha_i (A_i^\top P A_i - P)\right]\mathbf{x} \ge \mathbf{x}^\top\left[\left(\sum_{i=1}^p \alpha_i A_i^\top\right)P\left(\sum_{i=1}^p \alpha_i A_i\right) - P\right]\mathbf{x}.$$

As the left hand-side is negative for every $\mathbf{x} > 0$, so is the right hand-side. But this implies that $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ is a positive definite function such that $V((\sum_{i=1}^p \alpha_i A_i)\mathbf{x}) - V(\mathbf{x}) < 0$ for every $\mathbf{x} > 0$. So, as $\sum_{i=1}^p \alpha_i A_i$ is a positive matrix, this proves that it is Schur.

C0) $\Leftrightarrow$ C1) Set $A := \sum_{i=1}^p \alpha_i A_i$ and notice that $\sum_{i=1}^p \alpha_i I_n = I_n$. The equivalence is based on two facts: (1) A is nonnegative Schur if and only if $\tilde{A} := A - I_n$ is a Metzler Hurwitz matrix; (2) a Metzler matrix $\tilde{A}$ is Hurwitz if and only if [4], [22] there exists a vector $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top \tilde{A} \ll 0$.

C1) $\Rightarrow$ C2) From C1) it follows that, for every positive vector $\mathbf{x}$, one gets

$$\left[\mathbf{v}^\top \sum_{i=1}^p \alpha_i (A_i - I_n)\right] \mathbf{x} = \sum_{i=1}^p \alpha_i \left[\mathbf{v}^\top (A_i - I_n)\mathbf{x}\right] < 0,$$

whence $\min_{i\in[1,p]}\ \mathbf{v}^\top (A_i - I_n)\mathbf{x} < 0$.

C2) $\Rightarrow$ C1) By assumption C2), there exists a strictly positive vector $\mathbf{v}$ such that for every $\mathbf{x} > 0$ the vector

$$\begin{bmatrix} \mathbf{v}^\top (A_1 - I_n) \\ \vdots \\ \mathbf{v}^\top (A_p - I_n) \end{bmatrix} \mathbf{x} \quad \in \mathbb{R}^{p\times 1}$$

has at least one negative entry. So, once we set

$$W := \begin{bmatrix} \mathbf{v}^\top (A_1 - I_n) \\ \vdots \\ \mathbf{v}^\top (A_p - I_n) \end{bmatrix},$$

we can claim that no positive vector $\mathbf{x}$ can be found such that $W\mathbf{x} \ge 0$. But then, by Lemma 2, in the Appendix, a positive vector $\mathbf{y}$ exists such that $\mathbf{y}^\top W \ll 0$. As it entails no loss of generality rescaling $\mathbf{y}$ so that its entries sum up to 1, this means that nonnegative coefficients α_i exist, with $\sum_{i=1}^p \alpha_i = 1$, such that

$$0 \gg [\,\alpha_1 \quad \dots \quad \alpha_p\,]\, W = \mathbf{v}^\top \sum_{i=1}^p \alpha_i (A_i - I_n),$$

thus proving C1).

C2) $\Leftrightarrow$ C3) as well as C3) $\Rightarrow$ D2) and D1) $\Rightarrow$ D2) can be proved along the same lines as the proofs of the analogous conditions (c2) $\Leftrightarrow$ c3), c3) $\Rightarrow$ d2) and d1) $\Rightarrow$ d2)) in Theorem 1.

D2) $\Rightarrow$ D1) The reasoning is very similar to the one used in the proof of d2) $\Rightarrow$ d1) of Theorem 1, except that the two continuous functions we need now are

$$\begin{aligned} f(\mathbf{x}) &= \max_{i\in[1,p]} |\mathbf{x}^\top [A_i^\top A_i - I_n]\mathbf{x}|, \\ g(\mathbf{x}) &= \min_{i\in[1,p]} \mathbf{x}^\top [A_i^\top P A_i - P]\mathbf{x}. \end{aligned}$$

By proceeding as in d2) $\Rightarrow$ d1), we prove that D2) $\Rightarrow$ D1).

D1) $\Rightarrow$ E) is obvious. ∎

Remark 5: While condition B) implies C0), the converse is not true. Consider the $p = 2$ positive matrices

$$A_1 = \begin{bmatrix} 3/2 & 0 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 3/2 \end{bmatrix}.$$

The convex combination of the two matrices $\alpha A_1 + (1-\alpha)A_2$ is Schur for every $\alpha \in \left(\frac{1}{3}, \frac{2}{3}\right)$, and hence satisfies C0). However the pair does not satisfy B). Suppose there exists a quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ and $\alpha \in [0,1]$ such that for every $\mathbf{x} > 0$

$$\begin{aligned} &F(x_1,x_2) := \alpha \Delta V_1(\mathbf{x}) + (1-\alpha)\Delta V_2(\mathbf{x}) \\ =\ & \alpha \mathbf{x}^\top (A_1^\top P A_1 - P)\mathbf{x} + (1-\alpha)\mathbf{x}^\top (A_2^\top P A_2 - P)\mathbf{x} < 0. \end{aligned}$$

It entails no loss of generality rescaling P in such a way that

$$P = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix},$$

with $c > b^2$ and hence $c > 0$. One finds

$$\begin{aligned} A_1^\top P A_1 - P &= \begin{bmatrix} 5/4 & -b \\ -b & -c \end{bmatrix} \\ A_2^\top P A_2 - P &= \begin{bmatrix} -1 & -b \\ -b & 5c/4 \end{bmatrix}, \end{aligned}$$

and hence

$$\begin{aligned} F(x_1,x_2) &= \alpha\left(\frac{5}{4}x_1^2 - 2bx_1x_2 - cx_2^2\right) \\ &+ (1-\alpha)\left(-x_1^2 - 2bx_1x_2 + \frac{5}{4}cx_2^2\right) \\ &= \left(\frac{9}{4}\alpha - 1\right)x_1^2 - 2bx_1x_2 + \left(\frac{5}{4} - \frac{9}{4}\alpha\right)cx_2^2. \end{aligned}$$

Now, if parameters b, c and α could be found such that the previous expression $F(x_1,x_2)$ would take negative values for every choice of $\mathbf{x} > 0$, it should be $F(0,x_2) < 0$ for every $x_2 > 0$ as well as $F(x_1,0) < 0$ for every $x_1 > 0$. By the positivity of c, the first condition implies

$$\frac{5}{4} - \frac{9}{4}\alpha < 0,$$

namely $\alpha > \frac{5}{9}$, while the second condition implies

$$\frac{9}{4}\alpha - 1 < 0,$$

namely $\alpha < \frac{4}{9}$, which clearly contradicts the previous one. So, no choice of $b, c > b^2$ and $\alpha \in [0,1]$ makes $F(x_1,x_2) < 0$ for every $(x_1,x_2) > 0$.

