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Switched Positive Linear Systems

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Contents

Abstract

In this monograph we consider the class of continuous-time positive switched systems. We discuss several problems, including stability, performance analysis, stabilization via switching control, and optimization. The monograph starts with a chapter where several application examples are provided, to motivate the interest in this class of systems. The rest of the monograph is dedicated to the theory of stability, stabilization and performance optimization of positive switched systems. The main existing results are recalled, but also new challenging problems are proposed and solved. Special attention has been devoted to point out those results that specifically pertain to positive (linear) switched systems and do not find a counterpart for the wider class of (nonpositive) linear switched systems.

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1

Introduction

Positive systems are an important class of systems that frequently arise in application areas, such as in the chemical process industry, electronic circuit design, communication networks and biology.

Stability problems arising in the study of positive systems differ from those pertaining to standard systems. The main difference stems from the fact that the state variables are confined to the positive orthant. Thus, the whole analysis of these systems focuses only on the trajectories generated under positivity constraints, and consequently stability can be deduced from the existence of copositive Lyapunov functions whose derivatives are required to be negative only along the system trajectories in the positive orthant.

Switched positive systems also arise in a variety of applications. Examples can be found in TCP congestion control, in processes described by non-homogeneous Markov chains, in image processing, in biochemical networks, etc... Differently from general switched systems, that have received a lot of attention in the past years, the theory for positive switched systems is still in a relative infancy.

In this monograph we study the stability, performance evaluation, stabilization via switching control and optimal control of (continuous-time and linear) positive switched systems. We discuss results that have already been established in the literature, but other results, especially those regarding norm computation and optimization, are new and integrated with the previous ones.

In Chapter 2 we present many examples and motivations for studying positive switched systems. These examples include thermal systems, fluid networks, traffic systems, biological and epidemiological models and transmission networks. We present some specific problems that should be inspirational (at least we hope they are) for the subsequent chapters.

In Chapter 3, we consider the stability problem, namely the problem of determining stability under arbitrary switching. We show that this problem can be generalized to the problem of establishing a convergence (or divergence) rate. We characterize the stability property in terms of the existence of convex homogeneous Lyapunov functions. In general these functions can be extremely complex, so we provide some special classes of Lyapunov functions, including copositive linear and copositive quadratic Lyapunov functions, which are conservative but simpler to be computed. We also discuss a famous conjecture, now disproved in the general case, regarding the equivalence between stability under arbitrary switching and Hurwitz robustness, namely the fact that all the matrices in the convex hull of the family of system matrices are Hurwitz. The statement is true for 2-dimensional systems and false in general, since Hurwitz robustness is only necessary when the system dimension n is greater than 2. We also investigate the case when dwell time is imposed on the switching signals, namely a minimum amount of time has to elapse between any pair of consecutive switching times.

In Chapter 4 we discuss the performance evaluation of positive switched systems in terms of several input-output induced norms. Notwithstanding the fact that, for positive systems, it is often easy to establish the worst-case signal, namely the one providing the largest output norm, computing these norms is in general hard. Then we provide computationally tractable ways to generate upper bounds, for both arbitrary switching signals and dwell-time constrained ones.

In Chapter 5 we consider the stabilization problem for systems for which the switching signal represents a control input. This problem has some interesting properties that are the counterpart of some properties established in the stability analysis case. Stabilizability is equivalent to the existence of a concave homogeneous copositive control Lyapunov function. Again, finding any such function is in general hard, so we investigate special classes including linear and quadratic copositive functions. We also provide some sufficient stabilizability conditions in terms of Lyapunov Metzler inequalities. The disproved conjecture about stability has a counterpart for the stabilizabity case: is stabilizability equivalent to the existence of at least one Hurwitz element in the convex hull of the matrices? Again the is true for 2-dimensional systems and false in general, as the existence of a Hurwitz convex combination is only a sufficient condition for stabilization. It is interesting to note that the existence of a Hurwitz convex combination is a necessary and sufficient condition for the existence of a smooth homogeneous control Lyapunov function.

Finally, in Chapter 6, we consider the optimal control problem for positive systems with a controlled switching signal. We show how some of the material presented in the previous chapters, such as the Lyapunov Metzler inequalities technique, can be successfully exploited to derive some conditions that allow to design a guaranteed cost control.

In addition to the simple numerical examples provided in Chapters 3-5 to illustrate the developed theory, in this chapter simulations are provided for a couple of "realistic" examples presented in Chapter 2, dedicated to the motivational part, and specifically: the optimal therapy scheduling for mitigation of the HIV viral load, and the disease free control applied to a SIS (Susceptible-Infective-Susceptible) epidemiological system.

This survey does not aim at providing an exhaustive account of all the research problems investigated in the literature and concerned with positive switched systems. Important issues have been omitted here, due to page constraints. Among them, it is worth quoting the following ones: controllability/reachability (see Fornasini and Valcher [2011], Santesso and Valcher [2008], Valcher and Santesso [2010], Valcher [2009], Xie and Wang [2006]), observability of positive switched systems (Li et al. [2014]), positive switched systems with delays (Li et al. [2013a,b], Liu and Dang [2011], Liu and Lam [2012], Xiang and Xiang [2013]), and interesting characterizations like joint spectral properties and asymptotic properties of matrix semigroups (Guglielmi and Protasov [2013], Protasov et al. [2010], Jungers [2012], and extremal norms for linear inclusions, Mason and Wirth [2014]). For all these topics we refer the interested Reader to the previous references. On the other hand, for the topics specifically addressed in this monograph, no references are provided in this introduction, being them appropriately quoted when needed within the text.

1.1 Notation

The notation used throughout the monograph is standard for positive systems. The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively, while $\mathbb{R}_+$ is the set of nonnegative real numbers. Capital letters denote matrices, small (bold face) letters denote vectors. For matrices or vectors, $(^\top)$ indicates transpose. The (ℓ, j)th entry of a matrix A is denoted by $[A]_{\ell,j}$, while the ith entry of a vector $\mathbf{x}$ is x_i or $[\mathbf{x}]_i$. When the vector $\mathbf{x}$ is obtained as the result of some mathematical operation, e.g. $\mathbf{x} = A\mathbf{y}$, we will generally adopt the latter notation $[A\mathbf{y}]_i$. The symbol $\mathbf{e}_i$ denotes the ith canonical vector in $\mathbb{R}^n$, where n is always clear from the context, while $\mathbf{1}_n$ denotes the n-dimensional vector with all entries equal to 1. The symbol I_n denotes the identity matrix of order n.

A (column or row) vector $\mathbf{x} \in \mathbb{R}^n$ is said to be *nonnegative*, $\mathbf{x} \geq 0$, if all its entries x_i, $i = 1, 2, \ldots, n$, are nonnegative. It is *positive* if nonnegative and at least one entry is positive. In this case, we will use $\mathbf{x} > 0$. It is said to be strictly positive if all its entries are greater than 0, and in this case, we will use the notation $\mathbf{x} \gg 0$. The set of all n-dimensional nonnegative vectors is denoted by $\mathbb{R}^n_+$ and referred to as the *positive orthant*. The expressions $\mathbf{x} \gg \mathbf{y}$, $\mathbf{x} > \mathbf{y}$ and $\mathbf{x} \geq \mathbf{y}$ mean that the difference $\mathbf{x} - \mathbf{y}$ is strictly positive, positive and nonnegative, respectively. Similar notation is used for the (real) matrices.

The set of n-dimensional nonnegative vectors whose entries sum up to 1 is the simplex

$$\mathcal{A}_n := \{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} \alpha_i = 1\}.$$

A square matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Metzler*[1] if its off-diagonal entries $[A]_{ij}$, $i \neq j$, are nonnegative. Every Metzler matrix A has a real dominant eigenvalue $\lambda_F \in \sigma(A)$ satisfying $\mathrm{Re}(\lambda_F) > \mathrm{Re}(\lambda)$ for every $\lambda \in \sigma(A), \lambda \neq \lambda_F$. λ_F is called the Frobenius eigenvalue of A, see Farina and Rinaldi [2000]. Also, associated with λ_F there is always both a left and a right positive eigenvector, known as (left/right) Frobenius eigenvectors.

An $n \times n$ Metzler matrix A is *reducible* if there exists a permutation matrix Π such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square (nonvacuous) matrices, otherwise it is *irreducible*. It follows that 1×1 matrices are always irreducible.

A linear state space model described by the linear differential equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, where A is a Metzler matrix, is called a *positive system*, see Berman et al. [1989], Farina and Rinaldi [2000], Kaczorek [2002], Krasnoselskii [1964], Luenberger [1979], because it enjoys the property that any trajectory starting in the positive orthant remains confined in it.

A square matrix is *Hurwitz* if all its eigenvalues lie in the open left half plane. A Metzler matrix is Hurwitz if and only if there exists a vector $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top A \ll 0$, or, equivalently if and only there exists a vector $\mathbf{w} \gg 0$ such that $A\mathbf{w} \ll 0$, see e.g. Farina and Rinaldi [2000].

Given two matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$, the expression $C = A \otimes B \in \mathbb{R}^{np\times mq}$ stands for the usual *Kronecker product*. If $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{p\times p}$, their *Kronecker sum* is defined as $A \oplus B = A \otimes I_p + I_n \otimes B \in \mathbb{R}^{np\times np}$. Properties of Kronecker operators can be found in Graham [1981].

[1] A Metzler matrix is also known in the literature as "essentially nonnegative matrix" (see Berman et al. [1989], Horn and Johnson [1985]) or as the opposite of a "Z-matrix" (see Horn and Johnson [1991]).

The symbols $\succ, \succeq, \prec$ and $\preceq$ are used to denote order relations induced by definiteness properties. For instance, the expression $P = P^\top \succ 0 \in \mathbb{R}^{n\times n}$ means that P is a (symmetric and) positive definite matrix, i.e. $\mathbf{x}^\top P\mathbf{x} > 0$ for every $\mathbf{x} \neq 0$. $P_1 \succeq P_2$ means that $P_1 - P_2$ is a (symmetric and) positive semi-definite matrix.

1.2 Continuous-time positive switched systems

A *continuous-time positive switched system* is described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{1.1}$$

where $\mathbf{x}(t)$ denotes the value of the n-dimensional state variable at time t, and $\sigma(t)$ is a right-continuous and piece-wise constant mapping from $\mathbb{R}_+$ into the finite set $\{1, ..., M\}$. This latter property ensures that in any bounded time interval the map σ has always a finite number of discontinuities, known as *switching instants* and denoted in the following by $0 = t_0 < t_1 < t_2 < \dots$. This amounts to saying that $\sigma(t)$ takes some constant value $i_k \in \{1, 2, \dots, M\}$ at every $t \in [t_k, t_{k+1})$ and that $\sigma(t_k) \neq \sigma(t_{k+1})$. In the sequel, when we will refer to an "arbitrary switching signal" σ we will always mean an arbitrary switching signal endowed with the aforementioned properties and we will denote the set of such switching signals by the symbol $\mathcal{D}_0$. The reason for this notation will be clarified later on.

A function $\mathbf{x} : \mathbb{R}_+ \mapsto \mathbb{R}^n$ is a solution of (1.1) if, see Shorten et al. [2007], it is continuous and piecewise continuously differentiable and if there is a switching signal σ such that (1.1) holds at every $t \in \mathbb{R}_+$, except at the switching instants. For every value i taken by the switching signal σ (at t), $\dot{\mathbf{x}}(t) = A_i\mathbf{x}(t)$ is a (autonomous[2]) continuous-time positive system, which means that A_i is an $n \times n$ Metzler matrix. This ensures that if $\mathbf{x}(0)$ belongs to the positive orthant $\mathbb{R}^n_+$, then, for every choice of σ, the state evolution $\mathbf{x}(t) = \mathbf{x}(t; \mathbf{x}(0), \sigma)$ belongs to $\mathbb{R}^n_+$ for every $t \in \mathbb{R}_+$. It is worth noticing that also for switched systems the

[2]In this monograph by an autonomous system we will always mean a system with no inputs, see Khalil [2002], Sun and Ge [2005], Willems [1970].

Metzler property of the matrices $A_i, i \in \{1, 2, \dots, M\}$, is both necessary and sufficient to ensure that all the state trajectories starting in the positive orthant remain in $\mathbb{R}^n_+$ at all subsequent times, for every choice of the switching signal. Given any initial state $\mathbf{x}_i \in \mathbb{R}^n_+$, any switching signal $\sigma : \mathbb{R}_+ \mapsto \{1, 2, \dots, M\}$, and any pair of time instants $t \geq \tau \geq 0$, the state at time t can be expressed as

$$\mathbf{x}(t) = \Phi(t, \tau, \sigma)\mathbf{x}_i,$$

where $\Phi(t, \tau, \sigma)$ represents the *state transition matrix* of system (1.1) corresponding to the time interval $[\tau, t]$ and the switching signal σ. Clearly, if we denote by $\tau = t_1 < t_2 < \cdots < t_k < t_{k+1} = t$ the switching instants in the time interval $[\tau, t]$ and by i_h the value of the switching signal σ in the time interval $[t_h, t_{h+1}), h \in \{1, 2, \dots, k\}$, then

$$\Phi(t, \tau, \sigma) = e^{A_{i_k}(t-t_k)} \dots e^{A_{i_2}(t_3-t_2)} e^{A_{i_1}(t_2-\tau)}.$$

In the following, we will also consider non-autonomous positive switched systems, described, for instance (but not only), by the following equations:

$$\begin{array}{rcll} \dot{\mathbf{x}}(t) & = & A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{u}(t), & \\ \mathbf{y}(t) & = & C_{\sigma(t)}\mathbf{x}(t) + D_{\sigma(t)}\mathbf{u}(t), & \quad t \in \mathbb{R}_+, \end{array} \tag{1.2}$$

where $\mathbf{x}(t), \mathbf{u}(t)$ and $\mathbf{y}(t)$ are the n-dimensional state variable, the m-dimensional input variable and the p-dimensional output variable, respectively, at time t. For every value i taken by σ (at t), A_i is an $n \times n$ Metzler matrix, while B_i, C_i and D_i are nonnegative matrices. Under these conditions, the nonnegativity of the input at every time $t \geq 0$ and the nonnegativity of the initial condition $\mathbf{x}(0)$ ensure the nonnegativity of the state and output trajectories at every $t \geq 0$.

2

Motivating applications

Positive systems arise in the description of a good number of dynamical processes of practical significance. In this chapter we present some examples and problems that we hope will be inspiring and stimulate an interest in the subsequent theory.

2.1 Some considerations about positivity and monotonicity

We start with some general considerations about positive systems. As a general statement, an autonomous system is positive if it has the property that whenever its state variables are nonnegative at a given time, they remain nonnegative at every subsequent time.
When dealing with linear time invariant systems, this is equivalent to the Metzler property of the state matrix in the continuous time case, and to the nonnegativity of the state matrix in the discrete time one. It is well-known, see Smith [2008], that this is also equivalent to saying that the system is *monotone*. This means that given any two initial conditions $\mathbf{x}_A(0)$ and $\mathbf{x}_B(0)$, and upon denoting by $\mathbf{x}_A(t)$ and $\mathbf{x}_B(t)$ the corresponding solutions, we have

$$\mathbf{x}_A(0) \leq \mathbf{x}_B(0) \Rightarrow \mathbf{x}_A(t) \leq \mathbf{x}_B(t), \ \forall \ t \geq 0.$$

In the nonlinear case the concepts of positivity and monotonicity are not equivalent anymore. Indeed, a system may be positive but not monotone and vice versa. For instance, the well known Lotka-Volterra prey-predator model:

$$\begin{aligned}\dot{x}(t) &= \alpha x(t) - \beta x(t)y(t),\\ \dot{y}(t) &= -\gamma y(t) + \delta x(t)y(t),\end{aligned}$$

where x represents the prey population and y represents the predator population, is positive but not monotone. This model has no meaning unless we restrict our attention to the positive orthant.

On the other hand, a monotone system is not necessarily positive. For instance, given the first order nonlinear system

$$\dot{x}(t) = -(x(t)+1)^2, \qquad t \geq 0,$$

its solution can be explicitly written as

$$x(t) = \frac{x_0 - t(x_0+1)}{1+t(x_0+1)}, \qquad \forall\ t \geq 0.$$

Therefore for any positive initial condition x_0, the state variable becomes negative for $t > \frac{x_0}{x_0+1}$, and hence the system is not positive. However, the system is monotone, since the derivative of $x(t)$ with respect to x_0 is positive for any x_0 and any $t \geq 0$.

Under mild assumptions, see Smith [2008] and Sontag [2007], a monotone system has a Jacobian matrix that is Metzler when evaluated at any equilibrium point, and therefore it is "locally positive".

Even when dealing with the linear monotone (and hence positive) case, there are many examples of positive systems for which it is nonetheless reasonable to assume negative initial conditions and henceforth negative values for the state variables. For instance, the circuit in Figure 2.1 is a positive system, if we take the capacitors voltages as state variables (and assume that the input voltage is positive). Clearly, there is nothing wrong in assuming negative values for the initial state variables, since the capacitors may be negatively charged. In other cases, we are interested in positive state variables, and hence restricting our attention to the positive orthant is not restrictive.

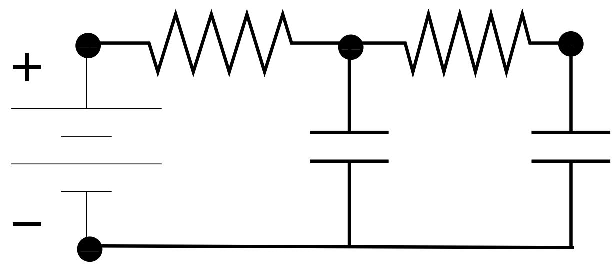

Figure 2.1: A positive linear system that admits negative initial conditions.

We point out that while positivity in the linear case and monotonicity in the nonlinear case ensure strong system properties, positive nonlinear systems that are not monotone exhibit no meaningful features that prevent them from displaying unexpected behaviors, like it happens for general nonlinear systems, see Hirsch [1988], Smale [1976].

We start with some examples of autonomous positive switched systems.

2.2 Modeling HIV virus therapy switching

One the fundamental problems in certain medical therapies is that viruses are subject to mutations, and drugs are often able to oppose the growth of some genotypes but not of others. Highly active antiretroviral therapies (HAARTs) provide a rapid drop in plasma viral load with a large reduction of infected cells in patients with HIV infection. Even after long periods of HAARTs, latently infected cells are still detectable. Therefore cellular reservoirs may contribute to HIV persistence promoting the emergence of resistant mutants, see Eisele and Siliciano [2012].

In the last treatment guidelines for HIV infection, AIDSinfo [2011], clinicians did not achieve a consensus on the optimal time to change therapy in the event of virological failure (inability to maintain HIV RNA levels at less than 50 copies/ml under HAART treatment). A widely accepted strategy (that we refer to as “switch on virological failure”) is to continue the current therapy until the viral load exceeds a fixed level (e.g. 500 - 1000 copies/ml). Using a mathematical approach, the paper D'Amato et al. [1998] claimed that alternating between therapies may delay the emergence of resistant mutant viruses. In this initial trial, alternating regimens appeared to outperform virological failure based treatment. Then switching between treatments may be crucial to minimize the risk of resistance, Ouattara et al. [2008], Craig and Xia [2005] and Luo et al. [2011]. Under several simplifying assumptions, Hernandez-Vargas et al. [2011] formalized the drug treatment scheduling problem of HIV infection as an optimal control problem for a specific class of autonomous positive switched systems. Here we briefly describe the linear model proposed for this phenomenon. The Reader is referred to Hernandez-Vargas et al. [2013] for more details.

Assume that there are n different viral genotypes, with viral populations, x_i, $i = 1, ..., n$, and M different possible drug therapies that can be administered. We denote by $\sigma(t)$ the specific therapy adopted at time $t \geq 0$. Clearly, at every time t, σ takes values in $\{1, ..., M\}$. Under the assumption that macrophage and CD4+T cell counts are approximately constant until virological failure, see e.g. Perelson and Nelson [1999], the following linear model can be written in order to represent the evolution of the populations of genotypes:

$$\dot{x}_i(t) = \rho_{i,\sigma(t)} x_i(t) - \delta x_i(t) + \sum_{j \neq i} \mu \, \zeta_{i,j} x_j(t), \tag{2.1}$$

where μ is a small positive parameter representing the mutation rate, $\rho_{i,\sigma(t)}$ is the replication rate of the ith viral species under the drug therapy $\sigma(t)$, $\delta > 0$ is the death or decay rate, and $\zeta_{ij} \in \{0, 1\}$ represents the genetic mutation between genotypes, that is, $\zeta_{i,j} = 1$ if and only if it is possible for genotype j to mutate into genotype $i \neq j$. Equation (2.1) can be rewritten in vector form as

$$\dot{\mathbf{x}}(t) = \Big(R_{\sigma(t)} - \delta I_n\Big) \mathbf{x}(t) + \mu Z \mathbf{x}(t), \tag{2.2}$$

where $[Z]_{ij} := \zeta_{ij}$ and $R_{\sigma(t)} := \mathrm{diag}\{\rho_{1,\sigma(t)}, \rho_{2,\sigma(t)}, \dots, \rho_{n,\sigma(t)}\}$. The previous model can be rewritten as

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)} \mathbf{x}(t),$$

where $A_{\sigma(t)} := R_{\sigma(t)} - \delta I_n + \mu Z$ is a Metzler matrix for every value of $\sigma(t) \in \{1, 2, \dots, M\}$, and hence it is an n-dimensional autonomous positive switched system.

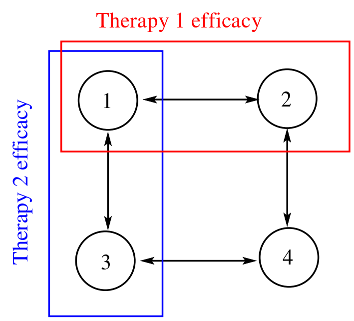

Figure 2.2: Genotype mutations and drug efficacy.

To investigate this issue a little deeper, we concentrate on the simple case (see Hernandez-Vargas et al. [2013]) where only 4 genetic variants are considered and two antiretroviral therapies are adopted, that is $n = 4$ and $M = 2$. The virus genotypes are the following ones:

- Wild type genotype (WTG): In the absence of any drug, this would be the most prolific variant. However, it is also the variant that both drug combinations are designed to oppose, and therefore is susceptible to both therapies.

- Genotype 1 (G1): A genotype that is resistant to therapy 1, but is susceptible to therapy 2.

- Genotype 2 (G2): A genotype that is resistant to therapy 2, but is susceptible to therapy 1.

- Highly resistant genotype (HRG): A genotype, with low proliferation rate, but resistant to both drug therapies.

The parameter values proposed in Hernandez-Vargas et al. [2013] are $\delta = 0.24\ day^{-1}$ for the decay rate and $\mu = 10^{-4}$, which is the order of magnitude typical of the viral mutation rate. These numbers are of course idealized. We assume a mutation graph that is symmetric and circular, since only the mutations $WTG \leftrightarrow G1$, $G1 \leftrightarrow HRG$, $HRG \leftrightarrow G2$ and $G2 \leftrightarrow WTG$ are possible. Other connections would correspond

Variant	Therapy 1	Therapy 2
Wild type genotype (x_1)	$\rho_{1,1} = 0.05$	$\rho_{1,2} = 0.05$
Genotype 1 (x_2)	$\rho_{2,1} = 0.27$	$\rho_{2,2} = 0.05$
Genotype 2 (x_3)	$\rho_{3,1} = 0.05$	$\rho_{3,2} = 0.27$
HR genotype (x_4)	$\rho_{4,1} = 0.27$	$\rho_{4,2} = 0.27$

Table 2.1: Replication rates for viral variants and therapy combinations for a symmetric case.

to double mutations, which are events of negligible probability. If we assume as state variables the ones described in Table 2.1, the resulting mutation matrix is

$$Z = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{2.3}$$

The replication rates in Table 2.1 are reported from Hernandez-Vargas et al. [2013].

Under the previous assumptions on the system parameters, the matrices $A_i \in \mathbb{R}^{4\times 4}$

involved in the system description are

$$A_i = \begin{bmatrix} \rho_{1,i} & 0 & 0 & 0 \\ 0 & \rho_{2,i} & 0 & 0 \\ 0 & 0 & \rho_{3,i} & 0 \\ 0 & 0 & 0 & \rho_{4,i} \end{bmatrix} + \mu \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad i = 1, 2.$$

The optimal control problem addressed in Hernandez-Vargas et al. [2013] consists of choosing a switching rule that minimizes the following finite-horizon cost function

$$J := \mathbf{c}^\top \mathbf{x}(t_f), \tag{2.4}$$

where $\mathbf{c}$ is a strictly positive cost vector, and $t_f > 0$ is an appropriate final time.

To conclude the example, note that this system is in general unstable or, in other words, it has a negative speed of convergence β

(see Chapter 3). As explained in Hernandez-Vargas et al. [2013] and references therein, keeping the virus population at a low level is fundamental in the therapy anyway. Simulation results are presented in Section 6.2.1.

2.3 Epidemiological models

This example is taken from Ait Rami et al. [2014] (see also Blanchini et al. [2014]). Consider the epidemiological model of a population divided into n groups. Each ith group is divided into two classes: infectives and susceptibles. Let $I_i(t)$ denote the number of infectives at time t and $S_i(t)$ the number of susceptibles at time t. Under the assumption that the total number $I_i(t)+S_i(t) = N_i$ is constant at every time $t \geq 0$, and by setting $x_i(t) := I_i(t)/N_i$ one can write, for $i = 1, 2, \ldots, n$:

$$\dot{x_i}(t) = (1 - x_i(t)) \sum_{j=1}^{N} \frac{\beta_{ij} N_j}{N_i} x_j(t) - (\gamma_i + \mu_i) x_i(t), \qquad (2.5)$$

where $\beta_{ij} > 0$ is the rate at which susceptibles in group i are infected by infectives in group j, $\gamma_i > 0$ is the rate at which an infective individual in group i is cured and $\mu_i > 0$ is the death rate in group i (that is equal to the birth rate appearing in the not written equation of $\dot{S}_i$, since the number of individuals in the each group is constant). Note that the set $\{\mathbf{x} \in \mathbb{R}^n : 0 \leq \mathbf{x} \leq \mathbf{1}_n\}$ is positively invariant for this system, i.e., if the initial condition $\mathbf{x}(0)$ belongs to this set, then the corresponding state trajectory remains in the set at all the subsequent times $t \geq 0$.

If we assume that M different therapies are introduced to fight the epidemic, the rate γ_i is not constant but it depends, at every time t, on some variable $\sigma(t) \in \{1, 2, \ldots, M\}$, that represents the value at t of the switching signal that orchestrates the different therapies for each population group. Therefore, we replace γ_i in (2.5) with $\gamma_{i,\sigma(t)}$. The introduction of the therapy scheduling preserves the positive invariance property of the set $\{\mathbf{x} \in \mathbb{R}^n : 0 \leq \mathbf{x} \leq \mathbf{1}_n\}$. We also introduce the simplifying assumption that the change of therapies does not affect the rates β_{ij}. Finally, we linearize the system around the *disease free* *equilibrium point* $\mathbf{x} = 0$. The linearized system is then given by

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \tag{2.6}$$

where $\sigma(t) \in \{1, 2, \dots, M\}$ and $A_i = \Lambda_i + \bar{A}$, for some Metzler matrix $\bar{A}$ and some diagonal matrix Λ_i, $i = 1, 2, \dots, M$. Specifically, $[\bar{A}]_{hk} = \beta_{hk} N_k / N_h$ and

$$[\Lambda_i]_{hk} = \begin{cases} -\gamma_{h,i} - \mu_h, & \text{if } h = k; \\ = 0, & \text{if } h \neq k. \end{cases}$$

We can embed system (2.6) into the bilinear system

$$\dot{\mathbf{x}}(t) = (\Lambda(\mathbf{u}(t)) + \bar{A})\mathbf{x}(t) = (\sum_{i=1}^{M} [\mathbf{u}(t)]_i \Lambda_i + \bar{A})\mathbf{x}(t), \tag{2.7}$$

where, at each time $t \geq 0$, the vector $\mathbf{u}(t)$ belongs to the simplex

$$\mathcal{A}_M := \{\mathbf{u} \in \mathbb{R}_+^M : \mathbf{1}_M^\top \mathbf{u} = 1\}.$$

Remark 2.1. This relaxation from $\{\mathbf{e}_i, i \in \{1, 2, \dots, M\}\}$, where $\mathbf{e}_i$ is the ith canonical vector, to $\mathcal{A}_M$ is fundamental in the investigation of switched systems, as we will see also in the following. Any state trajectory that can be achieved using some $\mathbf{u}$ taking values in $\mathcal{A}_M$ at every time t, i.e., $\mathbf{u} : \mathbb{R}_+ \to \mathcal{A}_M$, (equivalently, by means of a switching signal σ, taking values in $\{1, 2, \dots, M\}$) can be arbitrarily approximated by a solution achieved through some $\mathbf{u} : \mathbb{R}_+ \to \{\mathbf{e}_i, i \in \{1, 2, \dots, M\}\}$, provided that there are no upper bounds on the switching frequency. As we will discuss later, in order to define the solution of the switched system one has to resort to this relaxation, see Aubin [1991]. This relaxation is the cornerstone of the theory of differential inclusions and control theory. The main result basically states that the solution set of a switched system with (arbitrarily small) dwell time is dense in the set of the relaxed solutions (solutions of the associated bilinear system or equivalently of the associated differential inclusion). This result, known in the literature as Filippov-Wazewski relaxation theorem, can be regarded as a generalization of the bang-bang principle in linear control theory, see Aubin and Cellina [1984], Joo and Tallos [1999].

In Section 6.2.2 we will address an infinite horizon optimal control problem for this system, consisting in the minimization of the cost function

$$\int_0^{+\infty} \mathbf{1}^\top \mathbf{x}(t)dt,$$

and provide simulations results.

Often positive (especially linear) systems are of little interest if we do not consider external inputs. For instance, the capacitor circuit we previously discussed has no practical use without a voltage supply; the same can be said about thermal and fluid networks without an inflow, or traffic models with no external access, examples we will address in the following. Consequently, the model of general interest in many applications takes the form

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{u}(t), \tag{2.8}$$

where, for each value i taken by the switching signal $\sigma(t)$, A_i is Metzler and B_i is nonnegative. Moreover, $\mathbf{u}(t) \geq 0$ at every time $t \geq 0$. A lot of systems can be described, at least approximately, in this form.

When $\mathbf{u}(t) = \bar{\mathbf{u}}$ for every $t \geq 0$, namely the input is constant, the model can be written as an autonomous positive switched system provided that we augment the state by treating the variables $\mathbf{u}$ as fictitious state variables. In this way we get

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{u}(t), \\ \dot{\mathbf{u}}(t) &= 0, \end{aligned}$$

but we clearly have to impose $\mathbf{u}(0) = \bar{\mathbf{u}}$.

Alternatively, when the input is constant, and for the sake of simplicity scalar, we can also describe it as a *positive affine switched system* of the form

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}, \tag{2.9}$$

where $\sigma(t)$ takes values, as usual, in $\{1, ..., M\}$, the A_i and the B_i, $i \in \{1, 2, \dots, M\}$, are given Metzler matrices and nonnegative vectors, respectively.

2.4 Thermal models

Thermal systems are a popular class of positive systems. Their positivity nature is naturally enforced, once absolute Kelvin temperatures are considered, but they are in general positive under certain assumptions. For instance, the temperature inside a building cannot be lower than the external temperature, if a positive amount of heat is supplied. So, the difference between the internal temperature and the external temperature is always a nonnegative variable.

A building, regarded as a thermal system, is a switched system if the heat transfer coefficients between adjacent environments change for some reason.

Consider for instance the system in Figure 2.3, having three rooms: the first of them is heated, while the other two are not. Different thermal transmission coefficients have to be considered depending on whether the doors are closed or open.

