On the stabilizability and consensus of positive homogeneous multi-agent dynamical systems

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Abstract

In this note we consider a supervisory control scheme that achieves either asymptotic stability or consensus for a group of homogenous agents described by a positive state-space model. Each agent is modeled by means of the same SISO positive state-space model, and the supervisory controller, representing the information exchange among the agents, is implemented via a static output feedback. Necessary and sufficient conditions for the asymptotic stability, or the consensus of all agents, are derived under the positivity constraint.

Keywords: Positive linear system, asymptotic/simple stability, Metzler matrix, irreducible matrix, Hurwitz matrix, multi-agent system, consensus.

I. Introduction

The interest in multi-agents systems and consensus problems originated about a decade ago thanks to milestone contributions in [12], [19], [23]. These pioneering works triggered a rich flow of research, strongly stimulated by applications in transportation, telecommunications, and manufacturing, as well as in biology. Indeed, a large number of physical problems can be represented as networks of agents, interacting mutually and exchanging information. To quote a few, sensor networks, coordination of mobile robots or UAVs, flocking and swarming in animal groups, dynamics of opinion forming (see, e.g., [22], [24], [32]).

The intrinsic complexity of many physical networks makes the control of multi-agent systems a rather challenging task, both from the modeling and from the computational point of view.

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A simplifying and realistic assumption in a variety of practical situations is that of assuming that all agents have the same dynamics and identical local control. As a result, all the agents can be modeled by the same state-space model and hence exhibit the same transfer function. Moreover, the overall system dynamics can be mathematically described as the direct sum of several identical state-space realizations, and the communication exchange among the agents can be described as a static output-feedback connection [12], [15], [16], [17], [29].

Research on positive system theory, on the other hand, has a long history. Stimulated by application areas as economy, population dynamics, physiology, pharmacokinetic, etc., the literature on this topic has flourished, moving from fundamental system theoretic problems like stability, stabilizability, control and realization theory [1], [2], [7], [11] to more advanced topics such as robust stability, L_1-gain analysis, KYP lemma and dissipativity [4], [5], [9], [13], [14], [30], [31]. Furthermore, in recent times, positive systems have proved to be a convenient framework for a number of new applications. For example, positive systems have been used to model networks employing TCP and other congestion control protocols [28], to analyse the stability of the Foschini-Miljanic power control algorithm [6], [33], and to design optimal drug treatments to cope with viral mutation [18].

In several contexts where multi-agent systems are used, the positivity constraint on the state-space models describing the agents dynamics arises very naturally. For instance, the classical multi-agent system, whose agents are described either as integrators or as double integrators [12], [27], is an example of interconnected positive system. The emission control problem in a fleet of hybrid vehicles (a network of cars trying to agree on a common CO_2-emission level, by adjusting their speed and the balance between electric and combustion based propulsion) has been described as a consensus problem for positive agents [20], [21]. Sensor networks for greenhouse monitoring can be described as positive multi-agent systems, as each sensor collects, elaborates and exchanges information about physical parameters such as CO_2-concentration, humidity, PH values etc. that are intrinsically nonnegative [25]. Finally, positive multi-agent systems have been fruitfully employed to implement distributed filtering on grid sensor networks [8]. In all these contexts, it is eminently clear that the consensus has a clear practical meaning, since averaging among the measurements provided by the various sensors allows to implement an effective control strategy (see, also, [10], [26], where distributed control and output feedback interconnection of positive state-space models are investigated). Moreover, the positivity constraint on the state-space models is an intrinsic property of the agents' dynamics that needs to be preserved even by the overall controlled system, and hence it must be taken into account explicitly when dealing with consensus.

Motivated by these applications, in this paper we address the stabilizability and consensus problems for homogeneous multi-agents systems, mutually interacting through an output feedback control configuration, under the assumption that the common state-space description of the agents is a positive state model. Specifically, in section II we introduce the system model and we investigate conditions under which a static output feedback matrix can be found that preserves the positivity and makes the resulting system asymptotically stable. Section III addresses the irreducibility property of the overall system matrix, while the consensus problem for positive systems is posed and solved in section IV, where, using the results developed in section II and III, it is shown that consensus can be achieved if and only if static output feedback laws can be found, achieving stability and irreducibility for the resulting feedback system, meanwhile preserving positivity. This establishes a sort of separation principle, since the consensus problem is solvable if and only if the positive stabilization problem is solvable and irreducibility can be achieved via static output feedback. Finally, we suggest potential future research directions based on presented results.

Notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. For any $k, n \in \mathbb{Z}$, with $k \leq n$, $[k, n]$ is the set of integers $\{k, k+1, \dots, n\}$. The (i, j)th entry of a matrix A is denoted by $[A]_{ij}$, the ith entry of a vector $\mathbf{v}$ by $[\mathbf{v}]_i$. A matrix A with entries in $\mathbb{R}_+$ is called *nonnegative*, ($A \geq 0$); a nonnegative and nonzero matrix is *positive* ($A > 0$); a matrix with all positive entries is *strictly positive* ($A \gg 0$). We denote by $\mathbf{1}_n$ the n-dimensional vector with all unitary entries. A *Metzler matrix* is a real square matrix, whose off-diagonal entries, $[A]_{ij}, i \neq j$, are nonnegative. A Metzler matrix $A \in \mathbb{R}^{n \times n}, n > 1$, is *irreducible* if no permutation matrix P can be found such that

$$P^\top A P = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. By the Perron-Frobenius theorem [3], [11], an irreducible Metzler matrix A has a simple real dominant eigenvalue $\lambda_{\max}(A)$, and the corresponding (left or right) eigenvector is strictly positive.