Remark 6: While condition D1) implies E), the converse is not true, as shown by the following example. Consider the positive matrices

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

No quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ can be found such that in every point $\mathbf{x} > 0$ either $\Delta V_1(\mathbf{x}) = \mathbf{x}^\top(A_1^\top PA_1 - P)\mathbf{x}$ or $\Delta V_2(\mathbf{x}) = \mathbf{x}^\top(A_2^\top PA_2 - P)\mathbf{x}$ is negative. Indeed, if such a matrix would exist, it could be described w.l.o.g. in the form

$$P = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix},$$

with $c > b^2$, and in every nonzero point either one of the following inequalities would be satisfied:

$$\begin{aligned} \Delta V_1(\mathbf{x}) &= 3x_1^2 - 2bx_1x_2 - cx_2^2 < 0 \\ \Delta V_2(\mathbf{x}) &= -x_1^2 - 2bx_1x_2 + 3cx_2^2 < 0. \end{aligned}$$

Since for $x_1 = 0$ the first equation is obviously satisfied, we assume now $x_1 \neq 0$ and set $y := x_2/x_1$. So that the previous inequalities become:

$$p_1(y) := -cy^2 - 2by + 3 \quad < \quad 0 \tag{7}$$
$$p_2(y) := 3cy^2 - 2by - 1 \quad < \quad 0. \tag{8}$$

Upon observing that $c = b^2 + \varepsilon$ for some $\varepsilon > 0$, our goal is that of proving that, for every choice of $b \in \mathbb{R}$ and $\varepsilon > 0$, there exists $y > 0$ such that both $-cy^2 - 2by + 3 \geq 0$ and $3cy^2 - 2by - 1 \geq 0$. Indeed, the two zeros of the polynomial $p_1(y)$ are

$$\lambda_{-,+} := \frac{-2b \pm \sqrt{4b^2 + 12c}}{2c} = \frac{-b \pm \sqrt{4b^2 + 3\varepsilon}}{b^2 + \varepsilon},$$

and it is easy to prove that $\lambda_- < 0 < \lambda_+$. On the other hand, the polynomial $p_2(y)$ has zeros

$$\mu_{-,+} := \frac{2b \pm \sqrt{4b^2 + 12c}}{6c} = \frac{b \pm \sqrt{4b^2 + 3\varepsilon}}{3(b^2 + \varepsilon)}.$$

In order to ensure that in every $y > 0$ either (7) or (8) holds, it should be true that $\mu_- < 0$ and $\lambda_+ < \mu_+$. The first condition is easily proved to be verified, however condition $\lambda_+ < \mu_+$ amounts to

$$\frac{-b + \sqrt{4b^2 + 3\varepsilon}}{b^2 + \varepsilon} < \frac{b + \sqrt{4b^2 + 3\varepsilon}}{3(b^2 + \varepsilon)},$$

namely $2b > \sqrt{4b^2 + 3\varepsilon}$, a condition that, of course, is never verified. So, for every choice of b and $\varepsilon > 0$ all positive pairs (x_1, x_2) such that $\mu_+ < \frac{x_2}{x_1} < \lambda_+$ make both $\Delta V_1(\mathbf{x})$ and $\Delta V_2(\mathbf{x})$ positive.

On the other hand, one can verify, along the same procedure we just described, that the symmetric matrix of rank 2

$$P = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

defines a quadratic copositive function $V(\mathbf{x})$ such that, for every $\mathbf{x} > 0$, either

$$\mathbf{x}^\top(A_1^\top PA_1 - P)\mathbf{x} = 6x_1^2 - 6x_1x_2 - 2x_2^2 < 0$$

or

$$\mathbf{x}^\top(A_2^\top PA_2 - P)\mathbf{x} = -2x_1^2 - 6x_1x_2 + 6x_2^2 < 0.$$

The results of Theorem 2 are summarized in Figure 2. We ignore whether C0)−C3) $\Rightarrow$ D1)−D2) can be reversed.

When restricting our attention to DPSS's that switch between two subsystems, the search for special classes of Lyapunov functions brings to a new set of (equivalent) sufficient conditions for stabilizability that prove to be stronger than any of the conditions we presented in Theorem 2. This is mainly due to the fact that for a pair of matrices we can resort to the S-procedure [8], [16], which cannot be used for arbitrary $p$tuples of matrices A_i.

Proposition 2: Let A_1 and A_2 be $n \times n$ positive matrices. The following facts are equivalent:

A1) $\exists\ P = P^\top \succ 0$ such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a quadratic positive definite function that satisfies, for every $\mathbf{x} \neq 0$, $\min_{i\in[1,2]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,2]} \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < 0$;

A2) $\exists\ P = P^\top \succ 0$ and $\varepsilon > 0$ such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a quadratic positive definite function that satisfies, for every $\mathbf{x} \neq 0$, $\min_{i\in[1,2]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,2]} \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < -\varepsilon\mathbf{x}^\top\mathbf{x}$;

A3) $\exists\ P = P^\top \succ 0$ and $\alpha \in [0, 1]$ such that

$$\alpha(A_1^\top PA_1 - P) + (1 - \alpha)(A_2^\top PA_2 - P) \prec 0.$$

If A1)−A3) hold, then

B) $\exists\ P = P^\top \succ 0$ and $\alpha \in [0, 1]$, such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ satisfies, for every $\mathbf{x} > 0$, $\alpha\Delta V_1(\mathbf{x}) + (1 - \alpha)\Delta V_2(\mathbf{x}) = \mathbf{x}^\top[\alpha(A_1^\top PA_1 - P) + (1 - \alpha)(A_2^\top PA_2 - P)]\mathbf{x} < 0$.

