A generalized Tracking and Disturbance Rejection Problem for Multidimensional Behaviours

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Abstract

In this paper we study a generalized tracking and disturbance rejection problem for multidimensional behaviours. Given a multidimensional plant, our first goal is to design a compensator to be connected to the plant through regular partial interconnection, in such a way that the overall controlled system is autonomous and stable, when no exogenous signal acts on the system. On the other hand, when exogenous signals affect the controlled system evolution, we want to impose that a suitable linear combination of the overall system trajectories is "negligibile" in a sense we will clarify within the paper. This problem set-up formalizes a number of classical control problems, first of all tracking of some (reference) signal together with rejection of another (disturbance) signal. The adopted approach is extremely general and it is based on the idea of describing all behaviour trajectories as the sum of a "transient signal" and a "steady state" component, a decomposition that relies on Gabriel's localization theory. Necessary and sufficient conditions for the problem solvability are provided, and the compensators that satisfy the control goal are characterized in terms of an internal model condition. Furthermore, a parametrization of all such compensators is provided.

Keywords: Tracking, disturbance rejection, multidimensional linear system, behavioural approach.

Contents

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1 Introduction

Stabilization and regulation problems within the behavioural approach have a long history. Stimulated by two milestone contributions [20, 21], dealing with the control of one-dimensional behaviours, a long stream of research on these topics flourished, dealing either with one-dimensional (1D) behaviours (e.g. [1, 3, 4, 12]) or with the wider class of multidimensional (nD) behaviours [5, 6, 8, 13, 18].

In the behavioural framework, controlling a plant amounts to restricting the set of its trajectories to a proper subset, whose elements display desired properties (typically, but not solely, some form of convergence). So, stabilization, either by partial or by full interconnection, consists in designing a second system, the controller, that once connected with the original plant (either by means of all the plant variables or by a proper subset of them) makes it possible to achieve this goal. One of the main features of the behavioural approach lies in its capability of treating the system dynamics without imposing any input/output partition on the system variables. This is particularly appropriate when the plant under investigation has to become part of a larger, interconnected, system, since it is the specific interconnection structure that determines what are the inputs and what are the outputs. A simple but rather paradigmatic example is represented by passive circuits, for which the choice of assuming either the voltage or the current as input signal is strictly related to the way they are connected with the external generators. Consistently with this perspective, in most of the aforementioned references stabilization and regulation problems have been posed and solved without assuming any input/output partition of the system variables.

The tracking and disturbance rejection problem for 1D behaviours was first addressed in [3], where necessary and sufficient conditions for the problem solvability, under the assumption that the exogenous system generating both the reference signal and the disturbance is autonomous, have been provided. Interestingly enough, the solvability conditions involve the well-known internal model principle, first pointed out in the behavioural context for observers in [19]. An algorithm to explicitly construct controllers that achieve these goals was also proposed in [3].

In [2], the set-up introduced in [3] was generalized, to deal with more general stabilization goals (design of T-stabilizing compensators) and by introducing a target function that can formalize a number of classical control problems, first of all the tracking of a reference signal, meanwhile rejecting a disturbance acting on the system.

The aim of this paper is to extend the results derived in [2] to the multidimensional case to deal with stabilization and regulation problems for nD behaviours. The main results derived in this manuscript represent neat, but highly non-trivial, generalizations of the results provided in section 4 of [2]. Indeed, the mathematical set-up required to extend the analysis to the nD case deeply relies on advanced algebraic concepts, like Serre subcategories [16] and localization according to Gabriel [17]. Gabriel localization was first applied to system theoretic questions in [8], and later refined. In order to make the proofs accessible also to non-specialists, we have tried to briefly recall the main definitions and results about these topics in a preliminary section. We refer the reader to [15] for a more thorough description of the theory.

In detail, the paper is organized as follows: Section 2 recalls the basic concepts about behaviours, interconnection of behaviours, and negligibility of modules and signals, as well as some technical results about Gabriel localization and its use in defining the steady-state and the transient part of the behaviour trajectories. Two technical results, fundamental to develop the subsequent theory, are also presented. Section 3 addresses stabilization by partial and regular interconnection, by assuming that no exogenous signal acts on the overall controlled system. A parametrization of all such controllers is provided. Section 4 tackles the same stabilization problem in the presence of exogenous signals, which will later represent the reference signal and the disturbance. Lemma 18, at the end of the section, allows to extend the previous parametrization to this more general set-up. Finally, in Section 5, the general tracking and disturbance rejection problem for nD behaviours is posed and solved, and a parametrization of all controllers that achieve this goal is given. Sections 3 to 5 contain a comprehensive running example where the results and parametrizations are demonstrated.

A preliminary version of some of the results contained in this paper has been presented at the 21st International Symposium on Mathematical Theory of Networks and Systems (MTNS 2014), see [14].

2 Preliminaries

In this section we introduce the framework adopted in the paper to investigate the control design problems.

2.1 Main Ingredients

We consider a noetherian integral domain A, which is called the *ring of operators*, and a *signal space* $\mathscr{F}$, which is an injective cogenerator over A [7, p. 31]. From Section 2.10 onwards we will assume that the injective cogenerator $\mathscr{F}$ is large. Later on (see Assumption 5) we will introduce an additional condition on A and thus slightly restrict the set of eligible operator rings A. The scalar product of $f \in A$ with $w \in \mathscr{F}$ is denoted by $f \circ w$. Given an A-module (in the following simply a "module") $U = A^{1\times k}R \subseteq A^{1\times l}$ generated by the rows of a matrix $R \in A^{k\times l}$, we define the *behaviour* associated with U

as the set[1]

$$\mathscr{B} = U^{\perp} := \{w \in \mathscr{F}^l : \ U \circ w = 0, \text{ i.e., } \forall \eta \in U, \ \eta \circ w = 0\} = \{w \in \mathscr{F}^l : \ R \circ w = 0\}.$$

Conversely, given a behaviour $\mathscr{B}$, the *module of equations* associated with $\mathscr{B}$ is

$$\mathscr{B}^{\perp} := \{\eta \in A^{1\times l} : \ \eta \circ \mathscr{B} = 0, \text{ i.e., } \forall w \in \mathscr{B}, \ \eta \circ w = 0\}.$$

From the definition of behaviour, it follows immediately that $\mathscr{B}^{\perp\perp} = \mathscr{B}$, but since $\mathscr{F}$ is a cogenerator the identity $\mathscr{B}^{\perp} = U^{\perp\perp} = U$ holds, too [7, Cor. 47, Cor. 48, p. 29].

Standard examples are the following:

Example 1. 1. Let $A = F[s] = F[s_1, \dots, s_n]$ be the polynomial ring in n variables over a field F and consider the large injective cogenerator $\mathscr{F} = F^{\mathbb{N}^n}$ of sequences in F indexed by multi-indices $\mu \in \mathbb{N}^n$. The ring A acts on $\mathscr{F}$ via shifts, i.e., for $\mu \in \mathbb{N}^n$ and $w \in \mathscr{F}$ the signal $s^{\mu} \circ w$ is defined by

$$(s^{\mu} \circ w)(t) = w(t + \mu) \text{ for } t \in \mathbb{N}^n,$$

where we use the multi-index notation $s^{\mu} = s_1^{\mu_1} \cdots s_n^{\mu_n}$. The behaviours in this setting are the solution sets of systems of finitely many linear partial difference equations with constant coefficients.

Assume that F is algebraically closed. Another relevant signal module in this situation is the space of polynomial-exponential multisequences $\mathscr{F}_{\text{fin}} = \sum_{\lambda \in F^n} F[t] t^{\lambda}$, where $F[t] = F[t_1, \dots, t_n]$. This space is an injective cogenerator, but it is not large. We use the subscript "fin" to indicate that this is the space of those sequences w such that the A-module $A \circ w$ generated by w is finite dimensional as vector space over the base field F.

2. Again, assume that $A = F[s] = F[s_1, \dots, s_n]$ is the polynomial ring in n variables, but reduce the choices for F to $F = \mathbb{R}$ or $F = \mathbb{C}$. As signal space $\mathscr{F}$ choose either the space $\mathscr{F} = \mathscr{C}^{\infty}(\mathbb{R}^n, F)$ of smooth functions or $\mathscr{D}'(\mathbb{R}^n, F)$ of distributions (both spaces are large injective cogenerators), and the action of A on $\mathscr{F}$ is the one by differentiation

$$s_i \circ w = \frac{\partial w}{\partial t_i}.$$

Here, the behaviours are the sets of solutions of systems of finitely many linear partial differential equations with constant coefficients.

Similarly to the discrete case, for $F = \mathbb{C}$ the polynomial-exponential functions $\mathscr{F}_{\text{fin}} = \sum_{\lambda \in \mathbb{C}^n} \mathbb{C}[t]\, \mathrm{e}^{\lambda \bullet t}$, where $\lambda \bullet t := \sum_{i=1}^{n} \lambda_i t_i$, form the (not large) injective cogenerator of functions w for which $\mathbb{C}[s] \circ w$ is a finite dimensional vector space over $\mathbb{C}$.

[1] The notation $\perp$ was first used in this context in [7, p. 21]. The notions of orthogonal complement of a submodule $U \subseteq A^{1\times l}$ and of a behaviour $\mathscr{B} \subseteq \mathscr{F}^l$ are induced by the bilinear form

$$A^{1\times l} \times \mathscr{F}^l \longrightarrow A : \ (\eta, \ w) \longmapsto \eta \circ w,$$

i.e., $\eta \perp w$ if and only if $\eta \circ w = 0$.

2.2 The Image of a Behaviour

For a behaviour $\mathscr{B} \subseteq \mathscr{F}^l$ and a matrix $P \in A^{v\times l}$, the image of $\mathscr{B}$ under the map $P\circ$ is $P \circ \mathscr{B} = \{P \circ w \in \mathscr{F}^v :\ w \in \mathscr{B}\}$. Because of the injectivity of the signal module $\mathscr{F}^l$, the image $P \circ \mathscr{B}$ is again a behaviour, i.e., the solution set of finitely many equations. These equations can be obtained as follows:

Let $(X,\ Y) \in A^{k_1\times(l+v)}$ be a universal (or minimal) left annihilator of the block matrix $\binom{R}{P}$. This amounts to saying that

$$A^{1\times k_1}(X,\ Y) = \left\{(\eta,\ \eta_1) \in A^{1\times(k+v)} :\ (\eta,\ \eta_1)\binom{R}{P} = 0\right\}.$$

Then the image of $\mathscr{B}$ under $P\circ$ is [7, Thm. 34, p. 24]

$$P \circ \mathscr{B} = (A^{1\times k_1}Y)^{\perp} = \{\widetilde{w} \in \mathscr{F}^v :\ Y \circ \widetilde{w} = 0\}.$$

Consider the special case when

$$\mathscr{B} = \left\{ \binom{w_1}{w_2} \in \mathscr{F}^{l_1+l_2} :\ (R_1,\ R_2) \circ \binom{w_1}{w_2} = 0\right\} \tag{1}$$

and $P = (0,\ \mathrm{id}_{l_2})$. Then the map $P\circ$ is the projection on the variable w_2, i.e., $\mathrm{proj}_{w_2} : \binom{w_1}{w_2} \longmapsto w_2$, and

$$\begin{aligned}\mathrm{proj}_{w_2}(\mathscr{B}) = P \circ \mathscr{B} &= \{w_2 \in \mathscr{F}^{l_2} :\ \exists w_1 \in \mathscr{F}^{l_1} :\ \tbinom{w_1}{w_2} \in \mathscr{B}\}\\ &= \{w_2 \in \mathscr{F}^{l_2} :\ (XR_2) \circ w_2 = 0\},\end{aligned}$$

where, in this case, X is a universal left annihilator of R_1.

In this set-up, the variables w_2 are *free* for the behavior $\mathscr{B}$ if $\mathrm{proj}_{w_2}(\mathscr{B}) = \mathscr{F}^{l_2}$. This property is equivalent to the rank condition $\mathrm{rank}(R_1) = \mathrm{rank}(R_1,\ R_2)$ and implies that there exists a matrix $H \in K^{l_1\times l_2}$ with entries in the quotient field $K := \mathrm{quot}(A)$, such that $R_2 = R_1H$. A behaviour $\mathscr{B} = (A^{1\times k}R)^{\perp} \subseteq \mathscr{F}^l$ with no free variables is called *autonomous* and it is necessarily described by a full column rank matrix, namely $\mathrm{rank}(R) = l$.

2.3 Input/Output Behaviours

Let $\mathscr{B}$ be the behaviour described as in Equation (1). We denote the subbehaviour of $\mathscr{B}$ consisting of the trajectories of $\mathscr{B}$ whose components w_2 are identically zero by

$$\mathscr{N}_{w_1}(\mathscr{B}) := \{w_1 \in \mathscr{F}^{l_1} :\ (R_1,\ R_2) \circ \binom{w_1}{0} = R_1 \circ w_1 = 0\} = (A^{1\times k}R_1)^{\perp}.$$

A behaviour

$$\mathscr{B} = \left\{\binom{y}{u} \in \mathscr{F}^{p+m} : P \circ y = Q \circ u\right\}, \quad (P, -Q) \in A^{k\times(p+m)}, \tag{2}$$

is an *input/output (IO) behaviour* with input u and output y if u is maximally free in $\mathscr{B}$, i.e., if u is free and $\mathscr{B}^0 := \mathscr{N}_y(\mathscr{B}) = \{y \in \mathscr{F}^p :\ P \circ y = 0\}$ is autonomous. This is equivalent to the rank condition $p = \mathrm{rank}(P) = \mathrm{rank}(P,\ -Q)$ and implies the existence of a unique *transfer matrix* $H \in K^{p\times m}$ satisfying $Q = PH$.

2.4 Interconnection of Behaviours

The *(full) interconnection* of two behaviours $\mathscr{B}_i = U_i^{\perp} \subseteq \mathscr{F}^l$, $U_i = A^{1\times k_i} R_i$, $i = 1,2$, is their intersection

$$\begin{aligned}\mathscr{B}_1 \cap \mathscr{B}_2 &= \{w \in \mathscr{F}^l :\ U_1 \circ w = 0 \text{ and } U_2 \circ w = 0\} \\ &= \{w \in \mathscr{F}^l :\ (U_1 + U_2) \circ w = 0\} = (U_1 + U_2)^{\perp} = \left(A^{1\times(k_1+k_2)} \left(\begin{smallmatrix} R_1 \\ R_2 \end{smallmatrix}\right)\right)^{\perp}.\end{aligned}$$

Such an interconnection is *regular* if the sum $U_1 + U_2$ is a direct sum, i.e., if the intersection $U_1 \cap U_2$ consists of the zero element, or, equivalently,

$$\operatorname{rank} \left(\begin{smallmatrix} R_1 \\ R_2 \end{smallmatrix}\right) = \operatorname{rank}(R_1) + \operatorname{rank}(R_2).$$

Given two behaviours

$$\begin{aligned}\mathscr{B} &= \left\{ \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) \in \mathscr{F}^{l_1+l_2} :\ (R_1,\ R_2) \circ \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) = 0 \right\}, \qquad &&\text{with } (R_1,\ R_2) \in A^{k_1\times(l_1+l_2)}, \\ \widetilde{\mathscr{B}} &= \left\{ \left(\begin{smallmatrix} w_2 \\ w_3 \end{smallmatrix}\right) \in \mathscr{F}^{l_2+l_3} :\ (\widetilde{R_2},\ \widetilde{R_3}) \circ \left(\begin{smallmatrix} w_2 \\ w_3 \end{smallmatrix}\right) = 0 \right\}, \qquad &&\text{with } (\widetilde{R_2},\ \widetilde{R_3}) \in A^{k_2\times(l_2+l_3)},\end{aligned}$$

their *partial interconnection via* w_2 is the behaviour defined as

$$\begin{aligned}\mathscr{B} \wedge_{w_2} \widetilde{\mathscr{B}} &= \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \mathscr{F}^{l_1+l_2+l_3} :\ \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) \in \mathscr{B},\ \left(\begin{smallmatrix} w_2 \\ w_3 \end{smallmatrix}\right) \in \widetilde{\mathscr{B}} \right\} \\ &= \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \mathscr{F}^{l_1+l_2+l_3} :\ \begin{pmatrix} R_1 & R_2 & 0 \\ 0 & \widetilde{R_2} & \widetilde{R_3} \end{pmatrix} \circ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0 \right\}.\end{aligned}$$

The partial interconnection is called *regular* if the sum

$$A^{1\times k_1}(R_1,\ R_2,\ 0) + A^{1\times k_2}(0,\ \widetilde{R_2},\ \widetilde{R_3})$$

is a direct one.

2.5 Serre Subcategories and Negligibility

In order to introduce a general notion of stability, first we introduce negligibility of modules and signals via a Serre subcategory $\mathfrak{C}$ of modules with $\emptyset \neq \mathfrak{C} \subsetneq \mathrm{Mod}_A$, where Mod_A denotes the category of all A-modules. A *Serre subcategory* is a full subcategory closed under isomorphism, subobjects, factor objects, extensions and direct sums [16, Chap. I].