The diagonal (or block diagonal) matrix with diagonal entries (blocks) $M_i, i \in [1, n]$, is denoted by $\mathrm{diag}\{M_1, M_2, \ldots, M_n\}$. A diagonal matrix with all identical diagonal entries is called a *scalar matrix*. Given two matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$, the symbol $A \otimes B$ denotes the *Kronecker product* of A and B.

Given a positive matrix $Q \in \mathbb{R}_+^{n\times n}$, we associate with it a *digraph* $\mathcal{D}(Q)$, with vertices $1, \ldots, n$ [3], [11]. There is an arc (j, i) from j to i if and only if $[Q]_{ij} > 0$. A sequence $j_1 \to j_2 \to j_3 \to \ldots \to j_k \to j_{k+1}$ is a *path* of length k from j_1 to j_{k+1} provided that $(j_1, j_2), (j_2, j_3) \ldots, (j_k, j_{k+1})$ are arcs of $\mathcal{D}(Q)$. A digraph is said to be *strongly connected* if for every pair of distinct vertices $i, j \in [1, n]$ there is a path going from j to i. This is equivalent to the fact that for every $i, j \in [1, n]$ there exists $k \in \mathbb{Z}, k > 0$, such that $[Q^k]_{ij} > 0$. Similarly, given a Metzler matrix $A \in \mathbb{R}^{n\times n}$, we associate with it the digraph of the positive matrix $Q := A - \mathrm{diag}\{[A]_{11}, [A]_{22}, \ldots, [A]_{nn}\}$, and we denote it by $\mathcal{D}(A)$. A Metzler matrix A is irreducible if and only if the associated digraph is strongly connected.

II. System description and positive stabilization

We consider n SISO autonomous agents, each of them described by the same strictly proper continuous-time positive state-space model $\Sigma_h = (A_h, \mathbf{b}_h, \mathbf{c}_h^\top)$ of order d. This means that[1] $\mathbf{b}_h, \mathbf{c}_h \in \mathbb{R}_+^d$, and $A_h \in \mathbb{R}^{d\times d}$ is a Metzler matrix. Let $h(s) \in \mathbb{R}(s)$ be the strictly proper *scalar* transfer function of each agent. The input/output behavior of the n agents is represented by means of the scalar transfer function matrix $H(s) = h(s)I_n$. $H(s)$ has a positive state-space realization $\Sigma_p = (A, B, C)$, of order $N := nd$, given by the direct sum of the n realizations Σ_h:

$$
\begin{aligned}
A &= \mathrm{diag}\{A_h, A_h, \ldots, A_h\},\ B = \mathrm{diag}\{\mathbf{b}_h, \mathbf{b}_h, \ldots, \mathbf{b}_h\}, \\
C &= \mathrm{diag}\{\mathbf{c}_h^\top, \mathbf{c}_h^\top, \ldots, \mathbf{c}_h^\top\}. \qquad (1)
\end{aligned}
$$

We consider a static output feedback (n-dimensional supervisory controller) K, that acts on the system Σ_p as depicted in Figure 1.

[1]We assume that both $\mathbf{b}_h$ and $\mathbf{c}_h$ are nonzero, otherwise the analysis would be trivial. Consequently, $\mathbf{b}_h\mathbf{c}_h^\top > 0$.

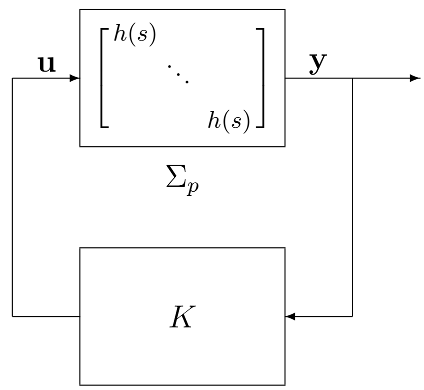

Fig. 1: Overall controlled system Σ_{sc}.

The state-space model of the overall system Σ_{sc} is given by (see [15], [16], [17], [29]):

$$\dot{\mathbf{x}}(t) = (A + BKC)\mathbf{x}(t), \tag{2}$$

$$\mathbf{y}(t) = C\mathbf{x}(t), \tag{3}$$

where $\mathbf{x}(t) = [\, \mathbf{x}_1(t)^\top \quad \mathbf{x}_2(t)^\top \quad \ldots \quad \mathbf{x}_n(t)^\top \,]^\top$ is the N-dimensional state vector of Σ_{sc} and $\mathbf{x}_i(t)$ is the state of the ith agent. In this context we introduce positive stabilization problem as follows:

Positive stabilization problem: *find an $n \times n$ matrix K such that the overall system matrix $\mathcal{A} := A + BKC$ is both Metzler and Hurwitz.*