Condition B) implies each of the following equivalent facts:

C0) $\exists\ \alpha \in [0, 1]$ such that $\alpha A_1 + (1 - \alpha)A_2$ is Schur;

C1) $\exists\ \mathbf{v} \gg 0$ and $\exists\ \alpha \in [0, 1]$ such that $\mathbf{v}^\top[\alpha(A_1 - I_n) + (1 - \alpha)(A_2 - I_n)] \ll 0$;

C2) $\exists\ \mathbf{v} \gg 0$ such that $V(\mathbf{x}) = \mathbf{v}^\top\mathbf{x}$ satisfies, for every $\mathbf{x} > 0$, $\min_{i\in[1,2]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,2]} \mathbf{v}^\top(A_i - I_n)\mathbf{x} < 0$;

C3) $\exists\ P = P^\top$ of rank 1 such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a quadratic copositive function that satisfies, for every $\mathbf{x} > 0$, $\min_{i\in[1,2]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,2]} \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < 0$.

If C0)−C3) hold, then any of the following two equivalent conditions holds:

D1) $\exists\ \tilde{P} = \tilde{P}^\top \succ 0$ such that $V(\mathbf{x}) = \mathbf{x}^\top\tilde{P}\mathbf{x}$ is a quadratic positive definite function that satisfies, for every $\mathbf{x} > 0$, $\min_{i\in[1,2]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,2]} \mathbf{x}^\top(A_i^\top\tilde{P}A_i - \tilde{P})\mathbf{x} < 0$;

D2) $\exists\ \tilde{P} = \tilde{P}^\top \succeq 0$ such that $V(\mathbf{x}) = \mathbf{x}^\top\tilde{P}\mathbf{x}$ is a quadratic copositive function that satisfies, for every $\mathbf{x} > 0$, $\min_{i\in[1,2]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,2]} \mathbf{x}^\top(A_i^\top\tilde{P}A_i - \tilde{P})\mathbf{x} < 0$.

If D1)−D2) hold, then

E) $\exists\ P = P^\top$ such that $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a quadratic copositive function that satisfies, for every $\mathbf{x} > 0$, $\min_{i\in[1,2]} \Delta V_i(\mathbf{x}) = \min_{i\in[1,2]} \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x} < 0$.

Proof: A1) $\Rightarrow$ A2) Both $\mathbf{x}^\top(A_1^\top PA_1 - P)\mathbf{x}$ and $\mathbf{x}^\top(A_2^\top PA_2 - P)\mathbf{x}$ are continuous functions, and so is $f(\mathbf{x}) := \min_{i\in[1,2]} \mathbf{x}^\top(A_i^\top PA_i - P)\mathbf{x}$. By Weierstrass' theorem, being

B) $\exists\, P = P^\top \succ 0$ and $\exists\, \alpha_1, \alpha_2, \ldots, \alpha_p \geq 0$, with $\sum_{i=1}^p \alpha_i = 1$, s.t. $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ satisfies $\sum_{i=1}^p \alpha_i \Delta V_i(\mathbf{x}) < 0,\ \forall \mathbf{x} > 0$

$\not\Uparrow \quad \Downarrow$

C0) $\exists\, \alpha_1, \alpha_2, \ldots, \alpha_p \geq 0$ with $\sum_{i=1}^p \alpha_i = 1$, s.t. $\sum_{i=1}^p \alpha_i A_i$ is Schur $\Longleftrightarrow$ **C1)** $\exists\, \mathbf{v} \gg 0$ and $\exists\, \alpha_1, \alpha_2, \ldots, \alpha_p \geq 0$ with $\sum_{i=1}^p \alpha_i = 1$, s.t. $\mathbf{v}^\top \sum_{i=1}^p \alpha_i (A_i - I_n) \ll 0$ $\Longleftrightarrow$ **C2)** $\exists\, \mathbf{v} \gg 0$ s.t. $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ satisfies $\min_{i \in [1,p]} \Delta V_i(\mathbf{x}) < 0, \forall \mathbf{x} > 0$ $\Longleftrightarrow$ **C3)** $\exists\, P = P^\top, \mathrm{rk} P = 1$ s.t. $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ is quadratic copositive and $\min_{i \in [1,p]} \Delta V_i(\mathbf{x}) < 0, \forall \mathbf{x} > 0$

$\Downarrow$

D1) $\exists\, \tilde{P} = \tilde{P}^\top \succ 0$ s.t. $V(\mathbf{x}) = \mathbf{x}^\top \tilde{P} \mathbf{x}$ satisfies $\min_{i \in [1,p]} \Delta V_i(\mathbf{x}) < 0, \forall \mathbf{x} > 0$ $\Longleftrightarrow$ **D2)** $\exists\, \tilde{P} = \tilde{P}^\top \succeq 0$ s.t. $V(\mathbf{x}) = \mathbf{x}^\top \tilde{P} \mathbf{x}$ is copositive, and $\min_{i \in [1,p]} \Delta V_i(\mathbf{x}) < 0, \forall \mathbf{x} > 0$

$\not\Uparrow \quad \Downarrow$

E) $\exists\, P = P^\top$ s.t. $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ is copositive, and $\min_{i \in [1,p]} \Delta V_i(\mathbf{x}) < 0,\ \forall \mathbf{x} > 0$

Figure 2: Stabilizability conditions based on the existence of certain CLFs .

$f(\mathbf{x})$ a negative and continuous function on the compact set $\partial \mathcal{B}_1 := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}$, it follows that

$$\max_{\mathbf{x} \in \partial \mathcal{B}_1} f(\mathbf{x}) < -\varepsilon < 0.$$

This implies that, for every $\mathbf{x} \neq 0$, $f(\mathbf{x}) < -\varepsilon \mathbf{x}^\top \mathbf{x}$.

