On the controllability and stabilizability of non-homogeneous multi-agent dynamical systems

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Abstract

In this paper we consider a supervisory control scheme for non-homogenous multi-agent systems. Each agent is modeled through an independent strictly proper SISO state space model, and the supervisory controller, representing the information exchange among the agents, is implemented in turn via a linear state-space model. Controllability and observability of the overall system are characterized, and some preliminary results about stability and stabilizability are provided. The paper extends to non-homogenous multi-agent systems some of the results obtained in [3], [4], [6] for the homogenous case.

Index Terms

Multi-agent system, supervisory control, controllability, asymptotic stability/stabilizability, polynomial matrices.

I. Introduction

Several systems in the areas of manufacturing, transportation, and telecommunications can be effectively represented as networks of agents, mutually interacting and exchanging information. Dynamical interactions among agents, and the intrinsic complexity of many physical networks, make the analysis and control of multi-agent network systems quite a challenging task, mostly

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due to complexity issues. Research efforts in this area have been quite significant in the last decade. Some fundamental contributions on this topic appeared in [2], [7], [10].

In order to make the analysis computationally more tractable, the simplifying assumption that the agents have common dynamics and identical local controllers is often introduced. As a consequence, they can all be described by the same state-space model and by the same transfer function. So, the overall formation dynamics can be represented as the interconnection of a scalar (diagonal) transfer matrix and of a feedback control block, that represents the communication exchange among the agents [2].

Under these assumptions, Hara and co-authors have been able to describe the overall homogeneous multi-agent system dynamics as a linear system with generalized frequency variable [3], [4], [5], [6], [12], and to derive powerful results regarding controllability, $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norm computation, stability and stabilizability of the overall system.

In this paper we consider the same control configuration as in [3], [4], but we drop the homogeneity assumption on the agents, thus considering the more realistic scenario when each agent is characterized by a distinct strictly proper transfer function. Consequently, the overall system is an interconnection of a diagonal transfer matrix and of a supervisory controller (also referred to in the literature as *cooperative output feedback*). This apparently small change has significant consequences in terms of complexity, as the analysis of the system properties turns out to be much more involved. In this paper we provide a complete characterization of the controllability (and, by duality, of the observability) property of the overall dynamic system, as well as some preliminary results about stability and stabilizability. Comparisons with the results derived in [3], [4], [6], [12] are performed, and results are illustrated with several examples.

In section II we introduce the system model: the agents dynamics will be the plant, while the information exchange among the agents will be described by the supervisory controller. Sections III and IV provide a complete characterization of the controllability property of the overall system. Comments and comparisons regarding this characterization are provided in section V. Finally, in section VI, the stability and stabilization problems are introduced and framed in the general setting of output feedback problem. Some either necessary or sufficient conditions are provided, together with counter-examples.

Before proceeding, we introduce some notation. Given two integers $k, n \in \mathbb{Z}_+$, with $k \leq n$, we denote by $[k, n]$ the set $\{k, k+1, \dots, n\}$. We let $\mathbf{e}_{i,n}$ denote the ith canonical vector of $\mathbb{R}^n$, namely the n-dimensional vector with all its entries equal to zero except for the ith which is 1. For $\mathbf{e}_{n,n}$ we will use the simpler notation $\mathbf{e}_n$. If M is a matrix, we denote by $[M]_{ij}$ its (i,j)th entry. The diagonal (or block diagonal) matrix with diagonal entries (blocks) $M_i, i \in [1,n]$, is denoted by $\mathrm{diag}\{M_1, M_2, \ldots, M_n\}$. A diagonal matrix with all identical diagonal entries is called a *scalar matrix*.

We let $\mathbb{C}_-$ and $\mathbb{C}_+$ be the open left half complex plane and the closed right half complex plane, respectively. We let $\mathbb{R}[s]$ and $\mathbb{R}(s)$ denote the ring of polynomials and the field of rational functions with real coefficients in the indeterminate s. A square polynomial matrix (in particular, a polynomial) is *Hurwitz* if it is of full rank at every point $s \in \mathbb{C}_+$. Given a polynomial matrix $P(s) \in \mathbb{R}[s]^{n\times k}$, we say that $P(s)$ is left prime if it has full row rank in every point s of the complex plane $\mathbb{C}$.

II. System description and corresponding state realization

We consider n SISO autonomous agents, each of them described by a strictly proper state-space model $\Sigma_i = (A_i, \mathbf{b}_i, \mathbf{c}_i^\top)$ of size k_i, $i \in [1,n]$. Let $h_i(s) \in \mathbb{R}(s), i \in [1,n]$, be the strictly proper scalar transfer function of the ith agent. We represent the transfer functions of the n agents by means of the diagonal transfer matrix

$$H(s) = \mathrm{diag}\{h_1(s), h_2(s), \ldots, h_n(s)\}. \tag{1}$$

Accordingly, the matrix $H(s)$ has a state space realization $\Sigma_p = (A,B,C)$ (referred to in the following as, "the plant"), of dimension $K := k_1 + k_2 + \ldots + k_n$, given by the direct sum of the n realizations Σ_i, i.e.

$$\begin{aligned} A &= \mathrm{diag}\{A_1, A_2, \ldots, A_n\},\ B = \mathrm{diag}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \\ C &= \mathrm{diag}\{\mathbf{c}_1^\top, \mathbf{c}_2^\top, \ldots, \mathbf{c}_n^\top\}. \end{aligned} \tag{2}$$

It is worthwhile to point out that the block diagonal structure of the matrices A, B and C in (2) allows us to say that $\Sigma_p = (A,B,C)$ is controllable (observable) if and only if each realization $\Sigma_i = (A_i, \mathbf{b}_i, \mathbf{c}_i^\top)$ is controllable (observable).

We consider a (proper rational) n-dimensional supervisory controller $\Sigma_c = (A_0, B_0, C_0, D_0)$ of transfer matrix $G_0(s) \in \mathbb{R}(s)^{p\times m}$, that acts on the plant in such a way that the overall interconnected scheme has desired properties (specifically, controllability and stability) and/or performances (normally expressed in terms of $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms [5]). The logical scheme describing the plant and the supervisory controller connection is given in Figure 1.

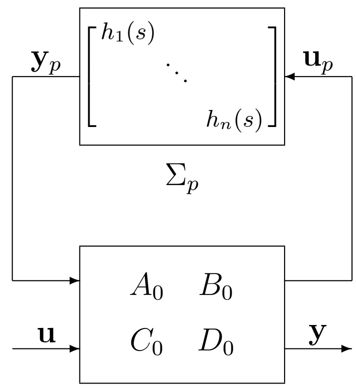

Fig. 1: Supervisory control scheme.