In order to solve this problem, we first identify all matrices K that make $\mathcal{A}$ Metzler. Since $\mathcal{A}$ has the following block structure

$$\mathcal{A} = \begin{bmatrix} A_h + \mathbf{b}_h\mathbf{c}_h^\top k_{11} & \mathbf{b}_h\mathbf{c}_h^\top k_{12} & \ldots & \mathbf{b}_h\mathbf{c}_h^\top k_{1n} \\ \mathbf{b}_h\mathbf{c}_h^\top k_{21} & A_h + \mathbf{b}_h\mathbf{c}_h^\top k_{22} & \ldots & \mathbf{b}_h\mathbf{c}_h^\top k_{2n} \\ \vdots & & \ddots & \vdots \\ \mathbf{b}_h\mathbf{c}_h^\top k_{n1} & \mathbf{b}_h\mathbf{c}_h^\top k_{n2} & \ldots & A_h + \mathbf{b}_h\mathbf{c}_h^\top k_{nn} \end{bmatrix} = I_n \otimes A_h + K \otimes \mathbf{b}_h\mathbf{c}_h^\top, \tag{4}$$

it is easy to see that $\mathcal{A}$ is Metzler if and only if all blocks $A_h + \mathbf{b}_h\mathbf{c}_h^\top k_{ii}, i \in [1, n]$, are Metzler and all blocks $\mathbf{b}_h\mathbf{c}_h^\top k_{ij}, i \neq j, i, j \in [1, n]$, are nonnegative. By the assumptions on the matrices $A_h, \mathbf{b}_h$ and $\mathbf{c}_h$, this is equivalent to requiring that the off-diagonal entries of K are nonnegative, while the diagonal entries of K are greater than or equal to k^*, where

$$k^* := \min\{k : A_h + \mathbf{b}_h\mathbf{c}_h^\top k \text{ is Metzler}\} = \max_{\substack{i,j: i \neq j, \\ [\mathbf{b}_h]_i[\mathbf{c}_h]_j \neq 0}} -\frac{[A_h]_{ij}}{[\mathbf{b}_h]_i[\mathbf{c}_h]_j}.$$

Therefore, the matrix K makes $\mathcal{A} = A + BKC$ Metzler if and only if $K \geq k^* I_n$. Note that k^* is well defined since the set $\{k : A_h + \mathbf{b}_h\mathbf{c}_h^\top k \text{ is Metzler}\}$ is always non-empty with 0 being an element of it. This also ensures that $k^* \leq 0$. Furthermore, the matrix K is necessarily Metzler, since its off-diagonal entries are always nonnegative. Based on the previous analysis, we can provide the solution to the positive stabilization problem.

Proposition 1: Introduce the two polynomials:

$$d_h(s) \;:=\; \det(sI_d - A_h) = s^d + \sum_{i=0}^{d-1} \alpha_i s^i, \tag{5}$$

$$n_h(s) \;:=\; \mathbf{c}_h^\top \mathrm{adj}(sI_d - A_h)\mathbf{b}_h = \sum_{i=0}^{d-1} \beta_i s^i, \tag{6}$$

associated with the state-space model $\Sigma_h = (A_h, \mathbf{b}_h, \mathbf{c}_h^\top)$. The following facts are equivalent:

i) the positive stabilization problem is solvable, i.e. there exists an $n \times n$ Metzler matrix K such that $\mathcal{A} = A + BKC$ is Metzler Hurwitz;

ii) the matrix $A_h + \mathbf{b}_h \mathbf{c}_h^\top k^*$ is Hurwitz;

iii) $[\,\alpha_{d-1} \quad \dots \quad \alpha_1 \quad \alpha_0\,] \gg k^* [\,\beta_{d-1} \quad \dots \quad \beta_1 \quad \beta_0\,]$.

Proof: i) $\Rightarrow$ ii) Suppose that that $\mathcal{A} = A + BKC$ is Metzler Hurwitz. The Metzler property of $\mathcal{A}$ implies that $K \geq k^* I_n$. On the other hand, given two Metzler matrices $\mathcal{A}_1$ and $\mathcal{A}_2$, condition $\mathcal{A}_2 \geq \mathcal{A}_1$ implies[2] $\lambda_{\max}(\mathcal{A}_2) \geq \lambda_{\max}(\mathcal{A}_1)$. Therefore, $\mathcal{A} \geq A + B(k^* I_n)C$ implies that (the Metzler matrix) $A + B(k^* I_n)C$ is Hurwitz, and this is equivalent to saying that $A_h + \mathbf{b}_h \mathbf{c}_h^\top k^*$ is Hurwitz.

ii) $\Rightarrow$ i) If $A_h + \mathbf{b}_h \mathbf{c}_h^\top k^*$ is Hurwitz, then, by definition of k^*, it is also Metzler, and this trivially implies that the choice $K = k^* I_n$ makes $\mathcal{A}$ both Metzler and Hurwitz.

ii) $\Leftrightarrow$ iii) Note, first, that the characteristic polynomial of $A_h + \mathbf{b}_h \mathbf{c}_h^\top k^*$ can be expressed as

$$\det(sI_d - A_h - \mathbf{b}_h \mathbf{c}_h^\top k^*) = \det(sI_d - A_h)\det(I_d - (sI_d - A_h)^{-1}\mathbf{b}_h \mathbf{c}_h^\top k^*)$$
$$= \det(sI_d - A_h)(1 - k^* \mathbf{c}_h^\top (sI_d - A_h)^{-1}\mathbf{b}_h) = d_h(s) - k^* n_h(s) = s^d + \textstyle\sum_{i=0}^{d-1} (\alpha_i - k^* \beta_i) s^i.$$