A2) $\Rightarrow$ A3) If either A_1 or A_2 is Schur, the result is obvious. So, we assume that neither of them is. Set $T_i := A_i^\top P A_i - P + \varepsilon I_n$ for $i \in [1, 2]$. Clearly, condition A2) implies that for every $\mathbf{x} \neq 0$ such that $\mathbf{x}^\top T_1 \mathbf{x} \geq 0$, one has $\mathbf{x}^\top T_2 \mathbf{x} < 0$. So, once we prove that there exists $\bar{\mathbf{x}} \neq 0$ such that $\bar{\mathbf{x}}^\top T_1 \bar{\mathbf{x}} > 0$, by making using of the S-procedure in the Appendix we can claim that there exists $\tau \geq 0$ such that $\tau T_1 + T_2 = \tau(A_1^\top P A_1 - P + \varepsilon I_n) + (A_2^\top P A_2 - P + \varepsilon I_n)$ is negative definite, and this immediately implies A3) for $\alpha = \tau/(1+\tau) \in [0, 1)$. To prove that there is a nonzero vector $\bar{\mathbf{x}}$ such that $\bar{\mathbf{x}}^\top T_1 \bar{\mathbf{x}} > 0$, we observe that as A_1 is not Schur, there exists $\bar{\mathbf{x}} \neq 0$ such that $\bar{\mathbf{x}}^\top (A_1^\top P A_1 - P)\bar{\mathbf{x}} \geq 0$. Consequently, $\bar{\mathbf{x}}^\top T_1 \bar{\mathbf{x}} > 0$.

A3) $\Rightarrow$ A1) and A3) $\Rightarrow$ B) are obvious.

The remaining conditions follow from Theorem 2 for the special case $p = 2$. ∎

Remark 7: A3) $\Rightarrow$ C0) has been proved in [17] for general discrete-time switched systems, by using similar arguments to the ones we used to prove B) $\Rightarrow$ C0).

We now investigate the possibility of implementing a state-feedback stabilizing switching law based on one of the Lyapunov functions we previously mentioned. Consider a DPSS (1) whose matrices $A_1, A_2, \ldots, A_p$ satisfy any of the conditions C), D) or E) of Theorem 2. For this system a Lyapunov function $V(\mathbf{x})$ can be found, endowed with one of the following properties: linear copositivity, quadratic copositivity or positive definiteness, and such that

$$\min_{i \in [1,p]} \Delta V_i(\mathbf{x}) < 0, \qquad \forall\ \mathbf{x} > 0. \tag{9}$$

We have the following result, independent of the special kind of Lyapunov function we are considering.

Proposition 3: Given a DPSS (1), if there exists a Lyapunov function $V(\mathbf{x})$, which is either linear copositive or quadratic copositive (in particular, positive definite), and that satisfies (9), then the state feedback switching rule

$$\sigma(\mathbf{x}(t)) := \min\{k : \Delta V_k(\mathbf{x}(t)) \leq \Delta V_i(\mathbf{x}(t)), \forall\ i \in [1, p]\} \tag{10}$$

stabilizes the system, i.e. it makes the state evolution goes to zero for every positive initial state.

Proof: Consider first the case when $V(\mathbf{x})$ is quadratic copositive (possibly positive definite) and hence takes the form $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$. The function

$$\Delta V(\mathbf{x}) := \min_{i \in [1,p]} \Delta V_i(\mathbf{x})$$

is a continuous function that takes negative values in every point of the compact set

$$\mathcal{S} := \mathbb{R}^n_+ \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top P \mathbf{x} = 1\}.$$

So, by Weierstrass' Theorem, $\max_{\mathbf{x}\in\mathcal{S}} \Delta V(\mathbf{x}) = -\nu$, with $0 < \nu \leq 1$, and this ensures that for every positive state $\mathbf{x}$, $\Delta V(\mathbf{x}) \leq -\nu \mathbf{x}^\top P \mathbf{x}$. This ensures that $V(\mathbf{x}(t+1)) = V(\mathbf{x}(t)) + \Delta V(\mathbf{x}(t)) \leq (1-\nu)\mathbf{x}^\top(t) P \mathbf{x}(t) \leq (1-\nu)^{t+1}\mathbf{x}^\top(0) P \mathbf{x}(0)$. Thus $V(\mathbf{x}(t))$ converges to zero, and $\mathbf{x}(t)$ converges to zero in turn.

The proof[3] in case of a linear copositive function $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ follows the same lines, upon assuming $\mathcal{S} := \mathbb{R}^n_+ \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{v}^\top \mathbf{x} = 1\}$. ∎

The existence of a quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, such that $\sum_{i=1}^p \alpha_i \Delta V_i(\mathbf{x}) = \sum_{i=1}^p \alpha_i (A_i^\top P A_i - P) \prec 0$ (9) holds for suitable $\alpha_1, \alpha_2, \dots, \alpha_p$, with $\sum_{i=1}^p \alpha_i = 1$, represents a stronger condition w.r.t. condition B) in Theorem 2, and coincides with any of the conditions A1)$-$A3) in Proposition 2 when $p = 2$.

If this is the case, we may resort to a stabilizing switching strategy based on multiple Lyapunov-Metzler inequalities, described by Geromel and Colaneri in [17] for arbitrary (namely non-positive) discrete-time switching systems. Such a strategy is completely equivalent to the one we have just illustrated.

Indeed, as described in the proof of Lemma 1 in [17], if the previous condition is fulfilled for suitable α_i's and P, then, for a suitably small $\varepsilon > 0$, each of the p matrices

$$P_i = A_i^\top P A_i + \varepsilon I_n, \qquad i \in [1, p],$$

satisfies the Lyapunov-Metzler inequality

$$A_i^\top \left(\sum_{j=1}^p \pi_{ji} P_j \right) A_i - P_i < 0,$$

where $\sum_{j=1}^n \pi_{ji} = 1$ for every index $i \in [1, p]$, and the switching strategy given in [17]

$$\sigma(\mathbf{x}(t)) = \arg \min_{i\in[1,p]} \mathbf{x}(t)^\top P_i \mathbf{x}(t)$$

is totally equivalent to the state feedback switching rule (10), since

$$\begin{aligned} & \arg \min_{i\in[1,p]} \mathbf{x}^\top(t) \left[A_i^\top P A_i - P\right] \mathbf{x}(t) \\ = \; & \arg \min_{i\in[1,p]} \mathbf{x}^\top(t) \left[A_i^\top P A_i + \varepsilon I_n\right] \mathbf{x}(t) \\ = \; & \arg \min_{i\in[1,p]} \mathbf{x}^\top(t) P_i \mathbf{x}(t). \end{aligned}$$