If we denote by $\mathbf{u}_p$ and $\mathbf{y}_p$ the input and the output of the plant, and by $\mathbf{u}$ and $\mathbf{y}$ the input and output of the controller (as well as of the overall system), the system in Figure 1 corresponds to feeding the plant state-space model $\Sigma_p = (A, B, C)$ with the output feedback signal

$$\mathbf{u}_p = A_0 \mathbf{y}_p(t) + B_0 \mathbf{u}(t), \tag{3}$$

and to represent as output, the measurement of the overall system

$$\mathbf{y}(t) = C_0 \mathbf{y}_p(t) + D_0 \mathbf{u}(t).$$

Equivalently, we can think of the system as one obtained by replacing the integrator block in the standard scheme describing the state space realization (A_0, B_0, C_0, D_0), with a state space realization of $H(s)$. In both cases, the block diagram of the overall interconnected system becomes

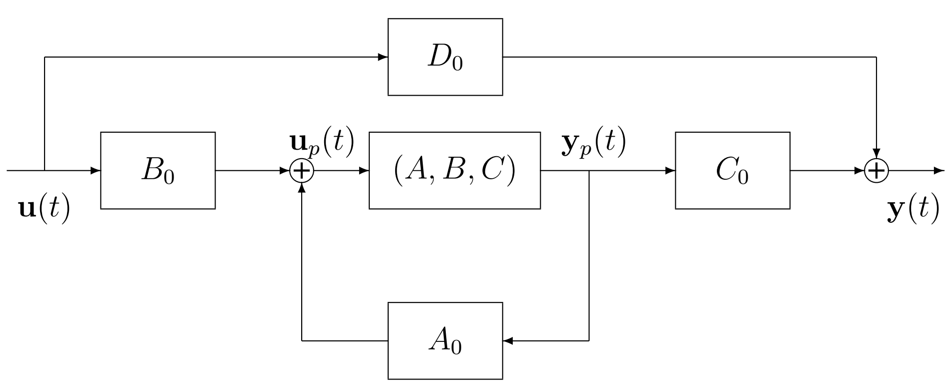

Fig. 2: Block diagram for the overall system Σ_{sc}.

and the state-space model of the overall system Σ_{sc} is

$$\dot{\mathbf{x}}(t) = (A + BA_0C)\mathbf{x}(t) + BB_0\mathbf{u}(t), \tag{4}$$

$$\mathbf{y}(t) = C_0C\mathbf{x}(t) + D_0\mathbf{u}(t), \tag{5}$$

where $\mathbf{x}(t)$ is the K-dimensional state-variable. Using standard computation, it is easy to show that the transfer matrix of the system Σ_{sc} is[1] $W_{sc}(s) = C_0(H(s)^{-1} - A_0)^{-1}B_0 + D_0 \in \mathbb{R}(s)^{p\times m}$. In the sequel we will refer to the overall system matrices as to $\Sigma_{sc} = (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, by this meaning that we will set

$$\begin{aligned} \mathcal{A} &:= A + BA_0C, & \mathcal{B} &:= BB_0, \\ \mathcal{C} &:= C_0C, & \mathcal{D} &:= D_0. \end{aligned}$$

In sections III and IV we investigate controllability of the overall system (4)-(5) in detail, and extend the results obtained to the observability analysis.

III. Controllability of the overall system: preliminaries

The following lemma extends Lemma 2.1 in [3], by resorting to a different approach.

Lemma 1: If the controlled system $\Sigma_{sc} = (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is controllable then all the realizations $\Sigma_i = (A_i, \mathbf{b}_i, \mathbf{c}_i^\top), i \in [1, n]$, are controllable.

Proof: Suppose, by contradiction, that at least one of the Σ_i's is not controllable and hence the pair (A, B) is not. Then $\lambda \in \mathbb{C}$ and a nonzero vector $\mathbf{v} \in \mathbb{R}^K$ can be found such that

$$\mathbf{v}^\top [\, \lambda I_K - A \;\mid\; B \,] = [\, \mathbf{0}^\top \;\mid\; \mathbf{0}^\top \,].$$

But then, it is easy to see that

$$\mathbf{v}^\top [\, \lambda I_K - \mathcal{A} \;\mid\; \mathcal{B} \,] = \mathbf{v}^\top [\, \lambda I_K - A - BA_0C \;\mid\; BB_0 \,] = [\, \mathbf{0}^\top \;\mid\; \mathbf{0}^\top \,].$$

This implies that $(\mathcal{A}, \mathcal{B})$ is not controllable. ∎

As a matter of fact, if B_0 has rank n, then the controllability of each realization Σ_i is also sufficient for the controllability of Σ_{sc}.

[1]Note that $H(s)$ is a square diagonal matrix, and hence it is always invertible.

Proposition 1: If $\operatorname{rank}(B_0) = n$, then Σ_{sc} is controllable if and only if all the realizations $\Sigma_i = (A_i, \mathbf{b}_i, \mathbf{c}_i^\top), i \in [1, n]$, are controllable.

Proof: The necessity has been proved in Lemma 1. So, we only need to prove the sufficiency. Suppose, by contradiction, that Σ_{sc} is not controllable. Then a number $\lambda \in \mathbb{C}$ and a nonzero vector $\mathbf{v} \in \mathbb{R}^K$ can be found such that

$$\mathbf{v}^\top [\,\lambda I_K - \mathcal{A} \quad | \quad \mathcal{B}\,] = \mathbf{v}^\top [\,\lambda I_K - A - BA_0C \quad | \quad BB_0\,] = [\,\mathbf{0}^\top \quad | \quad \mathbf{0}^\top\,].$$

But since B_0 is of full row rank, condition $\mathbf{v}^\top BB_0 = \mathbf{0}^\top$ implies $\mathbf{v}^\top B = \mathbf{0}^\top$. Therefore the previous identity implies $\mathbf{v}^\top [\,\lambda I_K - A \quad | \quad B\,] = [\,\mathbf{0}^\top \quad | \quad \mathbf{0}^\top\,]$. This means that (A, B) is not controllable and hence at least one of the pairs $(A_i, \mathbf{b}_i)$ is not controllable. ■

In view of Proposition 1, in the sequel we will assume that $\operatorname{rank}(B_0) < n$. Also, as a consequence of Lemma 1, we will assume that all the realizations $\Sigma_i, i \in [1, n]$, are controllable. Therefore, without loss of generality (w.l.o.g.) we assume (as in [3]), that they are in canonical controller form [8]:

$$A_i = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & \ddots & \\ \vdots & \vdots & \ddots & \ddots & 0 \\ & & & & 1 \\ -a_0^{(i)} & -a_1^{(i)} & -a_2^{(i)} & \dots & -a_{k_i-1}^{(i)} \end{bmatrix} \mathbf{b}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{6}$$

$$\mathbf{c}_i^\top = [\,c_0^{(i)} \quad c_1^{(i)} \quad c_2^{(i)} \quad \dots \quad c_{k_i-1}^{(i)}\,].$$

From the canonical controller form, it is clear that,

$$h_i(s) = \frac{c_{k_i-1}^{(i)} s^{k_i-1} + \ldots + c_1^{(i)} s + c_0^{(i)}}{s^{k_i} + a_{k_i-1}^{(i)} s^{k_i-1} + \ldots + a_1^{(i)} s + a_0^{(i)}}.$$