If we assume that the temperatures in the three rooms are x_1, x_2 and x_3, respectively, we can describe the various cases corresponding to the status (open/closed) of the two doors by means of a linear model of the form (2.9), with

$$A_i = \begin{bmatrix} -\alpha_i & \alpha_i & 0 \\ \alpha_i & -(\alpha_i + \beta_i) & \beta_i \\ 0 & \beta_i & -(\beta_i + \gamma) \end{bmatrix}, \qquad B_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

for $i \in \{1, 2, 3, 4\}$, corresponding to the four combinations open-closed of the two doors. Without restrictions, the external temperature is assumed to be $T_0 = 0$. All the coefficients are positive. The coefficients α_i

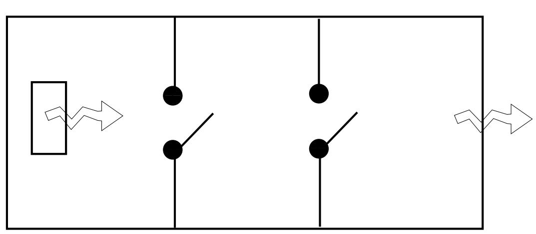

Figure 2.3: A switching thermal systems.

and β_i may have two values $\alpha_i \in \{\alpha_{\min}, \alpha_{\max}\}$ and $\beta_i \in \{\beta_{\min}, \beta_{\max}\}$ with $\alpha_{\min} < \alpha_{\max}$ and $\beta_{\min} < \beta_{\max}$. So there are four possibilities for (α_i, β_i). Precisely,

$$\begin{aligned} i = 1: &\quad (\alpha_{\min}, \beta_{\min}); \\ i = 2: &\quad (\alpha_{\max}, \beta_{\min}); \\ i = 3: &\quad (\alpha_{\min}, \beta_{\max}); \\ i = 4: &\quad (\alpha_{\max}, \beta_{\max}). \end{aligned}$$

Note that we have assumed that heat is supplied to room 1 and the only dissipation to the external environment is from room 3. The system is positive, and, for positive and fixed values of the parameters, it is exponentially stable. The question is whether this system remains exponentially stable under arbitrary switching, namely for every choice of σ. The answer is yes, as it will be shown later.

Thermal systems are usually stable and therefore a more interesting problem with respect to stability analysis is that of determining the rate of convergence to zero of their state variables. In the continuous time, we say that the switched system has *rate of convergence* $\beta > 0$ if there exists some constant $C > 0$ such that, for every initial condition $\mathbf{x}(0)$ and every switching signal σ, one has $\|\mathbf{x}(t; \mathbf{x}(0), \sigma)\| \leq Ce^{-\beta t}\|\mathbf{x}(0)\|$ for every $t \geq 0$ (see Definition 3.5). The problem of determining or ensuring a certain convergence speed can be rephrased as a modified stability analysis/stabilization problem, as we will see later.

So, an interesting question regarding the previous system is what is the convergence rate we can achieve corresponding to each switching law, by assuming that the two doors can be opened and closed in any time sequence. In general, the convergence rate β associated with a switching law σ satisfies the inequalities:

$$\beta_{worst} \leq \beta \leq \beta_{best},$$

where β_{worst} represents the worst case performance that can be obtained under arbitrary switching, while β_{best} is the best convergence rate one can achieve via a controlled switching.

For this thermal system one could expect that the fastest convergence rate, β_{best}, is achieved by selecting the higher values of the thermal exchange coefficients ($\sigma = 4$), and the slowest convergence rate β_{worst} by selecting the smaller values of the thermal exchange coefficients ($\sigma = 1$). In general, as we will see later, it is not true that β_{worst} and β_{best} are associated with two constant switching signals $\sigma(t)$.

The two numbers β_{worst} and β_{best} are in general an "infimum" and a "supremum", respectively. The question whenever they can be achieved, for positive switched systems, and hence they can be a "minimum" and a "maximum" is still unclear. This problem is strictly related to the finiteness conjecture for sets of matrices, see for instance Blondel et al. [2003].

Another interesting question is how the switching law can affect the system dynamics, leading to the question: what is the highest temperature that can be reached in any of the three rooms, starting from some given $\mathbf{x}(0)$? Suppose we start from $\mathbf{x}(0) = 0$. No one would doubt that the maximum (formally, the supremum) temperature x_1 in room 1 is achieved, at steady state, by keeping both doors closed. Conversely, what is the maximum temperature x_3 that can be reached in room 3? And how can it be reached? It is possible to show that the maximum value of the temperature x_3 is not obtained as the steady state temperature reached under a fixed configuration (a constant σ).

Assuming, for explanatory purposes, that high temperatures are undesirable, then the problem we address is that of finding the "worst control law" [1], namely the switching signal $\sigma(t), t \in [0, T]$, such that

$$J(T) := x_3(T)$$

is maximized at some given time $T > 0$. For the sake of simplicity, we consider a revised version of the previous problem with two rooms instead of three. This leads to a second order positive affine switched system described as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha_\sigma & \alpha_\sigma \\ \alpha_\sigma & -(\alpha_\sigma + \gamma) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and the cost function becomes

$$J(T) := x_2(T).$$

[1] The problem can be reversed, as one may want to establish the lowest possible temperature in each room.

It is not difficult to see that the "worst control" for this problem, namely the strategy that leads to the maximum temperature in the second room, consists in keeping the door closed for some time ($\sigma(t) = 1$ for $t \in [0, t_1]$), in order to heat up the first room, and then open it ($\sigma(t) = 2$ for $t \in [t_1, T]$), in order to release the accumulated heat to the second room. Assume $\alpha_1 = 0$ (perfect isolation between rooms, when the door is closed), $\alpha_2 = 1$ (all the heat is transferred from room 1 to room 2, when the door is open), $\gamma = 1$, and T =1. Then, we have that, starting from $\mathbf{x}(0) = 0$, as far as $\sigma(t) = 1$ (closed door), at any time $t \in [0, t_1)$, we have

$$x_1(t) = t, \quad x_2(t) = 0.$$

By resorting to elementary computations, one can see that if at time $t = t_1$ we switch to $\sigma = 2$, then we get

$$x_2(t - t_1) = f(t - t_1), \quad \text{with} \ \ f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + 3s + 1}\right],$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. The problem solution requires the maximization of $x_2(1 - t_1) = f(1 - t_1)$ with respect to t_1.

We see that switching changes the nature of the thermal system that, by its nature, would not overshoot. In other words, if the system is initialized at $\mathbf{x}(0) = 0$, with a fixed choice of σ, then the evolution of $x_2(t), t \in [0, 1]$, is strictly monotonically increasing.

2.5 Fluid network control

By a fluid model we mean any mathematical model adopted to describe the fluid level in a reservoir. Fluid models are quite similar to thermal ones. They are nonlinear but they can be reasonably approximated by linearized models. In this section we will consider the case when the fluid is stored in various reservoirs and transported from one reservoir to another by means of pipelines. This situation is typical, for instance, of water supply networks.

In a fluid network of this type (see Figure 2.4), the flow in each pipe typically depends on the difference between the levels in the reservoirs. On the other hand, in a fluid network of the open-channel type, the

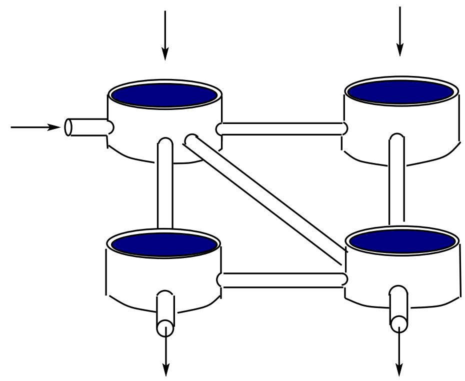

Figure 2.4: A fluid network.

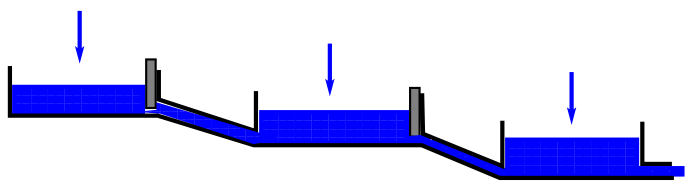

Figure 2.5: A flood control problem.

flow from one reservoir to another depends solely on the level of the upper reservoir (see Figure 2.5).

In this latter case, we can provide a linearized model whose equations, for a system described as in Figure 2.5, are affine of the form

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B,$$

with

$$A_\sigma = \begin{bmatrix} -\alpha_\sigma & 0 & 0 \\ \alpha_\sigma & -\beta_\sigma & 0 \\ 0 & \beta_\sigma & -\gamma \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

where B is the vector representing the natural inflow (e.g., the rain).

Suppose that the flow from the first to the second reservoir, and the flow from the second to the third reservoir are controlled by sluice gates, both of them having three positions, namely fully open, partially open or closed. Any intermediate position could be considered as well, but for simplicity we confine ourselves to this simple case.

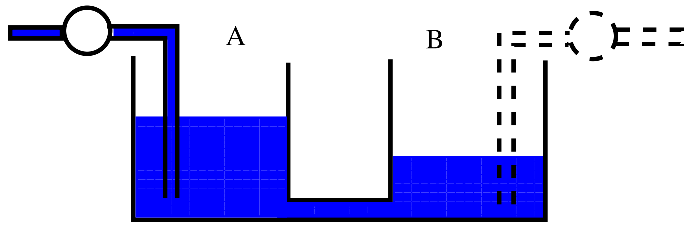

Figure 2.6: The well emptying problem.

We consider the problem of emergency emptying the reservoirs in case of a flood. When an abnormal rainfall occurs, a typical decision to take is how to manage the outflow of the system. For instance, if the upper reservoir has reached a dangerous level, one can control it by opening the first valve, and hence transferring the fluid to the downstream reservoir.

The emergency emptying strategy can be formulated as an optimal control problem by considering the following linear integral cost function:

$$\int_0^{+\infty} \mathbf{d}^\top \mathbf{x}(t) dt, \tag{2.10}$$

where $\mathbf{d}$ is a positive vector, penalizing each of the reservoir levels.

It is obvious that if we take $\mathbf{d} = [1\ 1\ 1]^\top$, the optimal strategy is the full opening (the choice of σ corresponding to all sluice gates completely open), because in this way we maximize the outflow. However, if the weights are taken in increasing order, i.e. $[\mathbf{d}]_1 < [\mathbf{d}]_2 < [\mathbf{d}]_3$, then, in general, it is not convenient to keep the gates constantly open, because this could cause undesirable flooding conditions to the downstream reservoirs. This type of optimal control problem will be considered later for the HIV therapy model.

Consider now the problem of fast emptying two wells that are connected by a pipe, as in Figure 2.6. We assume that only one of the pumps in A and B can be active. Note that this model can also be interpreted as a simplified mutating virus problem with two genotypes, each of them affected by one of the therapies, either A or B. A natural question one may pose is whether it is more effective to switch fast between the two configurations (assuming that switching from A to B

and conversely requires a negligible time) or to dwell for a long time in anyone of the two configurations before switching.

The model in this case takes the autonomous form: $\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t)$, with

$$A_\sigma = \begin{bmatrix} -(\alpha + \beta_1(\sigma - 1)) & \alpha \\ \alpha & -(\alpha + \beta_2(2 - \sigma)) \end{bmatrix},$$

and $\sigma = 1, 2$. For simplicity, assume $\beta_1 = \beta_2 = \beta$, and embed the previous system in the associated positive system

$$\dot{\mathbf{x}}(t) = A(u(t))\mathbf{x}(t) = (A_1(1 - u(t)) + A_2 u(t))\,\mathbf{x}(t), \tag{2.11}$$

where $u(t) \in [0, 1]$.

Assuming, temporarily, that $u(t)$ takes the constant value $\bar{u}$, let us write the expression of the dominant eigenvalue of the matrix $A(\bar{u})$, i.e. the Frobenius eigenvalue:

$$\lambda_F(\bar{u}) = -\frac{2\alpha + \beta}{2} + \sqrt{\frac{(2\alpha + \beta)^2 - 4\left(\alpha\beta + \bar{u}(1 - \bar{u})\beta^2\right)}{4}}.$$

The eigenvalue $\lambda_F(\bar{u})$ in this case is a convex function for $\bar{u} \in [0, 1]$, and it has a minimum for $\bar{u} = 1/2$, that is

$$\bar{\lambda}_F = -\frac{\beta}{2}.$$

So, if $u(t)$ were constant, then the highest convergence speed would be achieved for $\bar{u} = 1/2$, the value at which the Frobenius (dominant) eigenvalue achieves its minimum value $\bar{\lambda}_F$. Since keeping $\bar{u} = 1/2$ constant is not compatible with the switching rule $\sigma = \bar{u} + 1 \in \{1, 2\}$, the next question is whether this convergence speed can be ensured by means of a switching strategy.

To this aim let us consider the left Frobenius eigenvector $\mathbf{z}_F$ of $A(1/2)$, associated with the (minimal) Frobenius eigenvalue $\bar{\lambda}_F = -\beta/2$, that is $\mathbf{z}_F = [1\ 1]^\top$, and introduce the linear copositive function (see Chapter 3) $V(\mathbf{x}) = \mathbf{z}_F^\top \mathbf{x} = x_1 + x_2$. Consider the derivative of $V(\mathbf{x})$ along the trajectories of system (2.11) for $u = \bar{u}$:

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= \mathbf{z}_F^\top A(\bar{u})\mathbf{x} = \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -(\alpha + \beta\bar{u}) & \alpha \\ \alpha & -(\alpha + \beta(1 - \bar{u})) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= -\beta[\bar{u}\ \ 1 - \bar{u}]\mathbf{x}.
\end{aligned}$$

The value of $\dot{V}(\mathbf{x})$ as $\bar{u}$ varies in $[0,1]$ is necessarily less than or equal to the value of $\dot{V}(\mathbf{x})$ we get for $\bar{u} = 1/2$. Indeed, for any $\mathbf{x} > 0$, the derivative $\dot{V}(\mathbf{x})$ is an affine function of $\bar{u}$, and therefore the minimum of $\dot{V}(\mathbf{x})$ with respect to $\bar{u} \in [0,1]$ is achieved at one of the extrema, and therefore if we evaluate the derivative of $V(\mathbf{x})$ along the trajectories of the switched system we get

$$\min_{\sigma \in \{1,2\}} \dot{V}(\mathbf{x}) = \min_{0 \leq \bar{u} \leq 1} \dot{V}(\mathbf{x}) \leq \mathbf{z}_F^\top A(1/2)\mathbf{x} = \bar{\lambda}_F \mathbf{z}_F^\top \mathbf{x} = \bar{\lambda}_F V(\mathbf{x}) < 0.$$

Since $\dot{V}(\mathbf{x}) \leq \lambda_F V(\mathbf{x})$ implies $V(\mathbf{x}(t)) \leq e^{\lambda_F t} V(\mathbf{x}(0))$ (see Blanchini and Miani [2008]), this means that the switching strategy

$$\bar{\sigma}(\mathbf{x}) \in \arg \min_{\sigma \in \{1,2\}} \dot{V}(\mathbf{x}),$$

ensures at least the same convergence speed as the one associated with the "optimal eigenvalue" $\bar{\lambda}_F = -\beta/2$.

It is clear that the slowest convergence speed (worst case) is the one associated with either $\sigma = 1$ or $\sigma = 2$, namely, when only one of the two pumps keeps working and the other does not. In terms of eigenvalues, the worst convergence speed is associated with any of the Frobenius eigenvalues $\lambda_F(0) = \lambda_F(1) < 0$. This can be seen by considering the problem of maximizing the derivative $\dot{V}(\mathbf{x}) = \mathbf{z}_F^\top A(\bar{u})\mathbf{x}$ with respect to $\bar{u}$. The maximum is achieved either for $\bar{u} = 0$ or $\bar{u} = 1$. This basically means that the worst strategy is that of starting emptying the well with the lower level, and keeping the pump in the same position for ever.

Moving back to the optimal strategy

$$\sigma(t) \in \arg \min_{i \in \{1,2\}} \mathbf{z}_F^\top A_i \mathbf{x}(t),$$

the corresponding trajectory converges to a sliding mode, Utkin et al. [1999], as shown in Figure 2.7.

A legitimate question is whether this is a general result for positive switched systems, namely whether the procedure described here of minimizing the Frobenius eigenvalue and adopting the copositive linear function associated with the "optimized eigenvalue" always works well? We will show in the next example that this is not the case.

So far we have considered the case when the switching law is a control action. However, there are some interesting problems in which

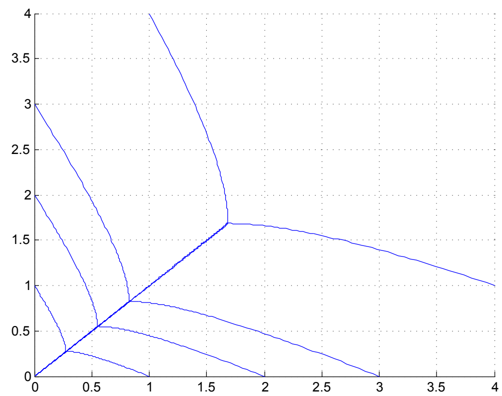

Figure 2.7: The trajectories for the well emptying problem.

the configuration cannot be chosen, i.e. σ is an arbitrary switching signal. Consider a network described as in Figure 2.4, in which the connections between reservoirs can change. We want to determine the worst case effect of a natural precipitation.

A problem of interest in fluid networks with switching configurations is the effect of a permanent flow such as rain or a flood. Consider the positive switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B\mathbf{d}(t),$$

where A_i are Metzler matrices and $\mathbf{d}$ is a positive and bounded unknown disturbance, i.e. $\|\mathbf{d}(t)\| \leq \bar{\mathbf{d}}$ for every $t \geq 0$, and B is a positive matrix. Let us assume that we take some output measurement on the system:

$$\mathbf{y}(t) = C\mathbf{x}(t),$$

with C a positive matrix. Also, suppose for simplicity, that $\mathbf{y}$ and $\mathbf{d}$ are both scalar (and hence in the following simply denoted by y and d). Then the question we investigate is the following one: Assuming $\mathbf{x}(0) = 0$, what is the worst output given the bound on the input, for all possible switching signals σ, i.e.,

$$\bar{y} = \sup_{t \geq 0, \sigma, |d| \leq \bar{d}} |y(t)|?$$

This is given by $\bar{y} = \mu\bar{d}$, where μ is the worst $\mathcal{L}_\infty$–induced norm (see Chapter 4), i.e.,

$$\mu := \sup_{t\geq 0, \sigma, |d|\leq 1} |y(t)|.$$

In order to determine the value of μ, or at least compute an upper bound on it, one can simply notice that for any switching signal $\sigma(t)$ and for $\mathbf{x}(0) = 0$, the following holds. Denote by $\mathbf{x}^+(t)$ and $\mathbf{x}^-(t)$ the solutions corresponding to the constant disturbances $d(t) = -1$ and $d(t) = +1$, for every $t \geq 0$, respectively. For any solution $\mathbf{x}(t)$ corresponding to some disturbance satisfying $-1 \leq d(t) \leq 1$, and every $t \geq 0$, we get

$$\mathbf{x}^-(t) \leq \mathbf{x}(t) \leq \mathbf{x}^+(t).$$

Indeed, $\mathbf{z}(t) = \mathbf{x}^+(t) - \mathbf{x}(t)$ is the solution corresponding to the nonnegative input $1 - d(t)$, while $\mathbf{w}(t) = \mathbf{x}(t) - \mathbf{x}^-(t)$ is the solution corresponding to the nonnegative input $d(t) - 1$. Both $\mathbf{z}(t)$ and $\mathbf{w}(t)$ are nonnegative solutions. This basically means that this problem can be solved by limiting our attention to the positive orthant, by considering only the upper value $d = 1$.

But then, using the monotonicity of the system, we can come to the conclusion that, for positive systems with positive outputs, the "worst case input", namely the input that provides the output of largest maximum modulus, is a constant signal equal to the upper bound $\bar{d}$ (or to the lower bound $-\bar{d}$). So, in the case of fluid systems (Figure 2.5), if the input is bounded as $0 \leq d(t) \leq \bar{d}$, the worst case input in terms of $\mathcal{L}_\infty$-induced norm is the constant one $d(t) = \bar{d}$. This issue about the $\mathcal{L}_\infty$-induced norm will be reconsidered in Section 4.3.

2.6 Congestion control and queueing models

Congestion control. Consider a traffic control problem in a junction, as sketched in Figure 2.8. There are three main roads (A, B and C) converging into a "triangular connection" governed by traffic lights. Three buffer variables, x_1, x_2 and x_3, represent the vehicles waiting at the three traffic lights inside the triangular loop. We assume that there are three symmetric configurations as far as the states of the 6 traffic lights are concerned. In the first configuration, described in Figure 2.8,

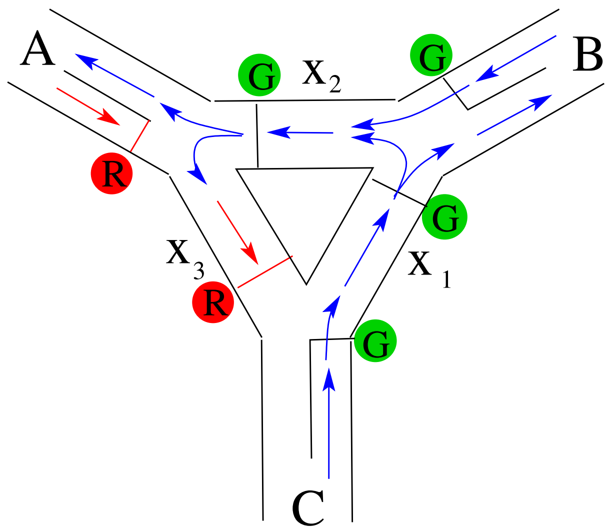

Figure 2.8: The traffic control problem.

we assume that traffic lights corresponding to x_1, x_2, B and C are green, while the ones corresponding to x_3 and A are red. Accordingly,

- x_3 increases proportionally ($\beta > 0$) to x_2;
- x_2 remains approximately constant, receiving inflow from B and buffer x_1, meanwhile giving outflow to A and buffer x_3.
- x_1 decays exponentially ($-\gamma < 0$), since the inflow from C goes all to x_2 and B. The exponential decay takes into account "approximately" the initial transient due to the traffic light switching.

The other two configurations are obtained by a circular rotation of x_1, x_2 and x_3 (as well as of A, B and C).

We model this problem by means of the switched system:

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t),$$

where the switching signal σ, regarded as a control variable, must select one of the three subsystems characterized by the matrices:

$$A_1 = \begin{bmatrix} -\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 & \beta \\ 0 & -\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & -\gamma \end{bmatrix}.$$

In the following we will assume $\gamma = 1$ and $\beta = 1$. This model, proposed in Blanchini et al. [2012], is quite interesting since it shows that the policy of minimizing the Frobenius eigenvalue and adopting a switching strategy based on the left Frobenius eigenvector does not work. Indeed no convex Hurwitz combination of the three matrices can be found, since the characteristic polynomial of the matrix $\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$, $\alpha \in \mathcal{A}_3$, the simplex of vectors in $\mathbb{R}^3_+$ summing up to 1, turns out to be

$$p(s, \alpha) = s^3 + (\alpha_1 + \alpha_2 + \alpha_3)s^2 + (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)s.$$

So, $p(s, \alpha)$ is not a Hurwitz polynomial for any choice of $\alpha \in \mathcal{A}_3$, and hence there are no Hurwitz convex combinations in the convex hull. Indeed the Frobenius eigenvalue is $\lambda_F = 0$ for any $\alpha_1, \alpha_2, \alpha_3$, thus convergence cannot be guaranteed (see Chapter 5).

However, the matrix product $e^{A_1 T} e^{A_2 T} e^{A_3 T}$ is Schur for $T = 1$ (the dominant eigenvalue is ≈ 0.69). So, the periodic switching law

$$\sigma(t) = \begin{cases} 3, & t \in [3k, 3k+1); \\ 2, & t \in [3k+1, 3k+2); \quad k \in \mathbb{Z}_+, \qquad\qquad t \geq 0, \\ 1, & t \in [3k+2, 3k+3); \end{cases}$$

drives the systems state to 0 from any positive initial condition (see Blanchini et al. [2012] for more details).

This fact shows an interesting difference with respect to the fluid dynamics case. In the fluid dynamics case, we could find a Hurwitz matrix in the convex hull of the system matrices A_i. Based on this matrix we have been able to find a stabilizing switching strategy by considering a Lyapunov function associated with its Frobenius eigenvector. In the traffic case, a Hurwitz matrix in the convex hull of the matrices does not exist. For any fixed value of $\alpha \in \mathcal{A}_3$, the corresponding system state would not converge to 0, hence no constant strategies

$$\mathbf{u}(t) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, \qquad t \geq 0,$$

would be effective in this case as in the fluid model. This strategy would correspond to a "sliding mode" for the system, namely a constant strategy for the associated system

$$\dot{\mathbf{x}}(t) = (A_1[\mathbf{u}]_3] + A_2[\mathbf{u}]_3] + A_3[\mathbf{u}]_3])\,\mathbf{x}(t).$$

Yet, an asymptotically stabilizing switching strategy exists.

Note that in the proposed stabilizing periodic switching law the "red" is imposed according to the circular order 3, 2, 1, 3, 2, 1 .... It may be surprising to notice that if the order is changed, not only the system performance deteriorates, but the system may even become unstable. Indeed, $e^{A_3}e^{A_2}e^{A_1}$ is an unstable matrix with spectral radius ≈ 1.90, which means that the switching order is fundamental and the order 1, 2, 3, 1, 2, 3 ... is unsuitable. A simple explanation is that switching the red light from 3 to 2 allows for a "fast recovery" from the congestion for x_3 (due to the exponential decay), while switching the red from 3 to 1 would leave such congestion unchanged. Also, from an intuitive point of view, in this context chattering, namely an infinitely fast switching among the various traffic light configurations, would be catastrophic, while it is quite predictable that to tackle this problem we must "dwell" on each configuration for a sufficiently long time. We will come back to this topic in Chapter 5.

We complete the example by considering the effect of a constant input (the incoming traffic) and hence by considering the positive affine switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B,$$

where $B = \mathbf{1}_3$, $\sigma(t) \in \{1, 2, 3\}$ at every time $t \geq 0$, and the matrices A_1, A_2 and A_3 are described as before, with $\gamma = 1$ and $\beta = 1.1$. It turns out that, with these values, $e^{A_1 T}e^{A_2 T}e^{A_3 T}$ is Schur for $T > 0.19$. This means that under a periodic strategy with $T > 0.19$, the system converges to a periodic trajectory as shown in Figure 2.9.

Note that it is possible to optimize T in order to achieve a strategy that reduces the buffer levels associated with the periodic trajectory as much as possible (see Blanchini et al. [2012] for details).

Finally, as we will see in Chapter 5, this example shows that the existence of a Hurwitz convex combination is sufficient but not necessary for stabilizability, Blanchini et al. [2012].

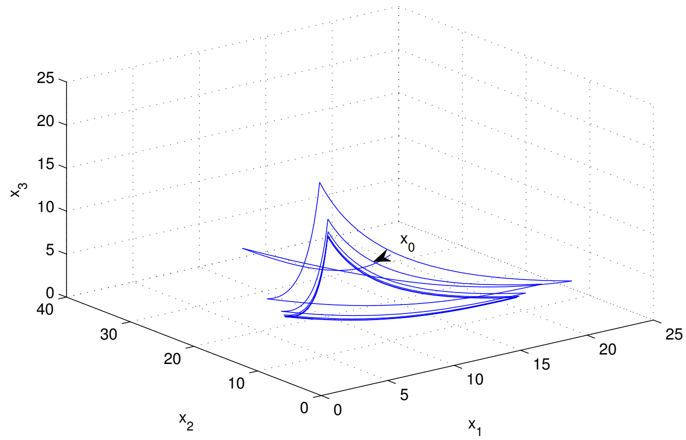

Figure 2.9: State trajectory corresponding to $T = 2.1$ and $x(0) = [10\ 10\ 10]^\top$.

Queueing model 1. Traffic congestion models are, in a broad sense, a special class of queueing systems that can be encountered in many problems, including production and data processing.

Consider the case of a production line of the type depicted in Figure 2.10. We assume that there are six processes. Raw materials or

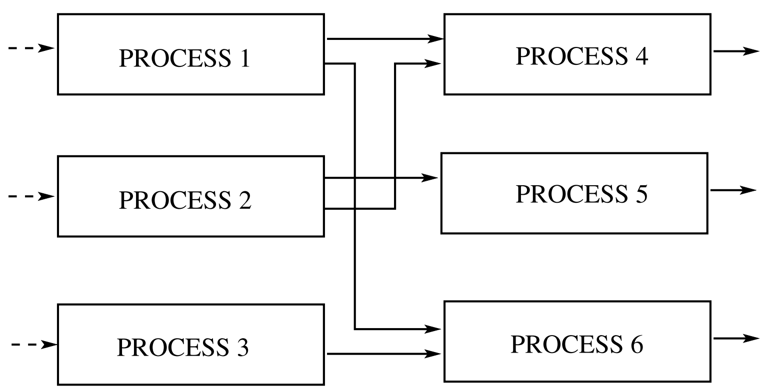

Figure 2.10: The queueing system.

data enter in processes 1, 2, 3, and once the processing is over they are transferred to one of the processes 4, 5, and 6. We assume that each part entering one process remains in the process for an integer time τ_i, necessary for the production. At the end, the process outcomes either are accumulated, or transferred to another machine, or leave the system.

Each process is associated with a delay-and-store block of the form

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \delta \end{bmatrix},$$

where

- $\delta = 1$ if the processed parts are not removed and consequently they remain accumulated at the end of the queue;
- $\delta = 0$ if the processed parts at the end of the queue are removed and transferred elsewhere or to the external system.

If the transferring to another processing phase or to the external environment (in case of finished products) is managed by agents, it is reasonable to adopt a *discrete-time positive switched* model

$$\mathbf{x}(k+1) = A_{\sigma(k)}\mathbf{x}(k), \quad k \in \mathbb{Z}_+, \tag{2.12}$$

where for every value i taken by the switching sequence $\sigma(k), k \in \mathbb{Z}_+$, the matrix A_i is nonnegative. In case of a two-machine system, we would have, for instance a matrix of the form

$$A_\sigma = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \sigma - 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 2-\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2-\sigma \end{array}\right]$$

with $\sigma \in \{1, 2\}$. We assume that the processing machines work automatically and an operator can either transfer the processed goods from the first machine to the second one, or remove the processed goods from the second machine. So,

- $\sigma = 0$ corresponds to the case when the operator is transferring the parts from the first process to the second one.
- $\sigma = 1$ corresponds to the operator removing the final goods from the second process.

More sophisticated models with several machines and transfer operators are clearly possible. Note that if the initial condition is a vector whose entries have integer values, the state-vector is integer-valued at every time $t \geq 0$, which can be useful when applying the model to a production context.

Queueing model 2. Another popular way to model queueing systems is by means of Markov chains. In this case one has to deal with discrete-time positive switched systems described as in (2.12), whose matrices A_i, $i \in \{1, 2, \ldots, M\}$, are transition probability matrices, and hence satisfy $\mathbf{1}_n^\top A_i = \mathbf{1}_n^\top$, $\forall i$.