Remark 8: If a convex combination of $A_1, A_2, \dots, A_p$ is Schur (i.e., if the DPSS (1) we are dealing with satisfies any of the equivalent conditions C0)$-$C3)), different state feedback switching strategies can be adopted.
Indeed, we may either resort to a linear copositive function, or to a quadratic copositive function (either or rank 1 or of higher rank) or to a quadratic positive definite function. Notice, however, that the switching strategies based on linear copositive functions and those based on quadratic copositive functions of rank 1 are just the same. In fact, as clarified in the proof of Theorem 2, a matrix $P = P^\top$ of rank 1 satisfies condition C3) if and only if it can be expressed as $P = \mathbf{v}\mathbf{v}^\top$, for some vector $\mathbf{v} \gg \mathbf{0}$. On the other hand, by the nonnegativity of the quantities involved,

$$\begin{aligned} & \min\{k : \mathbf{v}^\top (A_k - I_n)\mathbf{x} \leq \mathbf{v}^\top (A_i - I_n)\mathbf{x}, \forall\, i\} \\ = \; & \min\{k : \mathbf{v}^\top A_k \mathbf{x} \leq \mathbf{v}^\top A_i \mathbf{x}, \forall\, i\} \\ = \; & \min\{k : \mathbf{x}^\top A_k^\top \mathbf{v}\mathbf{v}^\top A_k \mathbf{x} \leq \mathbf{x}^\top A_i^\top \mathbf{v}\mathbf{v}^\top A_i \mathbf{x}, \forall\, i\} \\ = \; & \min\{k : \mathbf{x}^\top (A_k^\top \mathbf{v}\mathbf{v}^\top A_k - \mathbf{v}\mathbf{v}^\top)\mathbf{x} \\ & \qquad \leq \mathbf{x}^\top (A_i^\top \mathbf{v}\mathbf{v}^\top A_i - \mathbf{v}\mathbf{v}^\top)\mathbf{x}, \forall\, i\}, \end{aligned}$$

and hence the switching sequences based on $\mathbf{v}^\top \mathbf{x}$ and on $\mathbf{x}^\top \mathbf{v}\mathbf{v}^\top \mathbf{x}$ are just the same.

As the DPSS (1) fulfills condition E) of Theorem 2, too, we may design switching strategies based on the broader class of quadratic copositive Lyapunov functions (of arbitrary rank). Clearly, this class of switching laws encompasses those based on linear copositive functions and hence it ensures convergence performances at least as good as the previous ones.

Similarly, since the set of quadratic positive definite functions is included in the set of quadratic copositive functions, the stabilizing switching laws based on the former are a subset of those based on the latter. So, in order to optimize the converge performances, it is always convenient to resort to switching laws based on quadratic copositive functions.

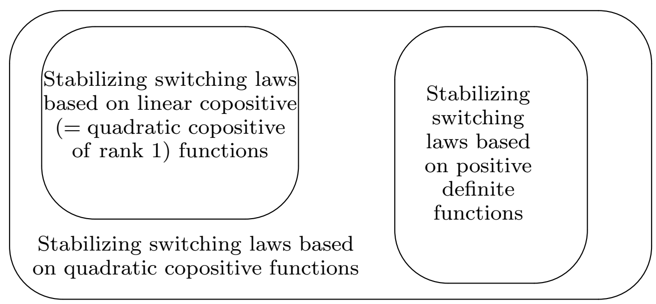

Figure 3: Stabilizing switching laws.

V. Patterns of stabilizing switching sequences

When looking for a stabilizing strategy, some natural questions arise regarding the patterns of the stabilizing switching sequences. For instance, is there a stabilizing switching sequence that eventually takes a constant value? Is there an upper bound on the number of consecutive time instants in which a stabilizing sequence takes the same value? What is the relative frequency with which a matrix A_i may appear in a stabilizing sequence? In this section we provide some insights into these problems, and relate the above mentioned features of the switching sequences to the specific structure of the (non-Schur) positive matrices $A_1, A_2, \dots, A_p$.

We first investigate whether, corresponding to some initial state $\mathbf{x}(0) > 0$, the DPSS (1) admits a switching sequence σ that leads to zero the state evolution and eventually takes a constant value $\ell \in [1, p]$. If A_ℓ is irreducible (and non-Schur), for every positive vector $\bar{\mathbf{x}}$ the sequence $A_\ell^k \bar{\mathbf{x}}, k \in \mathbb{Z}_+$, does

[3]It is worth noticing that this same reasoning would apply to every copositive homogeneous function, thus making this switching rule applicable when dealing with a broader class of Lyapunov functions.

not converge to zero[4]. So, if $\mathbf{x}(0)$ is positive, and the switching sequence σ drives the state evolution to zero and eventually takes (say for $t \geq N$) the constant value ℓ, then $\mathbf{x}(N) = 0$. This amounts to saying that σ drives to zero the state in a finite number of steps.

On the other hand, if all matrices A_i's are irreducible, no positive state can be driven to zero in a finite number of steps. In fact, for every choice of N and of the indices $i_0, i_1, \dots, i_{N-1}$, the equation $0 = \mathbf{x}(N) = A_{i_{N-1}} A_{i_{N-2}} \cdots A_{i_1} A_{i_0} \mathbf{x}(0)$ does not admit positive solutions.

So, to summarize, if all the matrices A_i's are irreducible, no stabilizing switching sequence can eventually take a constant value. If at least one of the matrices A_i is reducible, then a stabilizing switching sequence can eventually take the constant value ℓ only if either A_ℓ is reducible or there is a finite initial portion of the sequence that drives to zero the state evolution.

We want to further explore this issue. Suppose that A_ℓ is irreducible, and no positive state can be driven to zero in a finite number of steps, but the system (1) is stabilizable by resorting to some Lyapunov based switching strategy. We may wonder whether this strategy constrains the maximum number of consecutive time instants in which a stabilizing switching sequence σ takes the value ℓ. Let $V(\mathbf{x})$ be the Lyapunov function upon which the switching strategy is based, and define a slightly modified version of the switching rule, namely

$$\tilde{\sigma}(t) := \begin{cases} \tilde{\sigma}(t-1), & \text{if} \Delta V_{\tilde{\sigma}(t-1)}(\mathbf{x}(t)) \leq \Delta V_i(\mathbf{x}(t)), \forall\ i \in [1,p]; \\ \min\{k : \Delta V_k(\mathbf{x}(t)) \leq \Delta V_i(\mathbf{x}(t)), \forall i \in [1,p]\}, & \text{otherwise} \end{cases}$$

with initial condition

$$\tilde{\sigma}(0) := \min\{k : \Delta V_k(\mathbf{x}(0)) \leq \Delta V_i(\mathbf{x}(0)), \forall\ i \in [1,p]\}.$$