In the following we give some examples of Serre subcategories.

Example 2. 1. A multiplicatively closed subset $T \subseteq A$ induces the Serre subcategory

$$\mathfrak{C}(T) := \{M \in \mathrm{Mod}_A :\ M_T = 0\},$$

where $M_T = \{\frac{m}{t} :\ m \in M,\ t \in T\}$.

If A is a principal ideal domain, for example a polynomial ring in one variable over a field, every Serre subcategory of Mod_A is induced by a multiplicatively closed set T. For this reason, the theory of Serre categories and Gabriel localization is not necessary for one-dimensional systems theory. In higher dimensions, however, there are Serre subcategories which are not of the type $\mathfrak{C}(T)$ (see Example 6).

Nonetheless, Serre subcategories of the type $\mathfrak{C}(T)$ are important also in multidimensional systems theory, for example the one induced by the set of all monomials

$$T = \{as^{\mu} = as_1^{\mu_1} \cdots s_n^{\mu_n} : \ a \in F \setminus \{0\}, \ \mu \in \mathbb{N}^n\} \subseteq F[s_1, \ldots, s_n],$$

which leads to dead-beat stability. Also, time autonomy can be characterized via a multiplicatively closed set, see [10, Def. and Cor. 3.3, Thm. 3.7].

2. Let $A = \mathbb{C}[s] = \mathbb{C}[s_1, \ldots, s_n]$. For any point $\lambda \in \mathbb{C}^n$, denote by

$$\mathfrak{m}(\lambda) = \sum_{i=1}^{n} \mathbb{C}[s](s_i - \lambda_1) = \{f \in \mathbb{C}[s] : \ f(\lambda) = 0\}$$

the maximal ideal associated with λ and set $M_{\mathfrak{m}(\lambda)} = \{\frac{m}{t} : \ m \in M, \ t \in \mathbb{C}[s] \setminus \mathfrak{m}(\lambda)\}$. Any disjoint decomposition $\mathbb{C}^n = \Lambda_1 \uplus \Lambda_2$, with $\Lambda_2 \neq \emptyset$, induces the Serre subcategory

$$\mathfrak{C}(\Lambda_1) = \{M \in \mathrm{Mod}_A : \ M_{\mathfrak{m}(\lambda)} = 0 \text{ for all } \lambda \in \Lambda_2\}.$$

We will elaborate on the system theoretic relevance of Serre subcategories of the type $\mathfrak{C}(\Lambda_1)$ in Example 3, below.

3. Let $A = F[s_1, \ldots, s_n]$ and denote by $\mathrm{spec}(A)$ and $\max(A)$ the sets of prime ideals and maximal ideals of A, respectively. The Serre subcategory

$$\mathfrak{C}_{\mathrm{fin}} = \{M \in \mathrm{Mod}_A : \ M_{\mathfrak{p}} = 0 \text{ for all } \mathfrak{p} \in \mathrm{spec}(A) \setminus \max(A)\}$$

is used when it is acceptable that the control goal is achieved up to a finite dimensional error behaviour only (see [9, Sec. 2 and Thm. 3.3]).

We will use this Serre subcategory in the examples of Sections 3 to 5.

An A-module, in particular a behaviour, is $\mathfrak{C}$-*negligible* if it belongs to $\mathfrak{C}$, and a signal $w \in \mathscr{F}$ is $\mathfrak{C}$-*negligible* if the A-module generated by w, namely $A \circ w$ is in $\mathfrak{C}$. As a consequence, a behaviour is $\mathfrak{C}$-negligible if and only if all its trajectories are $\mathfrak{C}$-negligible.

Furthermore, from $\mathfrak{C} \neq \mathrm{Mod}_A$ it follows that $\mathfrak{C}$ consists of torsion modules only. Therefore, a $\mathfrak{C}$-negligible behaviour cannot have free variables and is always autonomous.

A behaviour $\mathscr{B} = (A^{1\times k}R)^{\perp}$ given by a matrix R is $\mathfrak{C}$-negligible if and only if the factor module $A^{1\times l}/A^{1\times k}R$ is in $\mathfrak{C}$, too. This equivalence is crucial since it provides a way to characterize the (analytical) properties of the behaviour's trajectories in algebraic terms, i.e., as properties of the factor module, the module of equations $A^{1\times k}R$ and the matrix R.

2.6 Transient Signals and Steady States

In the following, the $\mathfrak{C}$-*radical* $\mathrm{Ra}_{\mathfrak{C}}(M)$ of an arbitrary A-module M will play an important role. It is the largest $\mathfrak{C}$-negligible submodule of M, i.e.,

$$\mathrm{Ra}_{\mathfrak{C}}(M) := \sup\{N \subseteq M : \ N \in \mathfrak{C}\} \in \mathfrak{C}.$$

Clearly, M is $\mathfrak{C}$-negligible if and only if it is equal to its $\mathfrak{C}$-radical.

The Serre subcategory $\mathfrak{C}$ induces a direct sum decomposition of the signal space

$$\mathscr{F} = \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}) \oplus \mathscr{F}_2.$$

The radical is unique, however, in general, its direct complement is not, i.e., there are many possible choices of $\mathscr{F}_2 \cong \mathscr{F}/\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})$ and none of them is preferrable over the others. The signals $w \in \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})$ are $\mathfrak{C}$-negligible and often called *transient signals*, while the signals in $\mathscr{F}_2$ are called *steady states*. The latter ones form a system of representatives of the equivalence classes of signals of $\mathscr{F}$ (and all the other representatives differ from them by $\mathfrak{C}$-negligible signals). The direct sum decomposition carries over to vectors of signals

$$\mathscr{F}^l = \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})^l \oplus \mathscr{F}_2^l = \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}^l) \oplus \mathscr{F}_2^l$$

and to behaviours

$$\mathscr{B} = (\mathscr{B} \cap \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}^l)) \oplus (\mathscr{B} \cap \mathscr{F}_2^l) = \mathrm{Ra}_{\mathfrak{C}}(\mathscr{B}) \oplus (\mathscr{B} \cap \mathscr{F}_2^l). \tag{3}$$

From this decomposition it follows immediately that $\mathscr{B}$ is $\mathfrak{C}$-negligible if and only if $\mathscr{B} \cap \mathscr{F}_2^l = \{0\}$.

Example 3. 1. For Serre subcategories $\mathfrak{C}(T)$ induced by a multiplicatively closed set $T \subseteq A$, the signals w that are $\mathfrak{C}(T)$-negligible are those that are annihilated by some element in T, i.e., for which there is a $t \in T$ such that $t \circ w = 0$. In the dead-beat case the negligible signals are the nilpotent ones.

2. In the standard continuous case of Example 1, part 2, with $A = \mathbb{C}[s]$ and the signal space $\mathscr{F}_{\text{fin}}$ of polynomial-exponential functions, we consider a Serre subcategory $\mathfrak{C}(\Lambda_1)$ induced by a disjoint decomposition $\mathbb{C}^n = \Lambda_1 \uplus \Lambda_2$ as in Example 2, part 2. The $\mathfrak{C}(\Lambda_1)$-negligible signals are those polynomial-exponential functions whose exponents lie all in Λ_1, and they form the radical $\mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}(\mathscr{F}_{\text{fin}}) = \sum_{\lambda \in \Lambda_1} \mathbb{C}[t]\,\mathrm{e}^{\lambda \bullet t}$, where we use again the notation $\lambda \bullet t = \sum_{i=1}^{n} \lambda_i t_i$. Consequently, a behaviour $\mathscr{B}$ is $\mathfrak{C}(\Lambda_1)$-negligible if and only if all the exponents λ appearing in its trajectories lie in Λ_1. One possible choice for the direct complement of $\mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}(\mathscr{F}_{\text{fin}})$ is $\mathscr{F}_2 = \sum_{\lambda \in \Lambda_2} \mathbb{C}[t]\,\mathrm{e}^{\lambda \bullet t}$.

The *characteristic variety* of a module $M = A^{1\times l}/A^{1\times k}R$ is the set

$$\mathrm{char}(M) = \{\lambda \in \mathbb{C}^n :\ M_{\mathfrak{m}(\lambda)} = 0\}$$

and it is equal to the *variety of rank singularities*[2]

$$\{\lambda \in \mathfrak{C}^n :\ \mathrm{rank}(R(\lambda)) < \mathrm{rank}(R)\}$$

of R. Therefore, the behaviour $\mathscr{B} = (A^{1\times k}R)^\perp$ is $\mathfrak{C}(\Lambda_1)$-negligible if and only if the spectral condition $\mathrm{char}(M) \subseteq \Lambda_1$ is satisfied.

The decomposition $\mathbb{C}^n = \Lambda_1 \uplus \Lambda_2$ with $\Lambda_2 = \{z \in \mathbb{C} :\ \Re(z) \geqslant 0\}^n$ is often used. For one-dimensional systems, $\mathfrak{C}(\Lambda_1)$-negligibility with respect to this decomposition is the standard Hurwitz stability of autonomous systems, i.e., all the trajectories converge to zero as time goes to infinity.

[2] In case that R is a square matrix, the characteristic variety is the zero set of the characteristic polynomial $\det(R)$ of R.

For discrete systems the situation is similar, but different decompositions $\mathbb{C}^n = \Lambda_1 \uplus \Lambda_2$ are relevant – for example $\Lambda_2 = \{z \in \mathbb{C} : \ |z| \geqslant 1\}^n$, which is a generalization of one-dimensional Schur stability. $\mathfrak{C}(\Lambda_1)$-negligibility with respect to $\Lambda_1 = \mathbb{C}^n \setminus \left(\{z \in \mathbb{C} : \ |z| \geqslant 1\} \times \{z \in \mathbb{C} : \ |z| = 1\}^{n-1}\right)$ is related to L^2-stability of time autonomous systems (see [11, Eq. (14), Thm. 5.6]). In [13, Def. 2, Thm. 8] it is shown that asymptotic stability on sub-cones of the grid $\mathbb{Z}^2$ corresponds to the location of the exponents of the behaviour's polynomial-exponential trajectories, i.e., to a decomposition $\mathbb{C}^n = \Lambda_1 \uplus \Lambda_2$.

3. For $A = \mathbb{C}[s]$ in the continuous as well as in the discrete case the radical of the signal space $\mathscr{F}$ with respect to the Serre subcategory $\mathfrak{C}_{\text{fin}}$ from Example 2, part 3, consists of all polynomial-exponential functions or sequences, respectively, i.e. $\mathrm{Ra}_{\mathfrak{C}_{\text{fin}}}(\mathscr{F}) = \mathscr{F}_{\text{fin}}$.

It should be remarked that, in general, neither $\mathrm{Ra}_{\mathfrak{C}}(\mathscr{B})$ nor $\mathscr{B} \cap \mathscr{F}_2^l$ are behaviours in $\mathscr{F}^l$, i.e., in particular, that the A-submodule

$$\mathscr{B} \cap \mathscr{F}_2^l = \{w \in \mathscr{F}^l : \ R \circ w = 0\} \cap \mathscr{F}_2^l = \{w \in \mathscr{F}_2^l : \ R \circ w = 0\}$$

of $\mathscr{F}^l$, in general, cannot be written in the form

$$\mathscr{B} \cap \mathscr{F}_2^l = \{w \in \mathscr{F}^l : \ \widetilde{R} \circ w = 0\} = \left(A^{1 \times \widetilde{k}} \widetilde{R}\right)^{\perp}$$

for any matrix $\widetilde{R} \in A^{\widetilde{k} \times l}$.

Example 4. This phenomenon appears already in the one-dimensional case. In the continuous situation we use the polynomial ring in one variable $A = \mathbb{C}[s_1]$ and the signal space $\mathscr{F}_{\text{fin}}$ of polynomial-exponential functions. Set $\Lambda_1 = \{\lambda \in \mathbb{C} : \ \Re(\lambda) < 0\}$ and $\Lambda_2 = \mathbb{C} \setminus \Lambda_1$. We consider the Serre subcategory $\mathfrak{C}(\Lambda_1) = \mathfrak{C}(T)$, where $T = \{t \in \mathbb{C}[s_1] : t(\lambda) \neq 0 \text{ for all } \lambda \in \Lambda_2\}$ is the set of Hurwitz stable polynomials. The behaviour $\mathscr{B} = \{w \in \mathscr{F}_{\text{fin}} : \ 0 \circ w = 0\} = \mathscr{F}_{\text{fin}}$ is the unrestricted behaviour in one variable. The radical of $\mathscr{B}$ is

$$\mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}(\mathscr{B}) = \mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}(\mathscr{F}_{\text{fin}}) = \sum_{\lambda \in \Lambda_1} \mathbb{C}[t] \, \mathrm{e}^{\lambda t} \, .$$

Assume that $\mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}(\mathscr{F}_{\text{fin}})$ is a behaviour, i.e., that there is a matrix $\widetilde{R} \in A^{\widetilde{k} \times 1}$ such that $\mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}(\mathscr{F}_{\text{fin}}) = (A^{1 \times \widetilde{k}} \widetilde{R})^{\perp}$. Since A is a principal ideal domain and $\mathrm{Ra}_{\mathfrak{C}_{(\Lambda_1)}}(\mathscr{F})$ is a behaviour in one unknown we can assume without loss of generality that the matrix $\widetilde{R}$ is of size 1×1, i.e., a single polynomial. Since all the elements of Λ_1 appear as exponents in the trajectories of $\mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}(\mathscr{F}_{\text{fin}})$, all the elements of Λ_1 have to be zeros of the polynomial $\widetilde{R}$. Since Λ_1 is an infinite set, $\widetilde{R}$ must be the zero polynomial. But then $\mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}(\mathscr{F}_{\text{fin}}) = (A\widetilde{R})^{\perp} = \mathscr{B} = \mathscr{F}_{\text{fin}}$ which is a contradiction, because $\mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}(\mathscr{F}_{\text{fin}})$ is a proper subset of $\mathscr{F}_{\text{fin}}$.

Similarly, one can show that the direct complements $\mathscr{F}_2$ of $\mathrm{Ra}_{\mathfrak{C}(\Lambda_1)}(\mathscr{F}_{\text{fin}})$ are not behaviours.

In the one-dimensional case, the non-autonomous behaviours of various sizes are the only ones where this effect appears. In more than one dimension, however, it is easy to construct autonomous behaviours with this property. Take, for example, the two-dimensional behaviour $\mathscr{B} = \{w \in \mathscr{F}_{\text{fin}} : \ s_1 \circ w = 0\}$. As long as the two sets $\Lambda_i \cap \left(\{0\} \times \mathbb{C}\right)$, $i = 1, \ 2$, have infinite cardinality neither $\mathrm{Ra}_{\mathfrak{C}(\Lambda)}(\mathscr{B})$ nor $\mathscr{B} \cap \mathscr{F}_2$ are behaviours.

It is, however, still possible to interpret the sets $\mathscr{B} \cap \mathscr{F}_2^l$ as behaviours. In order to do this in Section 2.10, we need to introduce Gabriel localization, along with some of its properties (Sections 2.7, 2.8 and 2.9).

2.7 $\mathfrak{C}$-closed Modules and Gabriel Localization

Gabriel localization with respect to a Serre subcategory $\mathfrak{C}$ [17, Chap. IX] is a generalization of the usual localization of modules with respect to a multiplicatively closed subset $T \subseteq A$.

To show how the properties of the usual localization are generalized we start by assuming without loss of generality that T is saturated[3]. The modules Mod_{A_T} over the ring of fractions A_T are exactly those A-modules for which the map

$$\begin{aligned} M &\longrightarrow \mathrm{Hom}_A(At, M), \\ m &\longmapsto (at \mapsto atm), \end{aligned}$$

is an isomorphism for all $t \in T$. Thus Mod_{A_T} is a subcategory of Mod_A. This inclusion can be expressed via the the injection functor

$$\begin{aligned} \mathrm{Mod}_{A_T} &\xrightarrow{\subseteq} \mathrm{Mod}_A, \\ M &\longmapsto M, \end{aligned}$$

by which every A_T-module M is treated as an A-module. The injection functor has a right adjoint, which is uniquely determined up to an isomormphism. This right adjoint is the the localization functor

$$\begin{aligned} (-)_T : \ \mathrm{Mod}_A &\longrightarrow \mathrm{Mod}_{A_T}, \\ M &\longmapsto M_T = \{\tfrac{m}{t} : \ m \in M, \ t \in T\}. \end{aligned}$$

The set T induces the Serre subcategory $\mathfrak{C}(T) = \{M \in \mathrm{Mod}_A : \ M_T = 0\}$ and, since T is saturated, the identity

$$T = \{t \in A : \ A/At \in \mathfrak{C}(T)\} \tag{4}$$

holds, i.e., T can be retrieved from $\mathfrak{C}(T)$.

In the generalization to arbitrary Serre categories $\mathfrak{C}$, the role of the $t \in T$ and the cyclic ideals At is played by ideals $\mathfrak{a} \subseteq A$ whose factor module $A/\mathfrak{a}$ lies in $\mathfrak{C}$.

