The consensus problem in the behavioral approach

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To Jan: a mentor and a friend. Amazingly inspiring and entertaining in both roles.
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1 Introduction

The mathematical formulation of multi-agents systems and consensus problems was introduced several years ago in some pioneering papers such as [6, 17, 18]. But it was only a decade ago that a wide stream of research on these topics started, thanks to milestone contributions such as [7, 8, 10, 11, 12]. Aside from the theoretical challenges that these problems pose, strong motivations for such a widespread interest come from the numerous application problems that can be naturally stated as consensus problems. Indeed, when dealing with sensor networks, coordination of mobile robots or UAVs, flocking and swarming in animal groups, dynamics of opinion forming, etc., the main control target can be mathematically formalized as a consensus problem among agents, exchanging information and resorting to distributed algorithms that make use of the information collected from neighboring agents (see, e.g. [14, 16]).

While the first contributions on this subject focused on agents described as simple or double integrators, more recent works addressed the case of agents described by higher order models [7, 16, 19, 20, 21]. The vast majority of the literature on consensus, however, assumes that the homogeneous agents dynamics is described by a state-space model and that consensus is achieved through a static state- or output-feedback, that makes use of the weighted information collected from the neighboring agents, in a cooperative set-up (see [1] for consensus under antagonistic interactions).

The aim of this paper is to investigate the multi-agent consensus problem in a broader context, by assuming both for the agents and for the distributed controllers higher order input/output dynamic models. The behavioral approach developed by Jan Willems [13, 22, 23] seems to be a convenient set-up where to investigate this general problem. Since, to the best of our knowledge, this set-up has never been used before in this context, we have tried to make the paper as self-contained as possible, by recalling the few fundamental definitions and results that are necessary to understand the technical details of the paper. A comprehensive treatment of the behavior theory can be found in any of the three aforementioned references.

By making use of the behavioral approach, we will show that the consensus problem can be naturally rephrased as a variant of the stabilization problem: the stabilization pertains only to a part of the system variables (the outputs) and it is achieved through regular full interconnection of the agents models and of the controllers. We will prove that if the communication among agents is described by a weighted, undirected and connected graph, then a necessary and sufficient condition for the consensus problem to be solvable is that the output is stabilizable from the input in the agents model. In this respect, the theory here developed for higher-order input/output models naturally extends the results about consensus derived in the state-space approach (see [20], for instance).

The paper is organized as follows. In section 2 preliminary definitions, notation and results are given. In section 3 the consensus problem is posed. Section 4 provides a characterization of the controllers that make it possible for the agents to achieve consensus and section 5 provides a similar characterization under the additional assumption that the consensus is achieved by means of a regular interconnection. Section 6 provides a complete solution to the consensus problem. Section 7 concludes the paper by showing how the most classical result on consensus for agents described by state-space models easily follows from the present analysis.

A preliminary version of the results appearing in sections 3, 4 and 5 of this paper has appeablack, in a more general set-up, in [5]. However, no problem solution was provided: a characterization of the controllers that solve the problem was given, but no necessary and sufficient condition for the existence of such controllers, and hence for the consensus problem solvability, was provided.

2 Preliminaries

We introduce some notation and definitions that will be used in the following.

I_p denotes the $p \times p$ *identity matrix*. The p-dimensional vector with all entries equal to 1 is denoted by $\mathbf{1}_p$, while the ith *standard basis vector* in $\mathbb{R}^p$ (also known as the ith canonical vector) is denoted by e_i. The *spectrum* of a square matrix L is denoted by $\sigma(L)$. $\text{diag}\{v_1, v_2, \dots, v_p\}$ is the $p \times p$ diagonal matrix with diagonal entries $v_1, v_2, \dots, v_p$.

We let $\mathbb{R}[s]$ denote the ring of polynomials in the indeterminate s with real coefficients. A polynomial $p \in \mathbb{R}[s]$ is *Hurwitz* if all its zeros belong to $\{s \in \mathbb{C} : \text{Re}(s) < 0\}$. A polynomial matrix $P = P(s) \in \mathbb{R}[s]^{p \times q}$ is *right prime* if it is of full column rank q and the greatest common divisor of its maximal order minors is a unit, equivalently if rank $P(\lambda) = q$ for every $\lambda \in \mathbb{C}$. It is well-known [9] that $P(s)$

is right prime if and only if it admits a polynomial left inverse or, equivalently, the Bézout equation

$$XP = I_q$$

in the unknown polynomial matrix $X(s) \in \mathbb{R}[s]^{q\times p}$ is solvable. *Left prime* matrices are similarly defined and characterized. A square and nonsingular polynomial matrix $P = P(s) \in \mathbb{R}[s]^{q\times q}$ whose inverse P^{-1} is polynomial is called *unimodular*. Clearly a unimodular matrix is both right prime and left prime.

Every polynomial matrix $P \in \mathbb{R}[s]^{p\times q}$ of rank r factorizes over $\mathbb{R}[s]$ as $P = L\Delta R$, where L is $p\times r$ and right prime, Δ is $r\times r$ and nonsingular, and R is $r\times q$ and left prime.

The concepts of *left annihilator* and, in particular, of *minimal left annihilator* (MLA, for short) of a given polynomial matrix P have been originally introduced in [15] and can be summarized as follows: if P is a $p\times q$ polynomial matrix of rank r, a polynomial matrix H is a left annihilator of P if $HP = 0$. A left annihilator H_m of P is an MLA if it is of full row rank and for any other left annihilator H of P we have $H = QH_m$ for some polynomial matrix Q. It can be easily proved that, unless P is of full row rank, an MLA always exists (if P is of full row rank, its left annihilators are zero matrices with an arbitrary number of rows), it is a $(p-r)\times p$ left prime matrix and is uniquely determined modulo a unimodular left factor.

In the paper we consider (continuous-time) signals defined on the *time set* $\mathbb{R}$. Signals will be real valued and hence they will be, in general, elements of $(\mathbb{R}^q)^{\mathbb{R}}$, for some $q \in \mathbb{N}$. By $\mathscr{F}^q$ we will denote the set of arbitrarily often differentiable functions, i.e., $\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^q) \subseteq (\mathbb{R}^q)^{\mathbb{R}}$.

For every $P = \sum_{i=0}^n P_i s^i \in \mathbb{R}[s]^{p\times q}$, we associate with P the polynomial matrix differential operator $P\circ = \sum_{i=0}^n P_i \frac{d^i}{dt^i}$. The action of such a polynomial matrix differential operator P on any signal $w \in \mathscr{F}^q$ is denoted by $P\circ w$.

In this paper by a *system* we mean a triple $\Sigma = (\mathbb{R},\mathbb{R}^q,\mathscr{B})$, where $\mathbb{R}$ is the time set, $\mathbb{R}^q$ is the set where the system trajectories take values, and $\mathscr{B}$ is the *behavior*, namely the set of admissible trajectories of the system variable w. We will consider linear, time-invariant behaviors described as the kernels of polynomial matrix operators. This means that there exists a polynomial matrix $P \in \mathbb{R}[s]^{k\times q}$ such that

$$\mathscr{B} = \{w \in \mathscr{F}^q : P\circ w = 0\}. \tag{1}$$

It is always possible to find a matrix $\overline{P} \in \mathbb{R}[s]^{r\times q}$ of full row rank r such that $\mathscr{B} = \{w \in \mathscr{F}^q : \overline{P}\circ w = 0\}$.

A behavior $\mathscr{B} \subseteq \mathscr{F}^q$ is *autonomous* if it is a finite dimensional vector subspace of $\mathscr{F}^q$ as a vector space on $\mathbb{R}$. $\mathscr{B}$ described as in (1) is autonomous if and only if $P \in \mathbb{R}[s]^{k\times q}$ is of full column rank q.