Since a Metzler matrix is Hurwitz if and only if all the coefficients of its characteristic polynomial are positive [11], the Metzler matrix$A_h + \mathbf{b}_h \mathbf{c}_h^\top k^*$ is Hurwitz if and only if iii) holds. ∎

[2]Let ρ be a nonnegative number such that $P_1 := \mathcal{A}_1 + \rho I_n$ is a nonnegative matrix. Then $P_2 := \mathcal{A}_2 + \rho I_n \geq \mathcal{A}_1 + \rho I_n =: P_1$. Since P_1 and P_2 are nonnegative matrices, condition $P_1 \leq P_2$ implies [3] that the spectral radius of the two matrices satisfy the inequality $\rho(P_1) \leq \rho(P_2)$. On the other hand, the real dominant eigenvalue of $\mathcal{A}_i$ is related to the spectral radius of the corresponding positive matrix P_i by the identity $\lambda_{\max}(\mathcal{A}_i) = \rho(P_i) - \rho$, $i \in [1, 2]$. This implies $\lambda_{\max}(\mathcal{A}_1) = \rho(P_1) - \rho \leq \rho(P_2) - \rho = \lambda_{\max}(\mathcal{A}_2)$.

Remark 1: By the same reasoning adopted in the last part of the previous proof, we can claim that, in general, the Metzler matrix $A_h + \mathbf{b}_h \mathbf{c}_h^\top k, k \geq k^*$, is Hurwitz if and only if

$$[\,\alpha_{d-1} \quad \dots \quad \alpha_1 \quad \alpha_0\,] \gg k\,[\,\beta_{d-1} \quad \dots \quad \beta_1 \quad \beta_0\,]\,. \tag{7}$$

Note that condition (7) cannot be used to check the Hurwitz property of $A_h + \mathbf{b}_h \mathbf{c}_h^\top k$ unless this matrix is Metzler and hence unless $k \geq k^*$. Finally, note that the previous inequality could be satisfied for every $k \geq k^*$ if and only all β_i's are either negative or zero, a case that can never occur under the given assumptions on $A_h, \mathbf{b}_h$ and $\mathbf{c}_h$ (see the proof of Lemma 3). So, there always exists $\bar{k} \geq k^*$ such that (7) does not hold and this implies that $A_h + \mathbf{b}_h \mathbf{c}_h^\top k$ is not Hurwitz for every $k \geq \bar{k}$.

While Proposition 1 provides a characterization of the solvability of the positive stabilization problem, Proposition 2, below, characterizes all Metzler matrices K that make $\mathcal{A}$ Metzler and Hurwitz. Once the positivity constraint on the matrices has been incorporated, the result immediately follows from Lemma 1 in [29], and hence its proof is omitted.

Proposition 2: [29] Assume that the positive stabilization problem is solvable, and that $K \geq k^* I_n$. The Metzler matrix $\mathcal{A} = A + BKC$ is Hurwitz if and only if $A_h + \mathbf{b}_h \mathbf{c}_h^\top \lambda$ is Hurwitz for every eigenvalue $\lambda \in \sigma(K)$.

Remark 2: We observe that, as K and $k^* I_n$ are both Metzler matrices, condition $K \geq k^* I_n$ implies that $\lambda_{\max}(K) \geq k^*$. This ensures that $A_h + \mathbf{b}_h \mathbf{c}_h^\top \lambda_{\max}(K)$ is Metzler and hence its Hurwitz property can be checked by resorting to (7) for $k = \lambda_{\max}(K)$. On the other hand, the non-dominant eigenvalues $\lambda_i, i \in [2, n]$, of K are not necessarily real and the real ones among them are not necessarily greater than or equal to k^*. So, the Hurwitz stability of the matrices $A_h + \mathbf{b}_h \mathbf{c}_h^\top \lambda_i, i \in [2, n]$, in general cannot be tested by resorting to (7). Finally, note that, as clarified in Remark 1, if $\lambda_{\max}(K) \geq \bar{k}$, then surely $A_h + \mathbf{b}_h \mathbf{c}_h^\top \lambda_{\max}(K)$ cannot be Hurwitz and hence $\mathcal{A} = A + BKC$ is not Hurwitz, in turn.

III. Irreducibility property of the matrix $\mathcal{A}$

In the previous section we have investigated the positive stabilizability of the overall positive multi-agent system. As clarified in Proposition 1, if a static output feedback K can be found that makes the resulting system both positive and asymptotically stable, then $K = k^* I_n$ is a possible solution. This solution, however, corresponds to the case when the matrix $\mathcal{A}$ is block diagonal, and hence each agent does not interact with the other agents. As we will see in the next section, when addressing the consensus problem, communication among agents and hence the irreducibility of the matrix $\mathcal{A}$ is fundamental. Without the irreducibility of $\mathcal{A}$ we could not ensure that, all agents asymptotically converge to the same nontrivial (i.e., nonzero) decision, independently of the (nonnegative) initial conditions. For this reason, in this section we explore the necessary and sufficient conditions for the irreducibility of $\mathcal{A}$.