We define the subcategory $\mathrm{Mod}_{A,\mathfrak{C}} \subseteq \mathrm{Mod}_A$ of *$\mathfrak{C}$-closed modules* which are those A-modules M such that for all ideals $\mathfrak{a} \subseteq A$ whose factor module $A/\mathfrak{a}$ lies in $\mathfrak{C}$ the map

$$\begin{aligned} M &\longrightarrow \mathrm{Hom}_A(\mathfrak{a}, M), \\ x &\longmapsto (a \mapsto ax), \end{aligned}$$

is an isomorphism. Again, the injection functor $\mathrm{Mod}_{A,\mathfrak{C}} \xrightarrow{\subseteq} \mathrm{Mod}_A$ has a right adjoint, which is uniquely determined up to an isomorphism by this property and called the *Gabriel localization functor*

$$\mathscr{Q}_{\mathfrak{C}} : \ \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{A,\mathfrak{C}}.$$

If the Serre subcategory is of the form $\mathfrak{C}(T)$ then $\mathrm{Mod}_{A,\mathfrak{C}(T)} = \mathrm{Mod}_{A_T}$ and the Gabriel localization functor is equal to the usual one, i.e., $\mathscr{Q}_{\mathfrak{C}(T)}(-) = (-)_T$.

[3] *Saturated* means that if a product $t_1 t_2$ is in T then both factors have to lie in T, too.

2.8 Properties of Gabriel Localization

In the following, the Serre subcategory $\mathfrak{C}$ will be fixed. Therefore, we will omit the suffixes and write $\mathscr{Q}$ and Ra instead of $\mathscr{Q}_{\mathfrak{C}}$ and $\mathrm{Ra}_{\mathfrak{C}}$, respectively.

Every $\mathfrak{C}$-closed module is a $\mathscr{Q}(A)$-module. This holds in particular for the Gabriel localization $\mathscr{Q}(M)$ of an A-module M. Furthermore, the functor $\mathscr{Q}$ is such that, for every M, M_1, $M_2 \in \mathrm{Mod}_A$,

$$
\begin{aligned}
&M \in \mathrm{Mod}_{A,\mathfrak{C}} \Longleftrightarrow M = \mathscr{Q}(M), && \text{(5a)}\\
&\mathscr{Q}(\mathscr{Q}(M)) = \mathscr{Q}(M), && \text{(5b)}\\
&M \in \mathfrak{C} \Longleftrightarrow \mathscr{Q}(M) = \{0\}, && \text{(5c)}\\
&M_1 \subseteq M_2 \Longrightarrow \mathscr{Q}(M_1) \subseteq \mathscr{Q}(M_2), && \text{(5d)}\\
&\mathrm{Ra}_{\mathfrak{C}}(M) = \{0\} \Longrightarrow M \subseteq \mathscr{Q}(M). && \text{(5e)}
\end{aligned}
$$

In general, however, M cannot be embedded in $\mathscr{Q}(M)$, i.e., M cannot be interpreted as a submodule of $\mathscr{Q}(M)$. Also, Gabriel localization preserves direct sums and intersections, i.e.,

$$
\begin{aligned}
&\mathscr{Q}(M_1 \oplus M_2) = \mathscr{Q}(M_1) \oplus \mathscr{Q}(M_2), && \text{for } M_1, M_2 \in \mathrm{Mod}_A, && \text{(5f)}\\
&\mathscr{Q}(M_1 \cap M_2) = \mathscr{Q}(M_1) \cap \mathscr{Q}(M_2), && \text{for } M_1, M_2 \subseteq N \in \mathrm{Mod}_A, && \text{(5g)}
\end{aligned}
$$

but not arbitrary sums. Indeed, only the inclusion

$$
\mathscr{Q}(M_1 + M_2) \supseteq \mathscr{Q}(M_1) + \mathscr{Q}(M_2), \qquad \text{for } M_1, M_2 \subseteq N \in \mathrm{Mod}_A \qquad \text{(5h)}
$$

holds, in general.

2.9 The Multiplicatively Closed Set Induced by a Serre Category

Let $\mathfrak{C}$ be an arbitrary Serre subcategory of Mod_A. Equation (4) motivates the definition of the multiplicatively closed set $T(\mathfrak{C}) = \{t \in A : A/At \in \mathfrak{C}\}$ and the associated localization

$$
(-)_{T(\mathfrak{C})} : M \longmapsto M_{T(\mathfrak{C})} \text{ for } M \in \mathrm{Mod}_A .
$$

In general the two localizations $\mathscr{Q}_{\mathfrak{C}}(-)$ and $(-)_{T(\mathfrak{C})}$ do not coincide. Nevertheless, the two are related and the latter one is important for the former one, in particular when trying to design algorithms to test properties and parametrize solutions. We make the following assumption which holds in all standard situations.

Assumption 5. In the rest of the article we will make the steady assumption that the two localizations of the ring A are equal, i.e., $\mathscr{Q}_{\mathfrak{C}}(A) = A_{T(\mathfrak{C})}$ holds. This assumption holds, for instance, whenever the ring A is a unique factorization domain, in particular if it is a polynomial ring $A = F[s_1, \ldots, s_n]$ over some field F.

However, we do not assume that $\mathscr{Q}_{\mathfrak{C}}(-) = (-)_{T(C)}$, i.e., in general we will consider in this paper modules for which the two localizations do not coincide.

Example 6. 1. Let $A = \mathbb{C}[s]$ and consider a Serre subcategory $\mathfrak{C}(\Lambda_1)$ induced by a disjoint decomposition $\mathbb{C}^n = \Lambda_1 \uplus \Lambda_2$ with $\Lambda_2 \neq \emptyset$ as in Example 2, part 2. The multiplicatively closed set induced by this Serre subcategory is the set

$$
T(\mathfrak{C}(\Lambda_1)) = \{t \in \mathbb{C}[s] : t(\lambda) = 0 \Longrightarrow \lambda \in \Lambda_1\}
$$

of all polynomials which vanish only on a subset of Λ_1 – in the continuous standard case these are the Hurwitz polynomials.

Now we focus on the decomposition $\Lambda_1 = \{0\}$ and $\Lambda_2 = \mathbb{C}^n \setminus \{0\}$, where $n \geqslant 2$. The only polynomials which vanish only on a subset of the singleton set $\{0\}$ are the non-zero constants, i.e., we have $T(\mathfrak{C}(\Lambda_1)) = \mathbb{C} \setminus \{0\}$ and consequently $M_{T(\mathfrak{C}(\Lambda_1))} = M$ for all modules $M \in \mathrm{Mod}_{\mathbb{C}[s]}$.

On the other hand, it can be proved that the module $M = \mathbb{C}[s]/\mathfrak{m}(0)$ is $\mathfrak{C}(\Lambda_1)$-negligible and therefore, by Property 5c, $\mathscr{Q}_{\mathfrak{C}(\Lambda_1)}\big(\mathbb{C}[s]/\mathfrak{m}(0)\big) = 0$ in contrast to $\big(\mathbb{C}[s]/\mathfrak{m}(0)\big)_{T(\mathfrak{C}(\Lambda_1))} = \mathbb{C}[s]/\mathfrak{m}(0) \neq 0$.

Since the polynomial ring $\mathbb{C}[s]$ is a unique factorization domain, the identity

$$\mathscr{Q}_{\mathfrak{C}(\Lambda_1)}(\mathbb{C}[s]) = \mathbb{C}[s]_{T(\mathfrak{C}(\Lambda_1))} = \mathbb{C}[s]$$

holds.

In general, it is very difficult to determine for given decompositions $\mathbb{C}^n = \Lambda_1 \uplus \Lambda_2$ whether the two localization functors are the same. For example, for the generalized Hurwitz situation with $\Lambda_2 = \{z \in \mathbb{C} : \Re(z) \geqslant 0\}$ the two localizations coincide if $n = 1$ or $n = 2$ but it is not known whether this holds also for higher dimensions.

2. Let A be a noetherian integral domain which is not a principal ideal domain and consider the Serre subcategory

$$\mathfrak{C}_{\text{fin}} = \{M \in \mathrm{Mod}_A : \ M_{\mathfrak{p}} = 0 \text{ for all } \mathfrak{p} \in \operatorname{spec}(A) \setminus \max(A)\}$$

from Example 2, part 3. Similarly to the situation above, it holds that the multiplicatively closed set $T(\mathfrak{C}_{\text{fin}})$ is just the set of invertible elements of the operator ring[4] (see [8, Lem. 3.2] and [9, Eq. (2)]) and thus $M_{T(\mathfrak{C}_{\text{fin}})} = M$ for all A-modules M. But again, for any maximal ideal $\mathfrak{m}$ we have $A/\mathfrak{m} \in \mathfrak{C}$ and thus $\mathscr{Q}_{\mathfrak{C}_{\text{fin}}}(A/\mathfrak{m}) = 0$, i.e. $\mathscr{Q}_{\mathfrak{C}_{\text{fin}}}(-) \neq (-)_{T(\mathfrak{C}_{\text{fin}})}$.

As for the Gabriel localization functor and the $\mathfrak{C}$-radical, we will write $T := T(\mathfrak{C})$ from now on.

The Gabriel localization $\mathscr{Q}(U) \subseteq \mathscr{Q}(A^{1\times l}) = A_T^{1\times l}$ of a module of equations $U = A^{1\times k}R$, $R \in A^{k\times l}$, is an A_T-submodule of $A_T^{1\times l}$, and thus it is finitely generated by a matrix $R' \in A_T^{k'\times l}$, i.e., we have

$$\mathscr{Q}(U) = A_T^{1\times k'}R'. \tag{6}$$

Let K denote the quotient field of A. The chain of inclusions

$$\begin{aligned} U \subseteq U_T = A_T^{1\times k}R \subseteq \mathscr{Q}(U) = A_T^{1\times k'}R' = \mathscr{Q}(U_T) \\ \subseteq KU = K^{1\times k}R = K\mathscr{Q}(U) = K^{1\times k'}R' \subseteq K^{1\times l} \end{aligned} \tag{7}$$

does always hold, but in general the two sets U_T and $\mathscr{Q}(U)$ are not equal. As a consequence of properties (5b), (5d) and (7), one has the set of equivalences

$$\mathscr{Q}(U_1) \subseteq \mathscr{Q}(U_2) \Longleftrightarrow U_1 \subseteq \mathscr{Q}(U_2) \Longleftrightarrow (U_1)_T \subseteq \mathscr{Q}(U_2), \qquad \text{for } U_1,\, U_2 \subseteq A^{1\times l}. \tag{8}$$

The matrix R' in Equation (6) can be computed via the following algorithm.

[4] In the case that $A = F[s]$ is a polynomial ring over a field, this means that $T(\mathfrak{C}_{\text{fin}}) = F \setminus \{0\}$.

Algorithm 7 ([15, Alg. 3.9]). *Let $U = A^{1\times k}R \subseteq A^{1\times l}$ with $R \in A^{k\times l}$. To find a matrix $R' \in A^{k'\times l}$ such that $\mathscr{Q}(U) = A_T^{1\times k'}R'$ compute a reduced primary decomposition*

$$U = \bigcap_{\mathfrak{p}\in \operatorname{ass}(A^{1\times l}/U)} U(\mathfrak{p})$$

of U in $A^{1\times l}$, where $\operatorname{ass}(M)$ denotes the associator[5] *of a module M and the modules $U(\mathfrak{p}) \subseteq A^{1\times l}$ are $\mathfrak{p}$-primary, i.e., $\operatorname{ass}(A^{1\times l}/U(\mathfrak{p})) = \{\mathfrak{p}\}$. Then form the set*

$$\mathfrak{P} := \{\mathfrak{p} \in \operatorname{ass}(A^{1\times l}/U) : A/\mathfrak{p} \notin \mathfrak{C}\}$$

and compute a matrix $R' \in A^{k'\times l}$ such that $A^{1\times k'}R' = \bigcap_{\mathfrak{p}\in\mathfrak{P}} U(\mathfrak{p})$. This matrix satisfies $A^{1\times k}R' = \mathscr{Q}(U)$.

For polynomial rings A the computations of the primary decomposition and of the intersection are implemented in various computer algebra systems, for instance in SINGULAR[6]. The methods for testing whether $A/\mathfrak{p} \in \mathfrak{C}$ depend on the specific Serre subcategory.

Example 8.
1. If $\mathfrak{C} = \mathfrak{C}(T)$ for a multiplicatively closed set T then $\mathscr{Q}_{\mathfrak{C}(T)}(U) = U_T = A_T^{1\times k}R$, i.e., one can choose $R' = R$ and does not need Algorithm 7.
2. If $\mathfrak{C} = \mathfrak{C}(\Lambda_2)$ for a disjoint decomposition $\mathbb{C}^n = \Lambda_1 \uplus \Lambda_2$ then $A/\mathfrak{p} \in \mathfrak{C}(\Lambda_2)$ if and only if the zero set of $\mathfrak{p}$ is contained in Λ_1. In many cases this criterion can be checked with the algorithm described in the first paragraph of [15, Sec. 7].
3. If $\mathfrak{C} = \mathfrak{C}_{\text{fin}}$ then $A/\mathfrak{p} \in \mathfrak{C}_{\text{fin}}$ if and only if $\mathfrak{p}$ is a maximal ideal. This condition can be checked using a computer algebra system.

In the following technical lemma we use the properties of Gabriel localization to show an identity that will come in handy in Corollary 24.

Lemma 9. *Let U_1 and U_2 be two submodules of $A^{1\times l}$ or of $A_T^{1\times l}$. Then*

$$\mathscr{Q}(\mathscr{Q}(U_1) + U_2) = \mathscr{Q}(U_1 + U_2).$$

Proof. **$\supseteq$.** This inclusion follows from (7) ($U_1 \subseteq \mathscr{Q}(U_1)$) and from property (5d) (since $M_1 = U_1 + U_2 \subseteq M_2 = \mathscr{Q}(U_1) + U_2$).

$\subseteq$. From (7) and (5d), again, we infer $U_2 \subseteq \mathscr{Q}(U_2)$ and $\mathscr{Q}(\mathscr{Q}(U_1)+U_2) \subseteq \mathscr{Q}(\mathscr{Q}(U_1) + \mathscr{Q}(U_2))$. By using (5h), (5b) and (5d), we conclude

$$\mathscr{Q}(\mathscr{Q}(U_1)+U_2) \subseteq \mathscr{Q}(\mathscr{Q}(U_1)+\mathscr{Q}(U_2)) \overset{(5h),(5d)}{\subseteq} \mathscr{Q}(\mathscr{Q}(U_1+U_2)) \overset{(5b)}{=} \mathscr{Q}(U_1+U_2).$$

□

2.10 Steady State Behaviours and Duality

The steady states $\mathscr{F}_2$ form a $\mathscr{Q}(A) = A_T$ module. The direct sum decomposition of the signal space in transient signals and steady states allows us to write down the scalar multiplication explicitly:

$$\frac{a}{t} \circ w = y_2 \quad \text{for } a \in A,\ t \in T, \text{ and } w \in \mathscr{F}_2,$$

where $y = y_1 + y_2 \in \operatorname{Ra}_{\mathfrak{C}}(\mathscr{F}) \oplus \mathscr{F}_2$ is a solution of $t \circ y = a \circ w$.

[5] $\operatorname{ass}(M)$ is the set of all prime ideals $\mathfrak{p}$ such that there is a monomorphism from $A/\mathfrak{p}$ to M.

[6] http://www.singular.uni-kl.de/

Such a solution y does always exist since $\mathscr{F}$ is an injective A-module and hence divisible over A.

The fact that $\mathscr{F}$ is a large injective cogenerator in Mod_A implies that the module $\mathscr{F}_2$ is an injective cogenerator in the category $\mathrm{Mod}_{A,\mathfrak{C}}$ and this, in turn, induces the following duality theory between steady state behaviours in $\mathscr{F}_2^l$ and $\mathfrak{C}$-closed submodules of equations of $A_T^{1\times l}$. The A_T-scalar multiplication on $\mathscr{F}_2$ carries over to multiple components. It gives rise to the bilinear form

$$A_T^{1\times l} \times \mathscr{F}_2^l \longrightarrow A_T: \ (\eta,\ w) \longmapsto \eta \circ w,$$

and to a corresponding notion of orthogonality:

$$\begin{aligned} \widetilde{U}^{\perp_2} &= \{w \in \mathscr{F}_2^l: \ \widetilde{U} \circ w = 0\} && \text{for } A_T\text{-submodules } \widetilde{U} \subseteq A_T^{1\times l}, \\ \widetilde{\mathscr{B}}^{\perp_2} &= \{\eta \in A_T^{1\times l}: \ \eta \circ \widetilde{\mathscr{B}} = 0\} && \text{for } A_T\text{-submodules } \widetilde{\mathscr{B}} \subseteq \mathscr{F}_2^l. \end{aligned}$$

Also, $\widetilde{U}^{\perp_2\perp_2} = \widetilde{U}$ for $\mathfrak{C}$-closed $\widetilde{U} \subseteq A_T^{1\times l}$ and $\widetilde{\mathscr{B}}^{\perp_2\perp_2} = \widetilde{\mathscr{B}}$ for $\mathfrak{C}$-closed $\mathscr{F}_2$-behaviours $\widetilde{\mathscr{B}}$ over A_T. We denote this concept of orthogonality by $\perp_2$ to distinguish it from the earlier one given for submodules of $A^{1\times l}$ and of $\mathscr{F}^l$.