An autonomous behavior (1) is *stable* if the greatest common divisor of the maximal (i.e., qth) order minors of P is a Hurwitz polynomial. If P is of full row rank and hence, under the autonomy assumption, square and nonsingular, this amounts to requiring that $\det P$ is Hurwitz. A trajectory $w \in \mathscr{F}^q$ is called *small* if it belongs to some stable autonomous behavior or, equivalently, if it satisfies the equation $p \circ w = 0$ for some Hurwitz polynomial p. Clearly, small signals are the polynomial-exponential functions that converge to zero as the time approaches $+\infty$.

If we partition the system variables as $w = \binom{y}{u} \in \mathscr{F}^{p+m}$, and accordingly describe the behavior $\mathscr{B}$ as

$$\mathscr{B} = \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{p+m} : P_y \circ y = P_u \circ u \right\}, \quad (P_y \ \ -P_u) \in \mathbb{R}[s]^{k \times (p+m)},$$

we say that u is *free* in $\mathscr{B}$ if for any $u \in \mathscr{F}^m$ there exists $y \in \mathscr{F}^p$ such that $\binom{y}{u} \in \mathscr{B}$. This is the case if and only if $\operatorname{rank}(P_y \ \ -P_u) = \operatorname{rank}(P_y)$. If additionally the behavior

$$\mathscr{B}^0 = \{ y \in \mathscr{F}^p : P_y \circ y = 0 \}$$

is autonomous, we say that $\mathscr{B}$ is an *input/output behavior* with *input* u and *output* y. Clearly, this is the case if and only if $\operatorname{rank}(P_y \ \ -P_u) = \operatorname{rank}(P_y) = p$. If the matrix $(P_y \ \ -P_u)$ is of full row rank k, it follows that $\mathscr{B}$ is an input/output behavior with input u and output y if and only if $k = p$ and P_y is nonsingular.

If $\mathscr{B}$ and $\mathscr{C}$ are behaviors in $\mathscr{F}^q$, described as kernels of the polynomial matrix operators $P\circ$ and $C\circ$, respectively, we denote the *interconnection of $\mathscr{B}$ and $\mathscr{C}$* as follows:

$$\mathscr{B} \wedge \mathscr{C} := \{ w \in \mathscr{F}^q : w \in \mathscr{B}, w \in \mathscr{C} \} = \left\{ w \in \mathscr{F}^q : \begin{pmatrix} P \\ C \end{pmatrix} \circ w \right\}.$$

The interconnection of $\mathscr{B}$ and $\mathscr{C}$ is said to be *regular* if

$$\operatorname{rank} \begin{pmatrix} P \\ C \end{pmatrix} = \operatorname{rank}(P) + \operatorname{rank}(C).$$

If $\mathscr{B}$ is an input/output behavior, with input u and output y, and the interconnection of $\mathscr{B}$ and $\mathscr{C}$ is regular, the input/output structure of $\mathscr{B}$ is preserved even after interconnection: this means that it is still possible to add (free) signals u' to the components of u after interconnection [2]. More precisely, the components of u' are free in the behavior $\mathscr{B} \wedge \mathscr{C}$, illustrated in Figure 1.

The concept of regular interconnection is fundamental in the behavioral approach and will be used in the sequel. For an in-depth discussion of its meaning and relevance see [2, 24].

Given a (not necessarily an input/output) behavior $\mathscr{B}$, with signals $\binom{y}{u} \in \mathscr{F}^{p+m}$, we say that y is *stabilizable* from u in $\mathscr{B}$ [2] if for every pair of trajectories $(y,u) \in \mathscr{B}$ there exists $(\bar{y}, \bar{u}) \in \mathscr{B}$ such that $\bar{y}(t) = y(t), \forall\ t < 0$, and $\bar{y}$ converges to zero. By rephrasing the result obtained in [3] for zero-controllability of discrete-time systems, we obtain the following characterization.

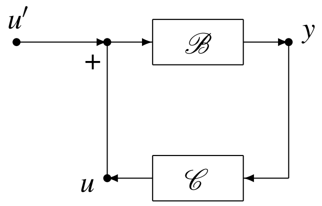


Figure 1: Feedback interconnection of the input/output behavior $\mathscr{B}$ and $\mathscr{C}$, with additional additive inputs u'.

Proposition 1. *Given a behavior*

$$\mathscr{B} = \left\{ \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathscr{F}^{p+m} : P_y \circ y = P_u \circ u \right\},$$

the following facts are equivalent:

i) *y is stabilizable from u in $\mathscr{B}$;*

ii) *there exists a behavior $\mathscr{C}$ in $\mathscr{F}^{p+m}$ such that the interconnection $\mathscr{B} \wedge \mathscr{C}$ is regular, and the* projection of $\mathscr{B} \wedge \mathscr{C}$ on the variable *y, i.e., $\{y \in \mathscr{F}^p : (y,u) \in \mathscr{B} \wedge \mathscr{C}, \exists u \in \mathscr{F}^m\}$, is an autonomous and stable behavior;*

iii) *either one of the following two cases applies:*

1. *P_u is of full row rank, or*
2. *P_u is not of full row rank, and if we let M_u denote an MLA for P_u, then $M_u P_y$ factorizes over $\mathbb{R}[s]$ as*

$$M_u P_y = L \Delta R,$$

where L is right prime, Δ is square and Hurwitz, and R is left prime.

A behavior $\mathscr{C}$ such that the interconnected system $\mathscr{B} \wedge \mathscr{C}$ is an autonomous stable behavior is called a *stabilizing controller* for $\mathscr{B}$ [2, 13]. Condition ii) in Proposition 1 means something weaker, namely that the controller $\mathscr{C}$ is stabilizing only for the variable y of the behavior $\mathscr{B}$.

3 The consensus problem

We assume that there are $N \geq 2$ agents whose dynamics is described by the same input/output behavior

$$\mathscr{P}_i := \mathscr{P} \quad := \quad \left\{ \begin{pmatrix} y_i \\ u_i \end{pmatrix} \in \mathscr{F}^{p+m} : P_y \circ y_i = P_u \circ u_i \right\}, \qquad (2)$$
$$\text{with} \quad (P_y \ -P_u) \in \mathbb{R}[s]^{p\times(p+m)}, \quad \det P_y \neq 0,$$

for $i = 1,\dots,N$. The variables y_i represent the target variables, i.e., those on which the agents should reach consensus. We assume that they are measurable (and hence available for feedback interconnection). Note that the assumption $\det(P_y) \neq 0$ ensures that the matrix $(P_y \ -P_u)$ is of full row rank p.

Upon setting

$$y := \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \in \mathscr{F}^{Np}, \quad \text{and} \quad u := \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \in \mathscr{F}^{Nm},$$

the *overall behavior of the N agents* can be described as follows:

$$\begin{aligned} \underline{\mathscr{P}} &:= \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{Np+Nm} : \begin{pmatrix} y_i \\ u_i \end{pmatrix} \in \mathscr{P}_i, \forall i \in \{1,\dots,N\} \right\} \\ &= \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{Np+Nm} : (I_N \otimes P_y) \circ y = (I_N \otimes P_u) \circ u \right\} \\ &\cong \mathscr{P}_1 \times \ldots \times \mathscr{P}_N, \end{aligned}$$

where $\otimes$ denotes the Kronecker product of matrices, and $\cong$ means that $\underline{\mathscr{P}}$ and $\mathscr{P}_1 \times \ldots \times \mathscr{P}_N$ are isomorphic, since their trajectories are mutually related by a simple entry permutation.