Consider the typical discrete-time Markov chain representing the buffer of a process with random arrivals. In a unitary time step a maximum of 2 parts can arrive and we denote by $p_i, i = 0, 1, 2$, the probability that i parts arrive in the queue. This implies that $\sum_{i=0}^{2} p_i = 1$. Also, we assume that the buffer saturates when it contains 5 parts, and hence assume as state variables

$$x_i = \text{Probability } \{i - 1 \text{ objects are in the queue}\}, \qquad i = 1, 2, 3, 4, 5.$$

If the unitary time step is the time necessary for the operator to process one part, when the operator is active the system can be described as follows:

$$\mathbf{x}(k+1) = A_{act}\mathbf{x}(k),$$

with

$$A_{act} = \begin{bmatrix} p_0 + p_1 & p_0 & 0 & 0 & 0 & 0 \\ p_2 & p_1 & p_0 & 0 & 0 & 0 \\ 0 & p_2 & p_1 & p_0 & 0 & 0 \\ 0 & 0 & p_2 & p_1 & p_0 & 0 \\ 0 & 0 & 0 & p_2 & p_1 & p_0 \\ 0 & 0 & 0 & 0 & p_2 & p_1 + p_2 \end{bmatrix}.$$

On the other hand, when the operator is idle, the Markov chain takes the form

$$\mathbf{x}(k+1) = A_{idl}\mathbf{x}(k),$$

with

$$A_{idl} = \begin{bmatrix} p_0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 & 0 & 0 \\ p_2 & p_1 & p_0 & 0 & 0 & 0 \\ 0 & p_2 & p_1 & p_0 & 0 & 0 \\ 0 & 0 & p_2 & p_1 & p_0 & 0 \\ 0 & 0 & 0 & p_2 & p_1+p_2 & 1 \end{bmatrix}.$$

This whole system can be described as in (2.12), with $\sigma \in \{1, 2\}$, $A_1 = A_{act}$ and $A_2 = A_{idl}$. One can switch between the two matrices A_{idl} and A_{act} in order to keep the buffer at a prescribed level. This problem can be formulated also in continuous-time.

2.7 Boundedness properties of positive biochemical systems

In general, the properties of positive nonlinear systems are rather different from those of positive linear systems. As previously remarked, in the non-linear case, a positive system is not necessarily monotone and vice versa. If we linearize a positive (not monotone) system around a positive equilibrium, we may find a Jacobian matrix which is not a Metzler matrix.

Nonetheless some fundamental problems for positive nonlinear systems can be tackled by resorting to the theory of positive (and linear) switched systems. For instance, this is the case for the *boundedness* of the solutions of certain biochemical systems, see Blanchini and Giordano [2014].
A typical biochemical system takes the form

$$\dot{\mathbf{x}}(t) = Sg(\mathbf{x}) + \mathbf{g}_0,$$

where S is a constant matrix (for instance the stoichiometric matrix) g is a nonlinear function, typically representing the reaction rate, and $\mathbf{g}_0$ is a constant input vector. Consider the network represented in Figure 2.11.

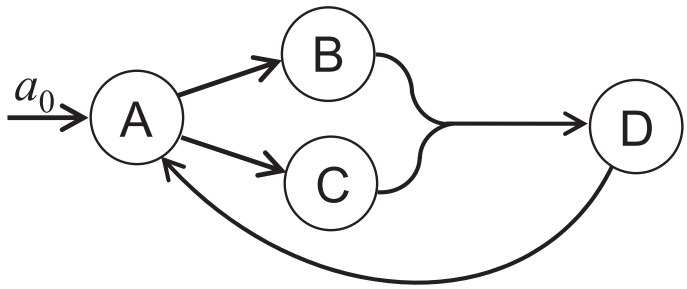

Figure 2.11: A chemical network.

The network is described by the following nonlinear differential equations:

$$\begin{cases} \dot{a}(t) = a_0 - g_a(a(t)) - g_a^*(a(t)) + g_d(d(t)) \\ \dot{b}(t) = g_a(a(t)) - g_{bc}(b(t), c(t)) \\ \dot{c}(t) = g_a^*(a(t)) - g_{bc}(b(t), c(t)) \\ \dot{d}(t) = g_{bc}(b(t), c(t)) - g_d(d(t)). \end{cases} \tag{2.13}$$

The functions labelled by g are the reaction rates. These functions are smooth and strictly increasing in each argument. They are zero if and only if at least one of the arguments is zero, otherwise they are positive. The term a_0 is a constant input. A typical choice of the functions is $g_a(a) = k_a a$ $g_a^*(a) = k_a^* a$, $g_{bc}(b, c) = k_{bc} bc$, $g_d(d) = k_d d$, known as mass action kinetics rate functions, see Feinberg [1987], Del Vecchio and Murray [2014].

Following Blanchini and Giordano [2014], we can write the equations as follows (where we have removed the time variable, to simplify the expressions):

$$\begin{cases} \dot{a} = a_0 - \frac{g_a(a)}{a} a - \frac{g_a^*(a)}{a} a + \frac{g_d(d)}{d} d, \\ \dot{b} = \frac{g_a(a)}{a} a - \frac{g_{bc}(b,c)}{b} b, \\ \dot{c} = \frac{g_a^*(a)}{a} a - \frac{g_{bc}(b,c)}{c} c, \\ \dot{d} = \frac{g_{bc}(b,c)}{2b} b + \frac{g_{bc}(b,c)}{2c} c - \frac{g_d(d)}{d} d. \end{cases}$$

Set $\alpha := g_a/a$, $\beta := g_a^*/a$, $\gamma := g_{bc}/b$, $\delta := g_d/d$, $\varepsilon := g_{bc}/c$, and assume $g_0 := [a_0\ 0\ 0\ 0]^\top$.

We can rewrite the previous system as

$$\dot{\mathbf{x}}(t) = M(t)\mathbf{x}(t) + \mathbf{g}_0, \tag{2.14}$$

with $\mathbf{g}_0 = [a_0\ 0\ 0\ 0]^T$ and

$$M(t) = \begin{bmatrix} -(\alpha(t)+\beta(t)) & 0 & 0 & \delta(t) \\ \alpha(t) & -\gamma(t) & 0 & 0 \\ \beta(t) & 0 & -\varepsilon(t) & 0 \\ 0 & \frac{\gamma(t)}{2} & \frac{\varepsilon(t)}{2} & -\delta(t) \end{bmatrix}, \tag{2.15}$$

for a suitable choice of the time-varying functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\epsilon(t)$. If we can assume that these coefficients are bounded as[2]

$$0 < \nu \leq \alpha, \beta, \gamma, \delta, \epsilon \leq \mu$$

for some positive constants ν and μ, then we can define the set $\mathcal{M}$ of all matrices having the form (2.15), with parameters $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\epsilon(t)$ subject to these bounds. Any solution of the original nonlinear system (2.13) is also a solution of the positive linear differential inclusion (2.14) with $M(t) \in \mathcal{M}$, see (Blanchini and Giordano [2014], Blanchini and Miani [2008]). The converse, however, is not true: the set of all possible solutions of the differential inclusion is a superset of the set of solutions of the nonlinear system (2.13).

Then the boundedness of the original system trajectories is ensured if the positive differential inclusion is exponentially stable. On the other hand, an important result due to Molchanov and Pyatnitskii [1986] states that the stability of a linear switched system is equivalent to the stability of the corresponding (relaxed) linear differential inclusion (2.14) obtained by allowing the parameters to take any value in the given intervals.

It can be shown, see Blanchini and Giordano [2014], that in this example the positive switched system is exponentially stable for every choice of $\alpha, \beta, \gamma, \delta, \epsilon \in \{\nu, \mu\}$. This implies the exponential stability for arbitrary $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\epsilon(t)$ varying in the interval $[\nu, \mu]$ and hence the global boundedness of the solutions of nonlinear biochemical systems.

[2] In view of the mean value theorem, this is equivalent to imposing bounds on the derivatives.

2.8 Convergence of algorithms

There are several algorithms that can be viewed as positive switched systems.

Synchronization. Consider the consensus problem in which there are some agents in a network that can communicate to each other according to a given topology. A typical case is the clock synchronization problem, Dörfler and Bullo [2014].

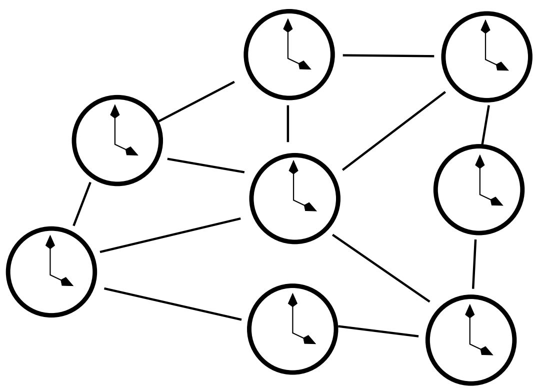

Figure 2.12: The clock synchronization.

Assuming pairwise communication, each pair of clocks, i and j, when communicating, update their time indications as follows:

$$\begin{aligned}\tau_i(k+1) &= \tau_i(k) - \gamma(\tau_i(k) - \tau_j(k)),\\ \tau_j(k+1) &= \tau_j(k) - \gamma(\tau_j(k) - \tau_i(k)),\end{aligned}$$

where $0 < \gamma < 1$. Therefore the average of their time indications remains the same, but the difference between the two indications decreases.

Consider the (undirected) graph that describes all the communications among n agents/clocks: there is an edge for each pair of clocks that communicate. Let B be the incidence matrix of such a graph. This is a matrix whose rows are associated with nodes and whose columns are associated with arcs. Each column of B has only two nonzero entries: -1 in the jth position and $+1$ in the ith position if i and j communicate (the other way round is possible and makes no difference). It can be shown that the overall dynamics can be written as

$$\tau(k+1) = (I - \gamma BB^\top)\tau(k) = (I - \gamma L)\tau(k).$$

The matrix $L = BB^\top$ is known as Laplacian matrix, Mohar [1991]. It is symmetric and $-L$ is Metzler. Its element $L_{ij}, i \neq j$, is -1 if j communicates with i, and 0 otherwise. Conversely, the diagonal elements L_{ii} are positive and equal to the number of clocks communicating with i. This system is positive if

$$\gamma \leq \frac{1}{\max_i [L]_{ii}}$$

and we assume that this is the case.

If we assume that the communication topology may vary with time, instead of being a fixed one, we can obtain the following discrete-time positive switched system describing the clock synchronization problem:

$$\tau(k+1) = (I - \gamma L_{\sigma(k)})\tau(k),$$

where σ takes values in the set of possible communication configurations. Note that the overall system has as equilibrium points all the vectors with identical entries. So, in particular, $\bar{\tau} = \mathbf{1}$ is an equilibrium point and it represents the synchronization (agreement) case.

If the time interval between two consecutive communications (the unit time in the previous discrete-time model) is small enough, we can assume a continuous-time model of the form

$$\dot{\tau}(t) = -\delta L_{\sigma(t)}\tau(t).$$

where $L_{\sigma(t)}$ is the same matrix involved in the discrete-time model and δ is a suitable positive constant. The natural question is whether the clocks asymptotically reach such an agreement, under arbitrary switching, provided that the network remains connected in each configuration.

The answer is positive: the times displayed by the clocks converge to their initial average values. Indeed one has to consider the fact that

$$\mathbf{1}_n^\top L_{\sigma(t)} = 0.$$

If we introduce $Ave(\tau) := \mathbf{1}_n^\top \tau / n$, the average value of the entries of τ, namely the average clock time, we have $\frac{d}{dt}\ Ave(\tau(t)) =$

$-\delta \mathbf{1}_n^\top L_{\sigma(t)}\tau(t)/n = 0$, hence the average is constant, and the state moves on the linear variety

$$Ave(\tau(t)) = Ave(\tau(0)) = const.$$

Consider the quadratic positive definite function $V(\tau) := \tau^\top\tau/2$. By the properties of the matrix $L_{\sigma(t)}$, the derivative of $V(\tau(t))$ along the system trajectories is

$$\dot{V}(\tau(t)) = -\delta\tau(t)^\top L_{\sigma(t)}\tau(t) \leq 0.$$

Since $\dot{V}(\tau(t))$ is negative semi-definite, we must resort to Lasalle's invariance principle, that we can apply in view of the fact that the Lyapunov function is non increasing, and hence the state is bounded in a set of the form $\{\tau : V(\tau) \leq \kappa\}$. The boundedness of the state implies that the state trajectory $\tau(t)$ necessarily converges to the set of points where $\dot{V}(\tau) = 0$. If the graph is connected for each σ, the only vectors τ for which $\dot{V}(\tau) = 0$ are those with equal components, i.e. $\tau = \lambda\mathbf{1}$, and hence $\tau(t)$ asymptotically converges, under arbitrary switching, to the vector $\bar{\tau}$ with all equal components satisfying $Ave(\bar{\tau}) = Ave(\tau(0))$.

Load balancing. An interesting application of positive switched systems is the Foschini-Miljanic algorithm for power regulation of transmitters, described in Zappavigna et al. [2012]. Essentially, each transmitter is required to regulate its transmitted power based on the presence of other transmitters. Precisely, if other transmitters are active, the interference noise increases and hence the transmitter should increase its power.

Upon denoting by p_i the power of the ith transmitter, the problem can be formalized as a continuous-time positive system. Indeed, the updating equation of the ith transmitter is

$$\dot{p}_i(t) = \kappa_i \left[-p_i(t) + \gamma_i \left(\sum_{j \in \mathcal{N}_i} \frac{g_{ij}}{g_{ii}} p_j(t) + \frac{\nu_i}{g_{ii}} \right) \right],$$

where $\kappa_i > 0$ is a proportionality constant, the terms g_{ij}, for all i and j, represent the proportionality coefficients between the amount of noise affecting the transmission i and the power p_j generated by the transmitter j, ν_i is the natural channel noise, $\mathcal{N}_i$ denotes all the other nodes different from i that interact with node i, and γ_i represents the SINR (Signal-to-Interference-and-Noise-Ratio), assigned to ensure a given quality of service. Notice that the equilibrium point is such that

$$\frac{p_i g_{ii}}{\sum_{j \in \mathcal{N}_i} g_{ij} p_j + \nu_i} = \gamma_i.$$

Of course, the equilibrium point needs to be exponentially stable. The system can be written as the positive system

$$\dot{\mathbf{x}}(t) = K[-\mathbf{x}(t) + C\mathbf{x}(t)] + \mathbf{r} = A\mathbf{x}(t) + \mathbf{r},$$

where $\mathbf{x}$ is the vector collecting the variables p_i, C is a nonnegative matrix depending on the coefficients g_{ij}, K is a positive diagonal matrix and $\mathbf{r}$ is a constant vector. Notice that $A = K(-I + C)$ is a Metzler matrix. This is an ideal model, since in practice two problems typically occur, see Zappavigna et al. [2012]:

- there are delays in the network;
- the network topology can switch.

Therefore, a more realistic model is (see Zappavigna et al. [2012], for the details)

$$\dot{\mathbf{x}}(t) = K[-\mathbf{x}(t) + \sum_{k=1}^{d} C_{\sigma(t)k}\mathbf{x}(t - \tau_k(t))] + \mathbf{r},$$

where $\sigma(t) \in \{1, 2, \dots, M\}$ represents the network configuration at time t (among the M possible), C_{ik} are nonnegative matrices for all $i \in \{1, 2, \dots, M\}$ and $k \in \{1, 2, \dots, d\}$, and $\tau_k(t)$, $k = 1, 2, \dots, d$, are (possibly time-varying) time delay coefficients. If we take all delays equal to zero, and set

$$B_i := \sum_{k=1}^{d} C_{ik},$$

then the model becomes a delay-free positive switched system $\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t)$, with $A_i = K(-I + B_i)$, $i = 1, 2, \dots, M$. In Zappavigna et al. [2012] it has been shown that if this delay-free positive switched system satisfies some stability condition under arbitrary switching, then delays do not affect its stability.

3

Stability

In this chapter we revise the stability properties of a *continuous-time positive switched system*, described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{3.1}$$

where $\mathbf{x}(t)$ denotes the value of the n-dimensional state variable at time t, $\sigma(t)$ is a (right-continuous and piece-wise constant) switching signal, mapping from $\mathbb{R}_+$ into the finite set $\{1, ..., M\}$, and A_i, $i \in \{1, ..., M\}$, are Metzler matrices.

It is far from the purpose of this chapter entering in too deep mathematical details. Yet, it is worth pointing out that a switched system is discontinuous and defining its solutions require some attention. The literature on discontinuous systems often resorts to the notion of differential inclusion, see Aubin [1991].

If the switching points t_k are all isolated, so that σ is constant in $[t_k, t_{k+1})$, then it is quite clear how to define the solution of (3.1). If we impose a dwell time, i.e. a minimum value $\tau > 0$ for the length of the interval $[t_k, t_{k+1})$, then no problem arises in the definition of a solution.

On the contrary, if we wish to consider solutions for which the switching points are not isolated, then the situation is different. The best way to define the solutions is to embed the system in the corresponding differential inclusion (see Remark 2.1 in Section 2.3), namely to consider the bilinear system

$$\dot{\mathbf{x}}(t) = \left(\sum_{i=1}^{M} A_i[\mathbf{u}(t)]_i\right)\mathbf{x}(t), \tag{3.2}$$

where the function $\mathbf{u}(t) \triangleq [[\mathbf{u}(t)]_1\ [\mathbf{u}(t)]_2\ \ldots\ [\mathbf{u}(t)]_M]^\top, t \in \mathbb{R}_+$, belongs to $\mathcal{U}_{li}^M$, the class of locally integrable M-dimensional vector functions taking values in the simplex $\mathcal{A}_M$, and claim that any absolutely continuous function that satisfies the bilinear system almost everywhere is a solution of the switched system, Aubin [1991]. This fact is consistent with the property that any solution of the bilinear system can be arbitrarily closely approximated by a solution of the switched system if we take τ small.

For instance the switched system $\dot{x}(t) = A_{\sigma(t)}x(t)$, with $A_{\sigma(t)} \in \{-1, 1\}$ for every $t \in \mathbb{R}_+$, admits the constant solutions $x(t) = c$, because at each time t its derivative $\dot{x} = 0$ is inside the interval $[-c, c]$, although neither of the extremal points has zero derivative.

We are not going to discuss this matter further, but we refer the interested Reader to specialized literature, e.g. Liberzon [2003], Sun and Ge [2005], Shorten et al. [2007].

This chapter is probably the most articulated of the whole survey, due to the fact that the majority of the results available in the literature on switched positive linear systems pertain their stability properties. We have tried to make this chapter comprehensive of the main available results, but unavoidably this may have affected its homogeneity, since several different approaches and specific problems have been addressed. In detail, we first investigate uniform exponential stability, corresponding to arbitrary switching functions. Necessary conditions are first presented, then equivalent conditions based on Lyapunov functions are given. Sufficient conditions based on special classes of copositive Lyapunov functions are also proposed. The relationship between uniform exponential stability and convergence to zero of all state trajectories corresponding to periodic switching signals is investigated. This issue is closely related to stability under dwell-time, which is the subject of subsection 3.2. The analysis of stability properties of positive switched systems whose matrices are obtained by means of a state feedback is finally explored.

3.1 Exponential stability of continuous-time positive switched systems

In this section we focus on the uniform exponential stability of the origin as an equilibrium point of (3.1). As it is well-known (see Proposition 2.13 of Sun and Ge [2011]), for switched linear systems the concepts of attractivity, uniform attractivity, asymptotic stability, uniform asymptotic stability, exponential stability and uniform exponential stability are equivalent, and there is no distinction between local and global properties. For this reason, in the sequel we will concentrate on uniform exponential stability, with the understanding that any other form of stability would lead to the same results.

Definition 3.1. The positive switched system (3.1) is said to be *uniformly exponentially stable* if there exist real constants $C > 0$ and $\beta > 0$ such that all the solutions of (3.1) satisfy

$$\|\mathbf{x}(t; \mathbf{x}(0), \sigma)\| \leq C e^{-\beta t} \|\mathbf{x}(0)\|, \tag{3.3}$$

for every $\mathbf{x}(0) \in \mathbb{R}^n_+, t \in \mathbb{R}_+$ and every switching signal $\sigma \in \mathcal{D}_0$.

It is worth noticing that even if uniform exponential stability of positive switched systems is defined by restricting the attention to initial conditions that belong to the positive orthant, it is equivalent to standard uniform exponential stability of (nonpositive) switched systems. Indeed, if condition (3.3) holds for every $\mathbf{x}(0) \in \mathbb{R}^n$ then, a fortiori, it holds true for positive initial conditions. Conversely, if (3.3) holds for positive initial conditions, then for every $\mathbf{x}(0) \in \mathbb{R}^n$ we can always adopt the decomposition

$$\mathbf{x}(0) = \mathbf{x}_+ - \mathbf{x}_-, \qquad \mathbf{x}_+, \mathbf{x}_- \in \mathbb{R}^N_+,$$

where

$$[\mathbf{x}_+]_i := \begin{cases} [\mathbf{x}(0)]_i, & \text{if } [\mathbf{x}(0)]_i > 0; \\ 0, & \text{otherwise;} \end{cases}$$

and

$$[\mathbf{x}_-]_i := \begin{cases} -[\mathbf{x}(0)]_i, & \text{if } [\mathbf{x}(0)]_i < 0; \\ 0, & \text{otherwise.} \end{cases}$$

The linearity and the norm properties allow to say that

$$\begin{aligned} \|\mathbf{x}(t;\mathbf{x}(0),\sigma)\| &= \|\mathbf{x}(t;\mathbf{x}_+,\sigma) - \mathbf{x}(t;\mathbf{x}_-,\sigma)\| \\ &\leq \|\mathbf{x}(t;\mathbf{x}_+,\sigma)\| + \|\mathbf{x}(t;\mathbf{x}_-,\sigma)\| \\ &\leq Ce^{-\beta t}\|\mathbf{x}_+\| + Ce^{-\beta t}\|\mathbf{x}_-\| \leq 2Ce^{-\beta t}\|\mathbf{x}(0)\|. \end{aligned}$$

Therefore (3.3) holds for every $\mathbf{x}(0) \in \mathbb{R}^n$, provided that we replace C with $2C$.

This simple remark allows one to inherit all the results already derived for standard switched systems (see, e.g. Liberzon [2003], Shorten et al. [2007], Sun and Ge [2005, 2011]). First of all, an obvious necessary condition for uniform exponential stability is that all matrices A_i are Hurwitz (since among all switching signals σ, we have to consider the constant ones).

Proposition 3.1. The continuous-time positive switched system (3.1) is uniformly exponentially stable only if each subsystem $\dot{\mathbf{x}}(t) = A_i\mathbf{x}(t), i \in \{1, 2, \ldots, M\}$, is (uniformly) exponentially stable, namely each A_i is Metzler Hurwitz.

Unfortunately, as in the standard nonpositive case, this condition is not sufficient. The following example, given in Mason and Shorten [2006], proves this fact.

Example 3.1. Consider the continuous-time positive switched system (3.1), with $M = 2$, $n = 3$ and

$$\begin{aligned} A_1 &= \begin{bmatrix} -1.1309 & 0.0087 & 0.8499 \\ 0.0222 & -1.0413 & 0.5865 \\ 0.4105 & 0.4817 & -0.8792 \end{bmatrix} \\ A_2 &= \begin{bmatrix} -2.9923 & 1.5069 & 2.9142 \\ 4.0681 & -3.9685 & 1.8570 \\ 0.1072 & 0.0618 & -0.7999 \end{bmatrix}. \end{aligned}$$

Both A_1 and A_2 are Metzler Hurwitz matrices, however the matrix product $e^{A_1 0.5}e^{A_2 0.5}$ has 1.0114 as an eigenvalue and the corresponding eigenvector, say $\mathbf{v}$, is strictly positive. This means that corresponding to the periodic switching signal

$$\sigma(t) := \begin{cases} 1, & \text{for } 0 \leq t < 0.5; \\ 2, & \text{for } 0.5 \leq t < 1; \\ \sigma(t-1), & \text{for } t \geq 1; \end{cases}$$

the state trajectory starting from $\mathbf{x}(0) = \mathbf{v}$ diverges, thus preventing asymptotic stability.

As a matter of fact, the necessary condition given in Proposition 3.1 can been strengthened, since a necessary condition for uniform exponential stability is that all convex combinations of the matrices $A_i, i \in \{1, 2, \ldots, M\}$, are (Metzler and) Hurwitz. To formalize it, we resort to the simplex $\mathcal{A}_M$ (see Chapter 2).

Proposition 3.2. Molchanov and Pyatnitskii [1986], Barabanov [1988, 1993] (see also Liberzon [2003]). The continuous-time positive switched system (3.1) is uniformly exponentially stable only if $A(\alpha)$ is Metzler Hurwitz, for every choice of $\alpha \in \mathcal{A}_M$.

This condition is easily proved to be not sufficient for general (non-positive) switched systems. On the contrary, it was initially conjectured by Mason and Shorten, and independently by David Angeli, that for positive switched systems the Hurwitz property of all convex combinations $A(\alpha), \alpha \in \mathcal{A}_M$, was also sufficient for uniform exponential stability. This conjecture was initially disproved in Gurvits et al. [2007], and later Fainshil, Margaliot and Chigansky provided (see Fainshil et al. [2009]) the following three-dimensional counterexample.

Example 3.2. Consider the continuous-time positive switched system (3.1), with $M = 2$, $n = 3$ and

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 0 & -10 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -10 & 0 & 10 \\ 0 & -10 & 0 \\ 0 & 10 & -1 \end{bmatrix}.$$

All convex combinations of A_1 and A_2 are easily proved to be Hurwitz. However, as shown in Fainshil et al. [2009], the positive switched system is not uniformly exponentially stable.

Remark 3.1. While in the general case the proposed conjecture turned out to be false (except for two-dimensional systems, as we will see later in this chapter), there are classes of systems for which it is true that uniform exponential stability is equivalent to the Hurwitz property of all convex combinations $A(\alpha), \alpha \in \mathcal{A}_M$. For example, certain classes of positive switched systems whose matrices $A_i, i \in \{1, 2, \ldots, M\}$, satisfy $\text{rank}(A_i - A_j) = 1$ for every $i \neq j$, Fornasini and Valcher [2014], or can be seen as the state matrices of uncertain feedback interconnections, Hinrichsen et al. [2003], Son and Hinrichsen [1998, 1996]. $\square$

Remark 3.2. The positive switched system (3.1) is exponentially stable if and only if the associated bilinear system (3.2) is exponentially stable, Molchanov and Pyatnitskii [1986], Barabanov [1988]. $\square$

Remark 3.3. It is important to note, see Molchanov and Pyatnitskii [1986], Barabanov [1988], Blanchini and Miani [2003], that uniform exponential stability of (3.2) is equivalent to uniform exponential stability of its transposed, i.e.

$$\dot{\mathbf{x}}(t) = \left(\sum_{i=1}^{M} A_i^\top [\mathbf{u}(t)]_i \right) \mathbf{x}(t). \tag{3.4}$$

Therefore dual conditions can be derived from those worked out for system (3.1) by transposition, i.e. considering the matrices $A_i^\top$, $i \in \{1, 2, \ldots, M\}$, instead of A_i, $i \in \{1, 2, \ldots, M\}$. $\square$

3.1.1 Lyapunov functions

A standard tool for investigating the uniform exponential stability of a switched system is represented by Lyapunov functions. In this context we are typically interested in global exponential stability and hence we search for positive definite functions whose derivatives along the system trajectories are decreasing, for every choice of the initial condition and of the switching signal.

Definition 3.2. A differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is said to be a *Lyapunov function* for the continuous-time (positive) switched system (3.1) if it is positive definite and

$$\nabla V(\mathbf{x}) A_i \mathbf{x} < 0, \qquad \forall\, \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0, \forall\, i \in \{1, 2, \dots, M\}. \tag{3.5}$$

A well-know result for general switched systems, switching among a finite number of subsystems (actually the result is even more general, but this is what we need for the class of systems we are considering), is the following, Brayton and Tong [1980], Molchanov and Pyatnitskii [1986], Dayawansa and Martin [1999], Blanchini and Miani [1999]:

Theorem 3.1. The following facts are equivalent:

i) the continuous-time positive switched system (3.1) is uniformly exponentially stable;

ii) there exists a (differentiable) Lyapunov function V for the switched system (3.1), homogeneous of order 2 (i.e., $V(\alpha\mathbf{x}) = \alpha^2 V(\mathbf{x})$ for every $\alpha > 0$ and every $\mathbf{x} \in \mathbb{R}^n$);

iii) there exists an infinitely differentiable (smooth) and convex Lyapunov function for the switched system (3.1).

A polyhedral function is any function that can be written in the form

$$V(\mathbf{x}) = \max_{i \in \{1,2,\dots,s\}} [F\mathbf{x}]_i,$$

for some $s \times n$ full column rank matrix F, see Blanchini and Miani [2008]. In particular, a symmetric polyhedral function $V(\mathbf{x})$ can be expressed as $V(\mathbf{x}) = \|W\mathbf{x}\|_\infty$, for some full column rank matrix W, or in dual form as $V(\mathbf{x}) = \min\{\|\mathbf{z}\|_1 : \mathbf{z} > 0 \text{ s.t. } \mathbf{x} = X\mathbf{z}\}$ [1], where X is a matrix whose columns represent the vertices of the unit ball of $V(\mathbf{x})$. Any such function is clearly continuous and positive definite. A necessary and sufficient condition for uniform exponential stability of system (3.1) can also be stated in terms of polyhedral Lyapunov functions, following the lines traced in Brayton and Tong [1980], Molchanov and Pyatnitskii [1986] for general switched systems (see also Blanchini [1999]). Indeed, the following result can be proven.

Proposition 3.3. The following facts are equivalent:

i) the continuous-time positive switched system (3.1) is uniformly exponentially stable;

ii) there exist $s \in \mathbb{Z}_+$, a full row rank nonnegative $n \times s$ matrix X and $s \times s$ square Metzler matrices $P_i, i \in \{1, 2, \dots, M\}$, such that
$$A_i X = X P_i, \quad \mathbf{1}_s^\top P_i \ll 0; \tag{3.6}$$

iii) there exist $s \in \mathbb{Z}_+$, a full column rank nonnegative $s \times n$ matrix W and $s \times s$ square Metzler matrices $Q_i, i \in \{1, 2, \dots, M\}$, such that
$$W A_i = Q_i W, \quad Q_i \mathbf{1}_s \ll 0. \tag{3.7}$$

Proposition 3.3 suggests how to compute a polyhedral Lyapunov function for system (3.1), by making use of either (3.6) or (3.7). In the first case we have $V(\mathbf{x}) = \min\{\|\mathbf{z}\|_1 : \mathbf{x} = X\mathbf{z}, \mathbf{z} > 0\}$ [2]. In the second

[1] For a column vector $\mathbf{x} \in \mathbb{R}^n$, the 1-norm is defined as $\|\mathbf{x}\|_1 = \sum_{i=1}^n |[\mathbf{x}]_i|$ whereas the ∞-norm is $\|\mathbf{x}\|_\infty = \max_i |[\mathbf{x}]_i|$. Of course, for positive vectors, i.e. $\mathbf{x} \in \mathbb{R}^n_+$, the absolute values can be omitted. The 1-norm and the ∞-norm for matrices are induced from those of vectors. Given an $s \times s$ matrix P, we have $\|P\|_1 = \max_j \sum_{i=1}^s |[P]_{ij}|$. Analogously, given an $s \times s$ matrix Q, we have $\|Q\|_\infty = \max_i \sum_{j=1}^s |[Q]_{ij}|$. When dealing with Metzler matrices P and Q, the absolute values can be omitted for the off diagonal entries in the above expressions.

[2] Indeed, the right-upper Dini derivative of $V(\mathbf{x})$, see Garg [1998], computed along the trajectories of the system, satisfies $D^+V(\mathbf{x}) \leq \mu_1(P_\sigma)V(\mathbf{x})$, where μ_1 denotes the 1-measure of matrix P_σ, i.e. $\lim_{h\to 0} \frac{\|P_\sigma h + I\|_1 - 1}{h} = \max_j \left([P_\sigma]_{jj} + \sum_{i=1, i\neq j}^s |[P_\sigma]_{ij}|\right)$. Since P_σ is a Metzler matrix, it turns out that $\mu_1(P_\sigma) = \max_j [\mathbf{1}_s^\top P_\sigma]_j$.

case $V(\mathbf{x}) = \|W\mathbf{x}\|_\infty$ [3]. Notice, however, that this characterization can be computationally demanding since s, the number of columns of X in (3.6) and the number of rows of W in (3.7), is not known a priori, not even for positive switched systems.