Lemma 1: Consider a Metzler matrix $K \in \mathbb{R}^{n\times n}$, with $K \geq k^* I_n$. If $\mathcal{A} = A + BKC$ is irreducible, then both $A_h + \mathbf{b}_h \mathbf{c}_h^\top$ and K are irreducible.

Proof: We prove the statement by contrapositive, namely we show that if either $A_h + \mathbf{b}_h \mathbf{c}_h^\top$ or K is reducible then $\mathcal{A}$ is reducible as well. Suppose that $A_h + \mathbf{b}_h \mathbf{c}_h^\top$ is reducible, and let P_1 be a permutation matrix such that

$$P_1^\top (A_h + \mathbf{b}_h \mathbf{c}_h^\top) P_1 = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix},$$

where D_{11} and D_{22} are square Metzler matrices, while D_{12} is nonnegative. Accordingly,

$$P_1^\top A_h P_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \text{and} \qquad P_1^\top (\mathbf{b}_h \mathbf{c}_h^\top) P_1 = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

and since A_{21} and T_{21} are both nonnegative matrices (because $P_1^\top A_h P_1$ is Metzler and $P_1^\top (\mathbf{b}_h \mathbf{c}_h^\top) P_1$ is nonnegative), this is possible if and only if $A_{21} = T_{21} = 0$. Therefore, for every K, we have

$$(I_n \otimes P_1^\top)\mathcal{A}(I_n \otimes P_1) = \text{diag}\{P_1^\top, P_1^\top, \dots, P_1^\top\}\ (A + BKC)\ \text{diag}\{P_1, P_1, \dots, P_1\} =$$

$$\begin{bmatrix}
A_{11}+k_{11}T_{11} & A_{12}+k_{11}T_{12} & k_{12}T_{11} & k_{12}T_{12} & \dots & k_{1n}T_{11} & k_{1n}T_{12} \\
0 & A_{22}+k_{11}T_{22} & 0 & k_{12}T_{22} & \dots & 0 & k_{1n}T_{12} \\
k_{21}T_{11} & k_{21}T_{12} & A_{11}+k_{21}T_{11} & A_{12}+k_{21}T_{12} & \dots & k_{2n}T_{11} & k_{2n}T_{12} \\
0 & k_{21}T_{22} & 0 & A_{22}+k_{21}T_{22} & \dots & 0 & k_{2n}T_{12} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
k_{n1}T_{11} & k_{n1}T_{12} & k_{n2}T_{11} & k_{n2}T_{12} & \dots & A_{11}+k_{nn}T_{11} & A_{12}+k_{nn}T_{12} \\
0 & k_{n1}T_{22} & 0 & k_{n2}T_{22} & \dots & 0 & A_{22}+k_{nn}T_{12}
\end{bmatrix}.$$

Set $q := \dim A_{11} = \dim T_{11}$, and introduce the $N \times N$ permutation matrix

$$\Pi_2 = \left[\begin{array}{ccc|ccc} \Pi_1 & & & \tilde{\Pi}_1 & & \\ & \ddots & & & \ddots & \\ & & \Pi_1 & & & \tilde{\Pi}_1 \end{array}\right], \text{ where } \quad \Pi_1 = \begin{bmatrix} I_q \\ 0_{(d-q)\times q} \end{bmatrix} \text{ and } \quad \tilde{\Pi}_1 = \begin{bmatrix} 0_{q\times(d-q)} \\ I_{d-q} \end{bmatrix}$$

It is a matter of simple computation to verify that

$$\Pi_2^\top (I_n \otimes P_1^\top)\mathcal{A}(I_n \otimes P_1)\Pi_2 = \begin{bmatrix} I_n \otimes A_{11} & I_n \otimes A_{12} \\ 0 & I_n \otimes A_{22} \end{bmatrix} + \begin{bmatrix} K \otimes T_{11} & K \otimes T_{12} \\ 0 & K \otimes T_{22} \end{bmatrix}, \tag{8}$$

and this matrix is easily seen to be reducible. Therefore, $\mathcal{A}$ is reducible as well.

Similarly, suppose that K is reducible, and let P be a permutation matrix such that

$$P^\top K P = \begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix}, \tag{9}$$

where $K_{12} \geq 0$ and $K_{11}, K_{22} \geq k^* I$. Then, it is easy to see that $\mathcal{A}$ is reducible, as that

$$\begin{aligned} (P^\top \otimes I_d)\mathcal{A}(P \otimes I_d) &= \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \\ &= \begin{bmatrix} A_{11} + B_1 K_{11} C_1 & B_1 K_{12} C_2 \\ 0 & A_{22} + B_2 K_{22} C_2 \end{bmatrix}. \end{aligned} \tag{10}$$

■

Lemma 2: A necessary and sufficient condition for the existence of an (irreducible) Metzler matrix $K \geq k^* I_n$ such that $\mathcal{A} = A + BKC$ is irreducible is that $A_h + \mathbf{b}_h \mathbf{c}_h^\top$ is irreducible.