The information flow between the agents is modeled by an undirected and connected *weighted graph* $\mathscr{G}$, with vertex set $\mathscr{V} = \{1,\dots,N\}$ (the ith vertex representing the ith agent) and edge set $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$. The *adjacency matrix* of $\mathscr{G}$ is nonnegative, symmetric and denoted by $A = (a_{ij})_{1\leq i,j\leq N} \in \mathbb{R}_+^{N\times N}$: a_{ij} is the (positive) weight of the edge from j to i if such an edge exists, and zero otherwise. We assume that $a_{ii} = 0$ for every $i \in \{1,2,\dots,N\}$. Note that the assumption that the graph is undirected corresponds to the rather common case when the ith agent gives to the information received from the jth agent the same weight that the jth agent gives to the information received from the ith agent. The adjacency matrix gives rise to the *Laplacian matrix* $L := \Delta - A$ with $\Delta := \mathrm{diag}(A\mathbf{1}_N)$. The Laplacian matrix L is a symmetric and positive semidefinite matrix, whose nonnegative real eigenvalues can be sorted in such a way that

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

The connectedness assumption on $\mathscr{G}$ ensures that 0 is a simple eigenvalue of L, and hence all the eigenvalues $\lambda_i, i = 2,3,\dots,N$ are positive. Also, by the way L

has been defined, $\mathbf{1}_N$ is an eigenvector of L corresponding to the zero eigenvalue. As L is symmetric and hence diagonalizable, there exist N linearly independent eigenvectors $v_i, i = 1,2,\dots,N$, with v_i eigenvector corresponding to λ_i, such that the matrix

$$V = \begin{pmatrix} v_1 & v_2 & \dots & v_N \end{pmatrix} \tag{3}$$

satisfies $LV = VD$, where D is the diagonal matrix $\operatorname{diag}\{0,\lambda_2,\dots,\lambda_N\}$. As v_1 is an eigenvector of L corresponding to the zero eigenvalue, it entails no loss of generality assuming that it coincides with $\mathbf{1}_N$.

Each ith agent receives the weighted information

$$\widetilde{y}_i := \sum_{j=1}^{N} a_{ij}(y_i - y_j)$$

from the other agents of the network, and designs a control strategy based on the signal $\widetilde{y}_i$. Specifically, we assume that the agents adopt (identical) input/output controllers (with input $\widetilde{y}_i$ and output u_i) described by

$$\begin{aligned} \mathscr{C}_i := \mathscr{C} := \quad & \left\{ \begin{pmatrix} \widetilde{y}_i \\ u_i \end{pmatrix} \in \mathscr{F}^{p+m} : C_y \circ \widetilde{y}_i = C_u \circ u_i \right\} \\ \text{where} \quad & (C_y \ \ -C_u) \in \mathbb{R}[s]^{m\times(p+m)}, \quad \det(C_u) \neq 0, \end{aligned} \tag{4}$$

for $i = 1,\dots,N$, and they give rise to the *overall controller*

$$\begin{aligned} \underline{\mathscr{C}} &:= \left\{ \begin{pmatrix} \widetilde{y} \\ u \end{pmatrix} \in \mathscr{F}^{Np+Nm} : \begin{pmatrix} \widetilde{y}_i \\ u_i \end{pmatrix} \in \mathscr{C}_i,\ \forall i \in \{1,\dots,N\} \right\} \\ &= \left\{ \begin{pmatrix} \widetilde{y} \\ u \end{pmatrix} \in \mathscr{F}^{Np+Nm} : (I_N \otimes C_y) \circ \widetilde{y} = (I_N \otimes C_u) \circ u \right\} \\ &\cong \mathscr{C}_1 \times \ldots \times \mathscr{C}_N \end{aligned}$$

where, as before, $\widetilde{y} := \begin{pmatrix} \widetilde{y}_1 \\ \vdots \\ \widetilde{y}_N \end{pmatrix} \in \mathscr{F}^{Np}$, $u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \in \mathscr{F}^{Nm}$.

The agents behaviors $\mathscr{P}_1,\dots \mathscr{P}_N$, the graph $\mathscr{G}$, and the controllers $\mathscr{C}_1,\dots,\mathscr{C}_N$ define the overall interconnected behavior

$$\begin{aligned} \mathscr{K} \ &:= \ \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{Np+Nm} : \forall i \in \{1,\dots,N\}, \right. \\ &\qquad \left. \begin{pmatrix} y_i \\ u_i \end{pmatrix} \in \mathscr{P}_i,\ \begin{pmatrix} \widetilde{y}_i \\ u_i \end{pmatrix} \in \mathscr{C}_i,\ \text{where } \widetilde{y}_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) \right\} \\ &= \ \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{Np+Nm} : \begin{pmatrix} y \\ u \end{pmatrix} \in \underline{\mathscr{P}},\ \begin{pmatrix} \widetilde{y} \\ u \end{pmatrix} \in \underline{\mathscr{C}},\ \text{where } \widetilde{y} = (L \otimes I_p) y \right\} \\ &= \ \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{Np+Nm} : \begin{pmatrix} I_N \otimes P_y & -I_N \otimes P_u \\ L \otimes C_y & -I_N \otimes C_u \end{pmatrix} \circ \begin{pmatrix} y \\ u \end{pmatrix} = 0 \right\}. \end{aligned}$$

This overall interconnection can also be interpreted as the interconnection of the overall plant $\underline{\mathscr{P}}$ with the compensator

$$\begin{aligned}\underline{\mathscr{C}}_L &:= \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{Np+Nm} : \begin{pmatrix} (L\otimes I_p)y \\ u \end{pmatrix} \in \underline{\mathscr{C}} \right\} \\ &= \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{Np+Nm} : (L\otimes C_y)\circ y = (I_N \otimes C_u)\circ u \right\},\end{aligned}$$

namely as

$$\mathscr{K} = \underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_L.$$

Definition 2. We say that the controllers $\mathscr{C}_i$, $i = 1,\dots,N$, *lead to consensus among the N agents* (described by $\mathscr{P}_1,\dots,\mathscr{P}_N$ and the graph $\mathscr{G}$) if for every trajectory $\begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{K}$ the deviations $y_j - y_1$, $j \in \{2,\dots,N\}$, are small.

4 Characterization of controllers leading to consensus

In order to characterize the controllers that lead to consensus, it is convenient to introduce a variable transformation that transforms the overall connected system $\mathscr{K}$ into the following isomorphic behavior:

$$\mathscr{K}_D := \left\{ \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} \in \mathscr{F}^{Np+Nm} : \begin{pmatrix} I_N\otimes P_y & -I_N\otimes P_u \\ D\otimes C_y & -I_N \otimes C_u \end{pmatrix} \circ \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} = 0 \right\}.$$

As before, we assume that $\bar{y} =: \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{pmatrix} \in \mathscr{F}^{Np}$ and $\bar{u} =: \begin{pmatrix} \bar{u}_1 \\ \vdots \\ \bar{u}_N \end{pmatrix} \in \mathscr{F}^{Nm}$.

The behavior $\mathscr{K}_D$ can be interpreted as the interconnection of the overall plant $\underline{\mathscr{P}}$ with the compensator

$$\underline{\mathscr{C}}_D := \left\{ \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} \in \mathscr{F}^{Np+Nm} | (D\otimes C_y)\circ \bar{y} = (I_N \otimes C_u)\circ \bar{u} \right\},$$

namely as

$$\mathscr{K}_D = \underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_D.$$

Note that, as the adjacency matrix A of $\mathscr{G}$ is symmetric, and hence the Laplacian matrix L is diagonalizable, the behavior $\mathscr{K}_D$ is the direct product of N independent behaviors (see Figure 2 for an illustration of the structures of $\mathscr{K}$ and $\mathscr{K}_D$).