The steady state behaviour $\mathscr{B} \cap \mathscr{F}_2^l$ is $\mathfrak{C}$-closed. It is therefore an A_T-module and

$$(\mathscr{B} \cap \mathscr{F}_2^l)^{\perp_2} = \{\eta \in A_T^{1\times l} : \eta \circ (\mathscr{B} \cap \mathscr{F}_2^l) = 0\} = \mathscr{Q}(A^{1\times k}R) = A_T^{1\times k'}R' \tag{9}$$

holds, i.e., the module of equations of $\mathscr{B} \cap \mathscr{F}_2^l$ is finitely generated as an A_T-module by the rows of R'. This means that, although in general $\mathscr{B} \cap \mathscr{F}_2^l$ it is not an A-behaviour in $\mathscr{F}^l$, it is an A_T-behaviour in $\mathscr{F}_2^l$.

As a consequence, given two behaviours $\mathscr{B} = (A^{1\times k}R)^\perp$ and $\widetilde{\mathscr{B}} = (A^{1\times \widetilde{k}}\widetilde{R})^\perp$, one has the following set of equivalent conditions:

$$\begin{aligned} \mathscr{B} \cap \mathscr{F}_2^l \subseteq \widetilde{\mathscr{B}} \cap \mathscr{F}_2^l &\Longleftrightarrow \{w \in \mathscr{F}_2^l: \ R \circ w = 0\} \subseteq \{w \in \mathscr{F}_2^l: \ \widetilde{R} \circ w = 0\} \\ &\Longleftrightarrow \mathscr{Q}(A^{1\times k}R) \supseteq \mathscr{Q}(A^{1\times \widetilde{k}}\widetilde{R}) \Longleftrightarrow \exists X \in A_T^{\widetilde{k}'\times k'}: \ \widetilde{R}' = XR' \\ &\overset{(8)}{\Longleftrightarrow} \mathscr{Q}(A^{1\times k}R) \supseteq A^{1\times \widetilde{k}}\widetilde{R} \Longleftrightarrow \exists X \in A_T^{\widetilde{k}\times k'}: \ \widetilde{R} = XR', \end{aligned} \tag{10}$$

where $\mathscr{Q}(A^{1\times k}R) = A_T^{1\times k'}R'$ and $\mathscr{Q}(A^{1\times \widetilde{k}}\widetilde{R}) = A_T^{1\times \widetilde{k}'}\widetilde{R}'$.

2.11 The Image of a Steady State Behaviour

Consider a behaviour $\mathscr{B} = (A^{1\times k}R)^\perp \subseteq \mathscr{F}^l$, a matrix $P \in A^{v\times l}$ and the associated map $P\circ$ as described in Section 2.2. The image of $\mathscr{B} \cap \mathscr{F}_2^l$ under $P\circ$ is

$$P \circ (\mathscr{B} \cap \mathscr{F}_2^l) = (P \circ \mathscr{B}) \cap \mathscr{F}_2^v.$$

In other words,

$$\{u \in \mathscr{F}^v: \ \exists w \in \mathscr{B} \cap \mathscr{F}_2^l : P \circ w = u\} = \{u \in \mathscr{F}_2^v : \exists w \in \mathscr{B}: \ P \circ w = u\}.$$

Now we take a look at the dual map between row spaces $A^{1\times \cdot}$. Let $V \subseteq A^{1\times v}$ and $U \subseteq A^{1\times l}$ be submodules with $VP \subseteq U$ – for example $V = (P \circ \mathscr{B})^\perp$ and $U = \mathscr{B}^\perp = A^{1\times k}R$ – and consider the homomorphism

$$\begin{aligned} \cdot P: V &\longrightarrow U \\ \eta &\longmapsto \eta P, \end{aligned}$$

induced by P. Because of Assumption 5, we have $\mathscr{Q}(V) \subseteq A_T^{1\times v}$ and $\mathscr{Q}(U) \subseteq A_T^{1\times l}$. The functor $\mathscr{Q}$ sends the homomorphism $\cdot P$ to

$$\begin{aligned} \mathscr{Q}(\cdot P) : \mathscr{Q}(V) &\longrightarrow \mathscr{Q}(U), \\ \eta &\longmapsto \eta P, \end{aligned} \tag{11}$$

i.e., the map $\mathscr{Q}(\cdot P)$ is again the multiplication of row vectors by P. As a consequence, $\mathscr{Q}(V)P \subseteq \mathscr{Q}(VP) = \mathscr{Q}(\mathrm{im}(\cdot P)) \subseteq \mathscr{Q}(U)$. The same holds for A_T-modules $V \subseteq A_T^{1\times v}$ and $U \subseteq A_T^{1\times l}$.

2.12 Free Variables and $\mathfrak{C}$-stable IO Behaviours

Since the free variables of a behaviour (in particular, of an IO behaviour) are only related to the ranks of the matrices involved in the behaviour representation, a variable is free in $\mathscr{B}$ if and only if the same variable is free in $\mathscr{B} \cap \mathscr{F}_2^l$.

An IO behaviour $\mathscr{B} = (A^{1\times k}(P,\ -Q))^\perp$ with $(P,\ -Q) \in A^{k\times(p+m)}$ is $\mathfrak{C}$-*stable* if its autonomous part $\mathscr{B}^0 = (A^{1\times k}P)^\perp$ is $\mathfrak{C}$-negligible, i.e., equivalently, $\mathscr{B}^0 \in \mathfrak{C}$ or $\mathscr{B}^0 \cap \mathscr{F}_2^p = \{0\}$ or $\mathscr{Q}(A^{1\times k}P) = A_T^{1\times p}$ [8, Thm. and Def. 4.2]. If this is the case, then the entries of the transfer matrix lie in A_T and

$$\mathscr{B} \cap \mathscr{F}_2^{p+m} = \left\{ \binom{y}{u} \in \mathscr{F}_2^{p+m} :\ y = H \circ u \right\}, \tag{12}$$

i.e., $H \in A_T^{p\times m}$ can be seen as an operator that maps every steady state input to the corresponding steady state output [15, Cor. 3.8].

Lemma 10. *Consider a behaviour $\mathscr{B} = \{\binom{w_1}{w_2} \in \mathscr{F}^{l_1+l_2} :\ (R_1,\ R_2) \circ \binom{w_1}{w_2} = 0\}$ with free variables w_2 and let $H \in K^{l_1\times l_2}$ be such that $R_2 = -R_1 H$. Assume that the entries of H lie in A_T, i.e., $H \in A_T^{l_1\times l_2}$. Let $R_1' \in A_T^{k'\times l_1}$ be such that $\mathscr{Q}(A^{1\times k}R_1) = A_T^{1\times k'}R_1'$. Then the module of equations of the steady state behaviour of $\mathscr{B}$, i.e., of $\mathscr{B} \cap \mathscr{F}_2^{l_1+l_2}$, is*

$$(\mathscr{B} \cap \mathscr{F}_2^{l_1+l_2})^{\perp_2} = A_T^{1\times k'} R_1' (\mathrm{id}_{l_1},\ -H) = A_T^{1\times k'}(R_1',\ -R_1' H).$$

In the special case when $\mathscr{B} = (A^{1\times k}(P,\ -Q))^\perp$ is a $\mathfrak{C}$-stable IO behaviour, we obtain

$$(\mathscr{B} \cap \mathscr{F}_2^{p+m})^{\perp_2} = A_T^{1\times p}(\mathrm{id}_p,\ -H). \tag{13}$$

Proof. Since

$$\begin{aligned} (\mathscr{B} \cap \mathscr{F}_2^{l_1+l_2})^{\perp_2} &\overset{(9)}{=} \mathscr{Q}(A^{1\times k}(R_1,\ R_2)) = \mathscr{Q}(A_T^{1\times k}(R_1,\ R_2)) \\ &= \mathscr{Q}(A_T^{1\times k} R_1 (\mathrm{id}_{l_1},\ -H)), \end{aligned}$$

we need to show that $\mathscr{Q}(A_T^{1\times k} R_1(\mathrm{id}_{l_1},\ -H)) = A_T^{1\times k'} R_1' (\mathrm{id}_{l_1},\ -H)$.

Consider the isomorphism

$$\begin{aligned} \phi :\ A_T^{1\times k} R_1 &\longrightarrow A_T^{1\times k} R_1 (\mathrm{id}_{l_1},\ -H) \\ \eta_1 &\longmapsto \eta_1 (\mathrm{id}_{l_1},\ -H) = (\eta_1,\ \eta_1 H), \end{aligned}$$

with inverse

$$\begin{aligned} \phi^{-1} :\ A_T^{1\times k} R_1 (\mathrm{id}_{l_1},\ -H) &\longrightarrow A_T^{1\times k} R_1 \\ (\eta_1,\ \eta_2) &\longmapsto \eta_1. \end{aligned}$$

Applying the functor $\mathscr{Q}$ leads to the isomorphism

$$\psi := \mathscr{Q}(\phi): \ \mathscr{Q}(A_T^{1\times k}R_1) \longrightarrow \mathscr{Q}(A_T^{1\times k}R_1(\mathrm{id}_{l_1},\ -H))$$
$$\eta_1 \longmapsto \eta_1(\mathrm{id}_{l_1},\ -H),$$

with inverse

$$\psi^{-1} = \mathscr{Q}(\phi^{-1}) = \mathscr{Q}(\phi)^{-1}: \ \mathscr{Q}(A_T^{1\times k}R_1(\mathrm{id}_{l_1},\ -H)) \longrightarrow \mathscr{Q}(A_T^{1\times k}R_1)$$
$$(\eta_1,\ \eta_2) \longmapsto \eta_1.$$

Notice that ψ and ψ^{-1} are given by the same matrices as the original maps. We use $\mathscr{Q}(A^{1\times k}R_1) = \mathscr{Q}(A_T^{1\times k}R_1) = A_T^{1\times k'}R_1'$ to infer that

$$\begin{aligned}\mathscr{Q}(A_T^{1\times k}R_1(\mathrm{id}_{l_1},\ -H)) &= \psi\left(\psi^{-1}\left(\mathscr{Q}(A_T^{1\times k}R_1(\mathrm{id}_{l_1},\ -H))\right)\right)\\ &= \psi(\mathscr{Q}(A_T^{1\times k}R_1)) = \psi(A_T^{1\times k'}R_1') = A_T^{1\times k'}R_1'(\mathrm{id}_{l_1},\ -H).\end{aligned}$$

If $\mathscr{B} = (A^{1\times k}(P,\ -Q))^{\perp}$ is a $\mathfrak{C}$-stable IO behaviour then its transfer matrix H has entries in A_T and $PH = Q$. Consequently, H satisfies the conditions of the lemma. Furthermore, $\mathscr{Q}(A^{1\times k}P) = A_T^{1\times p}$. So, the assertion follows as a special case of what has been proven above. □

3 Stabilization by Partial Interconnection

As a first step, we formally introduce the stabilization problem. Then we state and prove our main result on stabilization (Theorem 13), followed by a parametrization of the stabilizing compensators. We conclude the section with Example 15 which we will expand in Sections 4 and 5.

In the following, we will steadily assume that $\emptyset \neq \mathfrak{C} \subsetneq \mathrm{Mod}_A$ is a fixed Serre subcategory of modules satisfying Assumption 5, i.e., $\mathscr{Q}_{\mathfrak{C}}(A) = A_{T(\mathfrak{C})}$, and we will refer to it in order to define the concept of $\mathfrak{C}$-negligibility and hence of $\mathfrak{C}$-stabilizability.

Definition 11. Given a plant

$$\mathscr{P} = \left\{ \left(\begin{smallmatrix} w \\ c \end{smallmatrix}\right) \in \mathscr{F}^{l_w+l_c} : \ (R_w,\ R_c) \circ \left(\begin{smallmatrix} w \\ c \end{smallmatrix}\right) = 0 \right\}, \tag{14}$$

with $(R_w,\ R_c) \in A^{k_p\times(l_w+l_c)}$. The to-be-controlled variable w is of size l_w and the control variable c is of size l_c. We say that the compensator

$$\mathscr{C} = \{c \in \mathscr{F}^{l_c} : \ C_c \circ c = 0\}, \tag{15}$$

with $C_c \in A^{k_c\times l_c}$, $\mathfrak{C}$*-stabilizes the plant* $\mathscr{P}$ if the partial interconnection

$$\mathscr{P} \wedge_c \mathscr{C} = \left\{ \begin{pmatrix} w \\ c \end{pmatrix} \in \mathscr{F}^{l_w+l_c} : \ \begin{pmatrix} R_w & R_c \\ 0 & C_c \end{pmatrix} \circ \begin{pmatrix} w \\ c \end{pmatrix} = 0 \right\}$$

is

1. regular, i.e., $A^{1\times k_p}(R_w,\ R_c) \cap A^{1\times k_c}(0,\ C_c) = \{0\}$, and
2. $\mathfrak{C}$-negligible, i.e., $\mathscr{P} \wedge_c \mathscr{C} \in \mathfrak{C}$.

We call a plant $\mathfrak{C}$-*stabilizable* if it admits a (partial regular) $\mathfrak{C}$-stabilizing compensator.

The second item of the previous definition is the control goal, namely, that all trajectories in the interconnected behaviour should be $\mathfrak{C}$-negligible. If the interconnection were not regular, the equations of the compensator could conflict with those of the plant and this would leave no viable trajectory, whereas a regular interconnection can always be seen as an input-output feedback connection (see [21]). Furthermore, the the controller may only affect the control variable c directly and any influence on the variable to be controlled w must be exerted via the control variable.

The diagram illustrating the connection of a plant and a compensator is given in Figure 1.

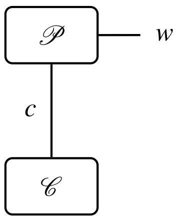

Figure 1: The interconnection diagram for the problem of stabilization by partial interconnection.

The first goal of this section is to find necessary and sufficient conditions for the existence of a $\mathfrak{C}$-stabilizing compensator for a given plant. To this end, we need a preliminary lemma.

Lemma 12. *Let $\mathscr{P}$ and $\mathscr{C}$ be a plant and compensator, described as in* (14) *and* (15)*, respectively, for suitable matrices $(R_w,\ R_c) \in A^{k_p\times(l_w+l_c)}$ and $C_c \in A^{k_c\times l_c}$. Also, let $R' = (R'_w,\ R'_c) \in A_T^{k'_p\times(l_w+l_c)}$ and $C'_c \in A_T^{k'_c\times l_c}$ be matrices such that*

$$\mathscr{Q}(A^{1\times k_p}(R_w,\ R_c)) = A_T^{1\times k'_p}R' \quad \text{and} \quad \mathscr{Q}(A^{1\times k_c}C_c) = A_T^{1\times k'_c}C'_c.$$

The behaviour $\mathscr{C}$ is a $\mathfrak{C}$-stabilizing compensator for the plant $\mathscr{P}$ if and only if the sum of $A_T^{1\times k'_p}R'$ and $A_T^{1\times k'_c}(0,\ C'_c)$ is a direct one and it coincides with $A_T^{1\times(l_w+l_c)}$, namely $A_T^{1\times k'_p}R' \oplus A_T^{1\times k'_c}(0,\ C'_c) = A_T^{1\times(l_w+l_c)}$.

Proof. By properties (5g) and (7) and the fact that $\mathscr{Q}(\{0\}) = \{0\}$ (which follows from $\{0\} \in \mathfrak{C} \neq \emptyset$ and (5c)) we have

$$\begin{aligned} &A^{1\times k_p}(R_w,\ R_c) \cap A^{1\times k_c}(0,\ C_c) = \{0\} \\ &\qquad\qquad \Longleftrightarrow \mathscr{Q}(A^{1\times k_p}(R_w,\ R_c)) \cap \mathscr{Q}(A^{1\times k_c}(0,\ C_c)) = \{0\}. \end{aligned}$$

Hence, the interconnection $\mathscr{P} \wedge_c \mathscr{C}$ is regular if and only if the sum of the Gabriel localizations of the modules of equations of $\mathscr{P}$ and $\mathscr{C}$ is direct.