In order to simplify the notation, we set $\mathbf{b}_i = \mathbf{e}_{k_i}$,

$$\mathbf{a}_i^\top := [\,a_0^{(i)} \quad a_1^{(i)} \quad a_2^{(i)} \quad \dots \quad a_{k_i-1}^{(i)}\,],$$

and we denote the matrix A_i of size $k_i \times k_i$ in companion form [8], having in the last row the coefficients $-\mathbf{a}_i^\top$ by $C_{k_i}(-\mathbf{a}_i^\top)$. Note that

$$\det(sI_{k_i} - A_i) = \det(sI_{k_i} - C_{k_i}(-\mathbf{a}_i^\top)) = s^{k_i} + a_{k_i-1}^{(i)} s^{k_i-1} + \ldots + a_1^{(i)} s + a_0^{(i)}.$$

As a consequence,

$$\begin{aligned} A &= \begin{bmatrix} C_{k_1}(-\mathbf{a}_1^\top) & & \\ & \ddots & \\ & & C_{k_n}(-\mathbf{a}_n^\top) \end{bmatrix} \\ B &= \operatorname{diag}\{\mathbf{e}_{k_1}, \dots, \mathbf{e}_{k_n}\} \quad C = \operatorname{diag}\{\mathbf{c}_1^\top, \dots, \mathbf{c}_n^\top\}. \end{aligned}$$

Before proceeding we notice that in order to study the controllability of the pair $(\mathcal{A}, \mathcal{B})$ we can introduce w.l.o.g. the simplifying assumption that B_0 is of full column rank and takes the following form

$$B_0 = \begin{bmatrix} I_m \\ \hline \hat{\mathbf{b}}_{m+1}^\top \\ \vdots \\ \hat{\mathbf{b}}_n^\top \end{bmatrix}, \qquad \hat{\mathbf{b}}_i^\top \in \mathbb{R}^{1\times m}, i \in [m+1, n].$$

Indeed, if $B_0 = B_l B_r$, with B_l of full column rank and B_r of full row rank, it is easy to see that $(\mathcal{A}, BB_0)$ is controllable if and only if $(\mathcal{A}, BB_l)$ is controllable. On the other hand, once we assume that B_0 is of full column rank m, we can always postmultiply it by the inverse of one of its nonsingular $m \times m$ submatrices, thus obtaining an identity submatrix. Finally, by applying suitable permutations to the blocks of BB_0 (and hence of $\mathcal{A} = A + BA_0C$), we can assume that

$$\mathcal{B} := BB_0 = \begin{bmatrix} \mathbf{e}_{k_1} & & & \\ & \mathbf{e}_{k_2} & & \\ & & \ddots & \\ & & & \mathbf{e}_{k_m} \\ \hline & \mathbf{e}_{k_{m+1}} \hat{\mathbf{b}}_{m+1}^\top & & \\ & \vdots & & \\ & \mathbf{e}_{k_n} \hat{\mathbf{b}}_n^\top & & \end{bmatrix}, \tag{7}$$

namely that in the upper part we have a block diagonal matrix whose diagonal blocks are canonical vectors, while the bottom part consists of $n - m$ blocks, each of them composed of all zero rows except for the last one that coincides with $\hat{\mathbf{b}}_i^\top, i \in [m+1, n]$. This amounts to assuming for B_0 the aforementioned structure.

Under the previous simplifying assumptions, $\mathcal{B}$ is described as in (7) while $\mathcal{A}$ is expressed as:

$$\mathcal{A}=\begin{bmatrix} C_{k_1}(-\mathbf{d}_{11}^\top) & -\mathbf{e}_{k_1}\mathbf{d}_{12}^\top & \dots & -\mathbf{e}_{k_1}\mathbf{d}_{1n}^\top \\ -\mathbf{e}_{k_2}\mathbf{d}_{21}^\top & C_{k_2}(-\mathbf{d}_{22}^\top) & \dots & -\mathbf{e}_{k_2}\mathbf{d}_{2n}^\top \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbf{e}_{k_n}\mathbf{d}_{n1}^\top & -\mathbf{e}_{k_n}\mathbf{d}_{n2}^\top & \dots & C_{k_n}(-\mathbf{d}_{nn}^\top) \end{bmatrix},$$

where

$$\mathbf{D} := \begin{bmatrix} \mathbf{d}_{11}^\top & \mathbf{d}_{12}^\top & \dots & \mathbf{d}_{1n}^\top \\ \mathbf{d}_{21}^\top & \mathbf{d}_{22}^\top & \dots & \mathbf{d}_{2n}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{d}_{n1}^\top & \mathbf{d}n2^\top & \dots & \mathbf{d}_{nn}^\top \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^\top & \mathbf{0}^\top & \dots & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{a}_2^\top & \dots & \mathbf{0}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^\top & \mathbf{0}^\top & \dots & \mathbf{a}_n^\top \end{bmatrix} - A_0 \begin{bmatrix} \mathbf{c}_1^\top & \mathbf{0}^\top & \dots & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{c}_2^\top & \dots & \mathbf{0}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^\top & \mathbf{0}^\top & \dots & \mathbf{c}_n^\top \end{bmatrix}.$$

It is worthwhile noticing that $\mathcal{A}$ is a multivariable companion matrix. Unfortunately, the system is not in multivariable canonical form and hence it is not necessarily controllable.

Since the pair $(\mathcal{A}, \mathcal{B})$ is controllable if and only if [1] $(\mathcal{A} + \mathcal{B}\mathcal{K}, \mathcal{B})$ is controllable for any choice of $\mathcal{K}$, we take advantage of the structure of $\mathcal{B}$ to simplify the problem. Specifically, the choice $\mathcal{K} = [\, I_m \quad 0\,]\,\mathbf{D}$ ensures that the matrix $\mathcal{A} + \mathcal{B}\mathcal{K}$ becomes as in

$$\mathcal{A}+\mathcal{B}\mathcal{K} = \left[\begin{array}{ccc|ccc} C_{k_1}(\mathbf{0}^\top) & & & & & \\ & \ddots & & & 0 & \\ & & C_{k_m}(\mathbf{0}^\top) & & & \\ \hline -\mathbf{e}_{k_{m+1}}\mathbf{r}_{m+11}^\top & \dots & -\mathbf{e}_{k_{m+1}}\mathbf{r}_{m+1m}^\top & C_{k_{m+1}}(-\mathbf{r}_{m+1m+1}^\top) & \dots & -\mathbf{e}_{k_{m+1}}\mathbf{r}_{m+1n}^\top \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\mathbf{e}_{k_n}\mathbf{r}_{n1}^\top & \dots & -\mathbf{e}_{k_n}\mathbf{r}_{nm}^\top & -\mathbf{e}_{k_n}\mathbf{r}_{nm+1}^\top & \dots & C_{k_n}(-\mathbf{r}_{nn}^\top) \end{array}\right], \tag{8}$$