As a matter of fact, we have

$$\begin{aligned}\mathscr{K}_D &= \left\{ \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} \in \mathscr{F}^{Np+Nm} : \begin{pmatrix} \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{P},\, \begin{pmatrix} \lambda_i \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{C},\, \forall i \in \{1,\dots,N\} \right\} \\ &= \left\{ \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} \in \mathscr{F}^{Np+Nm} : \begin{pmatrix} \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{P} \wedge \mathscr{C}(\lambda_i),\, \forall i \in \{1,\dots,N\} \right\} \\ &\cong (\mathscr{P}\wedge\mathscr{C}(\lambda_1)) \times \ldots \times (\mathscr{P}\wedge\mathscr{C}(\lambda_N)),\end{aligned}$$

where

$$\mathscr{C}(\lambda_i) := \left\{ \begin{pmatrix} \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{F}^{p+m} : \begin{pmatrix} \lambda_i \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{C} \right\}. \tag{5}$$

Lemma 3. *The behaviors $\mathscr{K}$ and $\mathscr{K}_D$ are related through the following isomorphism (variable transformation):*

$$\begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} V^{-1} \otimes I_p & 0 \\ 0 & V^{-1} \otimes I_m \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}, \tag{6}$$

where V is the matrix of eigenvectors of L defined in (3), with $v_1 = \mathbf{1}_N$, and satsfying $LV = VD$.

Proof. As $V \in \mathbb{R}^{N \times N}$ is nonsingular, it follows that also

$$\underline{V} := \begin{pmatrix} V \otimes I_p & 0 \\ 0 & V \otimes I_m \end{pmatrix}$$

is invertible over $\mathbb{R}[s]$. $\underline{V}^{-1}$ obviously defines an $\mathbb{R}[s]$-isomorphism from $\mathscr{F}^{Np+Nm}$ into $\mathscr{F}^{Np+Nm}$. We show that $\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathscr{K}$ if and only if $\left(\begin{smallmatrix} \bar{y} \\ \bar{u} \end{smallmatrix}\right) \in \mathscr{K}_D$. By definition, $\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathscr{K}$ if and only if

$$\underbrace{\begin{pmatrix} I_N \otimes P_y & -I_N \otimes P_u \\ L \otimes C_y & -I_N \otimes C_u \end{pmatrix}}_{=:K} \circ \begin{pmatrix} y \\ u \end{pmatrix} = 0,$$

or equivalently if and only if

$$\left(\begin{pmatrix} V^{-1} \otimes I_{n+p} & 0 \\ 0 & V^{-1} \otimes I_m \end{pmatrix} K \underline{V} \right) \circ \underline{V}^{-1} \begin{pmatrix} y \\ u \end{pmatrix} = 0$$

namely

$$\begin{pmatrix} I_N \otimes P_y & -I_N \otimes P_u \\ D \otimes C_y & -I_N \otimes C_u \end{pmatrix} \circ \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} = 0,$$

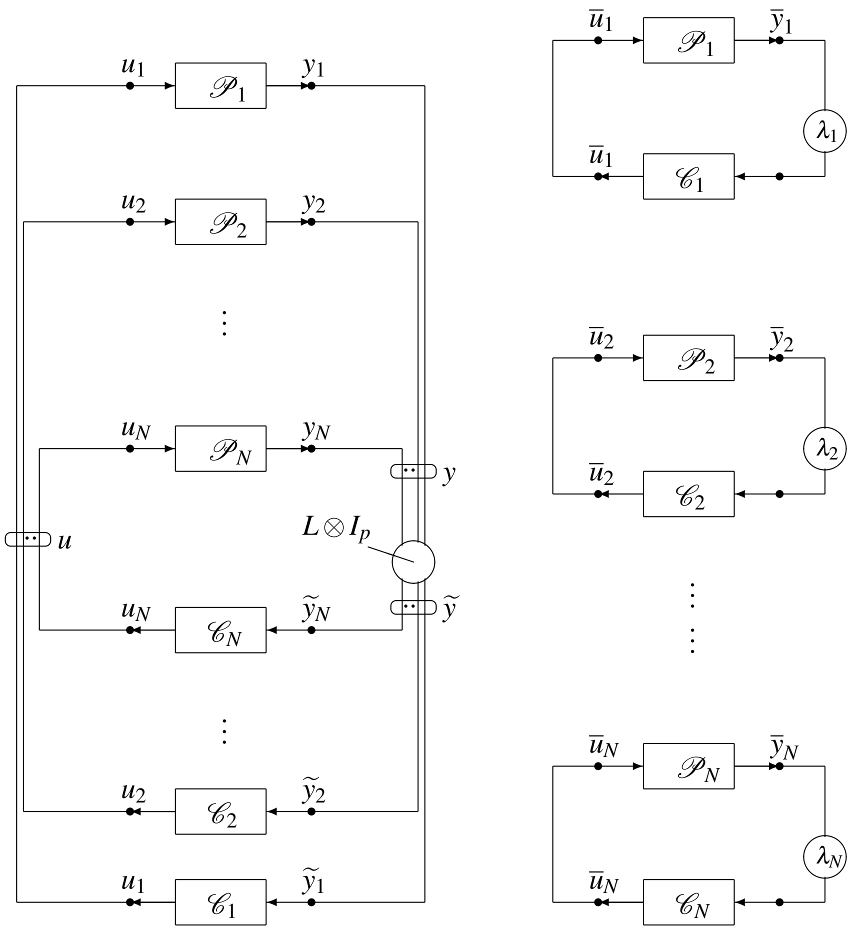
i.e., if and only if $\left(\begin{smallmatrix} \bar{y} \\ \bar{u} \end{smallmatrix}\right) \in \mathscr{K}_D$. $\square$

In light of Lemma 3 and of the decoupled expression of $\mathscr{K}_D$, we can derive the following characterization of the controllers that allow to solve the consensus problem.

Theorem 4. *The following statements are equivalent:*

1. *The N controllers $\mathscr{C}_i = \mathscr{C}, i = 1, 2, \ldots, N$, lead to consensus, i.e.,*

$$\forall \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{K} = \underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_L,\ \forall i \in \{2, \ldots, N\}, \quad y_i - y_1 \quad \textit{is small.}$$



(a) The interconnection $\underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_L = \mathscr{K}$.

(b) The interconnection $\underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_D = \mathscr{K}_D$.

Figure 2: The interconnections leading to $\mathscr{K}$ (original interconnected system) and $\mathscr{K}_D$ (after variable transformation).

2. *The behavior $\mathscr{K}_D$ satisfies*

$$\forall \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} \in \mathscr{K}_D = \underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_D,\ \forall i \in \{2,\dots,N\}, \quad \bar{y}_i \quad \textit{is small}.$$

3. $\forall i \in \{2,\dots,N\}$, $\forall \begin{pmatrix} \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{P} \wedge \mathscr{C}(\lambda_i)$, $\quad \bar{y}_i \quad$ *is small.*

4. $\forall i \in \{2,\dots,N\}$, $\forall \begin{pmatrix} \bar{y}_i \\ \hat{u}_i \end{pmatrix} \in \mathscr{P}(\lambda_i) \wedge \mathscr{C}$, $\quad \bar{y}_i \quad$ *is small, i.e., $\mathscr{C}$ stabilizes the variable $\bar{y}_i$ in $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$, where*

$$\mathscr{P}(\lambda_i) := \left\{ \begin{pmatrix} \bar{y}_i \\ \hat{u}_i \end{pmatrix} \in \mathscr{F}^{p+m} : \begin{pmatrix} \bar{y}_i \\ \lambda_i \hat{u}_i \end{pmatrix} \in \mathscr{P} \right\}, \tag{7}$$

and $\lambda_i, i = 2,3,\dots,N$ are the positive eigenvalues of L.