Moreover, due to the direct sum decomposition (3) of a behaviour and Property (5c), we have that $\mathscr{P} \wedge_c \mathscr{C} \in \mathfrak{C}$ if and only if

$$\mathscr{Q}(\mathscr{P} \wedge_c \mathscr{C}) \cong (\mathscr{P} \wedge_c \mathscr{C}) \cap \mathscr{F}_2^{l_w+l_c} = \left\{ \begin{pmatrix} w \\ c \end{pmatrix} \in \mathscr{F}_2^{l_w+l_c} :\ \begin{pmatrix} R_w & R_c \\ 0 & C_c \end{pmatrix} \circ \begin{pmatrix} w \\ c \end{pmatrix} = 0 \right\} = \{0\}.$$

This is equivalent to

$$
\begin{aligned}
A_T^{1\times(l_w+l_c)} &\overset{\text{ass. on } A}{=} \mathscr{Q}(A^{1\times(l_w+l_c)}) = \mathscr{Q}\left(A^{1\times(k_p+k_c)}\begin{pmatrix} R_w & R_c \\ 0 & C_c \end{pmatrix}\right) \\
&= \mathscr{Q}\left(A^{1\times k_p}(R_w,\, R_c) \oplus A^{1\times k_c}(0,\, C_c)\right) \\
&\overset{(5f)}{=} \mathscr{Q}\left(A^{1\times k_p}(R_w,\, R_c)\right) \oplus \mathscr{Q}\left(A^{1\times k_c}(0,\, C_c)\right) \\
&= A_T^{1\times k'_p} R' \oplus A_T^{1\times k'_c}(0,\, C'_c). \qquad \square
\end{aligned}
$$

The previous characterization allows us to derive a necessary and sufficient condition for the existence of a $\mathfrak{C}$-stabilizing compensator, given in Theorem 13 below.

For the proof of this theorem as well as for the resulting parametrization of all stabilizing compensators we need the following parametrization of all direct complements of $A_T^{1\times k'_p} R'$ in $A_T^{1\times l}$, where $l := l_w + l_c$. This technical result was first used in [8] for the parametrization of $\mathfrak{C}$-stabilizing *feedback* compensators.

First of all, there exists a direct complement of $A_T^{1\times k'_p} R'$ in $A_T^{1\times l}$ if and only if the inhomogeneous linear equation

$$R' G_0 R' = R' \tag{16}$$

in the unknown $G_0 \in A_T^{l\times k'_p}$ has a solution[7].

If $\widetilde{G}_0$ is a specific solution of (16), then the solutions of (16) are those and those only taking the form $G_0 = \widetilde{G}_0 - Y$, as Y varies in the set of solutions of the associated homogeneous equation $R'YR' = 0$. Accordingly, to every such solution G_0 of (16) we can associate the matrix $C_0 := \mathrm{id}_l - G_0 R' = (\mathrm{id}_l - \widetilde{G}_0 R') + YR' =: \widetilde{C}_0 + YR' \in A_T^{l\times l}$ that satisfies $A_T^{1\times k'_p} R' \oplus A_T^{1\times l} C_0 = A_T^{1\times l}$. More in detail, every direct complement of $A_T^{1\times k'_p} R'$ can be obtained via a solution Y of the homogeneous linear matrix equation $R'YR' = 0$ and there is the one-to-one correspondence

$$\{Y \in A_T^{l\times k'_p} :\ R'YR' = 0\}/\{Y :\ YR' = 0\} \longleftrightarrow \{U \subseteq A_T^{1\times l} :\ A_T^{1\times k'_p} R' \oplus U = A_T^{1\times l}\}$$

defined by the following map (see [8, Thms. 2.12, 2.13]):

$$\overline{Y} \longmapsto A_T^{1\times l}(\mathrm{id}_l - (\widetilde{G}_0 - Y)R') = A_T^{1\times l}(\widetilde{C}_0 + YR'), \tag{17}$$

where Y is any representative of the class $\overline{Y} = Y + \{\widetilde{Y} :\ \widetilde{Y}R' = 0\}$.

We are now in a position to state and prove the following result.

Theorem 13. *Given a plant $\mathscr{P}$, described as in* (14) *for some suitable matrix $(R_w,\, R_c) \in A^{k_p\times(l_w+l_c)}$, let $R' = (R'_w,\, R'_c) \in A_T^{k'_p\times(l_w+l_c)}$ be such that*

$$\mathscr{Q}(A^{1\times k_p}(R_w,\, R_c)) = A_T^{1\times k'_p} R'.$$

The following statements are equivalent:

1. *There exists a $\mathfrak{C}$-stabilizing compensator $\mathscr{C}$ for $\mathscr{P}$.*
2. *The module $A_T^{1\times k'_p} R'$ has a direct complement in $A_T^{1\times(l_w+l_c)}$, i.e., there exists a $G_0 \in A_T^{l\times k_p}$ such that $R'G_0R' = R'$, and $A_T^{1\times k'_p} R'_w = A_T^{1\times l_w}$, i.e., R'_w has a left inverse over A_T.*

[7] The same equation over the ring of operators A appeared already in [22, Cor. 5.2], where it was used to characterize those plants that are controllable to zero by interconnection.

Proof. $1 \Longrightarrow 2$. Let $\mathscr{C}$, described as in (15), be a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$, and let $C_c' \in A_T^{k_c' \times l_c}$ be such that $\mathscr{Q}(A^{1\times k_c} C_c) = A_T^{1\times k_c'} C_c'$.

Then, by Lemma 12, we have

$$A_T^{1\times k_p'}(R_w',\ R_c') \oplus A_T^{1\times k_c'}(0,\ C_c') = A_T^{1\times l},$$

i.e., $A_T^{1\times k_p'}(R_w',\ R_c') = A_T^{1\times k_p'} R'$ is a direct summand of $A_T^{1\times l}$ (equivalently, $A_T^{1\times k_p'} R'$ has a direct complement in $A_T^{1\times l}$). Furthermore, by focusing only on the first components in the previous identity, we get $A_T^{1\times k_p'} R_w' \oplus A_T^{1\times k_c'} 0 = A_T^{1\times l_w}$, i.e., $A_T^{1\times k_p'} R_w' = A_T^{1\times l_w}$, thus proving that R_w' has a left inverse over A_T.

$2 \Longrightarrow 1$. Since $A_T^{1\times k_p'} R'$ is a direct summand of $A_T^{1\times l}$, equation (16) has a solution $\widetilde{G}_0 \in A_T^{l\times k_p'}$ and the row space of the matrix $\widetilde{C}_0 := \mathrm{id}_l - \widetilde{G}_0 R' \in A_T^{l\times l}$ is a direct complement of $A_T^{1\times k_p'} R'$. Let $X \in A_T^{l_w \times k_p'}$ be a left inverse of R_w', and partition the matrix $\widetilde{C}_0$ as $\widetilde{C}_0 = (\widetilde{C}_{0w},\ \widetilde{C}_{0c})$, where the first block consists of l_w columns, while the second one of l_c columns. The matrix $Y := -\widetilde{C}_{0w} X = -\left(\begin{pmatrix} \mathrm{id}_{l_w} \\ 0 \end{pmatrix} - \widetilde{G}_0 R_w'\right) X$ satisfies

$$R' Y R' = -\underbrace{R' \begin{pmatrix} \mathrm{id}_{l_w} \\ 0 \end{pmatrix}}_{=R_w'} X R' + \underbrace{R' \widetilde{G}_0 R_w'}_{=R_w'} X R' = 0.$$

From the parametrization (17) it follows that the module $A_T^{1\times l} C_1$ with $C_1 := \widetilde{C}_0 + Y R'$ is also a direct complement of $A_T^{1\times k_p'} R'$. Furthermore, $C_{1w} = \widetilde{C}_{0w} - \widetilde{C}_{0w} X R_w' = \widetilde{C}_{0w} - \widetilde{C}_{0w} = 0$, i.e., the matrix $C_1 = (C_{1w},\ C_{1c}) = (0,\ C_{1c})$ has the correct block structure. Let $t \in T$ be a common denominator of the entries of C_c. Then $tC_1 = (0,\ tC_{1c}) \in A^{k_c \times (l_w + l_c)}$ and the module $A_T^{1\times k_c} t C_1 = A_T^{1\times k_c} C_1$ is a direct complement of $A_T^{1\times k_p'} R'$. By this argumentation, the behaviour

$$\mathscr{C} := \{c \in \mathscr{F}^{l_c} :\ (tC_{1c}) \circ c = 0\}$$

is a $\mathfrak{C}$-stabilizing compensator of $\mathscr{P}$. □

Remark 14. The second condition of item 2. in Theorem 13 is equivalent to the existence of a $\mathfrak{C}$-observer of w from c for the plant $\mathscr{P}$ [15, Thm. 4.4]. The existence of a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ by (regular) partial interconnection via c is thus equivalent to the existence of a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ by (regular) full interconnection and the existence of a $\mathfrak{C}$-observer of w from c for $\mathscr{P}$.

Now we give a parametrization of all $\mathfrak{C}$-stabilizing compensators for the plant $\mathscr{P}$. We assume that such a compensator exists and use the same notation as in the proof of Theorem 13. In particular, we let $\widetilde{G}_0$ denote a solution of $R' G_0 R' = R'$, and $X \in A_T^{l_w \times k_p'}$ a left inverse of R_w'. Hence the row space of

$$\begin{aligned} C_1 = (0,\ C_{1c}) &= (\mathrm{id}_l - \widetilde{G}_0 R') - \left(\begin{pmatrix} \mathrm{id}_{l_w} \\ 0 \end{pmatrix} - \widetilde{G}_0 R_w'\right) X R' \\ &= \mathrm{id}_l - \left(\widetilde{G}_0 + \left(\begin{pmatrix} \mathrm{id}_{l_w} \\ 0 \end{pmatrix} - \widetilde{G}_0 R_w'\right) X\right) R' \in A_T^{l\times l} \end{aligned} \tag{18}$$

represents a direct complement of $A_T^{1\times k'_p}R'$. Then the matrix C_{1c} defines a (partial regular) $\mathfrak{C}$-stabilizing compensators of the plant $\mathscr{P}$ (up to a multiplication by a suitable $t\in T$). To determine all the other compensators, we proceed as follows. We set $\widetilde{C}(Y) := C_1 + YR'$, where $Y\in A_T^{l\times k'_p}$ satisfies $R'YR' = 0$, and block-partition $\widetilde{C}(Y)$ as

$$\widetilde{C}(Y) = (\widetilde{C}_w(Y),\ \widetilde{C}_c(Y)).$$

We easily notice that

$$\widetilde{C}_w(Y) = \underbrace{C_{1w}}_{=0} + YR'_w = 0 \Longleftrightarrow YR'_w = 0,$$

i.e., the first block of $\widetilde{C}(Y)$ (corresponding to w) is zero if and only if $Y\in A_T^{l\times k'_p}$ is a left annihilator of R'_w. If $L\in A_T^{m\times k'_p}$ is a universal left annihilator of R'_w, all left annihilators of R'_w of size $l\times k'_p$ are described as $Y = ZL$ for some $Z\in A_T^{l\times m}$. So, we can associate with every $Z\in A_T^{l\times m}$ satisfying $R'ZLR' = 0$ the A_T-matrix

$$C(Z) := \widetilde{C}(ZL) = C_1 + ZLR' = (0,\ \widetilde{C}_c(ZL)) =: (0,\ C_c(Z)). \tag{19}$$

To obtain a stabilizing compensator one has to multiply the matrix $C_c(Z)$ by a common denominator $t\in T$ of its entries. Bijectivity of this parametrization can be obtained by factoring out the module $\{Z:\ ZLR' = 0\}$, similarly to what was done in (17).

Of course, there are many behaviours $\widetilde{\mathscr{C}}$ that satisfy $(\widetilde{\mathscr{C}}\cap\mathscr{F}_2^{l_c})^{\perp_2} = A_T^{1\times l}C_c(Z)$ for a given Z, thus we have not parametrized all stabilizing compensators for the plant $\mathscr{P}$. However, all those behaviours $\widetilde{\mathscr{C}}$ have the same steady state behaviour, namely $\{c\in\mathscr{F}_2^{l_c};\ C_c(Z)\circ c = 0\}$, and only their transient trajectories differ. In this sense, we have parametrized the $\mathfrak{C}$-stabilizing compensators of $\mathscr{P}$ up to their $\mathfrak{C}$-negligible, i.e., transient parts.

We conclude this section with a comprehensive example, which we will extend in the following sections.

Example 15. We use the complex polynomial ring $A = \mathbb{C}[s] = \mathbb{C}[s_1,s_2]$ in two variables and the Serre subcategory

$$\mathfrak{C} := \mathfrak{C}_{\mathrm{fin}} = \{M\in\mathrm{Mod}_A:\ M_{\mathfrak{p}} = 0 \text{ for all } \mathfrak{p}\in\mathrm{spec}(A)\setminus\max(A)\}$$

from Example 2, part 3. The signal space $\mathscr{F}$ is an arbitrary large injective cogenerator, for instance one of those from Example 1 – both the continuous as well as the discrete case are fine.

The only prime ideals in A are the maximal ideals $\mathfrak{m}(\lambda) = A(s_1-\lambda_1)+A(s_2-\lambda_2)$, $\lambda\in\mathbb{C}^2$, the principal ideals generated by the irreducible polynomials in A and the zero ideal, thus $\mathrm{spec}(A)\setminus\max(A) = \{Ap:\ p \text{ irreducible}\}\cup\{0\}$. Furthermore, recall from Assumption 5 and Example 6, part 2, that in this situation $T = T(\mathfrak{C}_{\mathrm{fin}}) = \mathbb{C}\setminus\{0\}$ and $\mathscr{Q}(A) = A_T = A$. This implies that all computations will take place over the polynomial ring $A = \mathbb{C}[s]$.

We consider the plant $\mathscr{P} = \left\{\binom{w}{c}\in\mathscr{F}^{1+3};\ R\circ\binom{w}{c} = 0\right\}$ with one to-be-controlled variable $w\in\mathscr{F}$ and three control variables $c = \begin{pmatrix} c_1\\ c_2\\ c_3\end{pmatrix}\in\mathscr{F}^3$ given by the matrix

$$R = \left(R_w \,\vdots\, R_c\right) = \left(\begin{array}{c:ccc} 0 & p(s_1-\lambda_1) & s_1-\lambda_1 & 0\\ 0 & p(s_2-\lambda_2) & s_2-\lambda_2 & 0\\ 0 & 0 & 0 & -1\\ 1 & -1 & 0 & 0\end{array}\right) \in \mathbb{C}[s]^{4\times 4}$$

of rank 3 with $\lambda = (\lambda_1,\ \lambda_2) = (-1,\ 2) \in \mathbb{C}^2$ and $p = s_1^3 + 3s_1^2 - s_2 \in \mathbb{C}[s]$ irreducible. Notice that these equations imply $w = c_1$ and $c_3 = 0$ (we will use c_3 in Section 5 for the disturbance). In the first two equations one can identify a $\mathfrak{C}$-negligible dynamics associated with the maximal ideal $\mathfrak{m}(\lambda)$ and a non-negligible one related to the principal prime ideal Ap.

The Gabriel localization of $A^{1\times 4}R$ is

$$\mathscr{Q}(A^{1\times 4}R) = A^{1\times 3}R' \quad \text{with} \quad R' = \left(R'_w \;\vdots\; R'_c\right) = \begin{pmatrix} 0 & \vdots & 0 & 0 & 1 \\ 1 & \vdots & -1 & 0 & 0 \\ 0 & \vdots & p & 0 & 0 \end{pmatrix}. \tag{20}$$

We solve the linear matrix equation (16) over A and obtain

$$\{G_0 :\ R'G_0R' = R'\} = \underbrace{\begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}}_{=\widetilde{G}_0} + A\begin{pmatrix} -1&0&0 \\ -1&0&0 \\ p&0&0 \\ 0&0&0 \end{pmatrix} + A\begin{pmatrix} 0&-1&0 \\ 0&-1&0 \\ 0&p&0 \\ 0&0&0 \end{pmatrix} + A\begin{pmatrix} 0&0&-1 \\ 0&0&-1 \\ 0&0&p \\ 0&0&0 \end{pmatrix}.$$

The matrix

$$C_0 = \mathrm{id}_4 - \widetilde{G}_0R' = \begin{pmatrix} 0 & \vdots & 1 & 0 & 0 \\ 0 & \vdots & 1 & 0 & 0 \\ 0 & \vdots & -p & 0 & 0 \\ 0 & \vdots & 0 & 0 & 0 \end{pmatrix} =: C_1$$

does already have the correct block structure and therefore we set $C_1 = \left(0 \;\vdots\; C_{1c}\right) = C_0$. If we use the compensator $\mathscr{C} = (A^{1\times 4}C_{1c})^{\perp}$ we obtain (after performing some simplifying row operations on the matrix) the system

$$(\mathscr{P} \wedge_c \mathscr{C})^{\perp} = A^{1\times 5}\begin{pmatrix} 1 & \vdots & 0 & 0 & 0 \\ 0 & \vdots & 1 & 0 & 0 \\ 0 & \vdots & 0 & s_1-\lambda_1 & 0 \\ 0 & \vdots & 0 & s_2-\lambda_2 & 0 \\ 0 & \vdots & 0 & 0 & 1 \end{pmatrix},$$

where all variables are zero except for c_2 which exhibits a $\mathfrak{C}$-negligible dynamics associated with $\mathfrak{m}(\lambda)$.