where the vector coefficients $\mathbf{r}_{ij}^\top$ can be expressed as

$$\mathbf{R} := \begin{bmatrix} \mathbf{r}_{m+11}^\top & \mathbf{r}_{m+12}^\top & \dots & \mathbf{r}_{m+1n}^\top \\ \mathbf{r}_{m+21}^\top & \mathbf{r}_{m+22}^\top & \dots & \mathbf{r}_{m+2n}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{r}_{n1}^\top & \mathbf{r}_{n2}^\top & \dots & \mathbf{r}_{nn}^\top \end{bmatrix} \tag{9}$$

$$= \left[\begin{array}{c|c} -\mathbf{b}_{m+1}^\top & \\ \vdots & I_{n-m} \\ -\mathbf{b}_n^\top & \end{array}\right] \left(\begin{bmatrix} \mathbf{a}_1^\top & \mathbf{0}^\top & \dots & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{a}_2^\top & \dots & \mathbf{0}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^\top & \mathbf{0}^\top & \dots & \mathbf{a}_n^\top \end{bmatrix} - A_0 \begin{bmatrix} \mathbf{c}_1^\top & \mathbf{0}^\top & \dots & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{c}_2^\top & \dots & \mathbf{0}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^\top & \mathbf{0}^\top & \dots & \mathbf{c}_n^\top \end{bmatrix} \right). \tag{10}$$

We are in a position now, to state the characterization of the controllability for the overall controlled system.

IV. Controllability characterization

Proposition 2: Suppose that $\mathrm{rank}(B_0) < n$ and all the realizations $\Sigma_i = (A_i, \mathbf{b}_i, \mathbf{c}_i^\top), i \in [1, n]$, are controllable. Then Σ_{sc} is controllable if and only if the polynomial matrix

$$\Psi(s) := \left[\begin{array}{c|c|c} \mathbf{R} & \begin{matrix} -\hat{\mathbf{b}}_{m+1}^\top \\ \vdots \\ -\hat{\mathbf{b}}_n^\top \end{matrix} & I_{n-m} \end{array}\right] \cdot \left[\begin{array}{c|c} \multicolumn{2}{c}{\mathrm{diag}\left\{ \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_1-1} \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_n-1} \end{bmatrix} \right\}} \\ \hline \mathrm{diag}\{s^{k_1}, \dots, s^{k_m}\} & 0 \\ \hline 0 & \mathrm{diag}\{s^{k_{m+1}}, \dots, s^{k_n}\} \end{array}\right] \tag{11}$$

is left prime.

Proof: To prove the result, we prove that Σ_{sc} is controllable if and only if the pair $(\mathcal{A} + \mathcal{B}\mathcal{K}, \mathcal{B})$, for the given feedback matrix $\mathcal{K}$, is controllable. This can be accomplished by resorting to the PBH matrix

$$\mathcal{P}(s) := [\, sI_K - (\mathcal{A} + \mathcal{B}\mathcal{K}) \;\;|\;\; \mathcal{B}\,], \tag{12}$$

namely by verifying that it has full row rank K for every $s \in \mathbb{C}$.

We first consider the case $s = 0$, which is clearly an eigenvalue of the matrix $\mathcal{A} + \mathcal{B}\mathcal{K}$. It is easily seen that $\mathrm{rank}(\mathcal{P}(0)) = \mathrm{rank}([\, \mathcal{A} + \mathcal{B}\mathcal{K} \;\;|\;\; \mathcal{B}\,])$ coincides with the rank of the matrix (13), obtained from $[\, \mathcal{A} + \mathcal{B}\mathcal{K} \;\;|\;\; \mathcal{B}\,]$ by means of elementary row operations,

$$\left[\begin{array}{ccc|ccc|ccc} C_{k_1}(\mathbf{0}^\top) & & & & & & \mathbf{e}_{k_1} & & \\ & \ddots & & & 0 & & & \ddots & \\ & & C_{k_m}(\mathbf{0}^\top) & & & & & & \mathbf{e}_{k_m} \\ \hline -\mathbf{e}_{k_{m+1}}\tilde{\mathbf{r}}_{m+11}^\top & \dots & -\mathbf{e}_{k_{m+1}}\tilde{\mathbf{r}}_{m+1m}^\top & C_{k_{m+1}}(-\tilde{\mathbf{r}}_{m+1m+1}^\top) & \dots & -\mathbf{e}_{k_{m+1}}\tilde{\mathbf{r}}_{m+1n}^\top & & & \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & 0 & \\ -\mathbf{e}_{k_n}\tilde{\mathbf{r}}_{n1}^\top & \dots & -\mathbf{e}_{k_n}\tilde{\mathbf{r}}_{nm}^\top & -\mathbf{e}_{k_n}\tilde{\mathbf{r}}_{nm+1}^\top & \dots & C_{k_n}(-\tilde{\mathbf{r}}_{nn}^\top) & & & \end{array}\right], \tag{13}$$

where the vectors $\tilde{\mathbf{r}}_{ij}^\top$ are obtained from the corresponding $\mathbf{r}_{ij}^\top$ by replacing all the entries, except for the first one, with zeros. In other words, $\tilde{\mathbf{r}}_{ij}^\top = \mathbf{r}_{ij}^\top \left[\, \mathbf{e}_{1,k_j} \quad \mathbf{0} \quad \dots \quad \mathbf{0} \,\right]$. By the structure of the previous matrix equation, it is clear that such a matrix is of full row rank if and only if the matrix $\mathbf{R}_0 = \mathbf{R} \cdot \mathrm{diag}\{\mathbf{e}_{1,k_1}, \mathbf{e}_{1,k_2}, \dots, \mathbf{e}_{1,k_n}\}$, with $\mathbf{R}$ defined as in (9), is of full row rank.

Next, we need to find necessary and sufficient conditions for the PBH matrix $\mathcal{P}(s)$ to be of full row rank for $s \in \mathbb{C}, s \neq 0$. Assume that s is a fixed nonzero complex number. Again, by applying elementary row operations to the matrix $\mathcal{P}(s)$, we can obtain the matrix (14) (see next page), whose rank coincides with $\mathrm{rank}(\mathcal{P}(s))$,

$$\left[\begin{array}{ccc|ccc|c}
sI_{k_1} - C_{k_1}(\mathbf{0}^\top) & & & & & & \mathbf{e}_{k_1} \\
 & \ddots & & & 0 & & \ddots \\
 & & sI_{k_m} - C_{k_m}(\mathbf{0}^\top) & & & & \mathbf{e}_{k_m} \\
\hline
 & & & D_{k_{m+1}}(\tilde{\mathbf{p}}_{m+1m+1}(s)^\top) & \dots & \mathbf{e}_{k_{m+1}}\tilde{\mathbf{p}}_{m+1n}(s)^\top & \mathbf{e}_{k_{m+1}}\mathbf{q}_{m+1}(s)^\top \\
 & 0 & & \vdots & \ddots & \vdots & \vdots \\
 & & & \mathbf{e}_{k_n}\tilde{\mathbf{p}}_{nm+1}(s)^\top & \dots & D_{k_n}(\tilde{\mathbf{p}}_{nn}(s)^\top) & \mathbf{e}_{k_n}\mathbf{q}_n(s)^\top
\end{array}\right] \tag{14}$$