Proof. We first prove the equivalence of 1. and 2. Upon introducing the $(N-1) \times N$ matrix $S := \begin{pmatrix} -\mathbf{1}_{N-1} & I_{N-1} \end{pmatrix}$, statement 1. can be rewritten as

$$\forall \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{K}, \quad (S \otimes I_n) y \quad \text{is small.}$$

On the other hand, by Lemma 3, $\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathscr{K}$ if and only if $\left(\begin{smallmatrix} \bar{y} \\ \bar{u} \end{smallmatrix}\right) \in \mathscr{K}_D$, where $\bar{u}$ and $\bar{y}$ are related to u and y via equation (6). In particular, $y = (V \otimes I_n)\bar{y}$, where V is the usual matrix of eigenvectors that diagonalizes the Laplacian L. So, condition 1. can be equivalently rewritten as

$$\forall \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} \in \mathscr{K}_D, \quad (S \otimes I_n)(V \otimes I_n)\bar{y} = (SV \otimes I_n)\bar{y} \quad \text{is small.} \tag{8}$$

The fact that V is invertible and the identity

$$V = \begin{pmatrix} 1 & V_{12} & \cdots & V_{1N} \\ 1 & V_{22} & \cdots & V_{2N} \\ \vdots & \vdots & & \vdots \\ 1 & V_{N2} & \cdots & V_{NN} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & V_{12} & \cdots & V_{1N} \\ 0 & V_{22} - V_{12} & \cdots & V_{2N} - V_{1N} \\ \vdots & \vdots & & \vdots \\ 0 & V_{N2} - V_{12} & \cdots & V_{NN} - V_{1N} \end{pmatrix}$$

imply that also

$$\widetilde{V} := \begin{pmatrix} V_{22} - V_{12} & \cdots & V_{2N} - V_{1N} \\ \vdots & & \vdots \\ V_{N2} - V_{12} & \cdots & V_{NN} - V_{1N} \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

is invertible over $\mathbb{R}$. Since $SV = (0\,\widetilde{V}) \in \mathbb{R}^{(N-1)\times(1+(N-1))}$, condition (8) is equivalent to

$$\forall \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} \in \mathscr{K}_D, \quad ((0\,\widetilde{V}) \otimes I_n)\bar{y} \quad \text{is small,}$$

and by the invertibility of $\widetilde{V}$ over $\mathbb{R}$, this is equivalent to

$$\forall \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} \in \mathscr{K}_D, \quad ((0\, I_{N-1}) \otimes I_n)\bar{y} \quad \text{is small},$$

i.e., statement 2. is satisfied.

The equivalence of 2. and 3. follows from the special form of $\mathscr{K}_D$ as D is diagonal. So, we are left with proving that the statements 3. and 4. are equivalent.

For every $i \in \{2,\dots,N\}$, the following equivalences hold:

$$\begin{aligned}
\begin{pmatrix} \bar{y}_i \\ \hat{u}_i \end{pmatrix} \in \mathscr{P}(\lambda_i) \wedge \mathscr{C} \quad &\Leftrightarrow \quad \begin{pmatrix} \bar{y}_i \\ \lambda_i \hat{u}_i \end{pmatrix} \in \mathscr{P}, \begin{pmatrix} \bar{y}_i \\ \hat{u}_i \end{pmatrix} \in \mathscr{C} \\
&\Leftrightarrow \quad \begin{pmatrix} \bar{y}_i \\ \lambda_i \hat{u}_i \end{pmatrix} \in \mathscr{P}, \begin{pmatrix} \lambda_i \bar{y}_i \\ \lambda_i \hat{u}_i \end{pmatrix} \in \mathscr{C} \\
&\Leftrightarrow \quad \begin{pmatrix} \bar{y}_i \\ \lambda_i \hat{u}_i \end{pmatrix} \in \mathscr{P}, \begin{pmatrix} \bar{y}_i \\ \lambda_i \hat{u}_i \end{pmatrix} \in \mathscr{C}(\lambda_i) \\
&\Leftrightarrow \quad \begin{pmatrix} \bar{y}_i \\ \lambda_i \hat{u}_i \end{pmatrix} \in \mathscr{P} \wedge \mathscr{C}(\lambda_i).
\end{aligned}$$

Consequently, $\forall \begin{pmatrix} \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{P} \wedge \mathscr{C}(\lambda_i)$, $\bar{y}_i$ is a small trajectory if and only if, $\forall \begin{pmatrix} \bar{y}_i \\ \hat{u}_i \end{pmatrix} \in \mathscr{P}(\lambda_i) \wedge \mathscr{C}$, $\bar{y}_i$ is a small trajectory. This proves that 3. is equivalent to 4. □

The main contribution of the previous theorem lies in the equivalence of conditions 1. and 4. Indeed, it shows that the N identical controllers $\mathscr{C}_i = \mathscr{C}, i = 1,2,\dots,N$, lead to consensus if and only if the controller $\mathscr{C}$ stabilizes the output variable $\bar{y}_i$ in all the plants $\mathscr{P}(\lambda_i) \wedge \mathscr{C}, i \in \{2,\dots,N\}$, where $\lambda_2,\dots,\lambda_N$ are the positive and real eigenvalues of L. Condition 4. allows us to make use of some results previously derived for the dead-beat control of discrete-time systems [3], but whose adaption to the case of a stabilizing control (of part of the system variables) of a continuous-time system is straightforward.

Proposition 5. *Given the behavior $\mathscr{P}(\lambda_i)$ described in (7), the controller $\mathscr{C}$ described in* (4) *stabilizes the output variable $\bar{y}_i$ in $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$, (i.e., for every $\left(\begin{smallmatrix} \bar{y}_i \\ \hat{u}_i \end{smallmatrix}\right) \in \mathscr{P}(\lambda_i) \wedge \mathscr{C}$ the variable $\bar{y}_i$ is small) if and only if*

$$\Gamma_i(s) := \begin{pmatrix} A(s) & \lambda_i B(s) \end{pmatrix} \begin{pmatrix} P_y(s) \\ C_y(s) \end{pmatrix} \tag{9}$$

is of full column rank and the greatest common divisor of its maximal order minors is Hurwitz, where $\begin{pmatrix} A(s) & B(s) \end{pmatrix}$ is an MLA of $\begin{pmatrix} P_u(s) \\ C_u(s) \end{pmatrix}$.