The main advantage of dealing with positive switched systems, however, is that one may test their asymptotic behavior by restricting the attention to trajectories originated and hence confined in the nonnegative orthant. This also allows to weaken the constraints on the Lyapunov functions that one needs to find in order to verify whether the systems is exponentially stable. Indeed, the (differentiable) function just needs to be *copositive* and hence to take positive values on the positive orthant.

Definition 3.3. A differentiable function[4] $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is said to be a *copositive Lyapunov function* for the continuous-time positive switched system (3.1) if it is copositive, namely $V(\mathbf{x}) > 0$ for every $\mathbf{x} > 0$, $V(0) = 0$, and condition (3.5) holds.

Even if we move our attention to the larger class of copositive functions, however, the general search for a copositive Lyapunov function without imposing any a priori structure is computationally intractable. So, easy computational tools for checking stability under arbitrary switching have been searched for, by focusing on two classes of copositive Lyapunov functions: the linear ones and the quadratic ones.

In the following we will consider also the case of time-varying linear or time-varying quadratic Lyapunov functions: these functions are linear or quadratic at every time $t \geq 0$, but the specific expressions they take depend on the value of the switching signal σ at t.

[3] Indeed, the right-upper Dini derivative of $V(\mathbf{x})$, computed along the trajectories of the system, satisfies $D^+V(\mathbf{x}) \leq \mu_\infty(Q_\sigma)V(\mathbf{x})$, where μ_∞ denotes the ∞-measure of matrix Q_σ, i.e. $\lim_{h\to 0} \frac{\|Q_\sigma h+I\|_\infty - 1}{h} = \max_i \left([Q_\sigma]_{ii} + \sum_{j=1, j\neq i}^{s} |[Q_\sigma]_{ij}|\right)$. Since Q_σ is a Metzler matrix it turns out that $\mu_\infty(Q_\sigma) = \max_i [Q_\sigma \mathbf{1}_s]_i$.

[4] The definition extends to the time-varying case, thus leading to the class of time-varying copositive Lyapunov functions. Since in this section we will mainly deal with the time-invariant case, we have chosen to focus on that case.

Definition 3.4. A (differentiable) copositive function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is

- *linear* if $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, for some $\mathbf{v} \in \mathbb{R}^n, \mathbf{v} \gg 0$;
- *quadratic copositive* if $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, for some matrix $P = P^\top \in \mathbb{R}^{n\times n}$ such that $\mathbf{x}^\top P \mathbf{x} > 0$ for every $\mathbf{x} > 0$;
- *quadratic positive definite* if $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, for some matrix $P = P^\top \succ 0 \in \mathbb{R}^{n\times n}$ (and hence $\mathbf{x}^\top P \mathbf{x} > 0$ for every $\mathbf{x} \neq 0$).

Remark 3.4. A symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that $\mathbf{x}^\top P \mathbf{x} > 0$ for every $\mathbf{x} \in \mathbb{R}^n_+, \mathbf{x} > 0$, is called a (strictly) copositive matrix, and has been the object of a good number of papers. Unfortunately, it has been proved that to decide whether a matrix is copositive is NP-hard, see Bomze [2012], Murty and Kabadi [1987]. When the size n satisfies $n \leq 4$, every weakly copositive matrix, namely every symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that $\mathbf{x}^\top P \mathbf{x} \geq 0$ for every $\mathbf{x} \in \mathbb{R}^n_+, \mathbf{x} > 0$, is the sum of a positive semi-definite matrix and a nonnegative matrix. For $n > 4$, it is still true that the sum of a positive semi-definite matrix and a nonnegative matrix is weakly copositive, but the converse is not true (see, for instance, Hiriart-Urruty and Seeger [2010]). □

A linear copositive function $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, with $\mathbf{v} \gg 0$, is a *linear copositive Lyapunov function (LCLF)* for the system (3.1) if and only if $\mathbf{v}^\top A_i \mathbf{x} < 0$ for every $i \in \{1, 2, \ldots, M\}$ and every $\mathbf{x} > 0$, which amounts to saying that

$$\mathbf{v}^\top A_i \ll 0, \qquad \forall\, i \in \{1, 2, \ldots, M\}.$$

Similarly, a quadratic copositive function $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, with $P = P^\top$, is a *quadratic copositive Lyapunov function (QCLF)* for the system (3.1) if and only if $\mathbf{x}^\top [A_i^\top P + P A_i] \mathbf{x} < 0$ for every $i \in \{1, 2, \ldots, M\}$ and every $\mathbf{x} > 0$, and a quadratic positive definite function $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, with $P = P^\top$, is a *quadratic positive definite Lyapunov function (QPDLF)* for the system (3.1) if and only if $\mathbf{x}^\top [A_i^\top P + P A_i] \mathbf{x} < 0$ for every $i \in \{1, 2, \ldots, M\}$ and every $\mathbf{x} > 0$. Equivalent conditions for the existence of an LCLF have been provided in Fornasini and Valcher [2009], Knorn et al. [2009], Mason and Shorten [2007] for continuous-time systems and in Fornasini and Valcher [2012] for discrete-time systems, see also the work of Bundfuss and Dür on this subject (Bundfuss and Dur [2009a,b], Sponsel et al. [2012]). The following proposition summarizes all of them and provides some new characterization.

Theorem 3.2. Given a continuous-time positive switched system (3.1), the following facts are equivalent:

1) $\exists\ \mathbf{v} \gg 0$ such that $\mathbf{v}^\top A(\alpha) = \mathbf{v}^\top \sum_{i=1}^M \alpha_i A_i \ll 0,\ \forall\ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M) \in \mathcal{A}_M$;

2) $\exists\ \mathbf{v} \gg 0$ such that $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ is an LCLF for (3.1);

3) $\exists\ P = P^\top$ of rank 1 such that $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ is a QCLF for (3.1);

4) for each map $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, M\}$, the matrix
$$A_\pi := \begin{bmatrix} \mathrm{col}_1(A_{\pi(1)}) & \mathrm{col}_2(A_{\pi(2)}) & \ldots & \mathrm{col}_n(A_{\pi(n)}) \end{bmatrix}$$
is Hurwitz;

5) the convex hull of the columns of $\mathbb{A} := \begin{bmatrix} A_1 & A_2 & \ldots & A_M \end{bmatrix} \in \mathbb{R}^{n \times nM}$ does not intersect[5] the positive orthant of $\mathbb{R}^n$.

6) for every choice of M nonnegative diagonal matrices $D_i, i \in \{1, 2, \ldots, M\}$, with $\sum_{i=1}^M D_i = I_n$, the matrix $\sum_{i=1}^M A_i D_i$ is Metzler Hurwitz.

Proof. 1) $\Leftrightarrow$ 2) Condition 2) is obtained from 1) for special values of the M-tuples $\alpha \in \mathcal{A}_M$. The reverse implication is obvious.

2) $\Rightarrow$ 3) Suppose that for some $\mathbf{v} \gg 0$ condition $\mathbf{v}^\top A_i \mathbf{x} < 0$ holds, $\forall i \in \{1, 2, \ldots, M\}$ and $\forall \mathbf{x} > 0$. This implies that $\mathbf{x}^\top (\mathbf{v}\mathbf{v}^\top) A_i \mathbf{x} + \mathbf{x}^\top A_i^\top (\mathbf{v}\mathbf{v}^\top) \mathbf{x} = 2(\mathbf{v}^\top \mathbf{x})^\top (\mathbf{v}^\top A_i \mathbf{x}) < 0$ holds, $\forall i \in \{1, 2, \ldots, M\}$ and $\forall \mathbf{x} > 0$. Therefore 3) is satisfied for $P := \mathbf{v}\mathbf{v}^\top$.

[5]This condition can be equivalently expressed by saying that the *positive kernel* of the matrix $\begin{bmatrix} I_n & -\mathbb{A} \end{bmatrix}$, i.e., $\ker \begin{bmatrix} I_n & -\mathbb{A} \end{bmatrix} \cap \mathbb{R}^n_+$, consists of the zero vector alone.

3) $\Rightarrow$ 2) If rank $P = 1$ and $P = P^\top$, then P can be expressed as $P = \mathbf{v}\mathbf{v}^\top$, for some vector $\mathbf{v}$. Moreover, as $\mathbf{x}^\top P\mathbf{x} = (\mathbf{v}^\top\mathbf{x})^2 > 0,\ \forall\mathbf{x} > 0$, all entries of $\mathbf{v}$ are nonzero and of the same sign, and it entails no loss of generality assuming that they are all positive. On the other hand, $\forall\mathbf{x} > 0$ and $\forall i \in \{1, 2, \dots, M\}$, condition

$$\mathbf{x}^\top[A_i^\top P + PA_i]\mathbf{x} = \mathbf{x}^\top A_i^\top \mathbf{v}\mathbf{v}^\top\mathbf{x} + \mathbf{x}^\top\mathbf{v}\mathbf{v}^\top A_i\mathbf{x} < 0,$$

can be rewritten as $2(\mathbf{v}^\top\mathbf{x})^\top(\mathbf{v}^\top A_i\mathbf{x}) < 0$, and from the positivity of $\mathbf{v}^\top\mathbf{x}$, one gets condition 2), namely:

$$\mathbf{v}^\top A_i\mathbf{x} < 0, \qquad \forall\mathbf{x} > 0,\ \forall i \in \{1, 2, \dots, M\}.$$

2) $\Leftrightarrow$ 5) It is well-known, Aliprantis and Tourky [2007] that, given some matrix $\mathbb{A}$, one and only one of the following alternatives holds:

$$\text{either} \quad \exists\ \mathbf{v} > 0 \quad \text{such that} \quad \mathbf{v}^\top\mathbb{A} \ll 0 \tag{3.8}$$

$$\text{or} \quad \exists\ \mathbf{z} > 0 \quad \text{such that} \quad \mathbb{A}\mathbf{z} \geq 0, \tag{3.9}$$

and in (3.9) the vector $\mathbf{z}$ can be assumed, without any loss of generality, to belong to the simplex $\mathcal{A}_{nM}$. If 2) (and hence (3.8)) holds true, (3.9) cannot be verified, and consequently no convex combination of the columns of $\mathbb{A}$ intersects the positive orthant $\mathbb{R}^n_+$. Vice versa, if no convex combination of the columns of $\mathbb{A}$ intersects $\mathbb{R}^n_+$, (3.9) does not hold, and hence (3.8) admits a positive solution $\mathbf{v} \in \mathbb{R}^n_+$. Clearly, there exists $\varepsilon > 0$ sufficiently small such that $\mathbf{v} + \varepsilon\mathbf{1}_n \gg 0$ satisfies $(\mathbf{v} + \varepsilon\mathbf{1}_n)^\top A_i \ll 0$ for every $i \in \{1, 2, \dots, M\}$.

2) $\Rightarrow$ 6) If there exists $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top A_i \ll 0$ for every $i \in \{1, 2, \dots, M\}$, then for every choice of the M diagonal matrices $D_i, i \in \{1, 2, \dots, M\}$, with $\sum_{i=1}^M D_i = I_n$, we have

$$\mathbf{v}^\top\left(\sum_{i=1}^M A_iD_i\right) = \sum_{i=1}^M(\mathbf{v}^\top A_i)D_i \ll 0.$$

This follows from the fact that each vector $\mathbf{v}^\top A_i$ is strictly negative and for every $j \in \{1, 2, \dots, n\}$ there exists at least one index i such that $[D_i]_{jj} > 0$. This proves that every matrix $\sum_{i=1}^M A_iD_i$ is (Metzler and) Hurwitz.

6) $\Rightarrow$ 4) If 6) holds, then in particular it holds for every choice of the diagonal matrices $D_i, i \in \{1, 2, \ldots, M\}$, with $\sum_{i=1}^{M} D_i = I_n$, satisfying the constraint that for every $k \in \{1, 2, \ldots, n\}$ there is only one index $i_k \in \{1, 2, \ldots, M\}$ such that $[D_{i_k}]_{kk} \neq 0$ and hence $[D_{i_k}]_{kk} = 1$. For any such choice of the matrices D_i we get

$$\sum_{i=1}^{M} A_i D_i = \begin{bmatrix} \mathrm{col}_1(A_{i_1}) & \mathrm{col}_2(A_{i_2}) & \ldots & \mathrm{col}_n(A_{i_n}) \end{bmatrix}. \tag{3.10}$$

This proves 4).

4) $\Rightarrow$ 2) Has been proved in Fornasini and Valcher [2009], Knorn et al. [2009]. □

The existence of an LCLF is, not unexpectedly, only a sufficient condition for the exponential stability of (3.1), as the following example clearly shows.

Example 3.3. Consider the 2-dimensional positive switched system (3.1), with $M = 2$ and matrices

$$A_1 = \begin{bmatrix} -1 & 1 \\ 1/2 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 1/2 \\ 1 & -1 \end{bmatrix}.$$

By a result of Akar *et al.* (Akar et al. [2006]), the system is exponentially stable. However it is easily seen that no LCLF can be found. Indeed, if $\mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^\top \gg 0$, then $\mathbf{v}^\top A_1$ implies $v_1 < v_2$, while $\mathbf{v}^\top A_2$ implies $v_2 < v_1$. So, a strictly positive vector $\mathbf{v}$ such that $\mathbf{v}^\top A_i \ll 0$ for $i = 1, 2$, does not exist. Alternatively, we can apply condition 4) of the previous theorem and simply notice that the matrix

$$A_\pi = \begin{bmatrix} \mathrm{col}_1(A_2) & \mathrm{col}_2(A_1) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

is not Hurwitz.

We now move to the analysis of how linear copositive, quadratic positive definite and quadratic copositive Lyapunov functions are mutually related, namely we explore how the existence of a Lyapunov function of an LCLF implies the existence of QPDLF which, in turn, implies the existence of a QCLF.

Theorem 3.3. Given a continuous-time positive switched system (3.1), the following implications hold: if there exists an LCLF for (3.1) then there exists a QPDLF for (3.1), and this in turn implies the existence of a QCLF for (3.1).

Proof. 1) $\Rightarrow$ 2) Let $\mathbf{v} \gg 0$ be such that $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ is an LCLF for (3.1). Define $P := \mathbf{v}\mathbf{v}^\top + \varepsilon I_n$, where ε is a positive parameter to be chosen. We want to show that $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ is a QPDLF for (3.1). First of all, $P = P^\top \succ 0$. Indeed, P is clearly symmetric and is positive definite since $\mathbf{x}^\top P \mathbf{x} = (\mathbf{v}^\top \mathbf{x})^2 + \varepsilon \|\mathbf{x}\|^2 \geq 0$ and

$$\mathbf{x}^\top P \mathbf{x} = (\mathbf{v}^\top \mathbf{x})^2 + \varepsilon \|\mathbf{x}\|^2 = 0 \qquad \Leftrightarrow \qquad \mathbf{x} = 0.$$

Finally, for every $\mathbf{x} > 0$ and every $i \in \{1, 2, \dots, M\}$,

$$\mathbf{x}^\top (A_i^\top P + P A_i)\mathbf{x} = (\mathbf{x}^\top A_i^\top \mathbf{v})(\mathbf{v}^\top \mathbf{x}) + (\mathbf{x}^\top \mathbf{v})(\mathbf{v}^\top A_i \mathbf{x}) + \mathbf{x}^\top [A_i \varepsilon + \varepsilon A_i]\mathbf{x}.$$

Now, set $\mathcal{K} := \{\mathbf{x} \in \mathbb{R}^n_+ : \|\mathbf{x}\| = 1\}$. $\mathcal{K}$ is a compact set and hence, by the Weierstrass theorem, there exist

$$-\alpha := \max_{\mathbf{x} \in \mathcal{K}, i \in \{1,2,\dots,M\}} (\mathbf{v}^\top \mathbf{x})(\mathbf{v}^\top A_i \mathbf{x}) < 0$$

and

$$\beta := \max_{\mathbf{x} \in \mathcal{K}, i \in \{1,2,\dots,M\}} |\mathbf{x}^\top A_i \mathbf{x}| \geq 0.$$

If $\varepsilon \in (0, \alpha/\beta)$, then for every $\mathbf{x} \in \mathcal{K}$

$$\mathbf{x}^\top (A_i^\top P + P A_i)\mathbf{x} = 2(\mathbf{v}^\top \mathbf{x})(\mathbf{v}^\top A_i \mathbf{x}) + 2\varepsilon \mathbf{x}^\top A_i \mathbf{x} \leq -2\alpha + 2\varepsilon\beta < 0.$$

On the other hand, for any $\mathbf{x} \in \mathbb{R}^n_+, \mathbf{x} \neq 0$, we have $\mathbf{x} = \|\mathbf{x}\| \cdot \bar{\mathbf{x}}$, with $\bar{\mathbf{x}} := \mathbf{x}/\|\mathbf{x}\| \in \mathcal{K}$. Therefore

$$\begin{aligned} \mathbf{x}^\top (A_i^\top P + P A_i)\mathbf{x} &= \|\mathbf{x}\|^2 \left(2(\mathbf{v}^\top \bar{\mathbf{x}})(\mathbf{v}^\top A_i \bar{\mathbf{x}}) + 2\varepsilon \bar{\mathbf{x}}^\top A_i \bar{\mathbf{x}}\right) \\ &\leq \|\mathbf{x}\|^2 (-2\alpha + 2\varepsilon\beta) < 0. \end{aligned}$$

This concludes the proof.

2) $\Rightarrow$ 3) is obvious. $\square$

The existence of an LCLF is a more restrictive condition with respect to the existence of a QPDLF.

Example 3.4. Consider the 2-dimensional positive switched system (3.1), with $M = 2$ and matrices

$$A_1 = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}.$$

It is easily seen that no LCLF can be found, since the matrix

$$A_\pi = \begin{bmatrix} \mathrm{col}_1(A_1) & \mathrm{col}_2(A_2) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

is not Hurwitz. However, it is also easy to verify that both $A_1^\top + A_1 \prec 0$ and $A_2^\top + A_2 \prec 0$, which means that $V(\mathbf{x}) = \|\mathbf{x}\|^2 = \mathbf{x}^\top \mathbf{x}$ is a QPDLF.

To the best of our knowledge, it is not clear yet whether the existence of a QPDLF is a stronger condition than the existence of a QCLF, or the two of them are equivalent conditions. The fact that, as recalled in Remark 3.4, a characterization of the matrices that define quadratic copositive functions is still missing has surely a high impact on this additional open problem. Some interesting results relating quadratic copositive matrices, P-matrices, and Z-transformations have been obtained by M.S. Gowda and co-authors (see Gowda [2012], Moldovan and Gowda [2010] and references therein).

Remark 3.5. The existence of a QCLF for the positive switched system (3.1) of size n is also induced by the existence of an LCLF in an extended space of size n^2. To show this, we first introduce the variable $\mathbf{X} := \mathbf{x} \otimes \mathbf{x}$, and note that, see [6] Graham [1981],

$$\mathbf{x}^\top P \mathbf{x} = \mathrm{vec}[\mathbf{x}^\top P \mathbf{x}] = (\mathbf{x}^\top \otimes \mathbf{x}^\top)\mathrm{vec}[P] = \mathbf{X}^\top \mathrm{vec}[P]$$

[6] The *vec* operator on a matrix, say $A \in \mathbb{R}^{n\times m}$, consists in constructing a column vector by ordinately stacking the columns of the matrix. It goes without saying that vec[A] is a column vector of size nm. For square symmetric matrices one can consider a "reduced" vec operator by stacking only the lower triangular part of the matrix.

and

$$\begin{aligned}\mathbf{x}^\top(A_i^\top P + PA_i)\mathbf{x} &= \mathrm{vec}[\mathbf{x}^\top(A_i^\top P + PA_i)\mathbf{x}] = (\mathbf{x}^\top \otimes \mathbf{x}^\top)\tilde{A}_i^\top \mathrm{vec}[P] \\ &= \mathbf{X}^\top \tilde{A}_i^\top \mathrm{vec}[P],\end{aligned}$$

where $\tilde{A}_i = A_i \oplus A_i$ [7]. Consider, now, the n^2-dimensional positive switched system

$$\dot{\mathbf{X}}(t) = \tilde{A}_{\sigma(t)}\mathbf{X}(t). \tag{3.11}$$

If there exists an LCLF for system (3.11), this means that there exists $\mathbf{V} \in \mathbb{R}_+^{n^2}$, with $\mathbf{V} \gg 0$, such that $\mathbf{V}^\top \tilde{A}_i \ll 0, \forall i \in \{1, 2, \ldots, M\}$. Consequently, for every $\mathbf{X} \in \mathbb{R}_+^{n^2}, \mathbf{X} > 0$, and every $i \in \{1, 2, \ldots, M\}$, we have

$$\mathbf{X}^\top \mathbf{V} > 0 \qquad \text{and} \qquad \mathbf{X}^\top \tilde{A}_i^\top \mathbf{V} < 0.$$

So, by making use of the relations we previously derived, we can show that the symmetric strictly positive matrix P [8]

$$P = \frac{1}{2}\left(\mathrm{vec}^{-1}(\mathbf{V}) + (\mathrm{vec}^{-1}(\mathbf{V}))^\top\right)$$

satisfies

$$\mathbf{x}^\top P\mathbf{x} > 0, \quad \mathbf{x}^\top(A_i^\top P + PA_i^\top)\mathbf{x} < 0, \quad \forall \mathbf{x} > 0, \quad \forall i \in \{1, 2, \ldots, M\},$$

and hence is a QCLF for the positive switched system (3.1).

As a result, stability under arbitrary switching can be checked via linear programming techniques by searching for LCLFs for system (3.11). Note that the dimension of such a system can be reduced if the Kronecker computations are considered by avoiding repetitions in the state variables and by computing the Metzler matrices $\tilde{A}_i$ accordingly. For the theory of polynomial homogeneous Lyapunov functions and the stability analysis via Gram matrices and SMRs (square matricial representations), see Chesi [2011] and Chesi et al. [2012]. □

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[7] The notation $\oplus$ indicates the Kronecker sum, defined, for two matrices A and B as $A \oplus B = A \otimes I + I \otimes B$.

[8] $\mathrm{vec}^{-1}(\mathbf{z})$, $\mathbf{z} \in \mathbb{R}^{n^2}$, denotes the unique $n \times n$ matrix P_z such that $\mathrm{vec}(P_z) = \mathbf{z}$.

Example 3.5. Consider the system of Example 3.4. We already verified that it does not admit an LCLF. Introduce the extended vector

$$\boldsymbol{\xi} = \begin{bmatrix} [\mathbf{x}]_1^2 \\ [\mathbf{x}]_1 [\mathbf{x}]_2 \\ [\mathbf{x}]_2^2 \end{bmatrix}$$

and define the 3rd order positive switched system

$$\dot{\boldsymbol{\xi}}(t) = \tilde{A}_{\sigma(t)}\boldsymbol{\xi}(t),$$

where

$$\tilde{A}_1 = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -4 & 1 \\ 0 & 2 & -6 \end{bmatrix}, \qquad \tilde{A}_2 = \begin{bmatrix} -6 & 2 & 0 \\ 1 & -4 & 1 \\ 0 & 2 & -2 \end{bmatrix}.$$

An LCLF for such a system can be easily found, for instance $\tilde{V}(\boldsymbol{\xi}) = \mathbf{V}^\top \boldsymbol{\xi}$ with $\mathbf{V} = \begin{bmatrix} 1 & 1.5 & 1 \end{bmatrix}^\top$. Correspondingly,

$$V(\mathbf{x}) = [\mathbf{x}]_1^2 + 1.5[\mathbf{x}]_1[\mathbf{x}]_2 + [\mathbf{x}]_2^2 = \mathbf{x}^\top \begin{bmatrix} 1 & 0.75 \\ 0.75 & 1 \end{bmatrix} \mathbf{x}$$

is a CQLF (in fact, a QPDLF) for the original switched system.

3.1.2 Periodic switching signals

A natural question arises: when testing exponential stability of a positive switched system, can we restrict our attention to periodic switching signals in $\mathcal{D}_0$? In other words, if all the trajectories generated by the positive switched system (3.1), corresponding to any periodic $\sigma \in \mathcal{D}_0$ and any nonnegative initial condition $\mathbf{x}_0 \in \mathbb{R}_+^n$, converge to zero exponentially, can one claim that the system is uniformly exponentially stable?

This problem has been extensively investigated for the class of standard switched systems, both in the discrete-time and in the continuous-time case (see Pyatnitskiy and Rapoport [1991], Blondel et al. [2003], Shorten et al. [2007], and references therein). In the discrete-time case this problem is equivalent to the finiteness conjecture introduced in Lagarias and Wang [1995], and this conjecture was disproved in Bousch and Mairesse [2001](see also the papers Blondel et al. [2003], Hare et al. [2011]). For continuous-time systems, there are several results showing that for low-dimensional systems periodic stability is sufficient for uniform exponential stability (see Pyatnitskii [1971], Pyatnitskiy and Rapoport [1996] as well as Barabanov [1993]). We refer the interested Reader to Shorten et al. [2007] and references therein. In Section 3.2, dedicated to stability under dwell time constraints, we will provide an example of a positive switched system that is stable under any switching signal, for which the length between two consecutive switching instants is nonzero, but that can be destabilized in a sliding mode. In such cases one can find an unstable convex combination in the convex hull of the matrices. Since the set of switching signals with dwell time includes the set of periodic switching signals, the example also demonstrates that exponential stability for any periodic switching signal does not imply exponential stability under arbitrary switching signal, see Remark 3.7.

Even if periodic stability does not ensure uniform exponential stability, a slightly weaker result holds true. If a switched system is periodically exponentially stable with some finite "robustness margin" ε, then it is uniformly exponentially stable. This result was proved in the discrete-time in Theorem 2.3 of Gurvits [1995], while a continuous-time version of the result can be found in Wulff et al. [2003].

Theorem 3.4. A switched linear system (1) is uniformly exponentially stable under arbitrary switching if and only if there exists $\varepsilon > 0$ such that, for every $T > 0$ and every periodic switching signal $\sigma \in \mathcal{D}_0$ of period T, the spectral radius of the corresponding transition matrix $\Phi(T, 0, \sigma)$ is smaller than $1 - \varepsilon$.

In the rest of this subsection we want to show that condition 6) in Theorem 3.2, namely the property that for every choice of M nonnegative diagonal matrices $D_i, i \in \{1, 2, \dots, M\}$, with $\sum_{i=1}^M D_i = I_n$, the matrix $\sum_{i=1}^M A_i D_i$ is (Metzler) Hurwitz, can be related to the uniform exponential stability of system (3.1) under periodic switching laws.

To clarify this relationship, we first provide the following lemma, that refers to the exponential stability of the positive switched system (3.1) corresponding to a specific switching signal.

Lemma 3.5. Assume that $\sigma = \bar{\sigma}$ is a specific switching signal, and let $\mathbf{q} \in \mathbb{R}^n_+$ be a strictly positive vector. The time-varying system obtained from (3.1) corresponding to $\bar{\sigma}$ is uniformly exponentially stable if and only if the differential inequality

$$\dot{\mathbf{r}}(t)^\top + \mathbf{r}(t)^\top A_{\bar{\sigma}(t)} \ll -\mathbf{q}^\top \tag{3.12}$$

has a solution $\mathbf{r}(t) \in \mathbb{R}^n_+$, differentiable almost everywhere, and such that

$$\bar{\mathbf{r}} < \mathbf{r}(t) < \hat{\mathbf{r}}, \quad t \geq 0,$$

for some $\bar{\mathbf{r}} \gg 0$ and $\hat{\mathbf{r}} \gg 0$.

Proof. Let $\mathbf{r}(t)$ be a solution of (3.12) with the afore mentioned properties, and let $\eta > 0$ be such that $\bar{\mathbf{r}} \ll \eta \mathbf{1}_n$. Introduce the time-varying copositive function $V(\mathbf{x}, t) = \mathbf{r}(t)^\top \mathbf{x}(t)$ and notice that it is well defined since $V(\mathbf{x}, t) \leq \hat{\mathbf{r}}^\top \mathbf{x}(t)$, $\forall t \geq 0$. Standard computations show that $\dot{V}(\mathbf{x}, t) < -\varepsilon \eta^{-1} V(\mathbf{x}, t)$, $\forall t \geq 0$, $\forall \mathbf{x} > 0$, where ε is any positive number such that $\mathbf{q} \gg \varepsilon \mathbf{1}_n$. Due the fact that $V(\mathbf{x}, t) \gg \bar{\mathbf{r}}^\top \mathbf{x}(t)$, for $\mathbf{x}(t) > 0$, uniform exponential stability of the time-varying system follows.
On the contrary, assume that system (3.1), for $\sigma = \bar{\sigma}$, is uniformly exponentially stable, and define

$$\mathbf{r}(t) := \int_t^{+\infty} \Phi(\tau, t, \bar{\sigma})(\mathbf{q} + \varepsilon \mathbf{1}_n) d\tau,$$

where $\Phi(\tau, t, \bar{\sigma})$ is the state transition matrix associated with $A_{\bar{\sigma}(t)}$, and $\varepsilon > 0$. The exponential stability of the time-varying system ensures that there exist $C > 0$ and $\beta > 0$ such that $\|\Phi(\tau, t, \bar{\sigma})\|_\infty < Ce^{-\beta(\tau - t)}$, for every $\tau > t \geq 0$. Taking the infinity norm of $\mathbf{r}(t)$, one gets $\|\mathbf{r}(t)\|_\infty \leq \frac{C}{\beta} \|\mathbf{q} + \varepsilon \mathbf{1}_n\|_\infty$ and hence $\mathbf{r}(t) < \hat{\mathbf{r}} := C/\beta\, (\mathbf{q} + \varepsilon \mathbf{1}_n)$, $t \geq 0$. Consequently, $\mathbf{r}(t)$ exists and is uniformly bounded. Also, as $A_{\bar{\sigma}(t)}$ is Metzler at every time $t \geq 0$, $\Phi(\tau, t, \bar{\sigma})$ is positive at every $\tau > t \geq 0$ (and devoid of zero rows) and there exists $\bar{\mathbf{r}} \gg 0$ such that $\mathbf{r}(t) > \bar{\mathbf{r}}$, $t \geq 0$. Finally, a straightforward computation shows that $\mathbf{r}(t)$ satisfies

$$\dot{\mathbf{r}}(t)^\top + \mathbf{r}(t)^\top A_{\bar{\sigma}(t)} = -(\mathbf{q}^\top + \varepsilon \mathbf{1}_n^\top) \ll -\mathbf{q}^\top. \tag{3.13}$$

$\square$

Note that a dual result can be stated, by referring again to a specific switching signal $\bar{\sigma}(t)$. The proof is analogous to the previous one and hence is omitted.

Lemma 3.6. Assume that $\sigma = \bar{\sigma}$ is a specific switching signal, and let $\mathbf{b} \in \mathbb{R}^n_+$ be a strictly positive vector. The time-varying system obtained from (3.1) corresponding to $\bar{\sigma}$ is uniformly exponentially stable if and only if the differential inequality

$$-\dot{\mathbf{d}}(t) + A_{\bar{\sigma}(t)}\mathbf{d}(t) + \mathbf{b} \ll 0 \tag{3.14}$$

has a solution $\mathbf{d}(t) \in \mathbb{R}^n_+$, differentiable almost everywhere, and such that

$$\bar{\mathbf{d}} < \mathbf{d}(t) < \hat{\mathbf{d}}, \quad t \geq 0$$

for some $\bar{\mathbf{d}} \gg 0$ and $\hat{\mathbf{d}} \gg 0$.