Now we derive the parametrization of all $\mathfrak{C}$-stabilizing compensators of $\mathscr{P}$. A universal left annililator of $R'_w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is

$$L = \begin{pmatrix} 1&0&0 \\ 0&0&1 \end{pmatrix} \in A^{2\times 3}. \tag{21}$$

Introducing the parameter matrix $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \\ z_5 & z_6 \\ z_7 & z_8 \end{pmatrix} \in A^{4\times 2}$ leads to

$$R'ZLR' = \begin{pmatrix} 0 & z_8p & z_8 & z_7 \\ 0 & (z_2-z_4)p & z_2-z_4 & z_1-z_3 \\ 0 & (z_4p+z_6)p & z_4p+z_6 & z_3p+z_5 \end{pmatrix}$$

and the condition $R'ZLR' = 0$ allows us to simplify the parameter matrix to

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_1 & z_2 \\ -z_1p & -z_2p \\ 0 & 0 \end{pmatrix}.$$

With Equation (19) we arrive at

$$C(Z) = C_1 + ZLR' = \begin{pmatrix} 0 & \vdots & 1+z_2p & z_2 & z_1 \\ 0 & \vdots & 1+z_2p & z_2 & z_1 \\ 0 & \vdots & -p-z_2p^2 & -z_2p & -z_1p \\ 0 & \vdots & 0 & 0 & 0 \end{pmatrix}.$$

The first and the second line of C_1 and Z (and therefore also of $C(Z)$) are equal, the third line is a multiple of the first one and the fourth line is zero, thus all but the first line can be omitted and, after renaming, we obtain

$$\begin{aligned} C_1 = \big(0 \,\vdots\, C_{1c}\big) = \big(0 \,\vdots\, 1 \quad 0 \quad 0\big), \qquad Z = \big(z_1 \quad z_2\big) \text{ and} \\ C(Z) = \big(0 \,\vdots\, C_1(Z)\big) = \big(0 \,\vdots\, 1 + z_2 p \quad z_2 \quad z_1\big)\,. \end{aligned} \tag{22}$$

The set of all $\mathfrak{C}$-stabilizing compensators is parametrized by

$$\begin{aligned} A^{1\times 2} &\longrightarrow \{\mathfrak{C}\text{-stabilizing compensators } \mathscr{C} \text{ of } \mathscr{P}\}, \\ Z = \big(z_1 \quad z_2\big) &\longmapsto \big(A\,\big(1 + z_2 p \quad z_2 \quad z_1\big)\big)^{\perp}. \end{aligned}$$

For example, the choice $z_1 = 0$, $z_2 = 1$ leads to

$$(\mathscr{P} \wedge_c \mathscr{C})^{\perp} = A^{1\times 5} \begin{pmatrix} 1 & \vdots & -1 & 0 & 0 \\ 0 & \vdots & s_1 - \lambda_1 & 0 & 0 \\ 0 & \vdots & s_2 - \lambda_2 & 0 & 0 \\ 0 & \vdots & -3 & 1 & 0 \\ 0 & \vdots & 0 & 0 & 1 \end{pmatrix},$$

where both the the control variables and the to-be-controlled variable $w = c_1$ exhibit $\mathfrak{C}$-negligible dynamics.

4 Stabilization in the presence of external signals

In the previous section we have addressed the stabilization problem by assuming that the only variables involved are the control variable c and the to be controlled variable w. A more realistic scenario is the one where also external signals, acting on the plant and the compensator (see Figure 2), appear in the system description. Specifically, we assume for the plant and the controller the descriptions:

$$\begin{aligned} \mathscr{P} &= \left\{ \begin{pmatrix} w \\ c \\ v \end{pmatrix} \in \mathscr{F}^{l_w + l_c + l_v} : \; (R_w,\, R_c,\, R_v) \circ \begin{pmatrix} w \\ c \\ v \end{pmatrix} = 0 \right\} \\ \mathscr{C} &= \left\{ \begin{pmatrix} c \\ u \end{pmatrix} \in \mathscr{F}^{l_c + l_u} : \; (C_c,\, C_u) \circ \begin{pmatrix} c \\ u \end{pmatrix} = 0 \right\} \end{aligned}$$

with $(R_w,\ R_c,\ R_v) \in A^{k_p \times (l_w + l_c + l_v)}$ and $(C_c,\ C_u) \in A^{k_c \times (l_c + l_u)}$. As before, l_w is the size of the variable to be controlled w and l_c denotes the size of the control variable c. The disturbance v has l_v components and the reference signal u is of size l_u. In the following, we use the notation $l_1 := l_w + l_c$, $l_2 := l_v + l_u$ and $l := l_1 + l_2$.

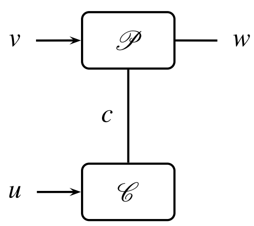

Figure 2: The interconnection diagram for the problem of stabilization by partial interconnection with additional exogenous signals.

Definition 16. We say that $\mathscr{C}$ is a $\mathfrak{C}$-*stabilizing compensator for* $\mathscr{P}$ *in the presence of exogenous signals* if

1. the partial interconnection $\mathscr{P} \wedge_c \mathscr{C}$ is regular,
2. $\binom{v}{u}$ is free in $\mathscr{P} \wedge_c \mathscr{C}$,
3. $\mathscr{N}_{w,c}(\mathscr{P}) \wedge_c \mathscr{N}_c(\mathscr{C}) \in \mathfrak{C}$.

The rationale behind items one and three is the same as in the previous section: the control shall be exerted only via the control variable c and not restrict what already is restricted (regular interconnection), and the resulting behaviour, for $v = 0$ and $u = 0$, consists only of $\mathfrak{C}$-negligible signals. As the ports u and v are used later to insert disturbances and to connect the system with a reference signal generator, respectively, v and u cannot be restricted by $\mathscr{P} \wedge_c \mathscr{C}$, i.e., they have to be free (item 2). Otherwise, it may happen for instance that the interconnected behaviour restricts the nature of disturbances that can affect it.

It is worthwhile remarking that, as a consequence of the previous definition, if $\mathscr{C}$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ then $\mathscr{P} \wedge_c \mathscr{C}$ is a $\mathfrak{C}$-stable IO behaviour with input $\binom{v}{u}$ and output $\binom{w}{c}$. The following technical lemma is of fundamental importance since it allows to extend the analysis of the previous section to the case of interconnection with exogenous signals.

Lemma 17. *The following facts are equivalent:*

1. $\mathscr{P} \wedge_c \mathscr{C}$ *is regular and* $\binom{v}{u}$ *is free in* $\mathscr{P} \wedge_c \mathscr{C}$*;*
2. $\mathscr{N}_{w,c}(\mathscr{P}) \wedge_c \mathscr{N}_c(\mathscr{C})$ *is regular,* v *is free in* $\mathscr{P}$ *and* u *is free in* $\mathscr{C}$*.*

Proof. $2 \Longrightarrow 1$**.** In the computation

$$\begin{aligned}
\operatorname{rank}\begin{pmatrix} R_w & R_c & R_v & 0 \\ 0 & C_c & 0 & C_u \end{pmatrix} &\leqslant \operatorname{rank}(R_w,\ R_c,\ R_v,\ 0) + \operatorname{rank}(0,\ C_c,\ 0,\ C_u) \\
&= \operatorname{rank}(R_w,\ R_c,\ R_v) + \operatorname{rank}(C_c,\ C_u) \\
&\overset{*}{=} \operatorname{rank}(R_w,\ R_c) + \operatorname{rank}(C_c) \\
&\overset{\dagger}{=} \operatorname{rank}\begin{pmatrix} R_w & R_c \\ 0 & C_c \end{pmatrix} \leqslant \operatorname{rank}\begin{pmatrix} R_w & R_c & R_v & 0 \\ 0 & C_c & 0 & C_u \end{pmatrix}
\end{aligned}$$

the equality indicated by $*$ holds because v and u are free in $\mathscr{P}$ and $\mathscr{C}$, respectively, while the regularity of $\mathscr{N}_{w,c}(\mathscr{P}) \wedge_c \mathscr{N}_c(\mathscr{C})$ implies the equality $\dagger$. As a consequence, all the relations are actually equalities. Since the expressions in the first two lines of the computation are equal, the interconnection $\mathscr{P} \wedge_c \mathscr{C}$ is regular. On the other hand, the fact that the expressions in the last line of the computation are equal implies that $\binom{v}{u}$ is free in $\mathscr{P} \wedge_c \mathscr{C}$.

$1 \Longrightarrow 2$**.** Using similar reasonings we obtain

$$\begin{aligned}
\operatorname{rank}\begin{pmatrix} R_w & R_c \\ 0 & C_c \end{pmatrix} &\leqslant \operatorname{rank}(R_w,\ R_c) + \operatorname{rank}(0,\ C_c) \\
&\leqslant \operatorname{rank}(R_w,\ R_c,\ R_v,\ 0) + \operatorname{rank}(0,\ C_c,\ 0,\ C_u) \\
&\overset{*}{=} \operatorname{rank}\begin{pmatrix} R_w & R_c & R_v & 0 \\ 0 & C_c & 0 & C_u \end{pmatrix} \overset{\dagger}{=} \operatorname{rank}\begin{pmatrix} R_w & R_c \\ 0 & C_c \end{pmatrix}.
\end{aligned}$$

Again, the equality due to the regularity is indicated by $*$, and the one following from the freeness by $\dagger$. All relations are equalities, therefore $\mathscr{N}_{w,c}(\mathscr{P}) \wedge_c \mathscr{N}_c(\mathscr{C})$ is regular. Furthermore, from

$$\begin{aligned} \operatorname{rank}(R_w,\ R_c) &\leqslant \operatorname{rank}(R_w,\ R_c,\ R_v), \\ \operatorname{rank}(C_c) &\leqslant \operatorname{rank}(C_c,\ C_u), \text{ and} \\ \operatorname{rank}(R_w, R_c) + \operatorname{rank}(C_c) &= \operatorname{rank}(R_w, R_c, R_v) + \operatorname{rank}(C_c, C_u) \end{aligned}$$

we conclude that $\operatorname{rank}(R_w,\ R_c) = \operatorname{rank}(R_w,\ R_c,\ R_v)$ and $\operatorname{rank}(C_c) = \operatorname{rank}(C_c,\ C_u)$, i.e., that v and u are free in $\mathscr{P}$ and $\mathscr{C}$, respectively. □

A direct consequence of this lemma is that $\mathscr{C}$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ in the presence of exogenous signals if and only if v is free in $\mathscr{P}$, u is free in $\mathscr{C}$ and the behaviour $\mathscr{N}_c(\mathscr{C})$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_{w,c}(\mathscr{P})$ in the previous set-up, namely without external signals. The next lemma formalizes this fact, by establishing the relationship between the $\mathfrak{C}$-stabilizing compensators for $\mathscr{N}_{w,c}(\mathscr{P})$ and the $\mathfrak{C}$-stabilizing compensators for $\mathscr{P}$, in the presence of external signals.

Lemma 18. *Assume that v is free in $\mathscr{P}$.*

1. *If $(A^{1\times k_c}C_c)^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_{w,c}(\mathscr{P})$, then for every $D \in A^{l_c\times l_u}$ and $t \in T$ we have that $(A^{1\times k_c}(tC_c,\ C_cD))^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$.*
2. *If $(A^{1\times k_c}(C_c,\ C_u))^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ then $(A^{1\times k_c}C_c)^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_{w,c}(\mathscr{P})$, and there exists a matrix $D \in A_T^{l_c\times l_u}$ such that $C_u = C_cD$.*

Proof. 1. Since $t \in T$, the element t is invertible in A_T. This implies that $A_T^{1\times k_c}tC_c = A_T^{1\times k_c}C_c$, and hence

$$\mathscr{Q}(A^{1\times k_c}tC_c) = \mathscr{Q}(A_T^{1\times k_c}tC_c) = \mathscr{Q}(A_T^{1\times k_c}C_c) = \mathscr{Q}(A^{1\times k_c}C_c).$$

This implies that also $(A^{1\times k_c}tC_c)^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_{w,c}(\mathscr{P})$. Furthermore, since the columns of C_cD belong to the column space of C_c over the quotient field K, we have $\operatorname{rank}(tC_c) = \operatorname{rank}(tC_c,\ C_cD)$. In other words, the variables u (corresponding to the right block) are free in $(A^{1\times k_c}(tC_c,\ C_cD))^\perp$. The assertion follows from Lemma 17.

2. $(A^{1\times k_c}C_c)^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_{w,c}(\mathscr{P})$ by point 3. of Definition 16 and Lemma 17. By assumption, $\mathscr{P} \wedge_c \mathscr{C}$ is a $\mathfrak{C}$-stable IO behaviour with input $\binom{v}{u}$ and output $\binom{w}{c}$. Thus by [8, Thm. and Def. 4.2] there exists a transfer matrix

$$\begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \in A_T^{l_1\times l_2} \text{ such that } \begin{pmatrix} R_w & R_c \\ 0 & C_c \end{pmatrix}\begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} = \begin{pmatrix} R_v & 0 \\ 0 & C_u \end{pmatrix}.$$

This implies, in particular, that $C_cH_4 = C_u$. So, the statement holds for $D = H_4$. □

From the previous lemma it follows that there are no constraints on the choice of the matrix $D \in A_T^{l_c\times l_u}$. Lemma 18 and Equation (19) lead to the following parametrization of the $\mathfrak{C}$-stabilizing compensators of the plant $\mathscr{P}$:

$$\begin{aligned}
\{(Z,\,D) \in A_T^{l_1\times m} \times A_T^{l_c\times l_u} :\ & R'ZLR' = 0\} \\
& \longrightarrow \{(\mathscr{C}\cap\mathscr{F}_2^{l_c+l_u})^{\perp_2} :\ \mathscr{C} \text{ is a stabilizing compensator for } \mathscr{P}\}, \\
(Z,\,D) & \longmapsto A_T^{1\times l_1}(C_c(Z),\ C_c(Z)D),
\end{aligned} \tag{23}$$

where $C_c(Z) := C_{1c} + ZLR'_c$ as in (19), $\mathscr{Q}(A^{1\times k_p}(R_w,\ R_c)) = A_T^{1\times k'_p}R'$, $R' = (R'_w,\ R'_c) \in A_T^{k'_p\times(l_w+l_c)}$ and $L \in A_T^{m\times k'_p}$ is a universal left annihilator of R'_w. In order to obtain the matrix of the compensator, one has to multiply the matrix $(C_c(Z),\ C_c(Z)D)$ by a common denominator $t \in T$ of its entries. Also, in this case, if we want to establish a bijective correspondence between the matrix parameters Z and D and the compensators, we have to keep into account the fact that all pairs of parameters $(Z_1,\ D_1)$ and $(Z_2,\ D_2)$ that satisfy $(Z_1 - Z_2)LR' = 0$ and $C_c(Z_1)D_1 = C_c(Z_2)D_2$ yield the same compensator. Furthermore, also in this context the parametrization we have obtained is up to $\mathfrak{C}$-negligible behaviours.

Example 19. We extend Example 15 and therefore use all the objects and notations introduced there. We add an external variable $v \in \mathscr{F}$ to the plant and equate it to the third control variable c_3, which was unused yet. To accommodate this, we modify the matrix R as

$$R = \begin{pmatrix} R_w & \vdots & R_c & \vdots & R_v \end{pmatrix} = \begin{pmatrix} 0 & \vdots & p(s_1-\lambda_1) & s_1-\lambda_1 & 0 & \vdots & 0 \\ 0 & \vdots & p(s_2-\lambda_2) & s_2-\lambda_2 & 0 & \vdots & 0 \\ 0 & \vdots & 0 & 0 & -1 & \vdots & 1 \\ 1 & \vdots & -1 & 0 & 0 & \vdots & 0 \end{pmatrix} \in A^{4\times(1+3+1)}.$$

It holds that $\operatorname{rank}\begin{pmatrix} R_w & \vdots & R_c & \vdots & R_v \end{pmatrix} = \operatorname{rank}\begin{pmatrix} R_w & \vdots & R_c \end{pmatrix}$, i.e., v is free in $\mathscr{P}$.

We also add an external variable $u \in \mathscr{F}$ to the compensator. The compensators without external signals are parametrized via $C_c(Z) = \begin{pmatrix} 1+z_2p & z_2 & z_1 \end{pmatrix}$ where $Z = \begin{pmatrix} z_1 & z_2 \end{pmatrix} \in A^{1\times 2}$ (recall that in this setting $A_T = A$ holds). According to Lemma 18, the columns of C_u have to be linear combinations of the columns of C_c and the coefficients can be freely chosen, thus with $D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \in A^{3\times 1}$ we obtain the parametrization

$$\begin{aligned}
A^{1\times 2} \times A^{3\times 1} &\longrightarrow \{\mathfrak{C}\text{-stabilizing compensators } \mathscr{C} \text{ of } \mathscr{P}\}, \\
(Z,\,D) &\longmapsto \left(A\left(C_c(Z) \ \vdots\ C_c(Z)D\right)\right)^{\perp}, \\
&\text{with } C_c(Z) = \begin{pmatrix} 1+z_2p & z_2 & z_1 \end{pmatrix} \in A^{1\times 3} \text{ and} \\
&\quad C_c(Z)D = (1+z_2p)d_1 + z_2d_2 + z_1d_3 \in A.
\end{aligned}$$

The choice $Z = 0$ and $D = 0$ leads to the compensator matrix

$$\begin{pmatrix} C_c & \vdots & C_u \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

and to the interconnected behaviour $\mathscr{P} \wedge_c \mathscr{C}$ resp. its module of equations

$$(\mathscr{P} \wedge_c \mathscr{C})^{\perp} = A^{1\times 5} \begin{pmatrix} 1 & \vdots & 0 & 0 & 0 & \vdots & 0 & \vdots & 0 \\ 0 & \vdots & 1 & 0 & 0 & \vdots & 0 & \vdots & 0 \\ 0 & \vdots & 0 & s_1-\lambda_1 & 0 & \vdots & 0 & \vdots & 0 \\ 0 & \vdots & 0 & s_2-\lambda_2 & 0 & \vdots & 0 & \vdots & 0 \\ 0 & \vdots & 0 & 0 & -1 & \vdots & 1 & \vdots & 0 \end{pmatrix}.$$

The external variable u does not appear in the system. The two blocks of variables $(w,\ c_1,\ c_2)$ and $(c_3,\ v)$ are separated which means, in particular, that the exogenous signal v has no influence on the to-be-controlled-variable w.