Proof: Necessity follows from Lemma 1, so we consider now the sufficiency. Suppose that $A_h + \mathbf{b}_h \mathbf{c}_h^\top$ is irreducible. If we consider $K = \mathbf{1}_n \mathbf{1}_n^\top$, then it is easy to see that

$$\mathcal{A} = \begin{bmatrix} A_h + \mathbf{b}_h \mathbf{c}_h & \mathbf{b}_h \mathbf{c}_h^\top & \dots & \mathbf{b}_h \mathbf{c}_h^\top \\ \mathbf{b}_h \mathbf{c}_h^\top & A_h + \mathbf{b}_h \mathbf{c}_h^\top & \dots & \mathbf{b}_h \mathbf{c}_h^\top \\ \vdots & & \ddots & \vdots \\ \mathbf{b}_h \mathbf{c}_h^\top & \mathbf{b}_h \mathbf{c}_h^\top & \dots & A_h + \mathbf{b}_h \mathbf{c}_h^\top \end{bmatrix},$$

is irreducible. To this end it is sufficient to notice that the corresponding digraph $\mathcal{D}(\mathcal{A})$ is strongly connected. This follows from the fact that: (1) all diagonal blocks in $\mathcal{A}$ are irreducible, and this means that in $\mathcal{D}(\mathcal{A})$, the vertices within each group $\{id+1, id+2, \dots, id+d\}, i \in [0, n-1]$, communicate with each other; (2) $\mathbf{b}_h \mathbf{c}_h^\top > 0$, which ensures that there exists at least one arc in $\mathcal{D}(\mathcal{A})$, connecting each group $\{id+1, id+2, \dots, id+d\}, i \in [0, n-1]$, with every other group

$\{jd+1, jd+2, \ldots, jd+d\}$, $j \in [1, n-1]$, with $i \neq j$. This ensures that for every pair of distinct vertices there is a path connecting them, and hence the digraph of $\mathcal{A}$ is strongly connected. ■

Remark 3: Note that $A_h + \mathbf{b}_h \mathbf{c}_h^\top$ is irreducible if and only if $A_h + \mathbf{b}_h \mathbf{c}_h^\top k$ is irreducible for every $k > k^*$. However, $A_h + \mathbf{b}_h \mathbf{c}_h^\top k^*$ is not necessarily irreducible. This is the case, for instance, for the triple

$$A_h = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{b}_h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{c}_h^\top = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The next result plays a fundamental role in solving the consensus problem in section IV.

Lemma 3: If

i) $A_h + \mathbf{b}_h \mathbf{c}_h^\top k^*$ is Hurwitz;

ii) $A_h + \mathbf{b}_h \mathbf{c}_h^\top$ is irreducible;

then there exists $k_0 > k^*$ such that $A_h + \mathbf{b}_h \mathbf{c}_h^\top k_0$ is Metzler and irreducible, with dominant eigenvalue equal to 0.

Proof: By assumption ii), for every $k > k^*$ the matrix $A_h + \mathbf{b}_h \mathbf{c}_h^\top k$ is irreducible. We want to prove that there exists $\bar{k} > k^*$ such that the dominant eigenvalue of $A_h + \mathbf{b}_h \mathbf{c}_h^\top k$ is positive for every $k > \bar{k}$. As a consequence, since $\lambda_{\max}(A_h + \mathbf{b}_h \mathbf{c}_h^\top k^*) < 0$ and, for $k > \bar{k}$, $\lambda_{\max}(A_h + \mathbf{b}_h \mathbf{c}_h^\top k) > 0$, the continuity of $\lambda_{\max}(A_h + \mathbf{b}_h \mathbf{c}_h^\top k)$ with respect to k ensures that there exists $k_0 \in (k^*, \bar{k})$ such that $\lambda_{\max}(A_h + \mathbf{b}_h \mathbf{c}_h^\top k_0) = 0$. Clearly, since k_0 is greater than k^*, $A_h + \mathbf{b}_h \mathbf{c}_h^\top k_0$ will be both Metzler and irreducible.

To prove the existence of such a $\bar{k}$, consider first the case when A_h is irreducible. In this case, for every $s \in \mathbb{R}$, with $s > \lambda_{\max}(A_h)$, $\mathrm{adj}(sI_d - A_h) \gg 0$ [3] [3]. Consequently, for every $s \in \mathbb{R}$, with $s > \lambda_{\max}(A_h)$, $n_h(s) = \mathbf{c}_h^\top \mathrm{adj}(sI_d - A_h)\mathbf{b}_h > 0$. This prevents the possibility that all the coefficients β_i's of $n_h(s) = \sum_{i=0}^{d-1} \beta_i s^i$ are either negative or zero. But then, by the same reasoning adopted in Remark 1, there exists $\bar{k} > 0$ such that $A_h + \mathbf{b}_h \mathbf{c}_h^\top \bar{k}$ is not Hurwitz, namely $\lambda_{\max}(A_h + \mathbf{b}_h \mathbf{c}_h^\top \bar{k}) \geq 0$. The irreducibility of $A_h + \mathbf{b}_h \mathbf{c}_h^\top k$ for $k > k^*$, and hence for $k \geq \bar{k}$, ensures that the dominant eigenvalue is strictly increasing with k [3], and hence $\lambda_{\max}(A_h + \mathbf{b}_h \mathbf{c}_h^\top k) > 0$ for every $k > \bar{k}$.

[3] As a matter of fact, the result has been proved in [3] for irreducible positive matrices P, and real numbers $s > \rho(P)$, $\rho(P)$ being the spectral radius of P. The extension of this result to the class of Metzler matrices is straightforward.