We are now in a position to clarify the aforementioned relationship between condition 6) in Theorem 3.2 and the exponential stability of system (3.1) under periodic switching laws. By following up on the previous Lemma 3.5, assume that the positive switched system (3.1) is uniformly exponentially stable, let $\bar{\sigma}$ be a periodic switching signal of period say $T > 0$, and consider the (continuous) function

$$\mathbf{r}(t) := \int_t^{+\infty} \Phi(\tau, t, \bar{\sigma})\mathbf{1}_n d\tau,$$

where $\Phi(\tau, t, \bar{\sigma})$ is again the transition matrix associated with $A_{\bar{\sigma}(t)}$. In this case the transition matrix is T-periodic, which means $\Phi(\tau + T, t + T, \bar{\sigma}) = \Phi(\tau, t, \bar{\sigma})$ for all τ and t, and $\mathbf{r}(t)$ is T-periodic as well. Notice that $\mathbf{r}(t)$ satisfies (3.12) for $\mathbf{q} = \mathbf{1}_n$, and therefore

$$\int_0^T \mathbf{r}(t)^\top A_{\bar{\sigma}(t)} dt \ll 0.$$

Let $\mathcal{S}_i$ denote the union of the time intervals included in $[0, T)$ where the ith subsystem is active, and introduce the M positive diagonal matrices D_i, $i \in \{1, 2, \ldots, M\}$, whose nonnegative diagonal entries are defined as follows

$$[D_i]_{kk} := \frac{\int_{\mathcal{S}_i} [\mathbf{r}(t)]_k dt}{[\bar{\mathbf{r}}]_k}, \qquad k \in \{1, 2, \ldots, n\},$$

where

$$\bar{\mathbf{r}} := \int_0^T \mathbf{r}(t)dt \gg 0.$$

Clearly, $\sum_{i=1}^M D_i = I_n$. It can be easily verified that

$$\bar{\mathbf{r}}^\top \sum_{i=1}^M A_i D_i \ll 0,$$

and hence $\sum_{i=1}^M A_i D_i$ is Metzler Hurwitz. This shows that corresponding to every periodic switching signal $\bar{\sigma}$ we can define positive diagonal matrices D_i, $i \in \{1, 2, \ldots, M\}$, with $\sum_{i=1}^M D_i = I_n$, such that $\sum_{i=1}^M A_i D_i$ is Metzler Hurwitz. Unfortunately, this result cannot be reversed, since there are choices of the diagonal matrices D_i, satisfying the previous assumptions, that cannot be related to any periodic switching signal.

3.1.3 Dual positive switched system

In this subsection we explore the uniform exponential stability of the positive switched system (3.1), by making use of its *dual system*, namely the positive switched system

$$\dot{\mathbf{z}}(t) = A_{\sigma(t)}^\top \mathbf{z}(t), \tag{3.15}$$

whose ith subsystem is described by the Metzler Hurwitz matrix $A_i^\top$, the transposed version of the one characterizing the ith subsystem of (3.1). By putting together Remarks 3.2 and 3.3, it immediately follows that the uniform exponential stability of (3.15) ensures that of (3.1) and conversely. As a consequence, the existence of an LCLF for the dual system (3.15) ensures the uniform exponential stability of the original system (3.1). An alternative proof, based on the polyhedral copositive Lyapunov function $V(\mathbf{x}) = \max_{i=1,2,\ldots,M} \frac{[\mathbf{x}]_i}{[\boldsymbol{\xi}]_i}$, will be provided in the next section, for systems with dwell-time.

Proposition 3.4. Given a continuous-time positive switched system (3.1), if there exists a strictly positive vector $\boldsymbol{\xi} \in \mathbb{R}^n_+$ such that

$$A_i \boldsymbol{\xi} \ll 0, \qquad \forall\, i \in \{1, 2, \ldots, M\}, \tag{3.16}$$

then the positive switched system (3.1) is uniformly exponentially stable.

In general, the existence of an LCLF for the positive switched system (3.1) does not ensure the existence of any such Lyapunov function for its dual (3.15), nor the converse is obviously true.

Example 3.6. Consider the 2-dimensional positive switched system (3.1), with $M = 2$ and the matrices

$$A_1 = \begin{bmatrix} -1 & 1/2 \\ 1 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -3 & 1 \\ 1/3 & -1 \end{bmatrix}.$$

It is easily seen that no LCLF can be found, but the dual system has the LCLF $\tilde{V}(\mathbf{z}) = [1 \ \ 1]^\top \mathbf{z}$.

An LCLF for system (3.1) corresponds to a polyhedral Lyapunov function for its dual (3.15), and vice versa. When LCLFs can be found for both system (3.1) and its dual (3.15), then a diagonal QPDLF can be found for (3.1). Indeed, we have the following result whose proof is a direct extension of the one provided by Araki for a single Metzler matrix (see Araki [1975]). For a discussion of how stability and Lyapunov functions of a switched system are related to stability and Lyapunov functions of its dual, see Plischke and Wirth [2008].

Proposition 3.5. Given a continuous-time positive switched system (3.1), if there exist strictly positive vectors $\mathbf{v}, \boldsymbol{\xi} \in \mathbb{R}^n_+$ such that

$$\begin{aligned} \mathbf{v}^\top A_i &\ll 0, \qquad \forall\, i \in \{1, 2, \dots, M\}, \\ A_i\, \boldsymbol{\xi} &\ll 0, \qquad \forall\, i \in \{1, 2, \dots, M\}, \end{aligned}$$

then the positive (and positive definite) diagonal matrix

$$D = \operatorname{diag}\left\{ \tfrac{[\mathbf{v}]_1}{[\boldsymbol{\xi}]_1} \quad \dots \quad \tfrac{[\mathbf{v}]_n}{[\boldsymbol{\xi}]_n} \right\}$$

satisfies

$$A_i^\top D + D A_i \prec 0, \qquad \forall\, i \in \{1, 2 \dots, M\}.$$

Example 3.7. Consider the two matrices

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 1/16 & -1 & 1 \\ 1/100 & 1/10 & -1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1/100 & -1 & 1 \\ 1/100 & 1/100 & -1 \end{bmatrix}.$$

It is easy to verify that for $\mathbf{v} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ and $\boldsymbol{\xi} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^\top$ we get both $\mathbf{v}^\top A_i \ll 0$ and $A_i \boldsymbol{\xi} \ll 0$ for $i = 1, 2$. On the other hand, $P = \text{diag}\{1, 1, 1/2\}$ is a QPDLF for the positive switched system described by these two matrices.

Remark 3.6. The existence of a diagonal QPDLF however does not ensure either the existence of an LCLF or of a vector $\boldsymbol{\xi} \gg 0$ satisfying (3.16). Indeed, one simply needs to consider Example 3.4. Clearly, in that case $P = I_2$, but there is no LCLF either for the system (3.1) or for its dual (just apply to $A_1^\top$ and $A_2^\top$ the same reasoning adopted for A_1 and A_2). □

3.1.4 Two-dimensional continuous-time positive switched systems

When dealing with two-dimensional systems, exponential stability admits much stronger characterizations. We will first consider the case of a two-dimensional positive switched system (3.1) switching between two subsystems (namely both n and M are equal to 2), and then extend the analysis to the case of two-dimensional positive switched systems, switching among $M \geq 2$ subsystems. Both results are due to Gurvits and co-authors (see Gurvits et al. [2007]).

Theorem 3.7. Given a two-dimensional continuous-time positive switched system (3.1), switching between $M = 2$ subsystems, the following facts are equivalent:

i) the switched system (3.1) is uniformly exponentially stable;

ii) there exists $P = P^\top \succ 0$ such that $A_i^\top P + P A_i \prec 0$ for $i = 1, 2$;

iii) there exists a QPDLF for (3.1);

iv) the matrix product $A_1 A_2^{-1}$ has no negative eigenvalues;

v) for every $\alpha \in \mathcal{A}_2$, the convex combination of the matrices A_1 and A_2, $A(\alpha)$, is Metzler Hurwitz.

Theorem 3.8. Gurvits et al. [2007]. Given a two-dimensional continuous-time positive switched system (3.1), switching between $M > 2$ subsystems, the following facts are equivalent:

i) the switched system (3.1) is uniformly exponentially stable;

ii) for every choice of $i, j \in \{1, 2, \ldots, M\}$, the switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad \text{with} \quad \sigma(t) \in \{i, j\},$$

is uniformly exponentially stable;

iii) for every $\alpha \in \mathcal{A}_M$, the convex combination of the matrices $A_1, A_2, \ldots, A_M$, $A(\alpha)$, is Metzler Hurwitz.

3.1.5 Rate of convergence

In some cases, verifying the exponential stability of the system is not meaningful. Most of the examples in Chapter 2 are clearly stable. What is more interesting, instead, is to understand whether the convergence to zero of the system trajectories is sufficiently fast.

Definition 3.5. Given an exponentially stable positive switched system (3.1), we say that the system has *rate of convergence* $\beta > 0$ if there exists some constant $C > 0$ such that for every switching signal $\sigma \in \mathcal{D}_0$ and every initial condition $\mathbf{x}(0) \in \mathbb{R}^n_+$, we have

$$\|\mathbf{x}(t; \mathbf{x}(0), \sigma)\| \leq Ce^{-\beta t}\|\mathbf{x}(0)\|.$$

Determining (or ensuring) a certain convergence speed can be reduced to a simple exponential stability analysis for a modified system.

Proposition 3.6. The positive switched system (3.1) is exponentially stable with rate of convergence $\bar{\beta}$ if and only if the perturbed positive switched system

$$\dot{\mathbf{x}}(t) = [\beta I_n + A_\sigma]\mathbf{x}(t)$$

is exponentially stable for any $\beta < \bar{\beta}$. Also, the solutions of the perturbed system and of the unperturbed one, corresponding to the same initial condition and to the same switching signal σ, say by $\mathbf{x}_\beta(t; \mathbf{x}_0, \sigma)$ and $\mathbf{x}(t; \mathbf{x}_0, \sigma)$, respectively, are related as follows:

$$\mathbf{x}_\beta(t; \mathbf{x}_0, \sigma) = e^{\beta t}\mathbf{x}(t; \mathbf{x}_0, \sigma), \qquad \forall\ t \geq 0.$$

Notice that $\bar{\beta}$ coincides with the largest Lyapunov exponent of the switched system and is connected with the existence of extremal trajectories along which a Barabanov norm is constant. For a thorough discussion on this topic the interested Reader is referred to Gaye et al. [2013].

3.2 Stability under dwell-time

Up to now, we have considered the exponential stability of the positive switched system (3.1) under the assumption that the switching signals σ take values in $\mathcal{D}_0$, namely they are right continuous and have a finite number of switching instants in every finite interval, but they are otherwise arbitrary. We now address the exponential stability problem with (hard) dwell-time, namely we investigate under what conditions exponential stability may be ensured for every switching signal σ belonging to the class

$$\begin{aligned} \mathcal{D}_T \quad := \quad & \{\sigma : \mathbb{R}_+ \to \{1, 2, \ldots, M\} : \text{ for every pair of consecutive} \\ & \text{switching times } t_k \text{ and } t_{k+1} \text{ one has } t_{k+1} - t_k > T\}, \end{aligned}$$

for some fixed $T \in \mathbb{R}_+, T > 0$. This leads to the following definition.

Definition 3.6. The positive switched system (3.1) is said to be *uniformly exponentially stable with dwell-time* T if there exist real constants $C > 0$ and $\beta > 0$ such that all the solutions of (3.1) satisfy

$$\|\mathbf{x}(t;\mathbf{x}(0),\sigma)\| \leq Ce^{-\beta t}\|\mathbf{x}(0)\|, \tag{3.17}$$

for every $\mathbf{x}(0) \in \mathbb{R}^n_+, t \in \mathbb{R}_+$ and every switching signal $\sigma \in \mathcal{D}_T$.

First of all, it is well-known already for nonpositive switched systems (3.1), see e.g. Hespanha and Morse [1999], that if all matrices $A_i, i \in \{1, 2, \dots, M\}$, are Hurwitz, then for large values of T uniform exponential stability with dwell-time T is always ensured. This is obviously related to the fact that for sufficiently large $t > 0$, each map $e^{A_i t}$ becomes a contraction. For positive switched systems (3.1) this amounts to saying that for every $\varepsilon > 0$ and every $i \in \{1, 2, \dots, M\}$ there exists $T_i > 0$ such that, for every $t \geq T_i$ and every $\mathbf{x}_0 > 0$, one has $0 < e^{A_i t}\mathbf{x}_0 < \varepsilon \mathbf{x}_0$. This ensures that stability with dwell-time $T := \max_i T_i$ is surely ensured. On the other hand, for every $T > 0$,

$$\{\sigma : \exists\ i \in \{1, 2, \dots, M\} \text{ such that } \sigma(t) = i, \forall\ t \in \mathbb{R}_+\} \subset \mathcal{D}_T,$$

and hence a necessary condition for stability with some dwell-time $T > 0$ is that the Metzler matrices A_i are Hurwitz $\forall i \in \{1, \dots, M\}$. So, the Hurwitz property of the matrices A_i, $\forall i \in \{1, \dots, M\}$, is a necessary and sufficient condition for the existence of some $T > 0$ such that the switched system (3.1) is exponentially stable with dwell-time T. Therefore in the following of this section we will assume that all matrices A_i are Metzler Hurwitz, and the only meaningful problem to address is that of determining the minimum dwell-time $T_{\min} > 0$, i.e. the infimum value of T for which the system is exponentially stable in $\mathcal{D}_T$. This is a rather challenging problem, and we will try to at least provide an upper bound on it, by verifying whether for a given $T > 0$ the positive switched system (3.1) is uniformly exponentially stable with dwell-time T. Following the same rationale as in Proposition 3.3, a necessary and sufficient condition for stability in $\mathcal{D}_T$ can be worked out in terms of time-varying polyhedral Lyapunov functions. Indeed (see Blanchini and Colaneri [2010]), the following result can be proven.

Theorem 3.9. The following statements are equivalent.

i) the continuous-time positive switched system (3.1) is uniformly exponentially stable for every $\sigma \in \mathcal{D}_T$;

ii) there exist $s \in \mathbb{Z}_+$, full row rank nonnegative $n \times s$ matrices X_i, $s \times s$ square Metzler matrices P_i and $s \times s$ square nonnegative matrices R_{ij} such that

$$\begin{aligned} A_i X_i &= X_i P_i, \quad P_i \mathbf{1}_s \ll 0, && (3.18) \\ e^{A_i T} X_j &= X_i R_{ij}, \quad R_{ij} \mathbf{1}_s \ll \mathbf{1}_s; && (3.19) \end{aligned}$$

iii) there exist $s \in \mathbb{Z}_+$, full column rank nonnegative $s \times n$ matrices W_i, $s \times s$ square Metzler matrices Q_i and $s \times s$ square nonnegative matrices Z_{ij} such that

$$\begin{aligned} W_i A_i &= Q_i W_i, \quad \mathbf{1}_s^\top Q_i \ll 0, && (3.20) \\ W_j e^{A_i T} &= Z_{ij} W_i, \quad \mathbf{1}_s^\top Z_{ij} \ll \mathbf{1}_s^\top. && (3.21) \end{aligned}$$

The rationale of Proposition 3.9 is to relate the exponential stability with dwell-time to the existence of a time-varying polyhedral Lyapunov function that decreases when no switchings occur and exhibits negative jumps between two consecutive switching instants (Branicky functions, see Branicky [2007]). It goes without saying that this proposition allows to compute (at the cost of demanding computational burden, though) the minimum dwell-time $T_{\min}$. Notice that for $T \to 0^+$ the conditions of Proposition 3.9 boil down to those of Proposition 3.3 for stability in $\mathcal{D}_0$. For converse Lyapunov theorems for linear parameter-varying systems and linear switched systems under dwell time the Reader is referred to Wirth [2005].

We will now discuss the stability problem using easy linear programming tools that produce upper bounds on the minimum dwell-time. A first simple way to study the stability properties under dwell time is to resort to well-known bounds on the norm of the exponential of a Hurwitz matrix. Indeed, for every $i \in \{1, 2, \ldots, M\}$, let $\alpha_i \geq 0$ and $\beta_i > 0$ be such that $\|e^{A_i t}\| < e^{\alpha_i - \beta_i t}$, for every $t \geq 0$. Then system (3.1) is stable for every $\sigma \in \mathcal{D}_T$, provided that $T > \max_{i \in \{1,2,\ldots,M\}} \frac{\alpha_i}{\beta_i}$.

Another, generally less conservative, way to explore the problem of stability under dwell-time is to investigate for which values of the dwell-time $T > 0$ we can find linear or time-varying linear copositive

Lyapunov functions that prove the exponential stability of the switched system (3.1) for every $\sigma \in \mathcal{D}_T$. In the following theorem, time-varying linear Lyapunov functions are considered.

Theorem 3.10. Assume that for some $T > 0$, there exist strictly positive vectors $\mathbf{v}_i \in \mathbb{R}^n_+, i \in \{1, 2, \dots, M\}$, such that

$$\begin{aligned} \mathbf{v}_i^\top A_i &\ll 0 \qquad \forall i \in \{1, 2, \dots, M\}, && (3.22) \\ \mathbf{v}_j^\top e^{A_i T} - \mathbf{v}_i^\top &\ll 0 \qquad \forall i, j \in \{1, 2, \dots, M\}, i \neq j. && (3.23) \end{aligned}$$

Then the switched system (3.1) is exponentially stable for each $\sigma \in \mathcal{D}_T$.

Proof. We first observe that condition (3.22) implies

$$\mathbf{v}_i^\top e^{A_i \tau} \ll \mathbf{v}_i^\top \qquad \forall \tau > 0, \forall i \in \{1, 2, \dots, M\}. \tag{3.24}$$

Since $\mathbf{v}_i \gg 0$ and (3.22) holds, $V(\mathbf{x}(t), \sigma(t)) = \mathbf{v}_{\sigma(t)}^\top \mathbf{x}(t)$ is a time-varying linear copositive function. We want to prove that this is a Lyapunov function for the system (3.1), once we restrict the switching signals σ to belong to $\mathcal{D}_T$. First, we note that the fact that the inequalities (3.22) and (3.23) hold true ensures that a sufficiently small $\varepsilon > 0$ can be found such that

$$\mathbf{v}_i^\top A_i \ll -\varepsilon \mathbf{v}_i^\top \qquad \forall i \in \{1, 2, \dots, M\}, \tag{3.25}$$

and

$$\mathbf{v}_j^\top e^{A_i T} \ll e^{-\varepsilon T} \mathbf{v}_i^\top, \qquad \forall i, j \in \{1, 2, \dots, M\}, i \neq j \tag{3.26}$$

hold true. For any $\sigma \in \mathcal{D}_T$ and any initial condition, at the switching times we have

$$\begin{aligned} V(\mathbf{x}(t_{k+1}), \sigma(t_{k+1})) &= \mathbf{v}_{\sigma(t_{k+1})}^\top \mathbf{x}(t_{k+1}) \\ &= \mathbf{v}_{\sigma(t_{k+1})}^\top e^{A_{\sigma(t_k)}(t_{k+1} - t_k)} \mathbf{x}(t_k) \\ &= \mathbf{v}_{\sigma(t_{k+1})}^\top e^{A_{\sigma(t_k)} T} e^{A_{\sigma(t_k)}(t_{k+1} - t_k - T)} \mathbf{x}(t_k) \qquad (3.27) \\ &< e^{-\varepsilon T} \mathbf{v}_{\sigma(t_k)}^\top e^{A_{\sigma(t_k)}(t_{k+1} - t_k - T)} \mathbf{x}(t_k) \\ &< e^{-\varepsilon T} \mathbf{v}_{\sigma(t_k)}^\top \mathbf{x}(t_k) = e^{-\varepsilon T} V(\mathbf{x}(t_k), \sigma(t_k)) \end{aligned}$$

where the first inequality follows from (3.26) and the second inequality from (3.24). This ensures that

$$V(\mathbf{x}(t_k),\sigma(t_k)) < e^{-\varepsilon T k} V(\mathbf{x}(0),\sigma(0)) \quad \forall k \in \mathbb{Z}_+. \tag{3.28}$$

On the other hand, by (3.24), for every $t \in [t_k, t_{k+1})$ one has

$$V(\mathbf{x}(t),\sigma(t)) = \mathbf{v}_{\sigma(t_k)}^\top e^{A_{\sigma(t_k)}(t-t_k)}\mathbf{x}(t_k) < \mathbf{v}_{\sigma(t_k)}^\top \mathbf{x}(t_k) = V(\mathbf{x}(t_k),\sigma(t_k)),$$

where we used again (3.24). This implies that the switched system (3.1) is uniformly exponentially stable for each $\sigma \in \mathcal{D}_T$. □

Notice that if $T \to 0^+$, the inequalities (3.23) become $\mathbf{v}_i^\top \leq \mathbf{v}_j^\top$ and they can be satisfied for every pair of indices i and j if and only if all vectors $\mathbf{v}_i$ coincide. So, if $\mathbf{v}_i = \mathbf{v}$ for every $i \in \{1,2,\dots,M\}$, conditions (3.22) become $\mathbf{v}^\top A_i \ll 0$ for every $i \in \{1,2,\dots,M\}$, and we obtain the usual linear copositive Lyapunov function, whose existence represents a sufficient condition for the exponential stability under arbitrary switching ($\sigma \in \mathcal{D}_0$).

On the other hand, as all the matrices A_i are Hurwitz, it is easily seen that for sufficiently large T both sets of inequalities are feasible, as they become $\mathbf{v}_i \gg 0$ and $\mathbf{v}_i^\top A_i \ll 0$, for each $i \in \{1,2,\dots,M\}$. This is in agreement with what we previously said about the fact that exponential stability with dwell-time is always possible provided that T is large enough.

Finally, notice that feasibility of (3.22) and (3.23) for $T = T_1$ implies their feasibility for $T = T_2 \geq T_1$. So, an upper bound on the minimum dwell-time[9] can be obtained as the solution of the optimization problem:

$$\begin{aligned} T_{\min}^{left} \;:=\; & \inf\,\{T \geq 0 : \exists\ \mathbf{v}_i \gg 0, i \in \{1,2,\dots,M\}, \text{ such that} \\ & (3.22) \text{ and } (3.23) \text{ hold}\}. \end{aligned} \tag{3.29}$$

To solve this problem one can start with a large value of T (an upper bound on it can be easily derived in terms of the exponential matrices

[9] Keep in mind that Theorem 3.10 provides only a sufficient condition, so this method will not necessarily lead to the minimum dwell-time $T_{\min}$. Sufficient conditions based on larger classes of copositive functions may lead to less conservative upper bounds on $T_{\min}$.

$e^{A_i t}, i \in \{1, 2, \dots, M\}$, as described before Theorem 3.10) and then check the feasibility of (3.22) and (3.23) when decreasing T via linear programming. Clearly, $T_{\min}^{left} \geq T_{\min}$.

In light of Remark 3.3, one can derive sufficient conditions for the system to be exponentially stable for each $\sigma \in \mathcal{D}_T$, by simply transposing the conditions of Theorem 3.10, and hence by ensuring that the dual system (3.15) is exponentially stable with dwell-time T. Here we give a proof of this result based on time-varying polyhedral copositive Lyapunov functions.

Theorem 3.11. Assume that for some $T > 0$, there exist strictly positive vectors $\boldsymbol{\xi}_i \in \mathbb{R}^n_+, i \in \{1, 2, \dots, M\}$, such that

$$A_i \boldsymbol{\xi}_i \ll 0 \qquad \forall i \in \{1, 2, \dots, M\}, \tag{3.30}$$

and

$$e^{A_i T} \boldsymbol{\xi}_j - \boldsymbol{\xi}_i \ll 0 \qquad \forall i, j \in \{1, 2, \dots, M\}, i \neq j. \tag{3.31}$$

Then the switched system (3.1) is exponentially stable for each $\sigma \in \mathcal{D}_T$.

Proof. Notice, first, that as in the proof of Theorem 3.10, a sufficiently small $\varepsilon > 0$ can be found such that the inequalities (3.30) and (3.31) imply also

$$A_i \boldsymbol{\xi}_i \ll -\varepsilon \boldsymbol{\xi}_i \qquad \forall i \in \{1, 2, \dots, M\}, \tag{3.32}$$

$$e^{A_i T} \boldsymbol{\xi}_j \ll e^{-\varepsilon T} \boldsymbol{\xi}_i \qquad \forall i, j \in \{1, 2, \dots, M\}, i \neq j. \tag{3.33}$$

Note, also that (3.32) implies

$$e^{A_i \tau} \boldsymbol{\xi}_i \ll e^{-\varepsilon \tau} \boldsymbol{\xi}_i, \quad \forall \tau > 0, \forall i \in \{1, 2, \dots, M\}, \tag{3.34}$$

and putting together (3.34) and (3.33) we can prove that (3.33) holds true also when T is replaced by any $t \geq T$. Consider an arbitrary $\sigma \in \mathcal{D}_T$ and an arbitrary initial condition $\mathbf{x}_0$. The function

$$V(\mathbf{x}(t), \sigma(t)) := \max_{r=1,2,\dots,n} \frac{[\mathbf{x}(t)]_r}{[\boldsymbol{\xi}_{\sigma(t)}]_r}$$

is a candidate time-varying polyhedral copositive Lyapunov function for system (3.1). Indeed, assume that $\sigma(t_k) = i$, $\sigma(t_{k-1}) = j$ and notice that

$$\mathbf{x}(t) \leq \boldsymbol{\xi}_{\sigma(t)} V(\mathbf{x}(t), \sigma(t)).$$

If

$$\bar{r} = \arg \max_{r=1,2,\ldots,n} \frac{[\mathbf{x}(t_{k+1}^-)]_r}{[\boldsymbol{\xi}_{\sigma(t_{k+1}^-)}]_r},$$

then

$$\begin{aligned}
V(\mathbf{x}(t_{k+1}^-), \sigma(t_{k+1}^-)) &= \frac{1}{[\boldsymbol{\xi}_i]_{\bar{r}}} \mathbf{e}_{\bar{r}}^\top e^{A_i(t_{k+1}-t_k-T)} e^{A_i T} \mathbf{x}(t_k) \\
&\leq \frac{1}{[\boldsymbol{\xi}_i]_{\bar{r}}} \mathbf{e}_{\bar{r}}^\top e^{A_i(t_{k+1}-t_k-T)} e^{A_i T} \boldsymbol{\xi}_j V(\mathbf{x}, \sigma, t_k^-) \\
&< \frac{1}{[\boldsymbol{\xi}_i]_{\bar{r}}} \mathbf{e}_{\bar{r}}^\top e^{A_i(t_{k+1}-t_k-T)} \boldsymbol{\xi}_i V(\mathbf{x}, \sigma, t_k^-) e^{-\varepsilon T} \qquad (3.35) \\
&< \frac{1}{[\boldsymbol{\xi}_i]_{\bar{r}}} \mathbf{e}_{\bar{r}}^\top \boldsymbol{\xi}_i V(\mathbf{x}, \sigma, t_k^-) e^{-\varepsilon T} \\
&= V(\mathbf{x}(t_k^-), \sigma(t_k^-)) e^{-\varepsilon T}.
\end{aligned}$$

On the other hand, for any $t \in [t_k, t_{k+1})$, if we set, again,

$$\bar{r} = \arg \max_{r=1,2,\ldots,n} \frac{[\mathbf{x}(t)]_r}{[\boldsymbol{\xi}_{\sigma(t)}]_r},$$

then

$$\begin{aligned}
V(\mathbf{x}(t), \sigma(t)) &= \frac{1}{[\boldsymbol{\xi}_i]_{\bar{r}}} \mathbf{e}_{\bar{r}}^\top e^{A_i(t-t_k)} \mathbf{x}(t_k) \\
&\leq \frac{1}{[\boldsymbol{\xi}_i]_{\bar{r}}} \mathbf{e}_{\bar{r}}^\top e^{A_i(t-t_k)} \boldsymbol{\xi}_i V(\mathbf{x}(t_k^-), \sigma(t_k^-)) \\
&< \frac{1}{[\boldsymbol{\xi}_i]_{\bar{r}}} \mathbf{e}_{\bar{r}}^\top \boldsymbol{\xi}_i V(\mathbf{x}(t_k^-), \sigma(t_k^-)) = V(\mathbf{x}(t_k^-), \sigma(t_k^-)) e^{-\varepsilon(t-t_k)}.
\end{aligned}$$

This ensures the exponential stability of the switched system (3.1), with dwell-time T. □

As for Theorem 3.10, the previous result suggests a way to find an upper bound on the minimum dwell-time $T_{\min}$, that can be obtained by minimizing the value of T for which strictly positive vectors $\boldsymbol{\xi}_i, i \in \{1, 2, \ldots, M\}$, can be found satisfying the inequalities

(3.30) and (3.31). If $T_{\min}^{right}$ is such a minimum value, then clearly $T_{\min} \leq \min\{T_{\min}^{left}, T_{\min}^{right}\}$.

The next theorem presents a sufficient condition for exponential stability with dwell-time by making use of time-varying quadratic Lyapunov functions $V(\mathbf{x}(t), \sigma(t)) = \mathbf{x}(t)^\top P_{\sigma(t)}\mathbf{x}(t)$, see Geromel and Colaneri [2006].

Theorem 3.12. Assume that for some $T > 0$, there exist positive definite matrices $P_i = P_i^\top \succ 0$, $i = 1, 2, \dots, M$, such that

$$A_i^\top P_i + P_i A_i \prec 0 \qquad \forall i \in \{1, \dots, M\}, \tag{3.36}$$

and

$$e^{A_i^\top T} P_j e^{A_i T} - P_i \prec 0 \qquad \forall i, j \in \{1, \dots, M\}, i \neq j. \tag{3.37}$$

Then the switched system (3.1) is exponentially stable for each $\sigma \in \mathcal{D}_T$.

Again, the minimum value of T, say $T_{\min}^{quad}$, for which the inequalities (3.36) and (3.37) are feasible is an upper bound on the minimum dwell-time. It is worth noticing, however, that one can exploit the results in Chesi et al. [2012] and find the minimum dwell-time by solving LMIs involving higher dimensional matrices coming from the description of the switched system (3.1) in an extended space obtained by using Kronecker products (see the previous Remark 3.5).

Example 3.8. Consider again Example 3.1. We have already seen that the system is not stable under arbitrary switching. The minimum time $T > 0$ for which strictly positive vectors $\mathbf{v}_i$ can be found such that conditions (3.22) and (3.23) in Theorem 3.10 hold is $T_{\min}^{left} = 1$. The minimum time $T > 0$ for which conditions (3.30) and (3.31) of Theorem 3.11 can be fulfilled is the same, i.e. $T_{\min}^{right} = 1$. On the other hand, as far as time-varying quadratic Lyapunov functions are concerned, one can check that conditions (3.36) and (3.37) of Theorem 3.12 can be satisfied for $T \geq 0.724$, that is $T_{\min}^{quad} = 0.724$. It is easy to check that this value is also the real minimum dwell-time, since a $2T$-periodic switching signal of the type $\sigma(t) = 1, t \in [0, T]$, $\sigma(t) = 2, t \in [T, 2T)$, for

< 0.724 destabilizes the system.
We can evaluate other upper bounds by considering $\lambda_1 = -0.1747$ and $\lambda_2 = -0.2134$, the Frobenius eigenvalues associated with A_1 and A_2, respectively. Let $\mathbf{v}_1^\top = \begin{bmatrix} 0.2209 & 0.2798 & 0.4994 \end{bmatrix}$ and $\mathbf{v}_2^\top = \begin{bmatrix} 0.1236 & 0.0630 & 0.8134 \end{bmatrix}$ be two left Frobenius eigenvectors corresponding to such eigenvalues. Therefore $\mathbf{v}_i^\top A_i = \lambda_i \mathbf{v}_i^\top$, $i = 1, 2$, imply $\mathbf{v}_i^\top e^{A_i t} = e^{\lambda_i t}\mathbf{v}_i^\top$, $i = 1, 2$, so that

$$\|e^{A_i t}\|_1 \le e^{\lambda_i t} \frac{\max_k [\mathbf{v}_i]_k}{\min_k [\mathbf{v}_i]_k}, \quad t \ge 0, \quad i = 1, 2.$$

So, by resorting to a reasoning similar to the one provided in Hespanha and Morse [1999], we can show that an upper bound on the minimum dwell-time is

$$T_1^* = \max_{i=1,2} \frac{\log(\frac{\max_k [\mathbf{v}_i]_k}{\min_k [\mathbf{v}_i]_k})}{-\lambda_i} = 11.98.$$

On the other hand, by making use of the right Frobenius eigenvectors, i.e. $A_i \boldsymbol{\xi}_i = \lambda_i \boldsymbol{\xi}_i$, $i = 1, 2$, we obtain

$$\|e^{A_i t}\|_\infty \le e^{\lambda_i t} \frac{\max_k [\boldsymbol{\xi}_i]_k}{\min_k [\boldsymbol{\xi}_i]_k}, \quad t \ge 0, \quad i = 1, 2$$

so that

$$T_\infty^* = \max_{i=1,2} \frac{\log(\frac{\max_k [\boldsymbol{\xi}_i]_k}{\min_i [\boldsymbol{\xi}_i]_k})}{-\lambda_i} = 6.44.$$

Finally, by optimizing the parameters $\alpha_i \ge 0$ and $\beta_i > 0$ in the bounds $\|e^{A_i t}\|_2 < e^{\alpha_i - \beta_i t}$, $i = 1, 2$, we obtain $T_2^* = \max_i \frac{\alpha_i}{\beta_i} \simeq 5$. All these bounds T_1^*, T_2^* and T_∞^* are greater than the upper bounds previously obtained through time-varying linear or quadratic Lyapunov functions.