where

$$\begin{aligned}
D_{k_i}(\tilde{\mathbf{p}}_{ii}(s)^\top) &:= \left[\begin{array}{ccccc}
s & -1 & & & \\
 & s & -1 & & \\
 & & \ddots & \ddots & \\
 & & & s & -1 \\
\hline
 & & \tilde{\mathbf{p}}_{ii}(s)^\top & &
\end{array}\right], \\
\tilde{\mathbf{p}}_{ij}(s)^\top &:= \left[\, 0 \quad 0 \quad \dots \quad 0 \quad p_{ij}(s) \,\right], \\
p_{ij}(s) &:= \begin{cases}
\mathbf{r}_{ij}^\top \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_j-1} \end{bmatrix}, & \text{if } i \neq j; \\
\mathbf{r}_{ij}^\top \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_j-1} \end{bmatrix} + s^{k_j}, & \text{if } i = j;
\end{cases}
\end{aligned}$$

$$\mathbf{q}_i(s)^\top \;:=\; \hat{\mathbf{b}}_i^\top \begin{bmatrix} s^{k_1} & & \\ & \ddots & \\ & & s^{k_m} \end{bmatrix} - [\,\mathbf{r}_{i,1}^\top \quad \mathbf{r}_{1,2}^\top \quad \dots \quad \mathbf{r}_{i,m}^\top\,] \cdot \mathrm{diag}\left\{ \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_1-1} \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_m-1} \end{bmatrix} \right\}$$

Due to its structure, it is clear that matrix (14) is of full row rank for every $s \neq 0$ if and only if the matrix

$$\left[\begin{array}{ccc|c} p_{m+1,m+1}(s) & \dots & p_{m+1,n}(s) & \mathbf{q}_{m+1}(s)^\top \\ \vdots & \ddots & \vdots & \vdots \\ p_{n,m+1}(s) & \dots & p_{n,n}(s) & \mathbf{q}_n(s)^\top \end{array}\right]$$

is of full row rank for every $s \neq 0$. If we refer to the previous matrix as $[\,P(s) \quad | \quad Q(s)\,]$, we can verify that $[\,P(0) \quad | \quad Q(0)\,]$ coincides with $\mathbf{R}_0$ up to a column permutation and the change of sign of some columns. So, we have shown that the system Σ_{sc} is controllable if and only if the matrix $[\,P(s) \quad | \quad Q(s)\,]$ is left prime.

Finally, it is a matter of simple computations to verify that

$$\Psi(s) = [\,P(s) \quad | \quad Q(s)\,] \begin{bmatrix} 0 & I_{n-m} \\ -I_m & 0 \end{bmatrix}.$$

■

As a matter of fact, the polynomial matrix $\Psi(s)$ in (11) can be rewritten in a more revealing way, by resorting to the expression of $\mathbf{R}$ given in (9)-(10). One easily sees that

$$\left[\begin{array}{c|c|c} \mathbf{R} & \begin{matrix} -\hat{\mathbf{b}}_{m+1}^\top \\ \vdots \\ -\hat{\mathbf{b}}_n^\top \end{matrix} & I_{n-m} \end{array}\right] = \left[\begin{array}{c|c} \begin{matrix} -\hat{\mathbf{b}}_{m+1}^\top \\ \vdots \\ -\hat{\mathbf{b}}_n^\top \end{matrix} & I_{n-m} \end{array}\right] \cdot$$

$$\cdot \left[\begin{array}{c|c} \begin{bmatrix} \mathbf{a}_1^\top & \mathbf{0}^\top & \dots & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{a}_2^\top & \dots & \mathbf{0}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^\top & \mathbf{0}^\top & \dots & \mathbf{a}_n^\top \end{bmatrix} - A_0 \begin{bmatrix} \mathbf{c}_1^\top & \mathbf{0}^\top & \dots & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{c}_2^\top & \dots & \mathbf{0}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^\top & \mathbf{0}^\top & \dots & \mathbf{c}_n^\top \end{bmatrix} & I_n \end{array}\right],$$

so if we set

$$n_i(s) := \mathbf{c}_i^\top \mathrm{adj}(sI_{k_i} - A_i)\mathbf{b}_i = \mathbf{c}_i^\top \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_i-1} \end{bmatrix}, \tag{15}$$

$$d_i(s) := \det(sI_{k_i} - A_i) = \mathbf{a}_i^\top \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_i-1} \end{bmatrix} + s^{k_i}, \tag{16}$$

we get

$$\Psi(s) = \left[\begin{array}{c|c} -\hat{\mathbf{b}}_{m+1}^\top & \\ \vdots & I_{n-m} \\ -\hat{\mathbf{b}}_n^\top & \end{array}\right] \cdot \left(\begin{bmatrix} d_1(s) & & \\ & \ddots & \\ & & d_n(s) \end{bmatrix} - A_0 \begin{bmatrix} n_1(s) & & \\ & \ddots & \\ & & n_n(s) \end{bmatrix} \right). \tag{17}$$

The results described above may be encapsulated in the following:

Theorem 1: Suppose that $\mathrm{rank}(B_0) < n$ and all the realizations $\Sigma_i = (A_i, \mathbf{b}_i, \mathbf{c}_i^\top), i \in [1, n]$, are controllable. Then Σ_{sc} is controllable if and only if the polynomial matrix $\Psi(s)$ in (17) is left prime.

V. Comments on Theorem 1

Remark 1: The characterization of controllability of Σ_{sc} provided in Theorem 1 relies on checking the primeness of a polynomial matrix. Appropriate caution must be exercised to take into account numerical difficulties encountered when dealing with large-order polynomial matrices.

Remark 2: It can be shown that the scalar plant case studied in [4], when $H(s) = h(s)I_n$, namely when all the agents have the same dynamics, is a special case of the characterizations obtained in the previous sections. For the case when $\mathrm{rank}(B_0) = n$ the result is quite obvious, since Proposition 1 provides precisely the same result as Proposition 3.1 (ii) in [4]. Hence, it suffices to prove that the necessary and sufficient condition given in Theorem 1 is equivalent to the one obtained in Proposition 3.1 (i) in [4], namely that, when $\mathrm{rank}(B_0) = m < n$, Σ_{sc} *is controllable if and only if the realization* $(A, \mathbf{b}, \mathbf{c}^\top)$ *of* $h(s)$ *is minimal and the realization* (A_0, B_0) *of* $G_0(s)$ *is controllable.*

To this end we have to prove that $\Psi(s)$ is left prime if and only if the triple $(A, \mathbf{b}, \mathbf{c}^\top)$ is controllable and observable and the pair (A_0, B_0) is controllable.