5 Consensus with regular interconnections

Up to now we have focused on the consensus target without imposing any constraint on the controller that allows to reach this goal. As a matter of fact, a natural constraint to introduce is that the connection of $\underline{\mathscr{P}}$ and $\underline{\mathscr{C}}_L$ is well-posed, i.e., that the input/output structure of $\underline{\mathscr{P}}$ is preserved even after interconnection with $\underline{\mathscr{C}}_L$. This means that each agent may have the possibility to correct the signal obtained from the other agents via feedback interconnection, by adding some completely free component. As previously recalled, in Section 2, this goal is achieved by imposing that the interconnection of the agents and the controllers is regular. By the assumptions we have introduced on the descriptions of the agents and the controllers, the regularity of the interconnection $\mathscr{K} = \underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_L$ amounts to requiring that the matrix

$$K = \begin{pmatrix} I_N \otimes P_y & -I_N \otimes P_u \\ L \otimes C_y & -I_N \otimes C_u \end{pmatrix}$$

is of full row rank, and K being a square matrix, this means that it must be nonsingular. Equivalently, the connected system $\mathscr{K}$ is autonomous. The following proposition shows that regularity of this interconnection is equivalent to regularity of the other interconnections appearing in Theorem 4. In order to prove it, it is convenient to preliminarily notice that the matrix

$$\begin{aligned} \overline{K} &= \begin{pmatrix} I_N \otimes P_y & -I_N \otimes P_u \\ D \otimes C_y & -I_N \otimes C_u \end{pmatrix} \\ &= \begin{pmatrix} V^{-1} \otimes I_p & 0 \\ 0 & V^{-1} \otimes I_m \end{pmatrix} K \begin{pmatrix} V \otimes I_p & 0 \\ 0 & V \otimes I_m \end{pmatrix} \end{aligned} \tag{10}$$

where V is the matrix of eigenvectors of L given in (3), can be rewritten, by resorting to suitable row and column permutations, to the block diagonal form

$$\overline{K}_{block} := \begin{pmatrix} K_1 & & \\ & \ddots & \\ & & K_N \end{pmatrix}, \tag{11}$$

where

$$K_i := \begin{pmatrix} P_y & -P_u \\ \lambda_i C_y & -C_u \end{pmatrix} \in \mathbb{R}[s]^{(p+m)\times(p+m)}.$$

This immediately leads to the following result, whose proof is straightforward and hence omitted.

Proposition 6. *The following conditions are equivalent:*

1. *The interconnection $\mathscr{K} = \underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_L$ is regular.*
2. *The interconnection $\mathscr{K}_D = \underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_D$ is regular.*
3. *For every $i \in \{1,\ldots,N\}$, the interconnection $\mathscr{P} \wedge \mathscr{C}(\lambda_i)$ is regular.*

4. *For every $i \in \{1,\dots,N\}$, the interconnection $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$ is regular.*

We can now restate the results of Theorem 4, by adding the requirement that the resulting interconnections are regular, i.e., that they are well-posed feedback interconnections. The proof is an immediate corollary of Theorem 4 and of Proposition 6, and hence is omitted.

Theorem 7 (consensus with regularity)**.** *The following statements are equivalent:*

1. (a) *The interconnection $\mathscr{K} = \underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_L$ is regular, and*
   (b) *the controllers $\mathscr{C}_i = \mathscr{C}, i = 1,2,\dots,N$, lead to consensus;*
2. (a) *The interconnection $\mathscr{K}_D = \underline{\mathscr{P}} \wedge \underline{\mathscr{C}}_D$ is regular, and*
   (b) $\forall \begin{pmatrix} \bar{y} \\ \bar{u} \end{pmatrix} \in \mathscr{K}_D$, $\forall i \in \{2,\dots,N\}$, $\quad \bar{y}_i \quad$ *is small.*
3. *For every $i \in \{2,\dots,N\}$,*
   (a) *the interconnections $\mathscr{P} \wedge \mathscr{C}(\lambda_i)$ are regular, and*
   (b) $\forall \begin{pmatrix} \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{P} \wedge \mathscr{C}(\lambda_i)$, $\quad \bar{y}_i \quad$ *is small.*
4. *For every $i \in \{2,\dots,N\}$,*
   (a) *the interconnections $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$ are regular, and*
   (b) $\forall \begin{pmatrix} \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{P}(\lambda_i) \wedge \mathscr{C}$, $\quad \bar{y}_i \quad$ *is small, i.e., $\mathscr{C}$ stabilizes the variable $\bar{y}_i$ in $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$.*

Clearly, condition 4. provides the characterization we want to explore in order to understand under what conditions on the original agents model the consensus problem is solvable. By making use of Proposition 5 and of the assumption on the matrices P_y and C_u involved in the agents and controller description, we can obtain algebraic characterizations of points (a) and (b) of condition 4. in Theorem 7. We will refer to these latter conditions as 4.(a) and 4.(b).

Proposition 8. *Given the behavior $\mathscr{P}(\lambda_i)$ described as in (7), with $\det P_y \neq 0$, a controller $\mathscr{C}$ described as in* (4)*, with $\det C_u \neq 0$, is such that*

4.(a) *the interconnection $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$ is regular, and*

4.(b) $\forall \begin{pmatrix} \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{P}(\lambda_i) \wedge \mathscr{C}, \bar{y}_i$ *is small, i.e. $\mathscr{C}$ stabilizes the variable $\bar{y}_i$ in $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$,*

if and only if

a) $\det[C_u - \lambda_i C_y P_y^{-1} P_u] \neq 0$*, and* *b)* $\Gamma_i = \begin{pmatrix} A & \lambda_i B \end{pmatrix} \begin{pmatrix} P_y \\ C_y \end{pmatrix} = AP_y + \lambda_i BC_y$ *is a square Hurwitz matrix, where* $\begin{pmatrix} A & B \end{pmatrix}$ *is an MLA of* $\begin{pmatrix} P_u \\ C_u \end{pmatrix}$.

Proof. We preliminarily notice that, by assumption, both P_y and C_u are square and nonsingular matrices, and the matrix M_i takes the form:

$$M_i = M_i(s) := \begin{pmatrix} P_y(s) & -\lambda_i P_u(s) \\ C_y(s) & -C_u(s) \end{pmatrix} \in \mathbb{R}[s]^{(m+p)\times(m+p)}.$$

[Necessity] If the interconnection $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$ is regular then

$$\mathrm{rank}(M_i) = \mathrm{rank}\begin{pmatrix} P_y & -\lambda_i P_u \end{pmatrix} + \mathrm{rank}\begin{pmatrix} C_y & -C_u \end{pmatrix} = p + m,$$

namely $\det M_i \neq 0$. As $\det P_y \neq 0$, we can write

$$\det M_i = \det P_y \cdot \det[C_u - \lambda_i C_y P_y^{-1} P_u] \neq 0,$$

and hence condition a) holds. On the other hand, if $\mathscr{C}$ stabilizes the variable $\bar{y}_i$ in $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$, by Proposition 5 we can claim that Γ_i is of full column rank and the greatest common divisor of its maximal order minors is Hurwitz. As we impose that C_u is square and nonsingular, then by [4] the matrix A must be square and nonsingular in turn, and hence Γ_i is a $p \times p$ square matrix. So, condition b) holds.

[Sufficiency] If a) holds, then $\det M_i \neq 0$. This ensures that M_i is square and nonsingular, and hence the the interconnection $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$ is regular. If b) holds, then, in particular, Γ_i is of full column rank and the greatest common divisor of its maximal order minors is Hurwitz. This ensures, by Proposition 5, that $\mathscr{C}$ stabilizes the variable $\bar{y}_i$ in $\mathscr{P}(\lambda_i) \wedge \mathscr{C}$. □

We want to prove that, as a consequence of the previous proposition, if the consensus problem is solvable, then the output y_i is stabilizable from the input u_i in $\mathscr{P}_i = \mathscr{P}$.

Corollary 9. *Suppose that the N agents are described by the model (2), with* $P_y(s) \in \mathbb{R}[s]^{p\times p}$ *and* $P_u(s) \in \mathbb{R}[s]^{p\times m}$, $\det P_y \neq 0$, *and let* $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$ *be the positive real eigenvalues of L. If the consensus problem is solvable, namely polynomial matrices* $C_y(s)$ *and* $C_u(s)$*, of suitable sizes, with* $\det C_u \neq 0$*, can be found such that conditions a) and b) of Proposition 8 hold, then* y_i *is stabilizable from* u_i *in* $\mathscr{P}_i = \mathscr{P}$.