It is easy to see that the modified switching law $\tilde{\sigma}$ is equivalent to the one, σ, defined in (10), except when, calling r the value of the switching sequence at time $t-1$, one has $\Delta V_r(\mathbf{x}(t)) \leq \Delta V_i(\mathbf{x}(t)), \forall\ i \in [1,p]$, but there exists an index $k < r$ such that $\Delta V_r(\mathbf{x}(t)) = \Delta V_k(\mathbf{x}(t))$. In fact, the former switching law (10) would impose to $\sigma(\mathbf{x}(t))$ a value smaller than r, while the new one would keep $\tilde{\sigma}(\mathbf{x}(t)) = r$. Consequently, if $\tilde{\sigma}(\mathbf{x}(t)) \neq \tilde{\sigma}(\mathbf{x}(t-1))$, it is also true that $\sigma(\mathbf{x}(t)) \neq \sigma(\mathbf{x}(t-1))$, but the converse is not necessarily true. So, if we prove that there exists an upper bound on the maximum number of consecutive time instants in which the switching rule $\tilde{\sigma}$ takes the value ℓ, this is also an upper bound for σ given in (10).

To find such a bound, we define the following compact sets:

$$\begin{aligned} \mathcal{S}_1 &:= \{\mathbf{x} > 0 : V(\mathbf{x}) = 1 \text{ and } \Delta V_\ell(\mathbf{x}) \leq \Delta V_i(\mathbf{x}), \forall\ i \in [1,p]\}, \\ \mathcal{S}_2 &:= \{\mathbf{x} > 0 : V(\mathbf{x}) = 1,\ \Delta V_\ell(\mathbf{x}) \leq \Delta V_i(\mathbf{x}) \text{ and} \\ &\qquad \Delta V_\ell(A_\ell \mathbf{x}) \leq \Delta V_i(A_\ell \mathbf{x}),\ \forall\ i \in [1,p]\}, \\ \mathcal{S}_3 &:= \{\mathbf{x} > 0 : V(\mathbf{x}) = 1,\ \Delta V_\ell(\mathbf{x}) \leq \\ &\qquad \Delta V_i(\mathbf{x}),\ \Delta V_\ell(A_\ell \mathbf{x}) \leq \Delta V_i(A_\ell \mathbf{x}) \text{ and} \\ &\qquad \Delta V_\ell(A_\ell^2 \mathbf{x}) \leq \Delta V_i(A_\ell^2 \mathbf{x}),\ \forall\ i \in [1,p]\}, \\ \dots &\qquad \dots \end{aligned}$$

[4]Indeed, A_ℓ has a strictly positive eigenvector corresponding to the real eigenvalue $\rho(A_\ell) \geq 1$, and every state $\bar{\mathbf{x}} > 0$ has a nonzero projection along that eigenvector. Consequently, the sequence $A_\ell^k \bar{\mathbf{x}}, k \in \mathbb{Z}_+$ does not go to zero asymptotically.

Clearly, $\mathcal{S}_k \supseteq \mathcal{S}_{k+1}, \forall\ k \in \mathbb{Z}_+$, and, by the irreducibility of A_ℓ, $\cap_{k=1}^{+\infty} \mathcal{S}_k = \emptyset$. Since all sets are compact, this ensures that $\mathcal{S}_N = \emptyset$ for some index $N \in \mathbb{Z}_+$, and hence no Lyapunov based stabilizing switching sequence may take the value ℓ for more than $N-1$ consecutive times.

Finally, we recall that in Theorem 2 we have shown that the existence of a Schur convex combination of the positive matrices $A_i \in [1,p]$:

$$\sum_{i=1}^{p} \alpha_i A_i, \quad \text{with } \alpha_1, \alpha_2, \dots, \alpha_p > 0 \text{ and } \sum_{i=1}^{p} \alpha_i = 1, \quad (11)$$

is a sufficient condition for the stabilizability of the DPSS (1). We aim to show how the coefficients of the Schur convex combination (11) can be related to the relative frequencies of $A_1, \dots, A_p$ in a stabilizing sequence.

Notice that as the eigenvalues of a matrix are continuous functions of its entries, the combination (11) remains Schur if the α_i's are slightly perturbed. This amounts to saying that there exists $\varepsilon > 0$ such that $[\alpha_i - \varepsilon, \alpha_i + \varepsilon] \subset (0,1), \forall\ i \in [1,p]$, and the matrix $\sum_{i=1}^p \tilde{\alpha}_i A_i$ is Schur for all $\tilde{\alpha}_i \in [\alpha_i - \varepsilon, \alpha_i + \varepsilon]$, $i \in [1,p]$.

On the other hand, by Proposition 1, we know that stabilizability is equivalent to the existence of some Schur matrix product $A_{i_{N-1}} A_{i_{N-2}} \cdots A_{i_1} A_{i_0}$, where $N \in \mathbb{N}$ and $i_0, i_1, \dots, i_{N-1} \in [1,p]$, namely, in the language of noncommutative algebra, to the existence of some word $w \in \Xi^*$ such that $w(A_1, A_2, \dots, A_p)$ is Schur. We want to investigate how these two facts are related.

To this end, we need a preliminary lemma regarding multinomial distributions. The multinomial distribution represents the generalization of the binomial distribution, and it is the joint distribution of a set of random variables which are the number of occurrences of the possible outcomes in a finite sequence of trials. At each trial, p outcomes are possible, each of them characterized by a nonzero probability α_i and clearly $\sum_{i=1}^p \alpha_i = 1$. The interested reader is referred to [13] for more details.