We first notice that, when $H(s) = h(s)I_n$ and all the realizations Σ_i are $(A, \mathbf{b}, \mathbf{c}^\top)$, the matrix $\Psi(s)$ becomes

$$\Psi(s) = \left[\begin{array}{c|c} -\hat{\mathbf{b}}_{m+1}^\top & \\ \vdots & I_{n-m} \\ -\hat{\mathbf{b}}_n^\top & \end{array}\right] \cdot (d(s)I_n - n(s)A_0)\,,$$

where $n(s) = \mathbf{c}^\top \mathrm{adj}(sI - A)\mathbf{b}$ and $d(s) = \det(sI - A)$. For every $s \in \mathbb{C}$, we distinguish three cases:

a) $d(s) = n(s) = 0$;

b) $d(s) \neq 0$, $n(s) = 0$;

c) $n(s) \neq 0$.

In case a) it is clear that $\Psi(s) = 0$, and this case occurs if and only if s is a common zero of $n(s)$ and $d(s)$, which implies that the realization $(A, \mathbf{b}, \mathbf{c}^\top)$ is either not controllable or not observable. In case b) it is easy to see that $\mathrm{rank}(\Psi(s)) = n - m$. Finally, in case c), it is a matter of easy computation to verify that $\mathrm{rank}(\Psi(s)) = n - m$ if and only if

$$\mathrm{rank}(\left[\begin{array}{c|c} \frac{d(s)}{n(s)} I_n - A_0 & B_0 \end{array}\right]) = n.$$

This corresponds to the controllability of the pair (A_0, B_0). So, since we have evaluated all possible cases, we have shown that the characterization given in [4] holds true, as a special case of Theorem 1.

Remark 3: It is worthwhile noticing that when dealing with the general diagonal case, namely $H(s)$ is not scalar, then the controllability of the overall system only requires that the agent state-space models $(A_i, \mathbf{b}_i, \mathbf{c}_i^\top)$ are controllable, and does not require any constraints on the realization of the controller $G_0(s)$. Indeed, by resorting to this supervisory controller we may ensure controllability of the overall system, even if the original plant is not observable, and the supervisory controller is not controllable. This result is quite different from the case of n identical models for the agents.

Example 1: Suppose that $n = 2, m = 1$ and that the plant has the following transfer matrix

$$H(s) = \begin{bmatrix} h_1(s) & \\ & h_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \\ & \frac{1}{s-1} \end{bmatrix}.$$

We assume that the realization of $h_2(s)$ is minimal, while the realization of $h_1(s)$ is not observable and it has a not observable eigenvalue at 0, so that $d_1(s) = s(s+1)$ and $n_1(s) = s$. We assume that the supervisory controller is described by the quadruple $\Sigma_c = (A_0, B_0, C_0, D_0)$ with

$$A_0 = \begin{bmatrix} 0 & 2 \\ 4 & 2 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

that is a not controllable pair. Then it easy to verify that

$$\begin{aligned} \Psi(s) &= \begin{bmatrix} -2 & 1 \end{bmatrix} \left(\begin{bmatrix} s(s+1) & 0 \\ 0 & s-1 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} -2s^2 - 6s & s+1 \end{bmatrix} \end{aligned}$$

is left prime, and hence the overall system is controllable.

Remark 4: The previous characterization of controllability has been obtained under the simplifying assumption that B_0 is of full column rank and its first $m \times m$ submatrix is the identity matrix. The generalization of the characterization to the case of an arbitrary B_0, however, is quite immediate. It is easy to see that, under our assumptions, the matrix

$$B_{perp} := \left[\begin{array}{c|c} -\hat{\mathbf{b}}_{m+1}^\top & \\ \vdots & I_{n-m} \\ -\hat{\mathbf{b}}_n^\top & \end{array} \right]$$

has rows which are a basis[2] for the vector space $(\mathrm{Im}(B_0))^\perp$. So, in the general case, we should just replace this matrix in $\Psi(s)$ with any full row rank matrix whose rows are a basis for the vector space $(\mathrm{Im}(B_0))^\perp$. This allows to immediately derive the analogous characterization for observability:

Theorem 2: If $\mathrm{rank}(C_0) = n$, Σ_{sc} is observable if and only if all the realizations $\Sigma_i = (A_i, \mathbf{b}_i, \mathbf{c}_i^\top), i \in [1, n]$, are observable. On the other hand, if $\mathrm{rank}(C_0) < n$, Σ_{sc} is observable if

[2] It is easy to see that $B_{perp} B_0 = 0$. So, all the rows of B_{perp} are orthogonal to the columns of B_0 and hence belong to $(\mathrm{Im}(B_0))^\perp$. On the other hand, $(\mathrm{Im}(B_0))$ is a subspace of $\mathbb{R}^n$ of rank m, and B_{perp} has $n - m$ linearly independent rows. This ensures that the rows of B_{perp} are a basis for $(\mathrm{Im}(B_0))^\perp$.

and only if all the realizations $\Sigma_i, i \in [1, n]$, are observable and the polynomial matrix

$$\left(\begin{bmatrix} d_1(s) & & \\ & \ddots & \\ & & d_n(s) \end{bmatrix} - A_0 \begin{bmatrix} n_1(s) & & \\ & \ddots & \\ & & n_n(s) \end{bmatrix}\right) \cdot H_c,$$

with H_c a full column rank matrix generating $\ker C_0$, and $d_i(s), n_i(s)$ defined as in (15)-(16), is right prime.

VI. Stability and stabilizability analysis

The aim of this section is to provide some preliminary results about the stability and stabilizability of the controlled system $\Sigma_{sc} = (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. Specifically, we will deal with two issues:
1) [Stability problem] assuming that A_0 is given, under what conditions the matrix $\mathcal{A}$ has all the eigenvalues within the open left half complex plane $\mathbb{C}_-$?
2) [Stabilizability problem] assuming that the supervisory controller is not given, under what conditions a matrix A_0 can be found such that $\mathcal{A}$ has all the eigenvalues in $\mathbb{C}_-$?
We first note that, by resorting to standard matrix computations, we can express the characteristic polynomial of $\mathcal{A}$ as

$$\Delta_{\mathcal{A}}(s) = \det(sI_K - A - BA_0C) = \frac{\prod_{i=1}^n \Delta_{A_i}(s)}{\prod_{i=1}^n \hat{d}_i(s)} \cdot \Delta^*(s),$$

where $\Delta^*(s) := \det M^*(s)$, with

$$M^*(s) := \begin{bmatrix} \hat{d}_1(s) & & \\ & \ddots & \\ & & \hat{d}_n(s) \end{bmatrix} - A_0 \begin{bmatrix} \hat{n}_1(s) & & \\ & \ddots & \\ & & \hat{n}_n(s) \end{bmatrix} \tag{18}$$

and each pair $(\hat{n}_i(s), \hat{d}_i(s))$ provides a coprime representation of the function $h_i(s)$. So, it is clear that $\mathcal{A}$ is Hurwitz if and only if all the eigenvalues of the non-controllable or non-observable part of each system Σ_i belong to $\mathbb{C}_-$ and the polynomial $\Delta^*(s)$ is Hurwitz. So, in the sequel we will always assume that all the systems Σ_i are minimal realizations (as a consequence, $\hat{n}_i(s)$ and $\hat{d}_i(s)$ will coincide with $n_i(s)$ and $d_i(s)$, defined in (15) and (16)), and we will focus on the conditions under which $\Delta^*(s)$ is (or may become) Hurwitz.