Remark 3.7. [10] Theorems 3.10, 3.11 and 3.12 provide sufficient conditions (and simple numerical algorithms) to check stability under dwell time $T > 0$. It is worth noticing that feasibility of such conditions for increasingly smaller values of T does not imply stability under arbitrary

[10] The Authors are indebted with an anonymous Reviewer for the example presented in this Remark, showing that stability corresponding to any periodic switching signal does not imply stability under arbitrary switching.

switching. Take, for instance, the switched system with matrices

$$A_1 = \begin{bmatrix} -1 & 1.5 \\ 0.5 & -1 \end{bmatrix}, \quad A_2 = A_1^\top.$$

It is easy to see that inequalities (3.36) and (3.37) in Theorem 3.12 are feasible for any $T > 0$. Therefore for any *periodic* switching signal the state trajectory converges to zero, independently of the initial condition. As $T \to 0$ the two positive definite solutions P_1 and P_2 tend to coincide with a positive definite matrix P solving $A_i^\top P + PA_i \leq 0$, but not solving inequality (3.37). In fact, inequality (3.37) is not satisfied by a single $P > 0$. Moreover, the system is not uniformly exponentially stable (under arbitrary switching) since the convex combination $0.5A_1 + 0.5A_2$ is not a Hurwitz matrix, recall Theorem 3.7. We want to show that the system is destabilized in a sliding mode. To this purpose let s be an arbitrary positive number, and consider the T/s-periodic switching signal whose restriction to the time interval $[0, T/s)$ is $\sigma(t) = 1$, for $t \in [0, T/(2s))$ and $\sigma(t) = 2$ for $t \in [T/(2s), T/s)$. The state of the system at time T corresponding to this periodic switching signal is

$$\mathbf{x}(T) = \left(e^{A_2 T/(2s)} e^{A_1 T/(2s)}\right)^s \mathbf{x}_0.$$

The limit for $s \to +\infty$ can be computed easily, see Cohen [1981], Elliott [2009], as

$$\lim_{s \to +\infty} x(T) = e^{0.5(A_1 + A_2)T},$$

and it coincides with the solution at time T of the bilinear system $\dot{\mathbf{x}}(t) = ([\mathbf{u}]_1 A_1 + [\mathbf{u}]_2 A_2)\mathbf{x}(t)$ with $\mathbf{u} = [0.5 \ 0.5]^\top$. The sliding mode corresponds to the previous periodic switching law for $s \to +\infty$.

3.3 Parametrization of state-feedback controllers

In this section we briefly discuss the effect of a memoryless state-feedback law applied to a positive switched system described as

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + G_{\sigma(t)}\mathbf{u}(t), \tag{3.38}$$

where the input signal $\mathbf{u} \in \mathbb{R}^{m_u}$ has been added together with the relevant matrices G_i, assumed to be nonnegative for every $i = 1, 2, \ldots, M$.

The state-feedback control takes the form

$$\mathbf{u}(t) = K_{\sigma(t)}\mathbf{x}(t),$$

where $K_i \in \mathbb{R}^{m_u \times n}$ for every i. Notice that both the switching signal $\sigma(t)$ and the state variable $\mathbf{x}(t)$ are supposed to be measurable. For brevity, we only focus on design problems that aim at preserving positivity and at ensuring stability under arbitrary switching. The problem of state-feedback stabilization, under the positivity constraint on the resulting feedback system, can be described as follows: find matrices K_i, $i = 1, 2, \dots, M$, of size $m_u \times n$, such that $A_i + G_i K_i$ $i = 1, 2, \dots, M$, are Metzler matrices, and the closed loop system

$$\dot{\mathbf{x}}(t) = \left(A_{\sigma(t)} + G_{\sigma(t)} K_{\sigma(t)}\right) \mathbf{x}(t) \tag{3.39}$$

is uniformly exponentially stable in $\mathcal{D}_0$.
A little thought reveals that this problem is ill-posed if one requires that K_i, $i = 1, 2, \dots, M$, are nonnegative matrices. If the matrices K_i are nonnegative, then $A_i + G_i K_i \geq A_i$ for every $i \in \{1, 2, \dots, M\}$, and by the monotonicity of positive systems this ensures that for every choice of $\mathbf{x}_0 \in \mathbb{R}^n_+$ and every switching signal $\sigma \in \mathcal{D}_0$

$$\mathbf{x}_K(t; \mathbf{x}_0, \sigma) \geq \mathbf{x}(t; \mathbf{x}_0, \sigma), \qquad \forall t \geq 0,$$

where $\mathbf{x}_K(t; \mathbf{x}_0, \sigma)$ and $\mathbf{x}(t; \mathbf{x}_0, \sigma)$ denote the state trajectories of the closed loop and the open loop systems, respectively, corresponding to $\mathbf{x}_0$ and σ. So, system (3.39) is exponentially stable in $\mathcal{D}_0$ only if the open loop system is already exponentially stable in $\mathcal{D}_0$, and this implies that it is not possible to stabilize an unstable positive switched system through a state-feedback law with nonnegative gains. However, one can relax the positivity constraint, by allowing matrices K_i, $i = 1, 2, \dots, M$, (and hence the input $\mathbf{u}(t)$) to have nonpositive entries, but requiring at the same time that the state trajectory remains in the positive orthant for any initial state and any σ. This amounts to impose only that the gains K_i, $i = 1, 2, \dots, M$, are such that the closed loop matrices $A_i + G_i K_i$ are Metzler for every i. This leads to the following linear constraints on the entries of K_i:

$$\mathbf{e}_r^\top G_i K_i \mathbf{e}_p \geq -[A_i]_{rp}, \quad i = 1, 2, \dots, M, \quad r \neq p = 1, 2, \dots, n. \tag{3.40}$$

Therefore, the following result can be stated:

Theorem 3.13. There exist K_i, $i = 1, 2, \ldots, M$, such that $A_i + G_i K_i$, $i = 1, 2, \ldots, M$, are Metzler matrices and the closed loop system (3.39) is exponentially stable in $\mathcal{D}_0$ if there exist a strictly positive vector $\boldsymbol{\xi} \in \mathbb{R}^n_+$ and vectors $\mathbf{h}_i^r \in \mathbb{R}^{m_u}$, $i = 1, 2, \ldots, M$, $r = 1, 2, \ldots, n$, such that

$$\begin{aligned} A_i \boldsymbol{\xi} + G_i \sum_{p=1}^{n} \mathbf{h}_i^p \quad &\ll \quad 0, && (3.41) \\ \mathbf{e}_r^\top G_i \mathbf{h}_i^p + [A_i]_{rp} \mathbf{e}_p^\top \boldsymbol{\xi} \quad &\geq \quad 0, && (3.42) \end{aligned}$$

for every $i = 1, 2, \ldots, M$, and $r \neq p = 1, 2, \ldots, n$. The matrices K_i, $i = 1, 2, \ldots, M$, are then obtained as

$$K_i \mathbf{e}_p = (\mathbf{e}_p^\top \boldsymbol{\xi})^{-1} \mathbf{h}_i^p, \quad p = 1, 2, \ldots, n. \tag{3.43}$$

Proof. Assume that (3.41) and (3.42) are feasible. Then, construct matrices K_i according to (3.43). Therefore, for $r \neq p$, it must hold

$$\begin{aligned} [A_i + G_i K_i]_{rp} &= \mathbf{e}_r^\top \left(A_i + G_i K_i\right) \mathbf{e}_p \\ &= [A_i]_{rp} + \mathbf{e}_r^\top G_i \; K_i \mathbf{e}_p \\ &= (\mathbf{e}_p^\top \boldsymbol{\xi})^{-1} \left([A_i]_{rp} (\mathbf{e}_p^\top \boldsymbol{\xi}) + \mathbf{e}_r^\top G_i \mathbf{h}_i^p \right). \end{aligned}$$

So, by (3.42), $[A_i + G_i K_i]_{rp} \geq 0$ for every $r \neq p$. This means that the matrices $A_i + G_i K_i$ are Metzler, for every i. Moreover,

$$\begin{aligned} 0 \quad \gg \quad & A_i \boldsymbol{\xi} + G_i \sum_{p=1}^{n} \mathbf{h}_i^p \\ = \quad & A_i \boldsymbol{\xi} + G_i \sum_{p=1}^{n} K_i \mathbf{e}_p \mathbf{e}_p^\top \boldsymbol{\xi} \\ = \quad & \left(A_i + G_i K_i\right) \boldsymbol{\xi} \end{aligned}$$

so that the closed loop system (3.39) is exponentially stable in $\mathcal{D}_0$ in view of Proposition 3.4. $\square$

It is worth noticing that Theorem 3.13 does not provide a full parametrization of all state-feedback control laws ensuring positivity and stability of the closed loop system, since it relies on the existence of a linear copositive Lyapunov function for the dual system (3.15). However, the advantage of the algorithm is the need for very simple linear programs, testing feasibility of (3.41) and (3.42). A full parametrization for positive time-invariant systems, i.e. for the case when $A_i = A$ and $B_i = B$, for every i, is immediately obtained from the theorem above by letting $K = K_i$, for every i. Indeed, in such a case the theorem provides a necessary and sufficient condition, in terms of linear programming, for the existence of K such that $A + BK$ is Hurwitz and Metzler, see Briat [2013].

4

Input-output performances

This chapter is devoted to the characterization of certain input/output norms of a positive switched system taking the form:

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{w}(t), \tag{4.1}$$

$$\mathbf{z}(t) = C_{\sigma(t)}\mathbf{x}(t) + D_{\sigma(t)}\mathbf{w}(t), \tag{4.2}$$

where the matrices A_i are Metzler and the matrices B_i, C_i and D_i are nonnegative, for every $i \in \{1, 2, \dots, M\}$. We let m denote the size of the nonnegative disturbance $\mathbf{w}(t)$ and p the size of the nonnegative output $\mathbf{z}$. If not differently specified, the initial condition $\mathbf{x}(0)$ is supposed to be zero and the nature of the disturbance $\mathbf{w}$ (the norm of the disturbance we are interested in) is specified from time to time. In particular, we consider (i) the $\mathcal{L}_1$ norm of the output associated with an impulse function, (ii) the induced norm in $\mathcal{L}_1$, (iii) the induced norm in $\mathcal{L}_\infty$, and (iv) the induced norm in $\mathcal{L}_2$. Both the sets $\mathcal{D}_0$ and $\mathcal{D}_T$, introduced in Chapter 3, will be considered.

In the following we will make use of the transfer functions

$$G_i(s) = C_i(sI - A_i)^{-1}B_i + D_i, \quad i = 1, 2, \dots, M, \tag{4.3}$$

of the time invariant systems corresponding to the constant switching signals $\sigma(t) = i$, $\forall\ t \geq 0$. Since constant signals $\sigma(t)$ belong to $\mathcal{D}_0$

and to $\mathcal{D}_T$, for each $T \geq 0$, it is clear that the worst performance associated with constant switching signals $\sigma(t)$ provides a lower bound on the worst performance of the switched system (4.1)-(4.2). The theory of performances of time-invariant positive systems is now well assessed and can be found in Son and Hinrichsen [1996] for the $\mathcal{L}_2$ induced norm (in that reference the stability radius of uncertain positive systems is studied), and Rantzer [2011], Briat [2013] for the $\mathcal{L}_1$ and $\mathcal{L}_\infty$ induced norms. In such papers it is shown that such performances are associated with appropriate numerical norms of $G_i(0)$ and algorithms are provided for their computation.

4.1 $\mathcal{L}_1$ norm of the impulse response

It is well-known that the state response of a system to an impulse in the input can be studied by investigating the unforced state response corresponding to a specific initial condition. Therefore, in order to investigate the $\mathcal{L}_1$ norm of the impulse response of (4.1)-(4.2), it is convenient to first consider the autonomous positive switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{4.4}$$

with A_i, $i \in \{1, 2, \ldots, M\}$, Metzler matrices and $\mathbf{x}_0$ a positive vector, and to evaluate the optimal value of certain linear functions associated with it. Specifically, we introduce the linear index

$$J(\sigma, \mathbf{x}_0) := \int_0^{+\infty} \mathbf{q}_{\sigma(t)}^\top \mathbf{x}(t) dt, \tag{4.5}$$

where $\sigma(t)$ is the same switching signal acting on system (4.4) and $\mathbf{q}_i$, $i \in \{1, 2, \ldots, M\}$, are positive vectors. If the system is exponentially stable under arbitrary switching, this index is finite for any $\sigma \in \mathcal{D}_0$. Therefore it makes sense to investigate the maximization problem

$$J_0(\mathbf{x}_0) := \sup_{\sigma \in \mathcal{D}_0} J(\sigma, \mathbf{x}_0). \tag{4.6}$$

Remark 4.1. Similarly to what we did in the previous chapter, we underline that problem (4.6), with $J(\sigma, \mathbf{x}_0)$ defined as in (4.5), can be investigated for the positive switched system (4.4) by solving the corresponding optimal control problem for the bilinear system

$$\dot{\mathbf{x}}(t) = A(\mathbf{u}(t))\mathbf{x}(t) = \left(\sum_{i=1}^{M} A_i[\mathbf{u}(t)]_i\right)\mathbf{x}(t), \tag{4.7}$$

corresponding to the (suitably adjusted) cost function

$$J_{li}(\mathbf{x}_0) := \sup_{\mathbf{u}\in\mathcal{U}_{li}^M} \int_0^{+\infty} \sum_{i=1}^{M} \mathbf{q}_i^\top [\mathbf{u}(t)]_i\, \mathbf{x}(t)dt,$$

where $\mathcal{U}_{li}^M$ is the class of locally integrable M-dimensional vector functions taking values in the simplex $\mathcal{A}_M$. This new problem in general has a solution that does not correspond to a switching signal, namely it is not generally true that $\mathbf{u}(t) = \mathbf{e}_{\sigma(t)}$, where $\mathbf{e}_i$ is the ith vector of the canonical basis. On the contrary, it often generates sliding trajectories obtained as limits of high frequency switching signals $\sigma \in \mathcal{D}_0$. $\square$

If the positive switched system (4.4) admits a linear copositive Lyapunov function $\tilde{V}(\mathbf{x}) = \tilde{\mathbf{v}}^\top\mathbf{x}$, with $\tilde{\mathbf{v}} \in \mathbb{R}^n_+, \tilde{\mathbf{v}} \gg 0$, then an upper bound on $J_0(\mathbf{x}_0)$ can be easily determined. Indeed, from $\tilde{\mathbf{v}}^\top A_i \ll 0$, $\forall i \in \{1, 2, \ldots, M\}$, it follows that there exists a sufficiently large positive scalar α such that

$$(\alpha\tilde{\mathbf{v}})^\top A_i + \mathbf{q}_i^\top \ll 0, \quad \forall i = 1, 2, \ldots, M.$$

Clearly, $V(\mathbf{x}) = \alpha\tilde{V}(\mathbf{x}) = (\alpha\tilde{\mathbf{v}})^\top\mathbf{x}$ is in turn an LCLF for the system (4.4). Moreover

$$\begin{aligned} J(\sigma, \mathbf{x}_0) &= \int_0^{+\infty} \mathbf{q}_{\sigma(t)}^\top \mathbf{x}(t)dt < -\alpha \int_0^{+\infty} \tilde{\mathbf{v}}^\top A_{\sigma(t)}\mathbf{x}(t)dt \\ &= -\alpha\tilde{\mathbf{v}}^\top \int_0^{+\infty} \dot{\mathbf{x}}(t)dt = \alpha\tilde{\mathbf{v}}^\top\mathbf{x}_0, \end{aligned}$$

where we made use of the fact that the existence of an LCLF ensures that $\mathbf{x}(t)$ exponentially converges to zero. Therefore we have proved the following result.

Theorem 4.1. Let $\mathbf{q}_i \in \mathbb{R}^n_+$, $i \in \{1, 2, \ldots, M\}$, be given positive vectors, and assume that there exists a strictly positive vector $\mathbf{v} \in \mathbb{R}^n_+$ such that:

$$\mathbf{v}^\top A_i + \mathbf{q}_i^\top \ll 0 \qquad \forall i = 1, \ldots, M. \tag{4.8}$$

Then system (4.4) is exponentially stable for each $\sigma \in \mathcal{D}_0$ and

$$J_0(\mathbf{x}_0) < \mathbf{v}^\top \mathbf{x}_0. \tag{4.9}$$

Remark 4.2. Notice that, as $\mathbf{q}_i \in \mathbb{R}^n_+$, $i \in \{1, 2, \ldots, M\}$, are positive vectors, condition (4.8) is *de facto* equivalent to the existence of an LCLF for the matrices $A_i, i \in \{1, 2, \ldots, M\}$. Moreover, due to the fact that the vector $\mathbf{v}$ does not depend on the running mode i, the inequalities (4.8) lend themselves to be used also for the cost associated with system (4.7) (see Remark 4.1). Indeed, from (4.8) one immediately gets $\mathbf{v}^\top(\sum_{i=1}^M A_i[\mathbf{u}(t)]_i) + \sum_{i=1}^M \mathbf{q}_i^\top[\mathbf{u}(t)]_i \ll 0$, for every $t \geq 0$, and hence the upper bound $\mathbf{v}^\top \mathbf{x}_0$ just obtained holds true also for the optimal control problem previously defined for system (4.7). □

A similar reasoning applies if we restrict the signal σ to the set $\mathcal{D}_T$ of switching signals with dwell-time $T > 0$, in which case the optimal control problem to be solved becomes

$$J_T(\mathbf{x}_0) := \sup_{\sigma \in \mathcal{D}_T} J(\sigma, \mathbf{x}_0).$$

We have already seen in the previous section that exponential stability for all switching signals in $\mathcal{D}_T$ calls for the use of time-varying linear copositive Lyapunov functions instead of linear copositive Lyapunov functions. The following result thus extends Theorem 3.10 to the case of systems with dwell-time and provides an upper bound on $J_T(\mathbf{x}_0)$.

Theorem 4.2. Let $\mathbf{q}_i \in \mathbb{R}^n_+$, $i \in \{1, 2, \ldots, M\}$, be given positive vectors, and assume that there exist strictly positive vectors $\mathbf{v}_i \in \mathbb{R}^n_+$, $i \in \{1, 2, \ldots, M\}$, such that:

$$\mathbf{v}_i^\top A_i + \mathbf{q}_i^\top \ll 0 \qquad \forall i = 1, \ldots, M, \tag{4.10}$$

and

$$\mathbf{v}_j^\top e^{A_i T} - \mathbf{v}_i^\top + \int_0^T \mathbf{q}_i^\top e^{A_i \xi} d\xi \ll 0 \qquad \forall i \neq j = 1, \ldots, M. \tag{4.11}$$

Then system (4.4) is exponentially stable for each $\sigma \in \mathcal{D}_T$ and

$$J_T(\mathbf{x}_0) < \max_{i \in \{1,2,\ldots,M\}} \mathbf{v}_i^\top \mathbf{x}_0. \tag{4.12}$$

Proof. By the nonnegativity of the vectors $\mathbf{q}_i$ and of the exponential matrices $e^{A_i\xi}$, if (4.10) and (4.11) hold then (3.22) and (3.23) hold, in turn. Therefore the system is exponentially stable with dwell-time T. Also, condition (4.10) implies

$$\mathbf{v}_i^\top e^{A_i\tau} + \int_0^\tau \mathbf{q}_i^\top e^{A_i\xi} d\xi \ll \mathbf{v}_i^\top, \quad \forall \tau > 0, \quad \forall i = 1, 2, \ldots, M. \tag{4.13}$$

Let σ be an arbitrary switching signal in $\mathcal{D}_T$, with switching times $0 = t_0 < t_1 < t_2 < \ldots$. Assume that $\sigma(t_k) = i$, $\sigma(t_{k+1}) = j$ and introduce $V(\mathbf{x}(t), \sigma(t)) = \mathbf{v}_{\sigma(t)}^\top \mathbf{x}(t)$. By making use of inequalities (4.10), (4.11) and (4.13), one can show that

$$\begin{aligned}
&V(\mathbf{x}(t_{k+1}), \sigma(t_{k+1})) - V(\mathbf{x}(t_k), \sigma(t_k)) = \left(\mathbf{v}_j^\top e^{A_i(t_{k+1}-t_k)} - \mathbf{v}_i^\top\right) \mathbf{x}(t_k) \\
&= \left(\mathbf{v}_j^\top e^{A_i T} e^{A_i(t_{k+1}-t_k-T)} - \mathbf{v}_i^\top\right) \mathbf{x}(t_k) \\
&< \left(\mathbf{v}_i^\top e^{A_i(t_{k+1}-t_k-T)} - \mathbf{v}_i^\top - \int_0^T \mathbf{q}_i^\top e^{A_i\xi} d\xi e^{A_i(t_{k+1}-t_k-T)}\right) \mathbf{x}(t_k) \\
&< -\left(\int_0^{t_{k+1}-t_k-T} \mathbf{q}_i^\top e^{A_i\xi} d\xi + \int_0^T \mathbf{q}_i^\top e^{A_i\xi} d\xi e^{A_i(t_{k+1}-t_k-T)}\right) \mathbf{x}(t_k) \\
&< -\int_{t_k}^{t_{k+1}} \mathbf{q}_i^\top e^{A_i(\xi-t_k)} d\xi \mathbf{x}(t_k) = -\int_{t_k}^{t_{k+1}} \mathbf{q}_{\sigma(\xi)}^\top \mathbf{x}(\xi) d\xi.
\end{aligned}$$

Summing up for k ranging from 0 to $+\infty$, one gets

$$\sum_{k=0}^{+\infty} [V(\mathbf{x}(t_{k+1}), \sigma(t_{k+1})) - V(\mathbf{x}(t_k), \sigma(t_k))] < -\int_0^{+\infty} \mathbf{q}_{\sigma(\xi)}^\top \mathbf{x}(\xi) d\xi.$$

By the exponential stability, this implies that

$$-\mathbf{v}_{\sigma(0)}^\top \mathbf{x}_0 = -V(\mathbf{x}(0), \sigma(0)) < -\int_0^{+\infty} \mathbf{q}_{\sigma(\xi)}^\top \mathbf{x}(\xi) d\xi = -J(\sigma, \mathbf{x}_0),$$

and hence the conclusion (4.12) follows. □

Remark 4.3. Note that if (4.10) and (4.11) are feasible for a certain $T > 0$, then they are also feasible for $\tau > T$ (and the same $\mathbf{v}_i$, $i \in \{1, 2, \ldots, M\}$). Indeed, (4.10) is independent of time. On the other hand, by making use of (4.11) first and of (4.13) then, we get

$$\begin{aligned}
&\mathbf{v}_j^\top e^{A_i\tau} - \mathbf{v}_i^\top + \int_0^\tau \mathbf{q}_i^\top e^{A_i\xi} d\xi \\
&= \mathbf{v}_j^\top e^{A_i T} e^{A_i(\tau - T)} - \mathbf{v}_i^\top + \int_0^\tau \mathbf{q}_i^\top e^{A_i\xi} d\xi \\
&\ll \mathbf{v}_i^\top e^{A_i(\tau-T)} - \int_0^T \mathbf{q}_i^\top e^{A_i\xi} d\xi \ e^{A_i(\tau-T)} - \mathbf{v}_i^\top + \int_0^\tau \mathbf{q}_i^\top e^{A_i\xi} d\xi \\
&\ll - \int_0^{\tau-T} \mathbf{q}_i^\top e^{A_i\xi} d\xi - \int_0^T \mathbf{q}_i^\top e^{A_i\xi} d\xi e^{A_i(\tau-T)} + \int_0^\tau \mathbf{q}_i^\top e^{A_i\xi} d\xi = 0
\end{aligned}$$

and hence (4.11) holds also at $\tau > T$. On the other hand, if the matrices A_i, $i \in \{1, 2, \ldots, M\}$, are Metzler Hurwitz, then for $T \to +\infty$ the inequalities are always feasible. Indeed, the fact that the matrices A_i are Hurwitz ensures that a family of vectors $\mathbf{v}_i$ satisfying (4.10), $i \in \{1, 2, \ldots, M\}$, can be found. Such vectors also satisfy (4.11). The vector $\bar{\mathbf{v}}_i^\top := \int_0^{+\infty} \mathbf{q}_i^\top e^{A_i\xi} d\xi$ solves the equation $\bar{\mathbf{v}}_i^\top A_i + \mathbf{q}_i^\top = 0$ and as such $\bar{\mathbf{v}}_i^\top = -\mathbf{q}_i^\top A_i^{-1} \ll (\mathbf{v}_i^\top A_i) A_i^{-1} = \mathbf{v}_i$. So, to conclude, $\mathbf{v}_i^\top \gg \int_0^{+\infty} \mathbf{q}_i^\top e^{A_i\xi} d\xi$, $\forall i \in \{1, 2, \ldots, M\}$, which shows the feasibility of (4.11) computed at $T = +\infty$. $\square$

By making use of the previous results, we can now move back to our original problem. Consider the switched system (4.1)-(4.2) and assume that $\mathbf{x}(0) = 0$ and $\mathbf{w}(t) = \delta(t)\mathbf{e}_h$, where $\delta(t)$ is the impulse function (Dirac distribution) and $\mathbf{e}_h$ the hth canonical vector. Let $\mathbf{x}^{[h]}(t)$ and $\mathbf{z}^{[h]}(t)$ denote the corresponding state and output (forced) responses, respectively. We define the $\mathcal{L}_1$ norm of the impulse response of the system, when switching in $\mathcal{D}_0$, as

$$J_{\mathcal{L}_1,0} := \sup_{\sigma \in \mathcal{D}_0} \sum_{h=1}^m \int_0^{+\infty} \mathbf{1}_p^\top \mathbf{z}^{[h]}(t) dt.$$

If we define

$$J_{\mathcal{L}_1}(\sigma, h) := \int_0^{+\infty} \mathbf{1}_p^\top \mathbf{z}^{[h]}(t) dt,$$

then

$$J_{\mathcal{L}_1,0} = \sup_{\sigma \in \mathcal{D}_0} \sum_{h=1}^m J_{\mathcal{L}_1}(\sigma, h).$$

In order to investigate this problem we make use of Theorem 4.1, presented in the first part of this section, upon noticing that the (forced) state evolution $\mathbf{x}^{[h]}(t)$ coincides with the unforced state response associated with the initial state $\mathbf{x}(0) = B_{\sigma(0)}\mathbf{e}_h$, and

$$\mathbf{z}^{[h]}(t) = C_{\sigma(t)}\mathbf{x}^{[h]}(t) + D_{\sigma(t)}\delta(t)\mathbf{e}_h, \qquad t \in \mathbb{R}_+.$$

This implies that for every switching signal σ we have

$$J_{\mathcal{L}_1}(\sigma, h) = \int_0^{+\infty} \mathbf{1}_p^\top C_{\sigma(t)}\mathbf{x}^{[h]}(t)dt + \mathbf{1}_p^\top D_{\sigma(0)}\mathbf{e}_h.$$

Theorem 4.3. Consider system (4.1)-(4.2) and assume that $\mathbf{x}(0) = 0$. If there exists a strictly positive vector $\mathbf{v} \in \mathbb{R}^n_+$ such that:

$$\mathbf{v}^\top A_i + \mathbf{1}_p^\top C_i \ll 0, \qquad \forall i = 1, \dots, M, \tag{4.14}$$

then system (4.4) is exponentially stable for each $\sigma \in \mathcal{D}_0$, and the $\mathcal{L}_1$ performance index $J_{\mathcal{L}_1,0}$ satisfies

$$\max_{i\in\{1,2,\dots,M\}} \mathbf{1}_p^\top G_i(0)\mathbf{1}_m \leq J_{\mathcal{L}_1,0} < \max_{i\in\{1,2,\dots,M\}} \left(\mathbf{v}^\top B_i + \mathbf{1}_p^\top D_i\right)\mathbf{1}_m, \tag{4.15}$$

where $G_i(s)$ is the transfer function of the time invariant system corresponding to $\sigma(t) = i, \forall\, t \geq 0$ (see (4.3)).

Proof. Exponential stability follows from Theorem 4.1, since (4.14) is a special case of (4.8). The lower bound in (4.15) is the maximum value of $J_{\mathcal{L}_1}(\sigma, h)$ over the constant switching signals (which obviously belong to $\mathcal{D}_0$). Indeed, corresponding to $\sigma(t) = i, \forall t \geq 0$, and $\mathbf{w}(t) = \delta(t)\mathbf{e}_h$, one gets

$$J_{\mathcal{L}_1}(\sigma, h) = \int_0^{+\infty} \mathbf{1}_p^\top g^{(i)}(t)\mathbf{e}_h dt = \mathbf{1}_p^\top G_i(0)\mathbf{e}_h,$$

where $g^{(i)}(t)$ is the (matrix) impulse response of the ith subsystem, and $G_i(s)$ its transfer matrix. Therefore

$$J_{\mathcal{L}_1,0} \geq \max_{i\in\{1,2,\dots,M\}} \sum_{h=1}^{m} \mathbf{1}_p^\top G_i(0)\mathbf{e}_h = \max_{i\in\{1,2,\dots,M\}} \mathbf{1}_p^\top G_i(0)\mathbf{1}_m.$$

Finally, by reasoning as in the proof of Theorem 4.1, we consider any switching signal $\sigma \in \mathcal{D}_0$, any assigned $h \in \{1, 2, \dots, m\}$, and introduce the LCLF $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, thus obtaining

$$\begin{aligned} J_{\mathcal{L}_1}(\sigma, h) &= \int_0^{+\infty} \mathbf{1}_p^\top C_{\sigma(t)} \mathbf{x}^{[h]}(t) dt + \mathbf{1}_p^\top D_{\sigma(0)} \mathbf{e}_h \\ &< -\int_0^{+\infty} \mathbf{v}^\top A_{\sigma(t)} \mathbf{x}^{[h]}(t) dt + \mathbf{1}_p^\top D_{\sigma(0)} \mathbf{e}_h \\ &= -\mathbf{v}^\top \int_0^{+\infty} \dot{\mathbf{x}}^{[h]}(t) dt + \mathbf{1}_p^\top D_{\sigma(0)} \mathbf{e}_h \\ &= \mathbf{v}^\top B_{\sigma(0)} \mathbf{e}_h + \mathbf{1}_p^\top D_{\sigma(0)} \mathbf{e}_h. \end{aligned}$$

By summing over all the indices $h \in \{1, 2, \dots, m\}$ and by taking the supremum over all the signals $\sigma \in \mathcal{D}_0$, one gets the second inequality in (4.15). □

Remark 4.4. Similarly to what we did for Theorem 4.1, we notice that the inequalities (4.8) imply $\mathbf{v}^\top (\sum_{i=1}^M A_i [\mathbf{u}(t)]_i) + \sum_{i=1}^M \mathbf{q}_i^\top [\mathbf{u}(t)]_i \ll 0, \forall\, t \geq 0$, thus leading to an upper bound on the linear index associated with the system

$$\dot{\mathbf{x}}(t) = \left(\sum_{i=1}^M A_i [\mathbf{u}(t)]_i \right) \mathbf{x}(t) + \left(\sum_{i=1}^M B_i [\mathbf{u}(t)]_i \right) \mathbf{w}(t), \quad (4.16)$$

$$\mathbf{z}(t) = \left(\sum_{i=1}^M C_i [\mathbf{u}(t)]_i \right) \mathbf{x}(t) + \left(\sum_{i=1}^M D_i [\mathbf{u}(t)]_i \right) \mathbf{w}(t). \quad (4.17)$$

□

Similar reasonings apply to the $\mathcal{L}_1$ performance index in $\mathcal{D}_T$, namely

$$J_{\mathcal{L}_1,T} := \sup_{\sigma \in \mathcal{D}_T} \sum_{k=1}^m J_{\mathcal{L}_1}(\sigma, k) = \sup_{\sigma \in \mathcal{D}_T} \sum_{k=1}^m \int_0^{+\infty} \mathbf{1}_p^\top \mathbf{z}^{[k]}(t) dt.$$

Indeed, by making use of Theorem 4.2, similarly to what we just did for switching signals in $\mathcal{D}_0$, we can derive the the following upper bound.