Lemma 1: Consider the multinomial distribution with p possible outcomes and positive probabilities $\alpha_1, \alpha_2, \dots, \alpha_p$, with $\sum_{i=1}^p \alpha_i = 1$. Let $\varepsilon > 0$ be small enough so that $[\alpha_i - \varepsilon, \alpha_i + \varepsilon] \subset (0,1)$ for every index $i \in [1,p]$, let N be a positive integer and let $\mathcal{P}(\alpha_1, \dots, \alpha_p, N, \varepsilon)$ denote the set of all nonnegative integer $p$tuples $(k_1, k_2, \dots, k_p)$ that satisfy

$$\begin{array}{rcl} N\alpha_1 - N\varepsilon & < k_1 < & N\alpha_1 + N\varepsilon \\ N\alpha_2 - N\varepsilon & < k_2 < & N\alpha_2 + N\varepsilon \\ \dots & \dots & \dots \\ N\alpha_p - N\varepsilon & < k_p < & N\alpha_p + N\varepsilon \\ k_1 + k_2 + \dots + k_p & = & N. \end{array} \quad (12)$$

Then, if $N > \frac{2p^3}{\varepsilon^3}$, we have

$$\sum_{(k_1,\dots,k_p) \in \mathcal{P}(\alpha_1,\dots,\alpha_p,N,\varepsilon)} \frac{N!}{k_1! k_2! \dots k_p!} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_p^{k_p} > 1 - \varepsilon \quad (13)$$

Proof: We first quote a classical result regarding the binomial coefficients

$$b(k,N,\alpha) := \binom{N}{k}\alpha^k(1-\alpha)^{N-k}. \quad (14)$$

Fact: [13] *Let* $\alpha \in (0,1)$ *and assume that* $\delta > 0$ *satisfies* $[\alpha-\delta,\alpha+\delta] \subset (0,1)$. *If* $N \in \mathbb{N}$ *is such that* $N > 2/\delta^3$, *then*

$$\sum_{k\in[N\alpha-N\delta,N\alpha+N\delta]} b(k,N,\alpha) > 1-\delta. \quad (15)$$

Consider now the multinomial distribution with p possible outcomes and probabilities $\alpha_1, \alpha_2, \ldots, \alpha_p$. When N independent trials are performed, the probability distributes according to the terms appearing in the right hand side of the following formula

$$\begin{aligned} 1 &= (\alpha_1+\alpha_2+\ldots+\alpha_p)^N \\ &= \sum_{\substack{k_1,k_2,\ldots,k_p\in\mathbb{N}\\ \sum_i k_i=N}} \frac{N!}{k_1!k_2!\cdots k_p!}\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p}. \quad (16)\end{aligned}$$

Note that, if one sets a fixed value for an index k_j in the above summation, say $k_j = \bar{k}_j$, then

$$\begin{aligned} &\sum_{\substack{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_p\in\mathbb{N}\\ \sum_{i\neq j} k_i=N-\bar{k}_j}} \frac{(N-\bar{k}_j)!}{k_1!\cdots k_{j-1}!\,k_{j+1}!\cdots k_p!}\cdot \\ &\cdot\quad \alpha_1^{k_1}\cdots\alpha_{j-1}^{k_{j-1}}\alpha_{j+1}^{k_{j+1}}\cdots\alpha_p^{k_p}\frac{N!}{\bar{k}_j!(N-\bar{k}_j)}\alpha_j^{\bar{k}_j} \\ &= \binom{N}{\bar{k}_j}\alpha_j^{\bar{k}_j}(\alpha_1+\ldots+\alpha_{j-1}+\alpha_{j+1}+\ldots+\alpha_p)^{N-\bar{k}_j} \\ &= \binom{N}{\bar{k}_j}\alpha_j^{\bar{k}_j}(1-\alpha_j)^{N-\bar{k}_j} = b(\bar{k}_j,N,\alpha_j),\end{aligned}$$

that, according to (14), represents a term of the binomial distribution with probabilities α_j and $1-\alpha_j$. By (15), if $N > \frac{2}{(\varepsilon/p)^3}$, the sum of all terms $b(k_j,N,\alpha_j)$ of the binomial distribution, as k_j varies outside the interval $[N\alpha_j - N\varepsilon, N\alpha_j + N\varepsilon] \supset \left[N\alpha_j - N\frac{\varepsilon}{p}, N\alpha_j + N\frac{\varepsilon}{p}\right]$, is less than $\frac{\varepsilon}{p}$, and consequently, in the multinomial distribution,

$$\begin{aligned} &\sum_{\substack{\sum_i k_i=N\\ k_j\notin[N\alpha_j-N\varepsilon,N\alpha_j+N\varepsilon]}} \frac{N!}{k_1!k_2!\cdots k_p!}\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p} \\ &= \sum_{k_j\notin[N\alpha_j-N\varepsilon,N\alpha_j+N\varepsilon]} b(k_j,N,\alpha_j) < \frac{\varepsilon}{p}.\end{aligned}$$

Therefore, for the $p$tuples $(k_1,k_2,\ldots,k_p)$ belonging to $\mathcal{P}(\alpha_1,\ldots,\alpha_p,N,\varepsilon)$ we have

$$\begin{aligned} &\sum_{(k_1,\ldots,k_p)\in\mathcal{P}(\alpha_1,\ldots,\alpha_p,N,\varepsilon)} \frac{N!}{k_1!k_2!\ldots k_p!}\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p} \\ &= 1 - \sum_{(k_1,\ldots,k_p)\notin\mathcal{P}(\alpha_1,\ldots,\alpha_p,N,\varepsilon)} \frac{N!}{k_1!k_2!\ldots k_p!}\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p} \\ &\geq 1 - \sum_{j=1}^{p}\Big(\sum_{\substack{\sum_i k_i=N\\ k_j\notin[N\alpha_j-N\varepsilon,N\alpha_j+N\varepsilon]}} \frac{N!}{k_1!k_2!\ldots k_p!}\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p}\Big) \\ &> 1-\varepsilon\end{aligned}$$

■

Proposition 4: Let $A_1, A_2, \ldots, A_p$ be $n\times n$ nonnegative matrices, and suppose that there exist $\alpha_1, \alpha_2, \ldots, \alpha_p \in (0,1)$, with $\sum_{i=1}^p \alpha_i = 1$, such that $\sum_{i=1}^p \alpha_i A_i$ is Schur.
Let $\varepsilon > 0$ be small enough so that

i) $[\alpha_i - \varepsilon, \alpha_i + \varepsilon] \subset (0,1)$ for every index $i \in [1,p]$, and
ii) for every choice of coefficients $\tilde{\alpha}_i \in [\alpha_i - \varepsilon, \alpha_i + \varepsilon]$, with $\sum_{i=1}^p \tilde{\alpha}_i = 1$, the convex combination $\sum_{i=1}^p \tilde{\alpha}_i A_i$ is a Schur matrix.

Then there exists $w \in \Xi^*$ such that

(a) $\frac{|w|_i}{\sum_{i=1}^p |w|_i} \in [\alpha_i - \varepsilon, \alpha_i + \varepsilon],\ i \in [1,p]$, and
(b) $w(A_1, A_2, \ldots, A_p)$ is Schur.