As far as stability analysis is concerned, the structure of $M^*(s)$ is more complicated than the structure of the analogous matrix for the case of homogeneous agents, thus making it significantly more difficult to extend the characterizations obtained in [3], [4], [12]. Indeed, when $H(s) = h(s)I_n$, and $n(s)/d(s)$ is a coprime representation of $h(s)$, then

$$\Delta^*(s) = \det[d(s)I_n - A_0 n(s)],$$

and it has been shown that $\Delta^*(s)$ is Hurwitz if and only if for every $\lambda \in \sigma(A_0), p(\lambda, s) := d(s) - \lambda n(s)$ is Hurwitz. This result can be partially extended to a necessary condition for $\Delta^*(s)$ to be Hurwitz in the general non-homogeneous case.

Proposition 3: A necessary condition for $\Delta^*(s) = \det M^*(s)$, with $M^*(s)$ given in (18), to be Hurwitz is that

$$\forall\ \lambda \in \sigma(A_0), \nexists\ \hat{s} \in \mathbb{C}_+ :\ p_i(\lambda, \hat{s}) := d_i(\hat{s}) - \lambda n_i(\hat{s}) = 0, \forall\ i \in [1, n]. \tag{19}$$

Proof: Suppose, by contradiction, that $\exists\ \lambda \in \sigma(A_0)$ and $\hat{s} \in \mathbb{C}_+$, such that $d_i(\hat{s}) - \lambda n_i(\hat{s}) = 0, \forall\ i \in \{1, 2, \ldots, n\}$. So, if $\mathbf{v}^\top$ is a left eigenvector of A_0 corresponding to λ, it is easily seen that $\mathbf{v}^\top$ lies in the left kernel of $M^*(\hat{s})$. This contradicts the fact that $M^*(s)$ and hence $\Delta^*(s)$ is Hurwitz. ■

Unfortunately, while necessary, condition (19) is not sufficient, as illustrated in the example below.

Example 2: Assume

$$d_1(s) = s^2 - s + 5,\ n_1(s) = 1,\ d_2(s) = s^2 + s,\ n_2(s) = s + 1,$$
$$A_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \rightarrow \sigma(A_0) = \{0, 0\}.$$

It is easily seen that $p_1(0, s) = d_1(s)$ and $p_2(0, s) = d_2(s)$ have no common zero in $\mathbb{C}_+$, however $\Delta^*(s) = (s + 1)(s^3 + 0s^2 + 3s + 5)$ is not Hurwitz. Note, however, that a matrix A_0 such that $M^*(s)$ is Hurwitz exists. For instance

$$A_0 = \begin{bmatrix} -12 & 6 \\ 3 & -2 \end{bmatrix}.$$

As far as the stabilization problem is concerned, due to the structure of the overall system matrix $\mathcal{A} = A + BA_0C$, it is immediately seen that this is a static output feedback problem, where A_0 is the static output feedback matrix. Notice that this is consistent with what we said in section II and, in particular, with equation (3). Unfortunately, it is well known that the static output feedback problem is difficult and still unsolved (see [11] for a survey).

When Kimura's condition [9] is satisfied, which in this specific case means that $K = \sum_{i=1}^{n} k_i \leq 2n-1$, then a solution can always be found. The case when all $h_i(s)$ are first order transfer functions trivially falls in this case, but it is interesting to notice that the structure of $M^*(s)$ allows to immediately find a diagonal solution A_0. Indeed, if we assume w.l.o.g. that $d_i(s) = s - \lambda_i$ and $n_i(s) = \beta_i$, then it is easily seen that for $[A_0]_{ii} = \frac{\lambda_i+1}{\beta_i}$, we get

$$d_i(s) - n_i(s)[A_0]_{ii} = s - \lambda_i + \beta_i \frac{\lambda_i + 1}{\beta_i} = s + 1,$$

and hence for

$$A_0 = \operatorname{diag}\left\{\frac{\lambda_1 + 1}{\beta_1}, \dots, \frac{\lambda_n + 1}{\beta_n}\right\}$$

we obtain $\Delta^*(s) = (s+1)^n$.

A similar reasoning applies to the case when all transfer functions $h_i(s)$ are of second order with a stable zero. If so, we may assume without loss of generality that $d_i(s) = s^2 + a_i s + b_i$ and $n_i(s) = c_i s + p_i$, with $c_i \cdot p_i > 0$. So, a sufficiently large $r_i > 0$ can be found such that

$$d_i(s) + \operatorname{sign}(c_i) r_i n_i(s) = s^2 + (a_i + |c_i| r_i)s + (b_i + |p_i| r_i)$$

has all positive coefficients. But then, by Descartes' rule of signs, the polynomial is Hurwitz and $A_0 = \operatorname{diag}\{\operatorname{sign}(c_1)r_1, \dots, \operatorname{sign}(c_n)r_n\}$ is the matrix that stabilizes the system.

Generally speaking, the stabilization by means of a diagonal A_0 is possible if and only if, for every $i \in [1, n]$, there exists $[A_0]_{ii}$ such that $d_i(s) - [A_0]_{ii} n_i(s)$ is Hurwitz, a condition that can be easily tested via the Routh-Hurwitz criterion, or by the root locus criterion (or by Nyquist criterion) by noticing that this is equivalent to find K_i such that the feedback system $\frac{K_i h_i(s)}{1-K_i h_i(s)}$ is BIBO stable. Indeed, a simple root-locus argument allows to say that if each $h_i(s)$ is minimum phase, and either one of the following two conditions holds: (a) the relative degree of each $h_i(s)$ (namely $\deg d_i(s) - \deg n_i(s)$) is not greater than 1 or (b) the relative degree of each $h_i(s)$ is 2 and the sum of the poles is smaller than the sum of the zeros; then a sufficiently large K_i can be found such that $\frac{K_i h_i(s)}{1-K_i h_i(s)}$ is BIBO stable [11].

Stabilization by means of a diagonal A_0 is referred to in [6] as *solely stabilization*, to mean that each single agent can be independently stabilized, as opposed to *cooperative stabilization*, obtained by means of a matrix A_0 whose off-diagonal entries are not all zeros. In the special case when $H(s)$ is a scalar matrix, solely stabilization and cooperative stabilization prove to be equivalent properties for certain classes of functions $h(s)$, but they are nonetheless distinct.

However, even when a diagonal scalar matrix A_0 cannot be found, the stabilization problem can be significantly simplified, as shown in the following proposition that extends a result given in [6].