Theorem 4.4. Consider system (4.1)-(4.2) and assume that $\mathbf{x}(0) = 0$. If there exist strictly positive vectors $\mathbf{v}_i \in \mathbb{R}^n_+$, $i \in \{1, 2, \dots, M\}$, such that:

$$\mathbf{v}_i^\top A_i + \mathbf{1}_p^\top C_i \ll 0 \qquad \forall i = 1, 2, \dots, M, \tag{4.18}$$

and

$$\mathbf{v}_j^\top e^{A_i T} - \mathbf{v}_i^\top + \int_0^T \mathbf{1}_p^\top C_i e^{A_i \xi} d\xi \ll 0 \qquad \forall i \neq j = 1, \dots, M, \tag{4.19}$$

system (4.4) is exponentially stable for each $\sigma \in \mathcal{D}_T$, and the $\mathcal{L}_1$ performance index $J_{\mathcal{L}_1,T}$ satisfies

$$\max_{i\in\{1,2,\dots,M\}} \mathbf{1}_p^\top G_i(0)\mathbf{1}_m \leq J_{\mathcal{L}_1,T} < \max_{i\in\{1,2,\dots,M\}} \left(\mathbf{v}_i^\top B_i + \mathbf{1}_p^\top D_i\right) \mathbf{1}_m. \tag{4.20}$$

Proof. Exponential stability follows from Theorem 4.2, since (4.18) is a special case of (4.10). The lower bound in (4.20) can be motivated as in the proof of Theorem 4.3 (since constant switching signals also belong to $\mathcal{D}_T$). Finally, by reasoning as in the proof of Theorem 4.2, we consider any switching signal $\sigma \in \mathcal{D}_T$, with switching times $0 = t_0 < t_1 < t_2$....., and any assigned $h \in \{1, 2, \dots, m\}$. Assume that $\sigma(t_k) = i$, $\sigma(t_{k+1}) = j$ and introduce the time-varying linear copositive Lyapunov function $V(\mathbf{x}(t), \sigma(t)) = \mathbf{v}_{\sigma(t)}^\top \mathbf{x}(t)$. In this way we show that

$$\begin{aligned}
&V(\mathbf{x}^{[h]}(t_{k+1}), \sigma(t_{k+1})) - V(\mathbf{x}^{[h]}(t_k), \sigma(t_k)) = \\
&= \left(\mathbf{v}_j^\top e^{A_i(t_{k+1}-t_k)} - \mathbf{v}_i^\top\right) \mathbf{x}^{[h]}(t_k) \\
&= \left(\mathbf{v}_j^\top e^{A_i T} e^{A_i(t_{k+1}-t_k-T)} - \mathbf{v}_i^\top\right) \mathbf{x}^{[h]}(t_k) \\
&< \left(\mathbf{v}_i^\top e^{A_i(t_{k+1}-t_k-T)} - \mathbf{v}_i^\top - \int_0^T \mathbf{1}_p^\top C_i e^{A_i\xi} d\xi e^{A_i(t_{k+1}-t_k-T)}\right) \mathbf{x}^{[h]}(t_k) \\
&< -\left(\int_0^{t_{k+1}-t_k-T} \mathbf{1}_p^\top C_i e^{A_i\xi} d\xi + \int_0^T \mathbf{1}_p^\top C_i e^{A_i\xi} d\xi e^{A_i(t_{k+1}-t_k-T)}\right) \mathbf{x}^{[h]}(t_k) \\
&\qquad < -\int_{t_k}^{t_{k+1}} \mathbf{1}_p^\top C_i e^{A_i(\xi-t_k)} d\xi \mathbf{x}^{[h]}(t_k) = -\int_{t_k}^{t_{k+1}} \mathbf{1}_p^\top C_{\sigma(\xi)} \mathbf{x}^{[h]}(\xi) d\xi.
\end{aligned}$$

Summing up for k ranging from 0 to $+\infty$ one gets

$$\sum_{k=0}^{+\infty}[V(\mathbf{x}^{[h]}(t_{k+1}),\sigma(t_{k+1}))-V(\mathbf{x}^{[h]}(t_k),\sigma(t_k))] < \\ -\int_0^{+\infty}\mathbf{1}_p^\top C_{\sigma(\xi)}\mathbf{x}^{[h]}(\xi)d\xi$$

By the exponential stability, this implies that

$$\begin{aligned} -V(\mathbf{x}^{[h]}(0),\sigma(0)) &= -\mathbf{v}_{\sigma(0)}^\top B_{\sigma(\mathbf{0})}\mathbf{e}_h < -\int_0^{+\infty}\mathbf{1}_p^\top C_{\sigma(\xi)}\mathbf{x}^{[h]}(\xi)d\xi \\ &= -J_{\mathcal{L}_1}(\sigma,h)+\mathbf{1}_p^\top D_{\sigma(0)}\mathbf{e}_h. \end{aligned}$$

By summing over all the indices $h\in\{1,2,\dots,m\}$ and by taking the supremum over all the signals $\sigma\in\mathcal{D}_T$, one gets the second inequality in (4.20). □

4.2 Guaranteed $\mathcal{L}_1$ induced norm

In this section we consider the $\mathcal{L}_1$ induced norm of the positive switched system (4.1)-(4.2) either for switching signals belonging to $\mathcal{D}_0$ or for switching signals belonging to $\mathcal{D}_T$. To this end, we assume that all the Metzler matrices A_i, $i\in\{1,2,\dots,M\}$, are Hurwitz and that $\mathbf{x}(0)=0$. In detail, we introduce the objective function

$$J_{\mathcal{L}_1}^{ind}(\sigma) := \sup_{\substack{\mathbf{w}\in\mathcal{L}_1,\mathbf{w}\neq 0\\ \mathbf{w}(t)\geq 0,\forall t\geq 0}} \frac{\int_0^{+\infty}\mathbf{1}_p^\top\mathbf{z}(t)dt}{\int_0^{+\infty}\mathbf{1}_m^\top\mathbf{w}(t)dt} \tag{4.21}$$

and search for an upper bound either on $\mathcal{D}_0$

$$J_{\mathcal{L}_1,0}^{ind} := \sup_{\sigma\in\mathcal{D}_0} J_{\mathcal{L}_1}^{ind}(\sigma),$$

or on $\mathcal{D}_T$

$$J_{\mathcal{L}_1,T}^{ind} := \sup_{\sigma\in\mathcal{D}_T} J_{\mathcal{L}_1}^{ind}(\sigma).$$

Remark 4.5. It is worth noticing that there is no loss of generality in assuming that the cost (4.21) is extended to nonpositive input signals $\mathbf{w} \in \mathcal{L}_1$. For any fixed σ, we can define this cost as

$$\bar{J}^{ind}_{\mathcal{L}_1}(\sigma) := \sup_{\mathbf{w}\in\mathcal{L}_1, \mathbf{w}\neq 0} \frac{\int_0^{+\infty} \mathbf{1}_p^\top |\mathbf{z}(t)| dt}{\int_0^{+\infty} \mathbf{1}_m^\top |\mathbf{w}(t)| dt}. \tag{4.22}$$

It is clear that $\bar{J}^{ind}_{\mathcal{L}_1}(\sigma) \geq J^{ind}_{\mathcal{L}_1}(\sigma)$, for each σ. Now, the forced state response of the system, corresponding to a given $\sigma(t)$, is

$$\mathbf{x}(t) = \int_0^t \Phi(t,\tau,\sigma) B_{\sigma(\tau)} \mathbf{w}(\tau) d\tau$$

and hence, being the transition matrix $\Phi(t,\tau,\sigma)$ nonnegative for $t \geq \tau$,

$$|\mathbf{x}(t)| \leq \int_0^t \Phi(t,\tau,\sigma) B_{\sigma(\tau)} |\mathbf{w}(\tau)| d\tau =: \tilde{\mathbf{x}}(t).$$

Finally

$$|\mathbf{z}(t)| = |C_{\sigma(t)}\mathbf{x}(t) + D_{\sigma(t)}\mathbf{w}(t)| \leq C_{\sigma(t)}|\mathbf{x}(t)| + D_{\sigma(t)}|\mathbf{w}(t)| =: \tilde{\mathbf{z}}(t),$$

where $\tilde{\mathbf{x}}(t)$ and $\tilde{\mathbf{z}}(t)$ denote the (nonnegative) state and output of the system

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}(\mathbf{t}) &= A_{\sigma(t)}\tilde{\mathbf{x}}(t) + B_{\sigma(t)}\tilde{\mathbf{w}}(t) \\ \tilde{\mathbf{z}}(t) &= C_{\sigma(t)}\tilde{\mathbf{x}}(t) + D_{\sigma(t)}\tilde{\mathbf{w}}(t) \end{aligned}$$

and $\tilde{\mathbf{w}}(t) = |\mathbf{w}(t)|$. It follows that, for each $\sigma(t)$:

$$\begin{aligned} \bar{J}^{ind}_{\mathcal{L}_1}(\sigma) &= \sup_{\mathbf{w}\in\mathcal{L}_1, \mathbf{w}\neq 0} \frac{\int_0^{+\infty} \mathbf{1}_p^\top |\mathbf{z}(t)| dt}{\int_0^{+\infty} \mathbf{1}_m^\top |\mathbf{w}(t)| dt} \\ &\leq \sup_{\substack{\mathbf{w}\in\mathcal{L}_1, \tilde{\mathbf{w}}\neq 0 \\ \tilde{\mathbf{w}}(t)\geq 0, \forall t\geq 0}} \frac{\int_0^{+\infty} \mathbf{1}_p^\top \tilde{\mathbf{z}}(t) dt}{\int_0^{+\infty} \mathbf{1}_m^\top \tilde{\mathbf{w}}(t) dt} = J^{ind}_{\mathcal{L}_1}(\sigma). \end{aligned}$$

Therefore $\bar{J}^{ind}_{\mathcal{L}_1}(\sigma) = J^{ind}_{\mathcal{L}_1}(\sigma)$ for any σ. □

The computation of the induced norm for a switched system is an extremely challenging problem that, to the best of our knowledge (see Blanchini and Miani [2008], Liberzon [2003], Sun and Ge [2005]), in the general case is still unsolved. However, for a given switching signal $\sigma = \bar{\sigma}$ the following necessary and sufficient condition can be proved.

Lemma 4.5. Let $\sigma = \bar{\sigma}$ be a fixed switching signal. The following statements are equivalent:

i) The time-varying system (4.1)-(4.2) obtained corresponding to $\bar{\sigma}$ is uniformly exponentially stable and such that $J_{\mathcal{L}_1}^{ind}(\bar{\sigma}) < \gamma$.

ii) There exist $\bar{\mathbf{r}} \gg 0, \hat{\mathbf{r}} \gg 0$, and a solution $\mathbf{r}(t), t \in \mathbb{R}_+$, of the differential inequality

$$\dot{\mathbf{r}}(t)^\top + \mathbf{r}(t)^\top A_{\bar{\sigma}(t)} + \mathbf{1}_p^\top C_{\bar{\sigma}(t)} \ll 0, \tag{4.23}$$

differentiable almost everywhere, and satisfying for every $t \geq 0$ conditions $\bar{\mathbf{r}} < \mathbf{r}(t) < \hat{\mathbf{r}}$, and

$$\mathbf{r}(t)^\top B_{\bar{\sigma}(t)} + \mathbf{1}_p^\top D_{\bar{\sigma}(t)} \ll \gamma \mathbf{1}_m^\top. \tag{4.24}$$

Proof. ii) $\Rightarrow$ i) Let $\mathbf{r}(t)$ be a strictly positive bounded solution of (4.23), with the given uniform bounding properties. Take $V(\mathbf{x}, t) = \mathbf{r}(t)^\top \mathbf{x}$ and $\mathbf{w}(t) = 0$, $t \geq 0$. Notice that $V(\mathbf{x}, t)$ is well defined and positive for $\mathbf{x} > 0$, because of the bounds on $\mathbf{r}(t)$. Standard computation shows that $\dot{V}(\mathbf{x}, t) < -\alpha V(\mathbf{x}, t)$, with $\alpha > 0$, $\forall t \geq 0$, $\forall \mathbf{x} > 0$. This, together with $0 \ll \bar{\mathbf{r}} < \mathbf{r}(t))$, imply uniform exponential stability. Now take $\mathbf{w} \in \mathcal{L}_1$, $\mathbf{w}(t) \geq 0$ for every $t \geq 0$, and $\mathbf{w} \neq 0$. From (4.23) it results

$$\dot{V}(\mathbf{x}, t) < -\mathbf{1}_p^\top \mathbf{z} + \mathbf{r}(t)^\top B_{\bar{\sigma}(t)} \mathbf{w} + \mathbf{1}_p^\top D_{\bar{\sigma}(t)} \mathbf{w}. \tag{4.25}$$

Integrating both members, by the uniform exponential stability and by the fact that $\mathbf{x}(0) = 0$ implies $V(\mathbf{x}, 0) = 0$, we conclude that

$$\int_0^{+\infty} \mathbf{1}_p^\top \mathbf{z}(t) dt < \int_0^{+\infty} \left(\mathbf{r}(t)^\top B_{\bar{\sigma}(t)} + \mathbf{1}_p^\top D_{\bar{\sigma}(t)} \right) \mathbf{w}(t) dt$$

for every positive $\mathbf{w} \in \mathcal{L}_1$. Therefore,

$$\begin{aligned} J_{\mathcal{L}_1}^{ind}(\sigma) \quad &\leq \sup_{\substack{\mathbf{w} \in \mathcal{L}_1, \mathbf{w} \neq 0 \\ \mathbf{w}(t) \geq 0, \forall t \geq 0}} \frac{\int_0^{+\infty} \left(\mathbf{r}(t)^\top B_{\bar{\sigma}(t)} + \mathbf{1}_p^\top D_{\bar{\sigma}(t)} \right) \mathbf{w}(t) dt}{\int_0^{+\infty} \mathbf{1}_m^\top \mathbf{w}(t) dt} \\ &= \left(\mathbf{r}(\bar{t})^\top B_{\bar{\sigma}(\bar{t})} + \mathbf{1}_p^\top D_{\bar{\sigma}(\bar{t})} \right) \mathbf{e}_{\bar{k}} < \gamma, \end{aligned}$$

where $\bar{t}$ and $\bar{k}$ are the time instant and the input index, respectively, associated with the maximum value of $\left(\mathbf{r}(t)^\top B_{\bar{\sigma}(t)} + \mathbf{1}_p^\top D_{\bar{\sigma}(t)} \right) \mathbf{e}_k$, and the supremum value $\left(\mathbf{r}(\bar{t})^\top B_{\bar{\sigma}(\bar{t})} + \mathbf{1}_p^\top D_{\bar{\sigma}(\bar{t})}\right)\mathbf{e}_{\bar{k}}$ is obtained by assuming $\mathbf{w}(t) = \delta(t-\bar{t})\mathbf{e}_{\bar{k}}$. Notice that the impulse is not in $\mathcal{L}_1$, but it can be approximated with arbitrary precision by a sequence of $\mathcal{L}_1$ functions.

i) $\Rightarrow$ ii) As a consequence of uniform exponential stability, the differential inequality $\dot{\mathbf{v}}(t)^\top + \mathbf{v}(t)^\top A_{\bar{\sigma}(t)} + \mathbf{q}^\top \ll 0$ has a positive uniformly bounded solution, uniformly strictly positive for any $t \geq 0$, for any $\mathbf{q} \in \mathbb{R}_+^n, \mathbf{q} \gg 0$, see Lemma 3.5. So, in particular, we can always find a solution $\mathbf{v}(t)$ with such properties satisfying $\dot{\mathbf{v}}(t)^\top + \mathbf{v}(t)^\top A_{\bar{\sigma}(t)} \ll 0$ for every $t \geq 0$ and such that $\bar{\mathbf{v}} < \mathbf{v}(t) < \hat{\mathbf{v}}$, for some $\bar{\mathbf{v}} \gg 0$, $\hat{\mathbf{v}} \gg 0$.

Let $\Phi(t,\tau,\bar{\sigma})$ be the transition matrix associated with $A_{\bar{\sigma}(\cdot)}$ and set

$$\tilde{\mathbf{r}}(t)^\top := \int_t^{+\infty} \mathbf{1}_p^\top C_{\bar{\sigma}(\tau)} \Phi(\tau, t, \bar{\sigma}) d\tau.$$

The exponential stability assumption ensures that $\tilde{\mathbf{r}}(t)$ exists, uniformly upper bounded by a strictly positive vector, say $\hat{\tilde{\mathbf{r}}}$. Moreover, the fact that $C_{\bar{\sigma}(\tau)}$ and $\Phi(\tau,t,\bar{\sigma})$ have nonnegative entries for $\tau \geq t$ ensures that $\tilde{\mathbf{r}}(t)$ is nonnegative at every time $t \geq 0$. For every $\varepsilon > 0$, the signal

$$\mathbf{r}(t) = \varepsilon \mathbf{v}(t) + \tilde{\mathbf{r}}(t)$$

satisfies (4.23) and $\bar{\mathbf{r}} < \mathbf{r}(t) < \hat{\mathbf{r}}$, with $\bar{\mathbf{r}} = \varepsilon\bar{\mathbf{v}}$, $\hat{r} = \varepsilon\hat{\mathbf{v}} + \hat{\tilde{\mathbf{r}}}$. Now, assume by contradiction that (4.24) is violated, i.e. there exists $k \in \{1, 2, \dots, m\}$ and $\bar{t} \geq 0$ such that $\left(\mathbf{r}(\bar{t})^\top B_{\bar{\sigma}(\bar{t})} + \mathbf{1}_p^\top D_{\bar{\sigma}(\bar{t})} - \gamma \mathbf{1}_m^\top\right)\mathbf{e}_k \geq 0$. Taking $V(\mathbf{x},t) := \tilde{\mathbf{r}}(t)^\top \mathbf{x}$ we have, after a computation similar to the one used for (4.25):

$$\begin{aligned}
\dot{V}(\mathbf{x},t) \quad = \quad & -\mathbf{1}_p^\top \mathbf{z}(t) + \gamma \mathbf{1}_m^\top \mathbf{w}(t) + \\
& -\varepsilon \mathbf{v}(t)^\top B_{\bar{\sigma}(t)} \mathbf{w}(t) + \left(\mathbf{r}(t)^\top B_{\bar{\sigma}(t)} + \mathbf{1}_p^\top D_{\bar{\sigma}(t)} - \gamma \mathbf{1}_m^\top\right)\mathbf{w}(t).
\end{aligned}$$

By integrating from 0 to $+\infty$, with $\mathbf{x}(0) = \varepsilon \mathbf{x}_0$, and by making use of the exponential stability and of the fact that $\mathbf{w}$ belongs to $\mathcal{L}_1$, we get

$$\begin{aligned}
& \int_0^{+\infty} \left(\mathbf{1}_p^\top \mathbf{z}(t) - \gamma \mathbf{1}_m^\top \mathbf{w}(t)\right) dt = \\
- \quad & \varepsilon \int_0^{+\infty} \mathbf{v}(t)^\top B_{\bar{\sigma}(t)} \mathbf{w}(t) dt + \varepsilon \tilde{\mathbf{r}}(0)^\top \mathbf{x}_0 + \\
+ \quad & \int_0^{+\infty} \left(\mathbf{r}(t)^\top B_{\bar{\sigma}(t)} + \mathbf{1}_p^\top D_{\bar{\sigma}(t)} - \gamma \mathbf{1}_m^\top\right)\mathbf{w}(t) dt.
\end{aligned}$$

Taking an approximation in $\mathcal{L}_1$ of the impulsive signal $\mathbf{w}(t) = \delta(t-\bar{t})\mathbf{e}_k$, we finally have

$$\int_0^{+\infty} \left(\mathbf{1}_p^\top \mathbf{z}(t) - \gamma \mathbf{1}_m^\top \mathbf{w}(t) \right) dt \geq -\varepsilon \mathbf{v}(\bar{t})^\top B_{\bar{\sigma}(\bar{t})} \mathbf{e}_k + \varepsilon \tilde{\mathbf{r}}(0)^\top \mathbf{x}_0.$$

Letting $\varepsilon \to 0^+$ we conclude that $J_{\mathcal{L}_1}^{ind}(\bar{\sigma}) \geq \gamma$, a contradiction. □

Based on the previous result, we can now postulate possible structures of $\mathbf{r}(t)$ that allow to handle all possible switching signals in $\mathcal{D}_0$ or in $\mathcal{D}_T$. Of course, only sufficient conditions will be given (based on easy linear programming tools) to establish whether either $J_{\mathcal{L}_1,0}^{ind}$ or $J_{\mathcal{L}_1,T}^{ind}$ is smaller than a given positive constant γ. Clearly, in order for this problem to be meaningful, γ should be greater than the $\mathcal{L}_1$ induced norm of the ith subsystem (A_i, B_i, C_i, D_i), which is known to be related to $G_i(0)$, Briat [2013], Rantzer [2011], where $G_i(s)$ is the ith subsystem transfer function. Indeed, it is a well-known fact that for the positive subsystem (A_i, B_i, C_i, D_i) the following facts are equivalent:

i) $J_{\mathcal{L}_1}^{ind}(i) < \gamma$;

ii) $\|G_i(0)\|_1 < \gamma$, where

$$\|G_i(0)\|_1 := \max_{r \in \{1,2,\ldots,m\}} \mathbf{1}_p^\top G_i(0) \mathbf{e}_r;$$

iii) there exists a strictly positive vector $\mathbf{v}_i$ such that $\mathbf{v}_i^\top A_i + \mathbf{1}_p^\top C_i \ll 0$ and $\mathbf{v}_i^\top B_i + \mathbf{1}_p^\top D_i \ll \gamma \mathbf{1}_m^\top$.

We are now in a position to extend this result to the $\mathcal{L}_1$ induced performance of a positive switched system (4.1)-(4.2) with $\sigma \in \mathcal{D}_0$.

Theorem 4.6. If there exists a strictly positive vector $\mathbf{v} \in \mathbb{R}_+^n$ such that

$$\mathbf{v}^\top A_i + \mathbf{1}_p^\top C_i \ll 0, \tag{4.26}$$

$$\mathbf{v}^\top B_i + \mathbf{1}_p^\top D_i \ll \gamma \mathbf{1}_m^\top, \tag{4.27}$$

hold for any $i = 1, 2, \ldots, M$, then the switched system (4.1)-(4.2) is exponentially stable and

$$\max_{i \in \{1,2,\ldots,M\}} \|G_i(0)\|_1 \leq J_{\mathcal{L}_1,0}^{ind} < \gamma.$$

Proof. As previously clarified, the inequality $\max_{i\in\{1,2,\dots,M\}} \|G_i(0)\|_1 \leq J^{ind}_{\mathcal{L}_1,0}$ follows from the fact that the set $\mathcal{D}_0$ also includes constant signals $\sigma(t)$. Taking now the LCLF $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, one gets

$$\begin{aligned}\dot V(\mathbf{x}(t)) &= \mathbf{v}^\top (A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{w}(t)) \\ &< -\mathbf{1}_p^\top \mathbf{z}(t) + \left(\mathbf{v}^\top B_{\sigma(t)} + \mathbf{1}_p^\top D_{\sigma(t)}\right)\mathbf{w}(t).\end{aligned}$$

From $\mathbf{x}(0) = 0$ and exponential stability, we have that $0 = \int_0^{+\infty} \dot V(\mathbf{x}(t))dt$, and hence

$$\int_0^{+\infty} \mathbf{1}_p^\top \mathbf{z}(t)dt < \int_0^{+\infty} \left(\mathbf{v}^\top B_{\sigma(t)} + \mathbf{1}_p^\top D_{\sigma(t)}\right)\mathbf{w}(t)dt.$$

Therefore, for each $\sigma \in \mathcal{D}_0$ we have

$$\begin{aligned}J^{ind}_{\mathcal{L}_1,0}(\sigma) &\leq \sup_{\substack{\mathbf{w}\in\mathcal{L}_1,\mathbf{w}\neq 0\\ \mathbf{w}(t)\geq 0,\forall t\geq 0}} \frac{\int_0^{+\infty}\left(\mathbf{v}^\top B_{\sigma(t)} + \mathbf{1}_p^\top D_{\sigma(t)}\right)\mathbf{w}(t)dt}{\int_0^{+\infty} \mathbf{1}_m^\top \mathbf{w}(t)dt} \\ &= \left(\mathbf{v}^\top B_{\sigma(\bar t)} + \mathbf{1}_p^\top D_{\sigma(\bar t)}\right)\mathbf{e}_{\bar k} < \gamma\end{aligned}$$

where $\bar t$ and $\bar k$ are the time instant and the input channel index, respectively, associated with the supremum value of $\left(\mathbf{v}^\top B_{\sigma(t)} + \mathbf{1}_p^\top D_{\sigma(t)}\right)\mathbf{e}_k$ and the worst disturbance is the limit of $\mathcal{L}_1$ functions, namely the impulse $\mathbf{w}(t) = \delta(t-\bar t)\mathbf{e}_{\bar k}$. The proof is concluded. □

Remark 4.6. The relation between Theorem 4.6 and Lemma 4.5 is immediate. As a matter of fact, the existence of $\mathbf{v}$ satisfying (4.26), (4.27) implies the existence of $\mathbf{r}(t) = \mathbf{v}$, $\forall t \geq 0$, satisfying (4.23)-(4.24) *for every* switching signal in $\mathcal{D}_0$. □

The case when $\sigma \in \mathcal{D}_T$, for some $T > 0$, namely the investigation of condition $J^{ind}_{\mathcal{L}_1,T} < \gamma$, where

$$J^{ind}_{\mathcal{L}_1,T} = \sup_{\sigma\in\mathcal{D}_T} J^{ind}_{\mathcal{L}_1}(\sigma),$$

is more complicated. Indeed, the derivation of Theorem 4.7, below, hinges on the solution of the differential equation

$$\dot{\bar{\mathbf{r}}}(t)^\top + \bar{\mathbf{r}}(t)^\top A_{\sigma(t)} + \mathbf{1}_p^\top C_{\sigma(t)} = 0, \tag{4.28}$$

corresponding to some *final* condition $\bar{\mathbf{r}}(t_f) \gg 0$ and some $\sigma \in \mathcal{D}_T$. Such a solution can be explicitly written as follows

$$\bar{\mathbf{r}}(t)^\top = \bar{\mathbf{r}}(t_f)^\top \Phi(t_f, t, \sigma) + \int_t^{t_f} \mathbf{1}_p^\top C_{\sigma(\xi)} \Phi(\xi, t, \sigma) d\xi, \qquad \forall\ t \leq t_f, \tag{4.29}$$

where $\Phi(\xi, t, \sigma)$ is the transition matrix associated with $A_{\sigma(\cdot)}$.

Notice that we have already encountered the differential equation (4.28) with solution $\bar{\mathbf{r}}(t)$ in Lemma 4.5, where a single switching signal was considered and the symbol $=$ was replaced by $\ll$. We are now allowing discontinuities not only in the switching signal but also in solutions, namely in the functions $\bar{\mathbf{r}}(t)$ satisfying (4.28), at the switching instants t_k, $k = 0, 1, ...$, of $\sigma \in \mathcal{D}_T$. Upon noticing that

$$-\frac{d}{dt}\left(\bar{\mathbf{r}}(t)^\top \mathbf{x}(t)\right) = -\left(\bar{\mathbf{r}}(t)^\top B_{\sigma(t)} + \mathbf{1}_p^\top D_{\sigma(t)}\right) \mathbf{w}(t) + \mathbf{1}_p^\top \mathbf{z}(t),$$

one gets, for every $t \in [t_k, t_{k+1})$

$$\begin{aligned} \int_{t_k}^{t_{k+1}} \mathbf{1}_p^\top \mathbf{z}(t) dt \quad &= \quad \bar{\mathbf{r}}(t_k)^\top \mathbf{x}(t_k) - \bar{\mathbf{r}}(t_{k+1})^\top \mathbf{x}(t_{k+1}) \\ &\quad + \int_{t_k}^{t_{k+1}} \left(\bar{\mathbf{r}}(t)^\top B_{\sigma(t_k)} + \mathbf{1}_p^\top D_{\sigma(t_k)}\right) \mathbf{w}(t) dt. \end{aligned} \tag{4.30}$$

We are in a position to prove the following result.