Proof. If P_u is of full row rank the result follows immediately from Proposition 1. If $r := \mathrm{rank} P_u < p$, then an MLA $\begin{pmatrix} A(s) & B(s) \end{pmatrix}$ of $\begin{pmatrix} P_u(s) \\ C_u(s) \end{pmatrix}$ can be obtained in the following way:

$$\begin{pmatrix} A(s) & B(s) \end{pmatrix} = \begin{pmatrix} M_u(s) & 0 \\ Y_p(s) & Y_c(s) \end{pmatrix},$$

where $M_u \in \mathbb{R}[s]^{(p-r)\times p}$ is an MLA of P_u, and Y_p and Y_c are suitably chosen polynomial matrices. So, w.l.o.g. we can assume that

$$\Gamma_i(s) = A(s)P_y(s) + \lambda_i B(s)C_y(s) = \begin{pmatrix} M_u(s)P_y(s) \\ Y_p(s)P_y(s) + \lambda_i Y_c(s)C_y(s) \end{pmatrix}.$$

Being of full row rank, the $(p-r)\times p$ matrix $M_u(s)P_y(s)$ factorizes as

$$M_u(s)P_y(s) = \Delta(s)R_y(s),$$

where Δ is square and nonsingular and R_y is left prime. So, if b) holds, and hence Γ_i is a square Hurwitz matrix, then Δ must be Hurwitz in turn, thus proving that y_i in stabilizable from u_i in $\mathscr{P}_i = \mathscr{P}$ (see Proposition 1). $\square$

6 The problem solution

In the previous section we have shown that the consensus problem among N identical agents, each of them described as in (2), with the regularity constraint on the agents/controllers interconnection, is equivalent to the problem of determining polynomial matrices $C_y(s) \in \mathbb{R}[s]^{m\times p}$ and $C_u(s) \in \mathbb{R}[s]^{m\times m}$, with $\det C_u \neq 0$, such that conditions a) and b) of Proposition 8 hold. Also, if this is the case, then for each ith agent $\mathscr{P}_i = \mathscr{P}$ the output y_i is stabilizable from the input u_i.

In this section we want to prove that this necessary assumption on the agents $\mathscr{P}_i, i = 1,\dots,N$, is also sufficient for the consensus problem to be solvable. To prove this, we need a preliminary lemma.

Lemma 10. *Given a nonzero polynomial $\gamma(s) \in \mathbb{R}[s]$ and positive real numbers $0 < \lambda_2 \leq \cdots \leq \lambda_N$, a polynomial $n(s) \in \mathbb{R}[s]$ can be found such that the $N-1$ polynomials $\gamma(s) + \lambda_i n(s), i = 2,\dots,N$, are Hurwitz.*

Proof. Factorize $\gamma(s)$ as $\gamma(s) = \gamma_s(s)\gamma_u(s)$, where $\gamma_s(s)$ is Hurwitz, while $\gamma_u(s)$ is monic and has all its roots in the closed right half-plane $\{s \in \mathbb{C} : \mathrm{Re}(s) \geq 0\}$. Also, set $d := \deg \gamma_u(s)$. It is easily seen, by resorting to the positive root locus plot, that there exists $\bar{K} > 0$ such that for every $K \geq \bar{K}$, the polynomial $\gamma_u(s) + K(s+1)^d$ is Hurwitz. This ensures that by assuming

$$n(s) = \frac{\bar{K}}{\lambda_2}\gamma_s(s)(s+1)^d$$

we have that all polynomials $\gamma(s) + \lambda_i n(s), i = 2,\dots,N$, are Hurwitz. $\square$

In the following, we address two cases: 1) the case when P_u is of full row rank; 2) the case when $\operatorname{rank} P_u =: r < p$.

6.1 Case 1: P_u is of full row rank

If P_u is of full row rank, then (see Proposition 1) y_i is necessarily stabilizable from u_i. Let P_{uc} be a completion of P_u to a square and nonsingular matrix, and set

$$C_u := \begin{pmatrix} P_u \\ P_{uc} \end{pmatrix} \in \mathbb{R}[s]^{m\times m}.$$

Clearly, $\det C_u \neq 0$. Moreover, it is immediately seen that

$$\begin{pmatrix} P_u \\ C_u \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ I_p & 0 \\ 0 & I_{m-p} \end{pmatrix} C_u,$$

and hence an MLA of $\begin{pmatrix} P_u \\ C_u \end{pmatrix}$ is

$$\begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} I_p & \vline & -I_p & 0_{p\times(m-p)} \end{pmatrix}.$$

This implies that

$$AP_y + \lambda_i BC_y = P_y - \lambda_i C_{yA},$$

where $C_{yA} \in \mathbb{R}[s]^{p\times p}$ is the block of the first p rows of C_y. Let $U \in \mathbb{R}[s]^{p\times p}$ be a unimodular matrix such that UP_y is in lower triangular Hermite form [9], and let $\gamma_1(s),\dots,\gamma_p(s)$ be its diagonal entries. By Lemma 10, polynomials $n_1(s),\dots,n_p(s)$ can be found such that the $p(N-1)$ polynomials $\gamma_j(s)+\lambda_i n_j(s)$, $j=1,\dots,p, i=2,\dots,N$, are Hurwitz. Consequently, by assuming

$$C_{yA} = -U^{-1}\mathrm{diag}\{n_1,\dots,n_p\}$$

(and by arbitrarily choosing the remaining $m-p$ rows of C_y) we ensure that condition b) of Proposition 8 holds. It is trivial to verify that also condition a) holds.

6.2 Case 2: P_u is of rank $r < p$

Assume that rank $P_u =: r < p$ and factorize P_u as $P_u = L_u R_u$, where $L_u \in \mathbb{R}[s]^{p\times r}$ is right prime and $R_u \in \mathbb{R}[s]^{r\times m}$ is of full row rank. Consider a completion R_{uc} of R_u to a nonsingular $m\times m$ matrix[1], and set

$$C_u := \begin{pmatrix} R_u \\ R_{uc} \end{pmatrix} \in \mathbb{R}[s]^{m\times m}.$$

Clearly, $\det C_u \neq 0$. Moreover, it is immediately seen that

$$\begin{pmatrix} P_u \\ C_u \end{pmatrix} = \begin{pmatrix} L_u & 0 \\ I_r & 0 \\ 0 & I_{m-r} \end{pmatrix} C_u,$$

[1] If $r=m$ then R_u is already square and hence we assume $C_u = R_u$. All the subsequent reasonings remain valid.

and hence an MLA of $\begin{pmatrix} P_u \\ C_u \end{pmatrix}$ is

$$\begin{pmatrix} A & B \end{pmatrix} = \left(I_p \;\middle|\; -L_u \quad 0_{p\times(m-r)}\right).$$

This implies that

$$AP_y + \lambda_i BC_y = P_y - \lambda_i L_u C_{yA},$$

where C_{yA} is the block of the first r rows of C_y. Let

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{matrix} \}r \\ \}p-r \end{matrix} \in \mathbb{R}[s]^{p\times p}$$

be a unimodular matrix such that

$$UL_u = \begin{pmatrix} I_r \\ 0_{(p-r)\times r} \end{pmatrix}.$$

Note that U_2 is an MLA of L_u and hence of P_u. On the other hand, set

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} P_y =: \begin{pmatrix} U_1 P_y \\ \Delta_2 R_y \end{pmatrix}.$$