Proposition 4: Given a scalar matrix $H(s) = h(s)I_n$, with strictly proper diagonal entries $h(s) = \frac{n(s)}{d(s)} \in \mathbb{R}(s)$, the following facts hold:
i) If n is odd, there exists A_0 such that $\det[d(s)I_n - A_0 n(s)]$ is Hurwitz if and only if there exists a scalar matrix A_0 for which this is true.
ii) If n is even, there exists A_0 such that $\det[d(s)I_n - A_0 n(s)]$ is Hurwitz if and only if there exists a block-diagonal A_0, with 2×2 identical diagonal blocks, for which this is true.

Proof: Clearly, for each item only one implication needs to be proved. According to [6], [12], if there exists A_0 such that $\det[d(s)I_n - A_0 n(s)]$ is Hurwitz, then, for every $\lambda \in \sigma(A_0)$, $p(\lambda, s) = d(s) - \lambda n(s)$ is Hurwitz.
i) If n is odd then at least one of the eigenvalues of A_0, say λ^*, is real, but then we have that $A_0 = \lambda^* I_n$ is the desired diagonal matrix.
ii) If n is even and at least one of the eigenvalues of A_0 is real, we can apply the same reasoning as in point i) (and the 2×2 diagonal blocks are, in fact, diagonal). If all the eigenvalues of A_0 are complex, then we can always assume that they are all distinct conjugate pairs $\sigma_i \pm j\omega_i$, $i \in [1, n/2]$ (indeed if A_0 makes $\Delta^*(s)$ Hurwitz, then so does a slightly perturbed version of it, and the eigenvalues of A_0 are a continuous function of its parameters). Let T be a nonsingular matrix such that

$$T^{-1}A_0T = \operatorname{diag}\left\{\begin{bmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{bmatrix}, \ldots, \begin{bmatrix} \sigma_{n/2} & \omega_{n/2} \\ -\omega_{n/2} & \sigma_{n/2} \end{bmatrix}\right\}.$$

Then the result is true, for instance, for

$$A_0 = \operatorname{diag}\left\{\begin{bmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{bmatrix}, \ldots, \begin{bmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{bmatrix}\right\}.$$

■

The previous result, for n odd, allows us to derive a sufficient condition for stabilizability in the general non-homogeneous case.

Proposition 5: Set $d(s) := \prod_{i=1}^n d_i(s)$ and $n(s) := \prod_{i=1}^n n_i(s)$. If $K = \sum_{i=1}^n k_i = \sum_{i=1}^n \deg d_i(s)$ is an odd number and there exists $\tilde{A}_0$ such that $\det[d(s)I_K - \tilde{A}_0 n(s)]$ is Hurwitz, then there exists A_0 such that $\Delta^*(s)$ is Hurwitz.

Proof: If the assumptions in the proposition hold, then, as in the proof of the previous proposition, there exists $\lambda \in \mathbb{R}$ such that $d(s) - \lambda n(s) = \prod_{i=1}^{n} d_i(s) - \lambda \prod_{i=1}^{n} n_i(s)$ is Hurwitz. But then for

$$A_0 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \lambda & & & 0 \end{bmatrix}$$

the matrix $M^*(s)$ has determinant $d(s) - \lambda n(s)$ and hence it is Hurwitz. ■

The converse, however, is not true, as shown by the simple example $h_i(s) = 1/(s-1)$, (namely $d_i(s) = (s-1), n_i(s) = 1$) for $i \in [1, 3]$, for which A_0 ($= -\rho I_3$, $\rho > 1$) can be found such that $\Delta^*(s)$ is Hurwitz, but the stabilization problem for $d(s) = (s-1)^3$ and $n(s) = 1$ is not solvable.

As a general statement, it is rather intuitive that solely stabilization is a stronger property with respect to cooperative stabilization for diagonal matrices $H(s)$. A guess one could make is that they are equivalent properties for transfer functions with no zeros, and hence described in the form

$$h_i(s) = \frac{\beta_i}{d_i(s)}, \qquad \beta_i \in \mathbb{R} \setminus \{0\}, \ \deg d_i \geq 1.$$

However, this is not generally true, as shown by the following

Example 3: Assume

$$d_1(s) = s^2 - s + 5, \ n_1(s) = 1, \ d_2(s) = s^2 + 2s, \ n_2(s) = 1.$$

It is easily seen that $d_1(s) - [A_0]_{11} n_1(s) = s^2 - s + (5 - [A_0]_{11})$ is never Hurwitz, and hence a diagonal solution A_0 does not exist. However, for

$$A_0 = \begin{bmatrix} 2 & 2 \\ 4 & -4 \end{bmatrix}$$

we get

$$\Delta^*(s) = s^4 + s^3 + 5s^2 + 2s + 4,$$

which can be verified to be Hurwitz by means of the Routh-Hurwitz criterion.

Up to now we have derived sufficient conditions for stabilizability. We now provide a necessary condition for diagonal matrices $H(s)$ whose diagonal entries $h_i(s) = \frac{n_i(s)}{d_i(s)}$ have all relative degree greater than or equal to 2. Assume without loss of generality, that each $d_i(s)$ is monic and hence can be described as $d_i(s) = s^{k_i} + \sum_{j=0}^{k_i-1} d_{ij} s^j$. By recalling the standard formula for the determinant of a matrix:

$$\Delta^*(s) = \sum_{\pi} (-1)^{\mathrm{sign}(\pi)} [M^*(s)]_{1\pi(1)} \dots [M^*(s)]_{n\pi(n)},$$

where the summation is taken over all possible permutations π of the first n positive integers, we can easily deduce that

$$\Delta^*(s) = \prod_{i=1}^{n} d_i(s) - \sum_{i=1}^{n} a_{ii} n_i(s) \cdot \left(\prod_{j \neq i} d_j(s) \right) + \Delta_{lo}(s),$$

where $\Delta_{lo}(s)$ is a polynomial of lower order with respect to the other two terms. As the relative degree of each $h_i(s)$ is greater than or equal to 2, then

$$\deg \left(\prod_{i=1}^{n} d_i(s) \right) \geq \deg \left(\sum_{i=1}^{n} a_{ii} n_i(s) \left(\prod_{j \neq i} d_j(s) \right) \right) + 2,$$

and hence the leading coefficient (the coefficient of s^K, $K = \sum_{i=1} k_i$), as well as the coefficient of s^{K-1} in $\Delta^*(s)$ depend uniquely on $\prod_{i=1}^{n} d_i(s)$. So, a necessary condition for $\Delta^*(s)$ to be Hurwitz is that the coefficient of s^{K-1} in $\Delta^*(s)$, $\sum_{i=1}^{n} d_{i,k_i-1}$, is strictly positive.

Example 4: Assume

$$d_1(s) = s^2 - s + 5, \ n_1(s) = 1, \ d_2(s) = s^2 + s, \ n_2(s) = 1.$$

It is easily seen that $d_1(s)d_2(s) = s^4 + 0s^3 + 4s^2 + 5s$, while $\deg n_i d_j = 2$ and $\deg n_i n_j = 1$. Consequently, for every choice of A_0, we have $\Delta^*(s) = s^4 + 0s^3 + as + b, \exists\ a, b \in \mathbb{R}$, which is never Hurwitz.

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