On the contrary, the choices $Z = \begin{pmatrix} -1 & 0 \end{pmatrix}$ and $D = 0$ lead to

$$\begin{pmatrix} C_c & \vdots & C_u \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & \vdots & 0 \end{pmatrix}$$

and to

$$(\mathscr{P} \wedge_c \mathscr{C})^{\perp} = A^{1\times 5} \left(\begin{array}{c|ccc|c|c} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & p(s_1-\lambda_1) & s_1-\lambda_1 & 0 & 0 & 0 \\ 0 & p(s_2-\lambda_2) & s_2-\lambda_2 & 0 & 0 & 0 \end{array}\right),$$

i.e., the to-be-controlled variable w is equal to the external signal v. If v is interpreted as a disturbance, then $w = v$ is a relation which is utterly undesirable. In Example 26 we will see that the choice $z_1 = -1$ which led to this situation is not permitted if disturbance rejection is part of the control goal.

5 Regulation by Partial Interconnection

Consider a plant $\mathscr{P}$ and a compensator $\mathscr{C}$, described as in Section 4. In addition, assume that the external signals v and u, acting on $\mathscr{P}$ and $\mathscr{C}$, are trajectories of two behaviours

$$\mathscr{E}_{\mathscr{P}} = \{v \in \mathscr{F}^{l_v} :\ V \circ v = 0\}, \quad V \in A^{k_1 \times l_v},$$
$$\mathscr{E}_{\mathscr{C}} = \{u \in \mathscr{F}^{l_u} :\ U \circ u = 0\}, \quad U \in A^{k_2 \times l_u},$$

(see Figure 3).

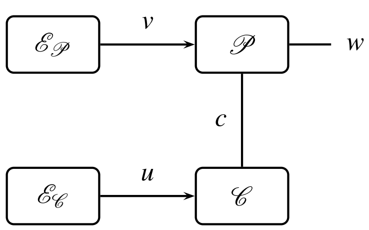

Figure 3: The interconnection diagram for the regulation problem.

Introduce the matrix

$$K = (K_w,\ K_c,\ K_v,\ K_u) \in A^{k\times(l_w+l_c+l_v+l_u)}.$$

The control goal we investigate in this section is that of finding a $\mathfrak{C}$-stabilizing compensator $\mathscr{C}$ such that $K \circ \begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix}$ is $\mathfrak{C}$-negligible for every trajectory $\begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix}$ of the *interconnected behavior*

$$\mathscr{B} := \mathscr{E}_{\mathscr{P}} \wedge_v \mathscr{P} \wedge_c \mathscr{C} \wedge_u \mathscr{E}_{\mathscr{C}} = \left\{ \begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} \in \mathscr{F}^l :\ \begin{pmatrix} R_w & R_c & R_v & 0 \\ 0 & C_c & 0 & C_u \\ 0 & 0 & V & 0 \\ 0 & 0 & 0 & U \end{pmatrix} \circ \begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} = 0 \right\}, \quad (24)$$

i.e., for every trajectory $\begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix}$ such that $\begin{pmatrix} w \\ c \\ v \end{pmatrix} \in \mathscr{P}$, $\begin{pmatrix} c \\ u \end{pmatrix} \in \mathscr{C}$, $v \in \mathscr{E}_{\mathscr{P}}$ and $u \in \mathscr{E}_{\mathscr{C}}$. For example, the choice $K = (\mathrm{id},\ 0,\ 0,\ -\mathrm{id})$ corresponds to designing a compensator in such a way that, in the interconnected behaviour, the trajectories of the tracking error variable $K \circ \begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} = w - u$ are $\mathfrak{C}$-negligible. This means that the variable to be controlled, w, coincides (modulo a transient signal) with the variable to be tracked, u, independently of the other variables, in particular of the disturbance v.

Definition 20. By referring to the set-up introduced in this and in the previous section as illustrated in Figure 3, we say that a behaviour $\mathscr{C}$ is a $\mathfrak{C}$-*regulator* for $\mathscr{P}$, with respect to $\mathscr{E}_{\mathscr{P}}$, $\mathscr{E}_{\mathscr{C}}$ and K, if the following conditions are satisfied:

1. $\mathscr{C}$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ in the presence of exogenous signals in the sense of Definition 16 in Section 4 and
2. $K \circ \mathscr{B}$ is $\mathfrak{C}$-negligible, where $\mathscr{B}$ is the interconnected behaviour.

The second part of Definition 20 clearly represents the main objective of the $\mathfrak{C}$-regulator. One may wonder whether the first requirement is really necessary or not. In principle, it may be possible that the $\mathfrak{C}$-regulation problem can be solved without introducing this constraint. However, it is clear that for example the tracking and disturbance rejection problem $\mathfrak{C}$-regulator problem with $K = (\mathrm{id},\, 0,\, 0,\, -\mathrm{id})$ only focuses on the behaviour of the to-be-controlled variable w. So, without the stabilization requirement, the case may occur that w is following the desired reference signal, meanwhile rejecting the disturbances, but the control variable c grows unpredictably, possibly with damaging effects on the system functioning.

Notice that from condition 1. of Definition 20 it follows that the interconnection $\mathscr{P} \wedge_c \mathscr{C}$ is regular and $\binom{v}{u}$ is free in this behaviour. Consequently, by the structure of the matrix description given in (24), it follows that $\mathscr{B} = \mathscr{E}_{\mathscr{P}} \wedge_v \mathscr{P} \wedge_c \mathscr{C} \wedge_u \mathscr{E}_{\mathscr{C}}$ is a regular interconnection, too.

Theorem 21. *Assume that $\mathscr{C}$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$. Then $K \circ \mathscr{B} \in \mathfrak{C}$, namely $K \circ \mathscr{B}$ is $\mathfrak{C}$-negligible, if and only if*

$$
\begin{aligned}
(\mathscr{E}_{\mathscr{P}} \times \mathscr{E}_{\mathscr{C}}) \cap \mathscr{F}_2^{l_2} &\subseteq \mathrm{proj}_{v,u}\left((\mathscr{P} \wedge_c \mathscr{C}) \cap (A^{1\times k}K)^{\perp} \right) \cap \mathscr{F}_2^{l_2} \\
&= \left\{ \binom{v}{u} \in \mathscr{F}_2^{l_2} : \exists \binom{w}{c} \in \mathscr{F}_2^{l_1} \text{such that } \begin{pmatrix} R_w & R_c & R_v & 0 \\ 0 & C_c & 0 & C_u \\ K_w & K_c & K_v & K_u \end{pmatrix} \circ \begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} = 0 \right\}.
\end{aligned}
$$

Proof. **Necessity.** Assume that $K \circ \mathscr{B} \in \mathfrak{C}$. Let $\binom{v}{u} \in (\mathscr{E}_{\mathscr{P}} \times \mathscr{E}_{\mathscr{C}}) \cap \mathscr{F}_2^{l_2}$. The behaviour $\mathscr{P} \wedge_c \mathscr{C}$ is an IO behaviour and this property is preserved when moving to $(\mathscr{P} \wedge_c \mathscr{C}) \cap \mathscr{F}_2^{l}$. Let thus $\binom{w}{c} \in \mathscr{F}_2^{l_1}$ be such that $\begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} \in (\mathscr{P} \wedge_c \mathscr{C}) \cap \mathscr{F}_2^{l}$. Then $\begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} \in \mathscr{B} \cap \mathscr{F}_2^{l}$. From $K \circ \mathscr{B} \in \mathfrak{C}$ we infer $(K \circ \mathscr{B}) \cap \mathscr{F}_2^{l} = K \circ (\mathscr{B} \cap \mathscr{F}_2^{l}) = 0$ and conclude that $K \circ \begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} \in K \circ (\mathscr{B} \cap \mathscr{F}_2^{l}) = 0$, i.e., $K \circ \begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} = 0$.

Sufficiency. Let $\begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} \in \mathscr{B} \cap \mathscr{F}_2^{l}$. Then $\binom{v}{u} \in (\mathscr{E}_{\mathscr{P}} \times \mathscr{E}_{\mathscr{C}}) \cap \mathscr{F}_2^{l_2}$ and thus, by assumption, there exists $\binom{\widetilde{w}}{\widetilde{c}} \in \mathscr{F}_2^{l_1}$ such that

$$
\begin{pmatrix} \widetilde{w} \\ \widetilde{c} \\ v \\ u \end{pmatrix} \in (\mathscr{P} \wedge_c \mathscr{C}) \cap \mathscr{F}_2^{l} \quad \text{and} \quad K \circ \begin{pmatrix} \widetilde{w} \\ \widetilde{c} \\ v \\ u \end{pmatrix} = 0.
$$

But from the definition of $\mathscr{B}$ it follows that also $\begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} \in (\mathscr{P} \wedge_c \mathscr{C}) \cap \mathscr{F}_2^l$. Thus the difference $\begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} - \begin{pmatrix} \widetilde{w} \\ \widetilde{c} \\ v \\ u \end{pmatrix}$ lies in $(\mathscr{P} \wedge_c \mathscr{C}) \cap \mathscr{F}_2^l$ too, and since the inputs of this trajectory are zero, $\begin{pmatrix} w-\widetilde{w} \\ c-\widetilde{c} \end{pmatrix} \in \mathscr{N}_{w,c}\left((\mathscr{P} \wedge_c \mathscr{C})\right) \cap \mathscr{F}_2^{l_1}$ is an element of the steady states of the autonomous part. But since $\mathscr{P} \wedge_c \mathscr{C}$ is $\mathfrak{C}$-stable, i.e., its autonomous part is $\mathfrak{C}$-negligible, this trajectory must be zero. Therefore, $w = \widetilde{w}$, $c = \widetilde{c}$ and

$$K \circ \begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} = K \circ \begin{pmatrix} \widetilde{w} \\ \widetilde{c} \\ v \\ u \end{pmatrix} = 0.$$

This ensures that $(K \circ \mathscr{B}) \cap \mathscr{F}_2^k = K \circ (\mathscr{B} \cap \mathscr{F}_2^l) = 0$, i.e., that $K \circ \mathscr{B}$ is $\mathfrak{C}$-negligible. □

The following algorithm provides a procedure to test whether the necessary and sufficient condition provided in the previous theorem is satisfied.

Algorithm 22. *Consider a plant $\mathscr{P}$ and a compensator $\mathscr{C}$, described as at the beginning of section 4, external behaviours $\mathscr{E}_{\mathscr{P}}$ and $\mathscr{E}_{\mathscr{C}}$, and a control goal K, described as at the beginning of this section. In order to test whether $\mathscr{C}$ is a $\mathfrak{C}$-regulator for this set-up, the following steps need to be taken:*

1. *Verify that $\mathscr{C}$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$. Check the conditions*

$$\begin{aligned} \operatorname{rank}(R_w,\, R_c,\, R_v) &= \operatorname{rank}(R_w,\, R_c), \\ \operatorname{rank}(C_c,\, C_u) &= \operatorname{rank}(C_c) \\ \operatorname{rank}\begin{pmatrix} R_w & R_c \\ 0 & C_c \end{pmatrix} &= \operatorname{rank}(R_w,\, R_c) + \operatorname{rank}(C_c) \end{aligned}$$

   to ascertain the freeness of v and u and the regularity of the interconnection. The methods to test if $\mathscr{N}_{w,c}(\mathscr{P}) \wedge_c \mathscr{N}_c(\mathscr{C})$ is $\mathfrak{C}$-negligible depend on the specific Serre subcategory $\mathfrak{C}$.

   For example, consider a Serre subcategory of the type $\mathfrak{C}(\Lambda_1)$ induced by a disjoint decomposition $\mathbb{C}^n = \Lambda_1 \uplus \Lambda_2$ as in Example 2, part 2. An autonomous behaviour $\mathscr{B} = \{w \in \mathscr{F}^l;\ R \circ w = 0\}$ is $\mathfrak{C}(\Lambda_1)$-negligible if its variety of rank singularities $\{\lambda \in \mathbb{C}^r :\ \operatorname{rank}(R(\lambda)) < l\}$, is contained in Λ_1, see Example 3, part 2. To test this, one computes the determinants of all $l \times l$-submatrices of R and checks whether their common zeros lie in Λ_1 using the algorithm described in the first paragraph of [15, Sec. 7].

2. *Compute a universal left annihilator $X \in A^{m \times (k_p + k_c + k)}$ of $\begin{pmatrix} R_w & R_c \\ 0 & C_c \\ K_w & K_c \end{pmatrix}$ and matrices $V' \in A_T^{k_1' \times l_v}$ and $U' \in A_T^{k_2' \times l_u}$ such that*

$$\mathscr{Q}(A^{1 \times k_1} V) = A_T^{1 \times k_1'} V' \text{ and } \mathscr{Q}(A^{1 \times k_2} U) = A_T^{1 \times k_2'} U',$$

   (see [15, Alg. 3.9]). Then, check if the inhomogeneous linear matrix equation

$$Y \begin{pmatrix} V' & 0 \\ 0 & U' \end{pmatrix} = X \begin{pmatrix} R_v & 0 \\ 0 & C_u \\ K_v & K_u \end{pmatrix}$$

   has a solution $Y \in A_T^{m \times (k_1' + k_2')}$. If this is the case, $\mathscr{C}$ is a $\mathfrak{C}$-regulator.

Solving systems of inhomogeneous linear equations over some quotient ring A_T can be difficult and it is still an unsolved problem in many cases. However, in [15, Sec. 7] an algorithm is given that performs this task in a number of important situations.

In the remainder of this section, we will extend the parametrization of $\mathfrak{C}$-stabilizing compensators provided in Section 4 to a parametrization of $\mathfrak{C}$-regulators.