The assumption that y_i is stabilizable from u_i in $\mathscr{P}_i = \mathscr{P}$, together with the fact that $U_2 P_y$ is of full row rank, ensure that $U_2 P_y$ factorizes as $U_2 P_y = \Delta_2 R_y$, where Δ_2 is square and nonsingular and Hurwitz, while R_y is left prime (see Proposition 1). This ensures that

$$\begin{aligned} U(AP_y + \lambda_i BC_y) &= U(P_y - \lambda_i L_u C_{yA}) = \begin{pmatrix} I_r & 0 \\ 0 & \Delta_2 \end{pmatrix} \begin{pmatrix} U_1 P_y \\ R_y \end{pmatrix} - \lambda_i \begin{pmatrix} C_{yA} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} I_r & 0 \\ 0 & \Delta_2 \end{pmatrix} \begin{pmatrix} U_1 P_y - \lambda_i C_{yA} \\ R_y \end{pmatrix}. \end{aligned}$$

We want to show that it is possible to choose C_{yA} in such a way that the matrix $U(P_y - \lambda_i L_u C_{yA})$, and hence $AP_y + \lambda_i BC_y$, will necessarily be square and Hurwitz for every $i = 2, \dots, N$. Let V be a $p\times p$ unimodular matrix such that

$$\begin{pmatrix} U_1 P_y \\ R_y \end{pmatrix} V = \begin{pmatrix} X_1 & X_2 \\ 0 & I_{p-r} \end{pmatrix},$$

and $X_1 \in \mathbb{R}[s]^{r\times r}$ is in lower triangular Hermite form, with $\gamma_1(s), \dots, \gamma_r(s)$ as diagonal entries. By Lemma 10, polynomials $n_1(s), \dots, n_r(s)$ can be found such that the $p(N-1)$ polynomials $\gamma_j(s) + \lambda_i n_j(s), j = 1, \dots, r, i = 2, \dots, N,$ are Hurwitz. Consequently, by assuming

$$C_{yA} = -\left(\operatorname{diag}\{n_1, \dots, n_r\} \quad 0_{r\times(p-r)}\right) V^{-1}$$

(while the lower block of rows in C_y is arbitrary), we ensure that condition b) of Proposition 8 holds. It is trivial to verify that also condition a) holds.

So, to summarize, by putting together the results of this section with Corollary 9, we obtain the following result.

Theorem 11. *Suppose that the N agents are described by the model (2), with $P_y(s) \in \mathbb{R}[s]^{p\times p}$ and $P_u(s) \in \mathbb{R}[s]^{p\times m}$, $\det P_y \neq 0$, and let $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$ be the positive real eigenvalues of L. The following facts are equivalent:*

i) y_i in stabilizable from u_i in $\mathscr{P}_i = \mathscr{P}$,

ii) the consensus problem is solvable, namely polynomial matrices $C_y(s)$ and $C_u(s)$, of suitable sizes, with $\det C_u \neq 0$, can be found such that conditions a) and b) of Proposition 8 hold.

7 Comparison with the state-space set-up

In this section we aim at comparing the set-up and the results derived in the previous sections with the most common set-up and related results available in the literature. We refer for the comparison to [20], but several other references could be mentioned (see, e.g., [7, 14, 21] and references therein). In [20] each agent is described by a state-space model and hence the dynamics takes the form

$$\begin{aligned}\mathscr{P}_i := \mathscr{P} &:= \left\{ \begin{pmatrix} x_i \\ u_i \end{pmatrix} \in \mathscr{F}^{n+m} : \dot{x}_i = Ax_i + Bu_i \right\} \\ &= \left\{ \begin{pmatrix} x_i \\ u_i \end{pmatrix} \in \mathscr{F}^{n+m} : (sI_n - A)\circ x_i = B \circ u_i \right\}\end{aligned}$$

for constant matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. This amounts to assuming

$$\begin{pmatrix} P_y & -P_u \end{pmatrix} = \begin{pmatrix} sI - A & -B \end{pmatrix},$$

and clearly $\det P_y = \det(sI - A) \neq 0$. The agents' network is described by a directed weighted graph. The information available to the ith agent is

$$\widetilde{x}_i := \sum_{j=1}^{N} a_{ij}(x_j - x_i) = -\sum_{j=1}^{N} a_{ij}(x_i - x_j)$$

(which differs from the present set-up just in the sign). The consideblack controllers are of the form

$$\begin{aligned}\mathscr{C}_i := \mathscr{C} &:= \left\{ \begin{pmatrix} \widetilde{x}_i \\ u_i \end{pmatrix} \in \mathscr{F}^{n+m} : u_i = K\widetilde{x}_i \right\} \\ &= \left\{ \begin{pmatrix} \widetilde{x}_i \\ u_i \end{pmatrix} \in \mathscr{F}^{n+m} : K \circ \widetilde{x}_i = I_m \circ u_i \right\}\end{aligned}$$

for a constant matrix $K \in \mathbb{R}^{m\times n}$. Note that, by the structure imposed on the feedback control law in [20], $C_u = I_m$, while C_y is replaced by a constant matrix. This automatically ensures that

$$\operatorname{rank}\begin{pmatrix} P_y & -\lambda_i P_u \\ C_y & -C_u \end{pmatrix} = \operatorname{rank}\begin{pmatrix} P_y & -\lambda_i P_u \end{pmatrix} + \operatorname{rank}\begin{pmatrix} C_y & -C_u \end{pmatrix},$$

namely the interconnection is regular, for every value of λ_i.

Lemma 3.1 in [20], which is quoted from [7, 21], states that consensus on x is reached if and only if the matrices $A - \lambda_i BK, i = 2, \ldots, N$, are Hurwitz. This is equivalent to the condition

$$\forall i \in \{2, \ldots, N\},\ \forall \begin{pmatrix} \bar{x}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{P} \wedge \mathscr{C}(-\lambda_i), \quad \bar{x}_i \text{ is small,}$$

since

$$\begin{aligned} \mathscr{P} \wedge \mathscr{C}(-\lambda_i) &= \left\{ \begin{pmatrix} \bar{x}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{F}^{n+m} : (sI_n - A) \circ \bar{x}_i = B \circ \bar{u}_i,\ \bar{u}_i = -\lambda_i K \circ \bar{x}_i \right\} \\ &= \left\{ \begin{pmatrix} \bar{x}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{F}^{n+m} : (sI_n - A) \circ \bar{x}_i = -\lambda_i BK \circ \bar{x}_i,\ \bar{u}_i = -\lambda_i K \circ \bar{x}_i \right\} \\ &= \left\{ \begin{pmatrix} \bar{x}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{F}^{n+m} : (sI_n - (A - \lambda_i BK)) \circ \bar{x}_i = 0,\ \bar{u}_i = -\lambda_i K \circ \bar{x}_i \right\}. \end{aligned}$$

Taking into account the different definition of $\widetilde{x}_i$ leading to the minus sign in $\mathscr{C}(-\lambda_i)$, and the fact that in [20] the target variables are x_i while here we refer to y_i, this result is in perfect agreement with Theorem 4. In addition, in Theorem 3.3 of [20] it is stated that, under the assumption that the communication graph is connected, the consensus problem is solvable if and only if the agent's state-space model is stabilizable. So, the theory derived in this paper in the behavioral context naturally extends and generalizes the state-space approach previously investigated in the literature.

It is worth mentioning that in general the fact that the N agents reach consensus, and hence that for every $\begin{pmatrix} \bar{y}_i \\ \bar{u}_i \end{pmatrix} \in \mathscr{P}(\lambda_i) \wedge \mathscr{C}$, $i = 2, \ldots, N$, the variable $\bar{y}_i$ is small, does not impose any constraint on the corresponding input $\bar{u}_i$ that, on the contrary, may diverge as $t \to +\infty$. Clearly, this situation never arises when dealing with state-space models since one always resorts to a static state-feedback, and hence the convergence to zero of the state trajectory naturally ensures that the corresponding input converges to zero, in turn. If we want to reach consensus also on the input trajectories, then we need to impose the stabilizability property on both variables in $\mathscr{P}_i = \mathscr{P}$.

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