We finally consider the case when A_h is reducible, a case that, by assumption ii) and Remark 3, can occur only if $k^* = 0$. If so, $\epsilon > 0$ can be found such that $A_h + \mathbf{b}_h\mathbf{c}_h^\top \epsilon$ is both Hurwitz and irreducible. Since $A_h + \mathbf{b}_h\mathbf{c}_h^\top k$ can be thought of as

$$A_h + \mathbf{b}_h\mathbf{c}_h^\top k = (A_h + \mathbf{b}_h\mathbf{c}_h^\top \epsilon) + \mathbf{b}_h\mathbf{c}_h^\top (k - \epsilon) =: \tilde{A} + \mathbf{b}_h\mathbf{c}_h^\top \tilde{k},$$

where $\tilde{A}$ is now irreducible, we can apply the same reasoning as in the first part of the proof, and deduce that there exists $\bar{k} > 0$ such that $\lambda_{\max}(A_h + \mathbf{b}_h\mathbf{c}_h^\top k) > 0$ for every $k > \bar{k}$. ■

IV. Positive consensus problem

We define the **positive consensus problem** as follows: Given a group of n identical agents, described by the same positive state-space model $\Sigma_h = (A_h, \mathbf{b}_h, \mathbf{c}_h^\top)$, we want to ensure that, for every choice of the (nonnegative) initial conditions, the states of the agents $\mathbf{x}_i(t), i \in [1, n]$, remain nonnegative and asymptotically converge to some nonnegative vector, which is the same for every agent (but depends on the agent's initial conditions), i.e.:

$$\lim_{t \to +\infty} \mathbf{x}_i(t) = \bar{\mathbf{x}} \geq 0, \qquad \forall\ i \in [1, n]. \tag{11}$$

Clearly, the trivial case when $\bar{\mathbf{x}} = 0$ for every choice of the initial conditions, corresponds to the positive stabilization problem, addressed in section II. We are interested in achieving consensus to constant, but nonzero, trajectories. To achieve this we impose the following conditions:

(1) the overall system is positive and simply stable, but not asymptotically stable;

(2) for every initial condition the state-trajectory converges to some block vector:

$$\begin{bmatrix} \bar{\mathbf{x}}^\top & \bar{\mathbf{x}}^\top & \dots & \bar{\mathbf{x}}^\top \end{bmatrix}^\top = \mathbf{1}_n \otimes \bar{\mathbf{x}}, \qquad \bar{\mathbf{x}} > 0.$$

It should be noted that condition (2) is a stronger requirement compared to what is normally imposed on consensus of nonpositive systems (see, e.g. [12], [32]). In general, consensus is achieved by requiring that the overall system is simply stable and that all agents asymptotically converge to the same trajectory. However, initial conditions exist for which all agents states converge to zero. When dealing with positive systems, it is possible to exploit the positivity constraint on the initial conditions to ensure that the common asymptotic evolution never converges to zero. To ensure this, it is sufficient to impose the irreducibility of the overall matrix $\mathcal{A}$. On the other hand, if $\mathcal{A}$ were simply stable but reducible, nonnegative initial conditions could be found such that $\lim_{t\to+\infty}[\mathbf{x}_i(t) - \mathbf{x}_j(t)] \neq 0$, for some $i \neq j, i, j \in [1, n]$. Therefore, for positive multi-agent systems, consensus is always achieved under this stronger requirement.

Theorem 1 below shows that positive consensus is achievable if and only if the two apparently independent problems addressed in sections II and III are solvable: i) the positive stabilization problem is solvable, namely there exists K such that $\mathcal{A} = A + BKC$ is Metzler and Hurwitz (see Proposition 1); ii) there exists K such that $\mathcal{A}$ is irreducible (see Lemma 2). By making use of the characterizations previously obtained, we can state the result in a form that allows for an immediate check on the matrices of Σ_h.

Theorem 1: The positive consensus problem is solvable if and only if the following two conditions hold:

i) $A_h + \mathbf{b}_h\mathbf{c}_h^\top k^*$ is Hurwitz;

ii) $A_h + \mathbf{b}_h\mathbf{c}_h^\top$ is irreducible.

Proof: [Sufficiency] If conditions i) and ii) hold, then, by Lemma 3, there exists $k_0 > k^*$ such that $A_h + \mathbf{b}_h\mathbf{c}_h^\top k_0$ is Metzler and irreducible, with dominant eigenvalue equal to 0. Consequently, $A_h+\mathbf{b}_h\mathbf{c}_h^\top k_0$ has a strictly positive dominant eigenvector $\mathbf{v}_h$. By assumption ii), for every $\epsilon > 0$ the matrix $A_h+\mathbf{b}_h\mathbf{c}_h^\top(k^*+\epsilon)$ is irreducible. So, in particular, $A_h+\mathbf{b}_h\mathbf{c}_h^\top\left(k^* + \frac{k_0-k^*}{n}\right)$ is irreducible. We want to prove that the (Metzler) feedback matrix

$$K := k^* I_n + \frac{k_0 - k^*}{n}\mathbf{1}_n\mathbf{1}_n^\top$$

makes the matrix $\mathcal{A} = A+BKC$ Metzler, irreducible, with dominant eigenvalue 0 and a strictly positive dominant eigenvector with all n blocks equal to $\mathbf{v}_h$.