Lemma 23. *Given the plant $\mathscr{P} = \left\{ \left(\begin{smallmatrix} w \\ c \\ v \end{smallmatrix}\right) \in \mathscr{F}^{l_1+l_v} : (R_w,\, R_c,\, R_v) \circ \left(\begin{smallmatrix} w \\ c \\ v \end{smallmatrix}\right) = 0 \right\}$, assume that v is a free variable for $\mathscr{P}$ and that $\mathscr{N}_{w,c}(\mathscr{P}) = (A^{1\times k_p}(R_w,\, R_c))^{\perp}$ is $\mathfrak{C}$-stabilizable. Let $R' \in A_T^{k'_p \times l_1}$ be such that $\mathscr{Q}(A^{1\times k_p}(R_w,\, R_c)) = A_T^{1\times k'_p} R'$. Then there exists a matrix $R'_v \in A_T^{k'_p \times l_c}$ such that*

$$(\mathscr{P} \cap \mathscr{F}_2^{l_1+l_v})^{\perp_2} = \mathscr{Q}(A^{1\times k_p}(R_w,\, R_c,\, R_v)) = A_T^{1\times k_p}(R',\, R'_v).$$

Proof. Let $\mathscr{C} = (A^{1\times k_c} C_c)^{\perp}$ be a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_{w,c}(\mathscr{P})$. Then

$$\mathscr{N}_{w,c}(\mathscr{P}) \wedge_c \mathscr{C} = \left(A^{1\times(k_p+k_c)} \begin{pmatrix} R_w & R_c \\ 0 & C_c \end{pmatrix} \right)^{\perp}$$

is $\mathfrak{C}$-negligible, and hence autonomous, and $\mathscr{P} \wedge_c \mathscr{C} = \left(A^{1\times(k_p+k_c)} \begin{pmatrix} R_w & R_c & R_v \\ 0 & C_c & 0 \end{pmatrix} \right)^{\perp}$ is a $\mathfrak{C}$-stable IO behaviour with input v and output $\left(\begin{smallmatrix} w \\ c \end{smallmatrix}\right)$. Let $H \in A_T^{l_1 \times l_v}$ be its transfer matrix. Then $\begin{pmatrix} R_w & R_c \\ 0 & C_c \end{pmatrix} H = \left(\begin{smallmatrix} R_v \\ 0 \end{smallmatrix}\right)$, in particular $(R_w,\, R_c)H = R_v$, and Lemma 10 implies

$$\mathscr{Q}(A^{1\times k_p}(R_w,\, R_c,\, R_v)) = A_T^{1\times k'_p} R'(\mathrm{id}_{l_1},\, H) = A_T^{1\times k'_p}(R',\, R'_v),$$

where $R'_v := R'H \in A_T^{k'_p \times l_v}$. ◻

Corollary 24. *Let $\mathscr{C}$ be a $\mathfrak{C}$-stabilizing compensator for*

$$\mathscr{P} = \left\{ \begin{pmatrix} w \\ c \\ v \end{pmatrix} \in \mathscr{F}^{l_1+l_v} : (R_w,\, R_c,\, R_v) \circ \begin{pmatrix} w \\ c \\ v \end{pmatrix} = 0 \right\}.$$

Assume, according to Lemma 23, that

$$\mathscr{Q}(A^{1\times k_p}(R_w,\, R_c,\, R_v)) = A_T^{1\times k'_p}(R'_w,\, R'_c,\, R'_v),$$

where we have split the matrix R' into two blocks as $R' = (R'_w,\, R'_c)$. Denote by $(W_{R'},\, W_C,\, W_K) \in A_T^{r\times(k'_p+k_c+k)}$ a universal left annihilator of $\begin{pmatrix} R'_w & R'_c \\ 0 & C_c \\ K_w & K_c \end{pmatrix}$. Finally, set

$$(\mathscr{E}_{\mathscr{P}} \cap \mathscr{F}_2^{l_v})^{\perp_2} = \mathscr{Q}(A^{1\times k_1} V) = A_T^{1\times k'_1} V' \text{ and } (\mathscr{E}_{\mathscr{C}} \cap \mathscr{F}_2^{l_u})^{\perp_2} = \mathscr{Q}(A^{1\times k_2} U) = A_T^{1\times k'_2} U',$$

for suitable matrices V' and U' with entries in A_T. Then $\mathscr{C}$ is a $\mathfrak{C}$-regulator for $\mathscr{P}$ with respect to $\mathscr{E}_{\mathscr{P}}$, $\mathscr{E}_{\mathscr{C}}$, and K if and only if

$$A_T^{1\times r}(W'_R,\, W_C,\, W_K) \begin{pmatrix} R'_v & 0 \\ 0 & C_u \\ K_v & K_u \end{pmatrix} \subseteq A_T^{1\times(k'_1+k'_2)} \begin{pmatrix} V' & 0 \\ 0 & U' \end{pmatrix}.$$

Proof. We introduce the notation

$$\mathscr{B}_2 := \left\{ \begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} \in \mathscr{F}_2^{l} : \begin{pmatrix} R'_w & R'_c & R'_v & 0 \\ 0 & C_c & 0 & C_u \\ K_w & K_c & K_v & K_u \end{pmatrix} \circ \begin{pmatrix} w \\ c \\ v \\ u \end{pmatrix} = 0 \right\}.$$

Then

$$\left\{ \begin{pmatrix} v \\ u \end{pmatrix} \in \mathscr{F}_2^{l_2} : \ (W_{R'},\, W_C,\, W_K) \begin{pmatrix} R'_v & 0 \\ 0 & C_u \\ K_v & K_u \end{pmatrix} \circ \begin{pmatrix} v \\ u \end{pmatrix} = 0 \right\} = \mathrm{proj}_{v,u}(\mathscr{B}_2).$$

From

$$\begin{aligned}
&\mathscr{Q}\left(A^{1\times(k_p+k_c+k)} \begin{pmatrix} R_w & R_c & R_v & 0 \\ 0 & C_c & 0 & C_u \\ K_w & K_c & K_v & K_u \end{pmatrix} \right) \\
&\qquad = \mathscr{Q}\left(A_T^{1\times k_p}(R_w,\, R_c,\, R_v,\, 0) + A_T^{1\times(k_c+k)} \begin{pmatrix} 0 & C_c & 0 & C_u \\ K_w & K_c & K_v & K_u \end{pmatrix} \right) \\
&\qquad \overset{*}{=} \mathscr{Q}\left(\mathscr{Q}\left(A_T^{1\times k_p}(R_w,\, R_c,\, R_v,\, 0) \right) + A_T^{1\times(k_c+k)} \begin{pmatrix} 0 & C_c & 0 & C_u \\ K_w & K_c & K_v & K_u \end{pmatrix} \right) \\
&\qquad = \mathscr{Q}\left(A_T^{1\times k'_p}(R'_w,\, R'_c,\, R'_v,\, 0) + A_T^{1\times(k_c+k)} \begin{pmatrix} 0 & C_c & 0 & C_u \\ K_w & K_c & K_v & K_u \end{pmatrix} \right) \\
&\qquad = \mathscr{Q}\left(A_T^{1\times(k'_p+k_c+k)} \begin{pmatrix} R'_w & R'_c & R'_v & 0 \\ 0 & C_c & 0 & C_u \\ K_w & K_c & K_v & K_u \end{pmatrix} \right),
\end{aligned}$$

where equality $*$ holds because of Lemma 9, it follows that $(\mathscr{P} \wedge_c \mathscr{C}) \cap (A^{1\times k}K)^{\perp} \cap \mathscr{F}_2^{l} = \mathscr{B}_2$. By Theorem 21, $\mathscr{C}$ is a $\mathfrak{C}$-regulator if and only if

$$(\mathscr{E}_{\mathscr{P}} \times \mathscr{E}_{\mathscr{C}}) \cap \mathscr{F}_2^{l_2} \subseteq \mathrm{proj}_{v,u}((\mathscr{P} \wedge_c \mathscr{C}) \cap (A^{1\times k}K)^{\perp} \cap \mathscr{F}_2^{l}) = \mathrm{proj}_{v,u}(\mathscr{B}_2).$$

By duality of $\mathscr{F}_2$-behaviours (10), this is equivalent to

$$\begin{aligned}
\mathscr{Q}\left(A_T^{1\times(k'_p+k_c+k)}(W'_R,\, W_C,\, W_K) \begin{pmatrix} R'_v & 0 \\ 0 & C_u \\ K_v & K_u \end{pmatrix} \right) &\subseteq \mathscr{Q}\left(A^{1\times(k_1+k_2)} \begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix} \right) \\
&= A_T^{1\times(k'_1+k'_2)} \begin{pmatrix} V' & 0 \\ 0 & U' \end{pmatrix},
\end{aligned}$$

and this is the case if and only if

$$A_T^{1\times(k'_p+k_c+k)}(W'_R,\, W_C,\, W_K) \begin{pmatrix} R'_v & 0 \\ 0 & C_u \\ K_v & K_u \end{pmatrix} \subseteq A_T^{1\times(k'_1+k'_2)} \begin{pmatrix} V' & 0 \\ 0 & U' \end{pmatrix}. \qquad \square$$

Corollary 25. *Under the same assumptions and notation as in Corollary 24, let* $W = (W_{R'},\, W_C,\, W_K) \in A_T^{r\times(k'_p+k_c+k)}$ *be a universal left annihilator of* $\begin{pmatrix} R'_w & R'_c \\ 0 & C_{1c} \\ K_w & K_c \end{pmatrix}$*, where* C_{1c} *is the matrix given in* (18).

Let $\mathscr{C} \subseteq \mathscr{F}^{l_c+l_u}$ *be a* $\mathfrak{C}$*-stabilizing compensator for* $\mathscr{P}$*, so that, by referring to the parametrization* (23)*, there exist matrix parameters* Z *with* $R'ZLR' = 0$ *(where* L *is a universal left annihilator of* R'_w*) and* D *such that*

$$(\mathscr{C} \cap \mathscr{F}_2^{l_c+l_u})^{\perp_2} = A_T^{1\times l_1}(C_c(Z),\, C_c(Z)D), \text{ where } C_c(Z) = C_{1c} + ZLR'_c.$$

Then $\mathscr{C}$ *is also a* $\mathfrak{C}$*-regulator if and only if there exist matrices* $B_1 \in A_T^{r\times k'_1}$ *and* $B_2 \in A_T^{r\times k'_2}$ *such that*

$$\begin{aligned}
W_{R'}R'_v - W_C ZLR'_v + W_K K_v &= B_1 V', \\
W_C C_{1c} D + W_C ZLR'_c D + W_K K_u &= B_2 U'.
\end{aligned} \tag{25}$$

Proof. Since $LR'_w = 0$, we have

$$\begin{pmatrix} R'_w & R'_c \\ 0 & C_c(Z) \\ K_w & K_c \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{id}_{k'_p} & 0 & 0 \\ ZL & \mathrm{id}_{l_1} & 0 \\ 0 & 0 & \mathrm{id}_k \end{pmatrix}}_{=:\, S \in A_T^{(k'_p+l_1+k)\times(k'_p+l_1+k)}} \begin{pmatrix} R'_w & R'_c \\ 0 & C_{1c} \\ K_w & K_c \end{pmatrix}.$$

The matrix S is invertible over A_T and therefore $WS^{-1} = (W_{R'} - W_C ZL,\ W_C,\ W_K)$ is a universal left annihilator of $\begin{pmatrix} R'_w & R'_c \\ 0 & C_c(Z) \\ K_w & K_c \end{pmatrix}$. The assertion follows now from Corollary 24.

□

Corollary 25 is the basis for modifying the parametrization of the $\mathfrak{C}$-stabilizing compensators given in Section 4 to a parametrization of the $\mathfrak{C}$-regulators with respect to the external behaviours $\mathscr{E}_{\mathscr{P}}$ and $\mathscr{E}_{\mathscr{C}}$ and the control goal K. The notation is the same as in Corollary 25 and (23). The following surjective map parametrises all regulators for the given setting, up to $\mathfrak{C}$-negligible behaviours:

$$\begin{aligned} \{(Z,\ D) \in A_T^{l_1 \times m} \times A_T^{l_c \times l_u} : R'ZLR' = 0,\ \exists (B_1,\ B_2) \in A_T^{r \times k'_1 + k'_2} \text{ s. t. Eq. (25) holds}\} \\ \longrightarrow \{(\mathscr{C} \cap \mathscr{F}_2^{l_c + l_u})^{\perp_2} :\ \mathscr{C} \text{ is a } \mathfrak{C}\text{-regulator for } \mathscr{P}\}, \\ (Z,\ D) \longmapsto A_T^{1 \times l_1}(C_c(Z),\ C_c(Z)D). \end{aligned} \tag{26}$$

Bijectivity of this parametrization is obtained by using, again, the equivalence relation for the parameters described in the text after (23).

We conclude the article with an example of a combined tracking and disturbance rejection regulation problem.

Example 26. We use all the data introduced in Examples 15 and 19, in particular the plant

$$\mathscr{P} = \left\{ \begin{pmatrix} w \\ c \\ v \end{pmatrix} \in \mathscr{F}^{1+3+1};\ R \circ \begin{pmatrix} w \\ c \\ v \end{pmatrix} = 0 \right\}$$

$$\text{with } R = \left(R_w \,\vdots\, R_c \,\vdots\, R_v\right) = \begin{pmatrix} 0 & p(s_1-\lambda_1) & s_1-\lambda_1 & 0 & 0 \\ 0 & p(s_2-\lambda_2) & s_2-\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix} \in A^{4 \times (1+3+1)}.$$

Let $q = s_1 s_2 + s_1 - 1 \in \mathbb{C}[s]$ and $\tau = (3,\ -4) \in \mathbb{C}^2$. We consider disturbances v in the external behaviour

$$\mathscr{E}_{\mathscr{P}} = \{v \in \mathscr{F};\ V \circ v = 0\} \text{ with } V = \begin{pmatrix} q(s_1-\tau_1) \\ q(s_2-\tau_2) \end{pmatrix} \in A^{2 \times 1}.$$

Let $r = s_1^2 + s_2^2 - 1 \in \mathbb{C}[s]$ and $\xi = (1,\ 0) \in \mathbb{C}^2$. The admissible reference signals u for tracking are the trajectories of

$$\mathscr{E}_{\mathscr{C}} = \{u \in \mathscr{F};\ U \circ u = 0\} \text{ with } U = \begin{pmatrix} r(s_1-\xi_1) \\ r(s_2-\xi_2) \end{pmatrix} \in A^{2 \times 1}.$$

Both the disturbance and the reference signal have a $\mathfrak{C}$-negligible part associated with the maximal ideals $\mathfrak{m}(\tau)$ and $\mathfrak{m}(\xi)$ as well a non-negligible part corresponding to the irreducible polynomials q and r, respectively. Consequently, the Gabriel localizations of the two modules of equations are

$$\begin{aligned} \mathscr{Q}(A^{1\times 2}V) &= AV', \quad \text{where } V' = q \in A^{1\times 1}, \text{ and} \\ \mathscr{Q}(A^{1\times 2}U) &= AU', \quad \text{where } U' = r \in A^{1\times 1}. \end{aligned}$$

The control goal is given by the matrix

$$K = \left(K_w \,\vdots\, K_c \,\vdots\, K_v \,\vdots\, K_u\right) = \left(1 \,\vdots\, 0 \quad 0 \quad 0 \,\vdots\, 0 \,\vdots\, -1\right) \in A^{1\times 6},$$

i.e., the to-be-controlled variable w should track u up to a $\mathfrak{C}$-negligible signal and the distrubance v should have no significant effect.

To be able to apply Equation (25), we use the matrices R'_c and R'_v from Equation (20) and the matrix C_{1c} from Equation (22) to form the block matrix

$$\begin{pmatrix} R'_w & R'_c \\ 0 & C_{1c} \\ K_w & K_c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & p & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

A universal left annihilator of this matrix is

$$\begin{pmatrix} W_{R'} & W_C & W_K \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & -1 & 1 \end{pmatrix}$$

The matrix R'_v is $R'_v = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$. We use the universal left annihilator $L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ of R'_w from Equation (21) and the parameter matrices $Z = \begin{pmatrix} z_1 & z_2 \end{pmatrix}$ and $D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$. Inserting everything into Equation (25), we obtain the two equations

$$\begin{aligned} -z_1 &= B_1 q \text{ and} \\ (1 + z_2 p)d_1 + z_2 d_2 + z_1 d_3 + 1 &= -B_2 r, \end{aligned}$$

where B_1, $B_2 \in A$. Here we can see that the choice $z_1 = -1$ which we made at the end of Example 19 is not compatible with the first equation and is therefore not suitable for disturbance rejection.

The choices

$$z_1 = q, \quad z_2 = 1, \quad d_1 = 0, \quad d_2 = r - 1, \quad d_3 = 0, \quad B_1 = -1 \quad \text{and} \quad B_2 = -1$$

satisfy the equations and lead to the matrix

$$\begin{pmatrix} C_c(Z) & C_c(Z)D \end{pmatrix} = \begin{pmatrix} C_{1c} + ZLR'_c & (C_{1c} + ZLR'_c)D \end{pmatrix} = \begin{pmatrix} 1+p & 1 & q & r-1 \end{pmatrix}$$

and the $\mathfrak{C}$-regulator

$$\mathscr{C} = \left\{ \begin{pmatrix} c \\ u \end{pmatrix} \in \mathscr{F}^{3+1};\ \begin{pmatrix} C_c(Z) & C_c(Z)D \end{pmatrix} \circ \begin{pmatrix} c \\ u \end{pmatrix} = 0 \right\}.$$

Finally, we check whether the compensator really satisfies the control goal. The module of equations of the interconnected behaviour $\mathscr{B}$ from Equation (24) is

$$\mathscr{B}^{\perp} = (\mathscr{E}_{\mathscr{P}} \wedge_v \mathscr{P} \wedge_c \mathscr{C} \wedge_u \mathscr{E}_{\mathscr{C}})^{\perp} = A^{1\times 9} \begin{pmatrix} 0 & p(s_1-\lambda_1) & s_1-\lambda_1 & 0 & 0 & 0 \\ 0 & p(s_2-\lambda_2) & s_2-\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1+p & 1 & q & 0 & r-1 \\ 0 & 0 & 0 & 0 & q(s_1-\tau_1) & 0 \\ 0 & 0 & 0 & 0 & q(s_2-\tau_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & r(s_1-\xi_1) \\ 0 & 0 & 0 & 0 & 0 & r(s_2-\xi_2) \end{pmatrix}.$$

The module of equations of the error behaviour $K \circ \mathscr{B}$ is

$$(K \circ \mathscr{B})^{\perp} = A^{1\times 3} \begin{pmatrix} s_2^2+12s_1+10s_2-12 \\ s_1s_2-8s_1-7s_2+8 \\ s_1^2+4s_1+4s_2-5 \end{pmatrix} = \mathfrak{m}(\lambda) \cap \mathfrak{m}(\tau) \cap \mathfrak{m}(\xi),$$

and the primary decomposition shows that it is $\mathfrak{C}$-negligible: the error consists only of a $\mathfrak{C}$-negligible influence of the dynamics of $\mathscr{P} \wedge_c \mathscr{C}$ associated with $\mathfrak{m}(\lambda)$, a $\mathfrak{C}$-negligible effect caused by the disturbance corresponding to $\mathfrak{m}(\tau)$ and a $\mathfrak{C}$-negligible deviation associated with $\mathfrak{m}(\xi)$ of the to-be-controlled variable from the tracking signal.

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