Proof: Given $\varepsilon > 0$ such that i) and ii) hold true, as $\sum_{i=1}^p \alpha_i A_i$ is Schur, there exists $\bar{N} > \frac{2p^3}{\varepsilon^3}$ such that $N > \bar{N}$ implies

$$\frac{1-\varepsilon}{n^3}\mathbf{1}_n\mathbf{1}_n^\top > \left(\sum_{i=1}^p \alpha_i A_i\right)^N. \quad (17)$$

Let $\mathcal{P}(\alpha_1,\ldots,\alpha_p,N,\varepsilon)$ be, as in the previous lemma, the set of all nonnegative integer $p$tuples $(k_1,k_2,\ldots,k_p)$ satisfying (12).
As the Nth power of $\sum_i \alpha_i A_i$ involves all matrix products of $A_1,\ldots A_p$ that include k_1 times the factor A_1, k_2 times the factor A_2, ..., k_p times the factor A_p, with $\sum_{i=1}^p k_i = N$, we get

$$\begin{aligned}\left(\sum_{i=1}^p \alpha_i A_i\right)^N &= \sum_{\substack{k_1,k_2,\ldots,k_p\in\mathbb{N}\\ \sum_i k_i=N}} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p}\cdot \\ &\cdot \sum_{\substack{|w|_1=k_1\\ \cdots\\ |w|_p=k_p}} w(A_1,A_2,\ldots A_p) \\ &\geq \sum_{(k_1,\ldots,k_p)\in\mathcal{P}(\alpha_1,\ldots,\alpha_p,N,\varepsilon)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p}\cdot \\ &\cdot \sum_{\substack{|w|_1=k_1\\ \cdots\\ |w|_p=k_p}} w(A_1,A_2,\ldots A_p)\end{aligned}$$

and hence, if $N > \bar{N}$, (17) implies

$$\begin{aligned}\mathbf{1}_n\mathbf{1}_n^\top\frac{1-\varepsilon}{n^3} &> \sum_{(k_1,\ldots,k_p)\in\mathcal{P}(\alpha_1,\ldots,\alpha_p,N,\varepsilon)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p} \\ &\cdot \sum_{\substack{|w|_1=k_1\\ \cdots\\ |w|_p=k_p}} w(A_1,A_2,\ldots A_p). \quad (18)\end{aligned}$$

We claim that, among all matrix words $w(A_1,A_2,\ldots,A_p)$ involved in (18), one at least is Schur. To prove this assertion, it is enough [38] to show that at least one matrix word satisfies the condition

$$w(A_1,A_2,\ldots,A_p) < \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top.$$

Suppose, by contradiction, that every matrix word in (18) includes at least one element greater than or equal to $\frac{1}{n}$, and consequently $\mathbf{1}_n^\top w(A_1,\ldots,A_p)\mathbf{1}_n \geq \frac{1}{n}$, for all $w \in$

$\Xi^*, |w|_i = k_i \in [N\alpha_i - N\varepsilon, N\alpha_i + N\varepsilon],\ i \in [1,p]$, with $\sum_{i=1}^p |w|_i = N$.
Then, by premultiplying by $\mathbf{1}_n^\top$ and by postmultiplying by $\mathbf{1}_n$ both members of (18), we get

$$\begin{aligned}
\frac{1-\varepsilon}{n} &> \sum_{(k_1,\dots,k_p)\in\mathcal{P}(\alpha_1,\dots,\alpha_p,N,\varepsilon)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p} \\
&\cdot \sum_{\substack{|w|_1=k_1\\ \cdots \\ |w|_p=k_p}} \mathbf{1}_n^\top w(A_1,A_2,\dots A_p)\mathbf{1}_n \\
&\geq \sum_{(k_1,\dots,k_p)\in\mathcal{P}(\alpha_1,\dots,\alpha_p,N,\varepsilon)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p} \sum_{\substack{|w|_1=k_1\\ \cdots \\ |w|_p=k_p}} \frac{1}{n} \\
&= \frac{1}{n} \sum_{(k_1,\dots,k_p)\in\mathcal{P}(\alpha_1,\dots,\alpha_p,N,\varepsilon)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_p^{k_p} \\
&\cdot \frac{N!}{k_1!k_2!\dots k_p!} > \frac{1-\varepsilon}{n},
\end{aligned}$$

which is a clear contradiction. So, condition (b) holds. To conclude, we have to prove condition (a). To this end, it is sufficient to notice that, if $(k_1,\dots,k_p) = (|w|_1,\dots|w|_p) \in \mathcal{P}(\alpha_1,\dots\alpha_p,N,\varepsilon)$, then, $\forall\ i \in [1,p]$,

$$\frac{|w|_i}{|w|_1+\ldots+|w|_p} \in \left[\frac{N\alpha_i - N\varepsilon}{N}, \frac{N\alpha_i + N\varepsilon}{N}\right] = [\alpha_i - \varepsilon, \alpha_i + \varepsilon].$$

■

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APPENDIX

Lemma 2 (see [1], Corollary 3.49): Let W be an $p\times n$ real matrix. Then one and only one of the following alternatives holds:

a) $\exists\ \mathbf{v} > 0$ such that $\mathbf{v}^\top W \ll \mathbf{0}^\top$;
b) $\exists\ \mathbf{z} > 0$ such that $W\mathbf{z} \geq \mathbf{0}$ (namely $W\mathbf{z} \in \mathbb{R}_+^{p\times 1}$).

The following lemma provides a restatement of the S-procedure, as it can be found, for instance, in [8], which is particularly convenient for the proof of A2) ⇒ A3) in Proposition 2.

Lemma 3 (S-procedure): Let T_1 and $T_2 \in \mathbb{R}^{n\times n}$ be two symmetric matrices, and suppose that there exists $\bar{\mathbf{x}} \neq 0$ such that $\bar{\mathbf{x}}^\top T_1 \bar{\mathbf{x}} > 0$. Then, the following facts are equivalent ones:

i) for every $\mathbf{x} \neq 0$ such that $\mathbf{x}^\top T_1 \mathbf{x} \geq 0$, one finds $\mathbf{x}^\top T_2 \mathbf{x} < 0$;
ii) there exists $\tau \geq 0$ such that $T_2 + \tau T_1$ is negative definite.

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