Since $K > k^* I_n$, clearly $\mathcal{A}$ is Metzler. On the other hand, in order to prove that $\mathcal{A}$ is irreducible, it is sufficient to follow the same reasoning adopted in the proof of Lemma 2.

It is easy to verify that $\mathbf{z}^\top := [\,\mathbf{v}_h^\top \quad \mathbf{v}_h^\top \quad \dots \quad \mathbf{v}_h^\top\,]$ is a strictly positive eigenvector of $\mathcal{A}$ corresponding to the eigenvalue 0. But since an irreducible Metzler matrix can have a strictly positive eigenvector only corresponding to the dominant eigenvalue, this ensures that $\lambda_{\max}(\mathcal{A}) = 0$. So, the fact that $\mathcal{A}$ is irreducible with $\lambda_{\max}(\mathcal{A}) = 0$ and has $\mathbf{z}$ as dominant eigenvector, ensures that for every positive initial condition and every $i \in [1, n]$, the state of the ith agent satisfies

$$\lim_{t\to+\infty} \mathbf{x}_i(t) = \alpha\mathbf{v}_h, \tag{12}$$

for some positive α [4]. Therefore, the consensus problem is solvable.

[Necessity] We first prove that condition ii) is necessary. If $A_h + \mathbf{b}_h\mathbf{c}_h^\top$ were reducible, we have shown (see proof of Lemma 1, and refer to the notation used within the proof) that $\mathcal{A}$ could be reduced, by means of suitable permutation matrices, to the block triangular form (8). We also notice that since $T_{21} = 0$, and T_{21} is the product of two nonnegative vectors, namely $[\,0 \quad I_{d-q}\,]\, P_1^\top \mathbf{b}_h$ and $\mathbf{c}_h^\top P_1 \begin{bmatrix} I_q \\ 0 \end{bmatrix}$, at least one of these two vectors is zero. But this implies that either $T_{11} = 0$ or $T_{22} = 0$. Consequently, it is easily seen also in this case that nonnegative initial conditions for the agents can be found such that condition (11) does not hold (namely different agents have states converging to different asymptotic vectors).

Consider now condition i). If $A_h + \mathbf{b}_h\mathbf{c}_h^\top k^*$ is not Hurwitz, we distinguish two situations: (a) $\lambda_{\max}(A_h + \mathbf{b}_h\mathbf{c}_h^\top k^*) > 0$; (b) $\lambda_{\max}(A_h + \mathbf{b}_h\mathbf{c}_h^\top k^*) = 0$.

In case (a), since for every $K \geq k^* I_n$, $\mathcal{A} \geq I_n \otimes (A_h + \mathbf{b}_h\mathbf{c}_h^\top k^*)$, it follows that $\lambda_{\max}(\mathcal{A}) \geq \lambda_{\max}(A_h + \mathbf{b}_h\mathbf{c}_h^\top k^*) > 0$, thus making it impossible to solve the positive consensus problem. In case (b), in order to make $\mathcal{A}$ irreducible, we have to choose $K > k^* I_n$ and irreducible. We want to show that in this case $\lambda_{\max}(\mathcal{A}) > \lambda_{\max}(A_h + \mathbf{b}_h\mathbf{c}_h^\top k^*)$ and hence $\lambda_{\max}(\mathcal{A}) > 0$. Let $\tilde{K}$ be any irreducible Metzler matrix satisfying $K > \tilde{K} > k^* I_n$. Then

$$A + BKC > A + B\tilde{K}C > A + B(k^* I_n)C.$$

On the other hand, we can choose $\tilde{K}$ in such a way that $A + B\tilde{K}C$ is irreducible, too. So, by keeping in mind the irreducibility of the first two matrices, we can say [3] that

$$\lambda_{\max}(A + BKC) > \lambda_{\max}(A + B\tilde{K}C) \geq \lambda_{\max}(A + B(k^* I_n)C) = 0.$$

This contradicts the fact that $\mathcal{A} = A + BKC$ is simply stable. So, consensus via static output feedback is achievable only if both condition i) and ii) are satisfied. ■

V. Future Research

The analysis presented in this paper could be specialized to the case when the communication among the agents, and hence the nonzero pattern of the matrix K, is chosen *a priori*. This is

[4]It is easy to determine the value of α in the limit in (12). For the specific choice of K we made, the overall state trajectory $\mathbf{x}(t)$ converges $\langle \mathbf{x}(0), \mathbf{z} \rangle \mathbf{z} = (\sum_{i=1}^n \mathbf{x}_i^\top(0)\mathbf{v}_h)\mathbf{z}$. Accordingly, the state of each agent converges to $\bar{\mathbf{x}} = (\sum_{i=1}^n \mathbf{x}_i^\top(0)\mathbf{v}_h)\mathbf{v}_h = \alpha \mathbf{v}_h$.

the case when the adjacency matrix describing the mutual relationships among the agents is pre-assigned, and we assume that the output feedback matrix matrix K is expressed in terms of the associated Laplacian [32]. Also, we believe that our results can be extended to the class of positive multi-agent systems with communication delay, thus making it possible to deal with a considerably wider set of applications and in particular with the Foschini-Miljanic algorithm [6], [33].

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