Theorem 4.7. If there exist strictly positive vectors $\mathbf{v}_i \in \mathbb{R}_+^n, i \in \{1, 2, \ldots, M\}$, and $T > 0$ such that

$$\mathbf{v}_i^\top A_i + \mathbf{1}_p^\top C_i \quad \ll \quad 0, \qquad i = 1, \ldots, M, \tag{4.31}$$

$$\mathbf{v}_j^\top e^{A_i T} + \int_0^T \mathbf{1}_p^\top C_i e^{A_i \tau} d\tau \quad \ll \quad \mathbf{v}_i^\top,\ i \neq j = 1, \ldots, M, \tag{4.32}$$

and

$$\mathbf{v}_i^\top B_i + (\mathbf{v}_j^\top - \mathbf{v}_i^\top) e^{A_i \tau} B_i + \mathbf{1}_p^\top D_i \ll \gamma \mathbf{1}_m^\top,\ i, j = 1, \ldots, M, \tag{4.33}$$

hold for any $\tau \in [0, T)$, then the switched system (4.1)-(4.2) is exponentially stable with dwell-time T and

$$\max_{i \in \{1,2,\ldots,M\}} \|G_i(0)\|_1 \leq J_{\mathcal{L}_1, T}^{ind} < \gamma.$$

Proof. The inequality $\max_{i\in\{1,2,\dots,M\}}\|G_i(0)\|_1 \le J^{ind}_{\mathcal{L}_1,T}$ follows by the same reasoning adopted in the proof of Theorem 4.6 (note that $\mathcal{D}_T \subseteq \mathcal{D}_0$ for all $T \ge 0$). The fact that the system is exponentially stable follows from Theorem 3.10. Now, consider a signal $\sigma \in \mathcal{D}_T$, with $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \dots$ as switching instants. Assume that $\sigma(t_k) = i$ and $\sigma(t_{k+1}) = j$, and consider the solution of (4.28) in the interval $[t_k\ t_{k+1})$ with final condition $\bar{\mathbf{r}}(t_{k+1}) = \mathbf{v}_{\sigma(t_{k+1})} = \mathbf{v}_j$. Comparing (4.28) with (4.31), one finds for every $t \in [t_k, t_{k+1})$:

$$\frac{d}{dt}(\bar{\mathbf{r}}(t)-\mathbf{v}_i)^\top = -\bar{\mathbf{r}}(t)^\top A_i - \mathbf{1}_p^\top C_i \gg -(\bar{\mathbf{r}}(t)-\mathbf{v}_i)^\top A_i,$$

and hence

$$\bar{\mathbf{r}}(t)^\top \ll \mathbf{v}_i^\top + (\mathbf{v}_j-\mathbf{v}_i)^\top e^{A_i(t_{k+1}-t)}, \quad t \in [t_k, t_{k+1}). \tag{4.34}$$

Consider now (4.31): multiply both members on the right by $e^{A_i\tau}$, $\tau > 0$, and integrate for τ ranging from 0 to $\xi > 0$. It results

$$\mathbf{v}_i^\top e^{A_i\xi} + \int_0^\xi \mathbf{1}_p^\top C_i e^{A_i\tau} d\tau \ll \mathbf{v}_i^\top,\ \forall i = 1,2,\dots,M,\ \forall \xi > 0.$$

Using this fact and the position $\bar{\mathbf{r}}(t_{k+1}) = \mathbf{v}_j$ we have

$$\begin{aligned}
\bar{\mathbf{r}}(t_k)^\top &= \mathbf{v}_j^\top e^{A_i(t_{k+1}-t_k)} + \int_{t_k}^{t_{k+1}} \mathbf{1}_p^\top C_i e^{A_i(\xi-t_k)} d\xi \\
&= \mathbf{v}_j^\top e^{A_iT} e^{A_i(t_{k+1}-t_k-T)} + \int_{t_k}^{t_{k+1}} \mathbf{1}_p^\top C_i e^{A_i(\xi-t_k)} d\xi \\
&\ll \mathbf{v}_i^\top e^{A_i(t_{k+1}-t_k-T)} - \int_0^T \mathbf{1}_p^\top C_i e^{A_i(t_{k+1}-t_k-T+\tau)} d\tau \\
&+ \int_{t_k}^{t_{k+1}} \mathbf{1}_p^\top C_i e^{A_i(\xi-t_k)} d\xi \\
&\ll \mathbf{v}_i^\top - \int_0^{t_{k+1}-t_k-T} \mathbf{1}_p^\top C_i e^{A_i\tau} d\tau - \int_0^T \mathbf{1}_p^\top C_i e^{A_i(t_{k+1}-t_k-T+\tau)} \\
&+ \int_{t_k}^{t_{k+1}} \mathbf{1}_p^\top C_i e^{A_i(\xi-t_k)} d\xi \\
&= \mathbf{v}_i^\top.
\end{aligned}$$

Finally, from (4.30), (4.33), (4.34), and $\bar{\mathbf{r}}(t_k) \ll \mathbf{v}_i$, we get

$$
\begin{aligned}
&\int_{t_k}^{t_{k+1}} \mathbf{1}_p^\top \mathbf{z}(t)dt - \gamma \int_{t_k}^{t_{k+1}} \mathbf{1}_m^\top \mathbf{w}(t)dt = \bar{\mathbf{r}}(t_k)^\top \mathbf{x}(t_k) - \mathbf{v}_j^\top \mathbf{x}(t_{k+1}) \\
&+ \int_{t_k}^{t_{k+1}} \left(\mathbf{r}(t)^\top B_i + \mathbf{1}_p^\top D_i - \gamma \mathbf{1_m}^\top\right) \mathbf{w}(t)dt \\
&< \mathbf{v}_i^\top \mathbf{x}(t_k) - \mathbf{v}_j^\top \mathbf{x}(t_{k+1}) \\
&+ \int_{t_k}^{t_{k+1}} \left[\left(\mathbf{v}_i^\top + (\mathbf{v}_j - \mathbf{v}_i)^\top e^{A_i(t_{k+1}-t)}\right) B_i + \mathbf{1}_p^\top D_i - \gamma \mathbf{1_m}^\top\right] \mathbf{w}(t)dt \\
&< \mathbf{v}_i^\top \mathbf{x}(t_k) - \mathbf{v}_j^\top \mathbf{x}(t_{k+1}) = \mathbf{v}_{\sigma(t_k)}^\top \mathbf{x}(t_k) - \mathbf{v}_{\sigma(t_{k+1})}^\top \mathbf{x}(t_{k+1})
\end{aligned}
$$

The above inequality holds for any $\mathbf{w}(t), t \in [t_k, t_{k+1}]$, including the worst one (by regarding, again, the impulse as the limit of $\mathcal{L}_1$ functions). Taking the sum for $k = 0$ to $k = +\infty$ and recalling that $\mathbf{x}(0) = 0$, we have, for any $\sigma \in \mathcal{D}_T$

$$
\sup_{\mathbf{w}>0, \mathbf{w}\in\mathcal{L}_1} \int_0^{+\infty} \left(\mathbf{1}_p^\top \mathbf{z}(t) - \gamma \mathbf{1}_m^\top \mathbf{w}(t)\right) dt < 0
$$

and the proof is concluded. □

Notice that the feasibility of (4.31), (4.32) and (4.33) for $T \to 0^+$ leads to $\mathbf{v}_i = \mathbf{v}_j$ and hence to the feasibility of (4.26) and (4.27) in Theorem 4.6.

Example 4.1. Let us consider system (4.1)-(4.2) with $M = 2$, $n = 2$, and the matrices

$$
A_1 = \begin{bmatrix} -3 & 1 \\ 2 & -8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix},
$$

that are both Metzler and Hurwitz. Moreover, assume $B_1 = B_2 = [1\ 1]^\top$, $C_1 = C_2 = [1\ 1]$, $D_1 = D_2 = 1$. Notice that A_1 and A_2 share a common linear copositive Lyapunov function (e.g. $V(\mathbf{x}) = [2 \quad 1]\mathbf{x}$). The system is therefore exponentially stable under arbitrary switching. It is also possible to find $\mathbf{v}$ such that inequalities (4.26) are satisfied. For instance, take $\mathbf{v} = (1+\varepsilon)[1.2512\ 0.7507]^\top$, with $\varepsilon > 0$ and small, and verify that (4.26) and (4.27) are satisfied for $\gamma \geq 3$. On the other hand, $\max\{G_1(0), G_2(0)\} = 3$, so that one can conclude that $J_{\mathcal{L}_1,0}^{ind} =$

3. In this case the worst signal $\mathbf{w}(t)$ can be approximated, again, by an impulse and the worst switching signal is $\sigma(t) = 2, t \geq 0$, since $\int_0^{+\infty} z(t)dt = \int_0^{+\infty}(C_2 e^{A_2 t} B_2 + D_2\delta(t))dt = G_2(0) = 3$. The fact that the worst case switching signal is constant is a phenomenon that we have already encountered in Section 2.5.

Example 4.2. Consider the 3×3 matrices A_1 and A_2 of Example 3.1, and assume $B_1 = B_2 = C_1^\top = C_2^\top = \mathbf{1}_3$, $D_1 = D_2 = 1$. Notice that $\max\{G_1(0), G_2(0)\} = 25.514$. We know that the system is not exponentially stable under arbitrary switching. Feasibility of (4.31) and (4.32) can be checked as a function of T, whereas (4.33) gives for each T a bound on the available attenuation value. Let us call $\gamma^*(T)$ this value. Therefore it is possible to plot in the plane (T, γ^*) the feasible points, see Figure 4.1. For $T \to +\infty$ the value $\gamma = 376.5249 > 25.514$ is obtained. The reason is that the possibility of slow switching signals enforces the feasibility of $\mathbf{v}_j^\top B_i + \mathbf{1}_p^\top D_i \ll \gamma \mathbf{1}_m$ obtained from (4.33) by assuming $\tau = 0$. A lower bound on the attenuation level can be obtained by taking a periodic switching signal of period $2T$, with $\sigma(t) = 2$ for $t \in [kT, kT + T)]$ and $\sigma(t) = 1$ for $t \in [kT + T, kT + 2T)]$, $k = 0, 1, \ldots$. In this case an easy necessary condition for $J_{\mathcal{L}_1,T}^{ind} < \gamma$ is

$$g(T) = C \int_0^{2T} \Phi(\tau, 0, \sigma) d\tau (I - e^{A_1 T} e^{A_2 T})^{-1} B + D < \gamma,$$

where $\Phi(\tau, 0, \sigma) = e^{A_2 \tau}$ for $\tau \in [0, T)$ and $\Phi(\tau, 0, \sigma) = e^{A_1(\tau - T)} e^{A_2 T}$, for $\tau \in [T, 2T)$. In Figure 4.1 the curve $(T, g(T))$ is plotted (dashed line).

The sufficient condition given in Theorem 4.6 for $J_{\mathcal{L}_1,0}^{ind} < \gamma$ becomes also necessary in the particular case where matrices A_i and C_i are not affected by the switching signal, i.e. $A_i = A$, $C_i = C$, $i = 1, 2, \ldots, M$. This case corresponds to the possibility of switching among different actuators, i.e. *actuator switching*, and one wants to have a certain performance level even in the worst case. Based on the separation of the two inequalities in Theorem 4.6, the following necessary and sufficient condition can be stated.

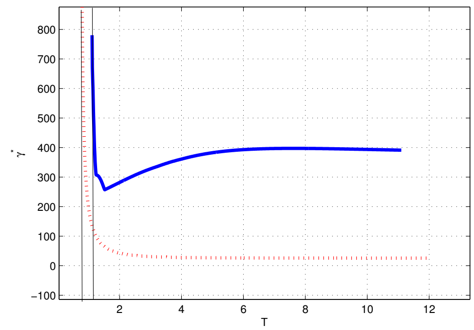

Figure 4.1: Curves for feasibility of the conditions of Theorem 4.7. Upper bound on $\bar{J}_1^*$ as a function of T is drawn as a continuous line, while a lower bound obtained with periodic switching is drawn as a dashed line.

Theorem 4.8. Assume that $A_i = A$, $C_i = C$, $i = 1, 2, \dots, M$. Then $J^{ind}_{\mathcal{L}_1,0} < \gamma$ if and only if there exists a strictly positive vector $\mathbf{v} \in \mathbb{R}^n_+$ such that

$$\mathbf{v}^\top A + \mathbf{1}_p^\top C \ll 0, \tag{4.35}$$

$$\mathbf{v}^\top B_i + \mathbf{1}_p^\top D_i \ll \gamma \mathbf{1}_m^\top, \tag{4.36}$$

hold for any $i = 1, 2, \dots, M$. In this case, the switched system (4.1)-(4.2) is exponentially stable,

$$J^{ind}_{\mathcal{L}_1,0} = \max_{i \in \{1,2,\dots,M\}} \|G_i(0)\|_1$$

and $\sigma(t) = \arg\max_{i \in \{1,2,\dots,M\}} \|G_i(0)\|_1$, $t \geq 0$, is the worst switching signal.

Proof. Exponential stability and $J^{ind}_{\mathcal{L}_1,0} < \gamma$ follow from Theorem 4.6 upon noticing that conditions (4.35) and (4.36) coincide with (4.26) and (4.27) in Theorem 4.6, when $A_i = A$, and $C_i = C$, for every $i = 1, 2, \dots, M$. Vice versa, assume that A is Hurwitz and $J^{ind}_{\mathcal{L}_1,0} < \gamma$. Then, $\|G_i(0)\|_1 < \gamma$ as well, which means that $\mathbf{1}_p^\top G_i(0) < \gamma \mathbf{1}_m^\top$, $i = 1, 2, \dots, M$. Take $\mathbf{v}^\top := -\mathbf{1}_p^\top C A^{-1} + \epsilon \mathbf{1}_n^\top$, with $\epsilon > 0$, and notice that

$\mathbf{v}$ is strictly positive and satisfies (4.35). Moreover,

$$\begin{aligned}\mathbf{v}^\top B_i + \mathbf{1}_p^\top D_i &= \mathbf{1}_p^\top \left(D_i - CA^{-1}B_i\right) + \epsilon \mathbf{1}_n^\top B_i \\ &= \mathbf{1}_p^\top G_i(0) + \epsilon \mathbf{1}_n^\top B_i \\ &< \gamma \mathbf{1}_m^\top + \epsilon \mathbf{1}_n^\top B_i\end{aligned}$$

so that (4.36) is satisfied for every i, provided that ϵ is small enough. The exact computation of $J_{\mathcal{L}_1,0}^{ind}$ proceeds as follows. Assume that A is Hurwitz, take again $\mathbf{v}^\top = -\mathbf{1}_p^\top CA^{-1} + \epsilon \mathbf{1}_n^\top$, with $\epsilon > 0$, and the function $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$. Computing the derivative along the trajectories of the system, one gets

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \mathbf{v}^\top(A\mathbf{x} + B_\sigma \mathbf{w}) = -\mathbf{1}_p^\top C\mathbf{x} + \epsilon \mathbf{1}_n^\top A\mathbf{x} + \mathbf{v}^\top B_\sigma \mathbf{w} \\ &= -\mathbf{1}_p^\top \mathbf{z} + \epsilon \mathbf{1}_n^\top A\mathbf{x} + \left(\mathbf{v}^\top B_\sigma + \mathbf{1}_p^\top D_\sigma\right)\mathbf{w}.\end{aligned}$$

As a consequence of stability

$$\begin{aligned}\int_0^{+\infty} \mathbf{1}_p^\top \mathbf{z}(t)dt &= \epsilon \mathbf{1}_n^\top A \int_0^{+\infty} \mathbf{x}(t)dt + \int_0^{+\infty} \left(\mathbf{v}^\top B_{\sigma(t)} + \mathbf{1}_p^\top D_{\sigma(t)}\right)\mathbf{w}(t)dt \\ &\leq \epsilon \mathbf{1}_n^\top A \int_0^{+\infty} \mathbf{x}(t)dt + \max_{j,s}(\mathbf{v}^\top B_j + \mathbf{1}_p^\top D_j)\mathbf{e}_s\end{aligned}$$

and equality is attained with arbitrary precision by taking an approximation in $\mathcal{L}_1$ of $\mathbf{w}(t) = \delta(t)\mathbf{e}_k$, where $(i,k) = \arg\max_{j,s}(\mathbf{v}^\top B_j + \mathbf{1}_p^\top D_j)\mathbf{e}_s$. Letting $\epsilon \to 0^+$, it follows that $J_{\mathcal{L}_1,0}^{ind} = \max_{i\in\{1,2,\ldots,M\}} \|G_i(0)\|_1$. □

4.3 Guaranteed induced $\mathcal{L}_\infty$ norm

In this section we investigate the $\mathcal{L}_\infty$ induced norm of the switched system (4.1)-(4.2). Again, we assume that all matrices $A_i, i \in \{1,2,\ldots,M\}$, are Metzler and Hurwitz, and that $\mathbf{x}(0) = 0$. For a given switching signal σ, the objective function is

$$J_{\mathcal{L}_\infty}^{ind}(\sigma) = \sup_{\substack{\mathbf{w}\in\mathcal{L}_\infty, \mathbf{w}\neq 0 \\ \mathbf{w}(t)\geq 0, \forall t\geq 0}} \frac{\max\{[\mathbf{z}(t)]_k \ : k \in \{1,2,\ldots,p\}, t\geq 0\}}{\max\{[\mathbf{w}(t)]_k \ : k\in\{1,2,\ldots,m\}, t\geq 0\}}. \quad (4.37)$$

Our scope is to determine sufficient conditions that allow us to evaluate whether

$$J_{\mathcal{L}_\infty,0}^{ind} := \sup_{\sigma\in\mathcal{D}_0} J_{\mathcal{L}_\infty}^{ind}(\sigma)$$

is smaller than a given positive constant γ.

In this case we recall from Briat [2013] and Rantzer [2011] that for each positive subsystem (A_i, B_i, C_i, D_i) the $\mathcal{L}_\infty$ induced norm is equal to $\|G_i(0)\|_\infty = \max_{r\in\{1,2,\dots,p\}} \mathbf{e}_r^\top G_i(0)\mathbf{1}_m$, and the following facts are equivalent:

i) $J_{\mathcal{L}_\infty}^{ind}(i) < \gamma$;

ii) $\|G_i(0)\|_\infty < \gamma$;

iii) there exists a positive vector $\boldsymbol{\xi}_i$ such that $A_i\boldsymbol{\xi}_i + B_i\mathbf{1}_m \ll 0$ and $C_i\boldsymbol{\xi}_i + D_i\mathbf{1}_m \ll \gamma\mathbf{1}_p$.

Before proceeding, a necessary and sufficient condition for $J_{\mathcal{L}_\infty}^{ind}(\sigma) < \gamma$ to hold can be given, by referring to a specific switching signal $\bar{\sigma}$, along the lines traced in Lemma 4.5.

Lemma 4.9. Let $\sigma = \bar{\sigma}$ be a given switching signal. The following statements are equivalent:

i) The time-varying system (4.1)-(4.2) obtained corresponding to $\sigma = \bar{\sigma}$ is uniformly exponentially stable and $J_{\mathcal{L}_\infty}^{ind}(\bar{\sigma}) < \gamma$.

ii) There exist $\bar{\mathbf{d}} \gg 0, \hat{\mathbf{d}} \gg 0$ and a solution $\mathbf{d}(t) \in \mathbb{R}_+^n$, $t \geq 0$, of the differential inequality

$$-\dot{\mathbf{d}}(t) + A_{\bar{\sigma}(t)}\mathbf{d}(t) + B_{\bar{\sigma}(t)}\mathbf{1}_m \ll 0 \tag{4.38}$$

differentiable almost everywhere, and satisfying for every $t \geq 0$ conditions $\bar{\mathbf{d}} < \mathbf{d}(t) < \hat{\mathbf{d}}$, and

$$C_{\bar{\sigma}(t)}\mathbf{d}(t) + D_{\bar{\sigma}(t)}\mathbf{1}_m \ll \gamma\mathbf{1}_p. \tag{4.39}$$

Proof. ii) $\Rightarrow$ i) Let $\mathbf{d}(t)$ be a strictly positive, bounded solution of (4.38) satisfying all the additional requirements, and introduce the function $V(\mathbf{x}(t), t) = \max_i \frac{[\mathbf{x}(t)]_i}{[\mathbf{d}(t)]_i}$. This function is well defined and strictly positive for each $\mathbf{x} > 0$, due to the bounds on $\mathbf{d}(t)$. At any time t, let i be any index such that $\frac{[\mathbf{x}(t)]_i}{[\mathbf{d}(t)]_i} = V(\mathbf{x}(t), t)$ and let j be any analogous index for $t = t^+$. Since $V(\mathbf{x}, t)$ is continuous we have $\frac{[\mathbf{x}(t)]_i}{[\mathbf{d}(t)]_i} = \frac{[\mathbf{x}(t)]_j}{[\mathbf{d}(t)]_j}$. On the other hand, by the nonnegativity of $B_{\bar{\sigma}(t)}\mathbf{1}_m$, condition (4.38) also implies the inequality $-\dot{\mathbf{d}}(t) + A_{\bar{\sigma}(t)}\mathbf{d}(t) \ll 0$. So, for $\mathbf{w}(t)$ identically zero, we get

$$\begin{aligned}
\dot{V}(\mathbf{x}, t^+) &= \sum_{k=1}^{n}[A_{\bar{\sigma}(t)}]_{jk}\frac{[\mathbf{x}(t)]_k}{[\mathbf{d}(t)]_j} - \frac{[\mathbf{x}(t)]_j}{([\mathbf{d}(t)]_j)^2}[\dot{\mathbf{d}}(t)]_j \\
&< \sum_{k=1}^{n}[A_{\bar{\sigma}(t)}]_{jk}\frac{[\mathbf{x}(t)]_k}{[\mathbf{d}(t)]_k}\frac{[\mathbf{d}(t)]_k}{[\mathbf{d}(t)]_j} - \sum_{k=1}^{n}[A_{\bar{\sigma}(t)}]_{jk}[\mathbf{d}(t)]_k\frac{[\mathbf{x}(t)]_j}{([\mathbf{d}(t)]_j)^2} \\
&= \frac{1}{[\mathbf{d}(t)]_j}\sum_{k\neq j}^{n}[A_{\bar{\sigma}(t)}]_{jk}[\mathbf{d}(t)]_k\left(\frac{[\mathbf{x}(t)]_k}{[\mathbf{d}(t)]_k} - \frac{[\mathbf{x}(t)]_j}{[\mathbf{d}(t)]_j}\right) \\
&< \frac{1}{[\mathbf{d}(t)]_j}\sum_{k\neq j}^{n}[A_{\bar{\sigma}(t)}]_{jk}[\mathbf{d}(t)]_k\left(\frac{[\mathbf{x}(t)]_i}{[\mathbf{d}(t)]_i} - \frac{[\mathbf{x}(t)]_j}{[\mathbf{d}(t)]_j}\right) = 0.
\end{aligned}$$

Therefore uniform asymptotic stability is proved. Such a stability is exponential since the system is piecewise linear. To prove the bound, note that for every $k \in \{1, 2, \ldots, p\}$

$$\begin{aligned}
[\mathbf{z}(t)]_k &= \mathbf{e}_k^\top[C_{\bar{\sigma}(t)}\mathbf{x}(t) + D_{\bar{\sigma}(t)}\mathbf{w}(t)] \\
&= \mathbf{e}_k^\top[C_{\bar{\sigma}(t)}\mathbf{d}(t) + D_{\bar{\sigma}(t)}\mathbf{1}_m] \\
&+ \mathbf{e}_k^\top[C_{\bar{\sigma}(t)}(\mathbf{x}(t) - \mathbf{d}(t)) + D_{\bar{\sigma}(t)}(\mathbf{w}(t) - \mathbf{1}_m)] \\
&< \gamma + \mathbf{e}_k^\top[C_{\bar{\sigma}(t)}(\mathbf{x}(t) - \mathbf{d}(t)) + D_{\bar{\sigma}(t)}(\mathbf{w}(t) - \mathbf{1}_m)]
\end{aligned}$$

Now, notice that

$$\dot{\mathbf{d}}(t) - \dot{\mathbf{x}}(t) \gg A_{\bar{\sigma}(t)}(\mathbf{d}(t) - \mathbf{x}(t)) + B_{\bar{\sigma}(t)}(\mathbf{1}_m - \mathbf{w}(t))$$

and that $\mathbf{d}(0) - \mathbf{x}(0) = \mathbf{d}(0) \gg 0$. Therefore, with the worst disturbance $\mathbf{w}(t) = \mathbf{1}_m$ we have $[\mathbf{z}(t)]_k < \gamma = \gamma[\mathbf{w}(t)]_k$, and this completes the proof of the implication.

i) $\Rightarrow$ ii) By Lemma 3.6, due to uniform exponential stability, we can claim that there exists a solution $\boldsymbol{\xi}(t)$, differentiable almost everywhere, of $-\dot{\boldsymbol{\xi}}(t) + A_{\bar{\sigma}(t)}\boldsymbol{\xi}(t) + \mathbf{b} \ll 0$ for any given $0 \ll \mathbf{b} \in \mathbb{R}_+^n$, and such that $\bar{\boldsymbol{\xi}} < \boldsymbol{\xi}(t) < \hat{\boldsymbol{\xi}}$ with $\bar{\boldsymbol{\xi}} \gg 0$, $\hat{\boldsymbol{\xi}} \gg 0$. Then $\mathbf{d}(t) = \varepsilon\boldsymbol{\xi}(t) + \int_{-\infty}^{t}\Phi(t, \tau, \bar{\sigma})B_{\bar{\sigma}(t)}\mathbf{1}_m d\tau$, with $\varepsilon > 0$. $\Phi(t, \tau, \sigma)$, the transition matrix associated with $A_{\bar{\sigma}(\cdot)}$, satisfies (4.38) and is uniformly bounded from above and below by suitable strictly positive vectors. Consider any signal $\mathbf{w} \in \mathcal{L}_\infty$. Set $\bar{w} := \max\{[\mathbf{w}(t)]_s : s \in \{1, 2, \ldots, m\}, t \geq 0\}$, so that $\mathbf{w}(t) \leq \mathbf{1}_m \bar{w}$, $\forall t \geq 0$. Take again the Lyapunov function $V(\mathbf{x}(t), t) = \max_i \frac{[\mathbf{x}(t)]_i}{[\mathbf{d}(t)]_i}$, and assume that i is the index such that $\frac{[\mathbf{x}(t)]_i}{[\mathbf{d}(t)]_i} = V(\mathbf{x}(t), t)$ while j is the analogous index for $t = t^+$, so that $\frac{[\mathbf{x}(t)]_i}{[\mathbf{d}(t)]_i} = \frac{[\mathbf{x}(t)]_j}{[\mathbf{d}(t)]_j}$. By resorting to a computation similar to the previous one (where, however, $\mathbf{w} \neq 0$), we get

$$\dot{V}(\mathbf{x}, t^+) < \frac{1}{[\mathbf{d}(t)]_j} \sum_{k=1}^{n} [B_{\bar{\sigma}(t)}]_{jk} \left(\bar{w} - V(\mathbf{x}, t^+) \right)$$

so that $V(\mathbf{x}, t) < \bar{w}$, $\forall t \geq 0$. Moreover, by (4.39) we have, for any $s \in \{1, 2, \ldots, p\}$

$$[\mathbf{z}(t)]_s - \gamma \bar{w} < \sum_{k=1}^{n} \left([D_{\bar{\sigma}(t)}]_{sk} - \gamma \right) (\bar{w} - V(\mathbf{x}(t), t))$$

Therefore, since $[D_{\bar{\sigma}(t)}]_{sk} - \gamma < 0$, $t \geq 0$, it turns out that, for each $t \geq 0$ and each bounded, nonnegative and not identically zero disturbance $\mathbf{w}(t)$:

$$\frac{\max_{s=1,2,\ldots,p} [\mathbf{z}(t)]_s}{\max_{s=1,2,\ldots,m} [\mathbf{w}(t)]_s} < \gamma$$

and this concludes the proof. □

We are now ready to state a sufficient condition for a positive switched system (4.1)-(4.2)

to have $\mathcal{L}_\infty^{\inf}(\sigma) < \gamma$ for any $\sigma \in \mathcal{D}_0$. This is achieved by making use of Lemma 4.9 in the special case when there exists a constant solution $\mathbf{d}(t) = \boldsymbol{\xi}$ satisfying (4.38) and (4.39) for any $\bar{\sigma} \in \mathcal{D}_0$.

Theorem 4.10. If there exists a strictly positive vector $\boldsymbol{\xi} \in \mathbb{R}_+^n$ such that

$$A_i \boldsymbol{\xi} + B_i \mathbf{1}_m \ll 0, \tag{4.40}$$

$$C_i \boldsymbol{\xi} + D_i \mathbf{1}_m \ll \gamma \mathbf{1}_p, \tag{4.41}$$

hold for any $i = 1, 2, \ldots, M$, then the switched system (4.1)-(4.2) is exponentially stable and

$$\max_{i \in \{1,2,\ldots,M\}} \|G_i(0)\|_\infty \leq J_{\mathcal{L}_\infty, 0}^{ind} < \gamma.$$

Proof. The proofs of exponential stability and of the first inequality follow the same lines as in previous proofs. Assume now

$$V(\mathbf{x}) = \max_{i \in \{1,2,\dots,n\}} \frac{[\mathbf{x}]_i}{[\boldsymbol{\xi}]_i},$$

and $\alpha := \max_{k,t\geq 0}[\mathbf{w}(t)]_k$. At any time t, let j be the index such that $V(\mathbf{x}(t)) = \frac{[\mathbf{x}(t)]_j}{[\boldsymbol{\xi}]_j}$. If we set $g(\mathbf{x}(t)) := V(\mathbf{x}(t)) - \alpha$, then

$$\begin{aligned}
\dot{g}(\mathbf{x}(t)) &= \sum_{k=1}^{n}[A_{\sigma(t)}]_{jk}\frac{[\mathbf{x}(t)]_k}{[\boldsymbol{\xi}]_j} + \sum_{k=1}^{m}[B_{\sigma(t)}]_{jk}\frac{[\mathbf{w}(t)]_k}{[\boldsymbol{\xi}]_j} \\
&\leq \sum_{k=1}^{n}[A_{\sigma(t)}]_{jk}\frac{[\boldsymbol{\xi}]_k}{[\boldsymbol{\xi}]_j}V(\mathbf{x}(t)) + \sum_{k=1}^{m}[B_{\sigma(t)}]_{jk}\frac{[\mathbf{w}(t)]_k}{[\boldsymbol{\xi}]_j} \\
&< -\sum_{k=1}^{m}[B_{\sigma(t)}]_{jk}\frac{1}{[\boldsymbol{\xi}]_j}V(\mathbf{x}(t)) + \sum_{k=1}^{m}[B_{\sigma(t)}]_{jk}\frac{\alpha}{[\boldsymbol{\xi}]_j} \\
&= -\sum_{k=1}^{m}[B_{\sigma(t)}]_{jk}\frac{1}{[\boldsymbol{\xi}]_j}g(\mathbf{x}(t)),
\end{aligned}$$

where we made use of (4.40). Consequently, since $g(\mathbf{x}(0)) = g(0) = -\alpha$ and $\dot{g}(\mathbf{x}(t)) < 0$ at every time $t \geq 0$, we have $g(\mathbf{x}(t)) < 0$, for every $t \geq 0$. On the other hand, for every $s \in \{1,2,\dots,p\}$, $t \geq 0$ and $\mathbf{w}(t)$, we have

$$\begin{aligned}
[\mathbf{z}(t)]_s &= \sum_{k=1}^{n}[C_{\sigma(t)}]_{sk}[\mathbf{x}(t)]_k + \sum_{k=1}^{m}[D_{\sigma(t)}]_{sk}[\mathbf{w}(t)]_k \\
&\leq \sum_{k=1}^{n}[C_{\sigma(t)}]_{sk}[\boldsymbol{\xi}]_k V(\mathbf{x}(t)) + \sum_{k=1}^{m}[D_{\sigma(t)}]_{sk}\alpha \\
&< \sum_{k=1}^{n}[C_{\sigma(t)}]_{sk}[\boldsymbol{\xi}]_k g(\mathbf{x}(t)) + \gamma\alpha \\
&< \gamma\alpha,
\end{aligned}$$

where we made use of (4.41). This is equivalent to saying that

$$\sup_{t\geq 0}\max_{s\in\{1,2,\dots,p\}}[\mathbf{z}(t)]_s < \gamma \cdot \sup_{t\geq 0}\max_{s\in\{1,2,\dots,m\}}[\mathbf{w}(t)]_s,$$

and this proves the second inequality. □

The case when $\sigma \in \mathcal{D}_T$, with $T > 0$, can be derived along dual lines, following the same rationale as in Theorem 4.7. In this case we are interested in the index

$$J^{ind}_{\mathcal{L}_\infty,T} := \sup_{\sigma\in\mathcal{D}_T} J^{ind}_{\mathcal{L}_\infty}(\sigma).$$

The following result holds.

Theorem 4.11. If there exist strictly positive vectors $\boldsymbol{\xi}_i$, $i \in \{1, 2, \dots, M\}$, and $T > 0$ such that

$$A_i\boldsymbol{\xi}_i + B_i\mathbf{1}_m \ll 0, \qquad i = 1, \dots, M, \tag{4.42}$$

$$e^{A_iT}\boldsymbol{\xi}_j + \int_0^T e^{A_i\tau}B_i\mathbf{1}_m d\tau \ll \boldsymbol{\xi}_i, \qquad i \neq j = 1, \dots, M, \tag{4.43}$$

and

$$C_i[\boldsymbol{\xi}_i + e^{A_i\tau}(\boldsymbol{\xi}_j - \boldsymbol{\xi}_i)] + D_i\mathbf{1}_m \ll \gamma\mathbf{1}_p, \qquad i, j = 1, \dots, M, \tag{4.44}$$

hold for any $\tau \in [0, T)$, then the switched system (4.1)-(4.2) is exponentially stable and

$$\max_{i\in\{1,2,\dots,M\}} \|G_i(0)\|_\infty \leq J^{ind}_{\mathcal{L}_\infty,T} < \gamma.$$

Proof. The proofs of exponential stability and of the first inequality follow the same lines as in previous proofs. Now, consider a signal $\sigma \in \mathcal{D}_T$, with $0 = t_0 < t_1 < \dots < t_k < t_{k+1} < ...$ as switching instants. Assume that $\sigma(t_k) = i$ and $\sigma(t_{k+1}) = j$. Note, also, that (4.42) implies

$$e^{A_it}\boldsymbol{\xi}_i + \int_0^t e^{A_i\tau}B_i\mathbf{1}_m d\tau \ll \boldsymbol{\xi}_i, \qquad i = 1, \dots, M, \ \forall t > 0. \tag{4.45}$$

Consider the differential equation

$$\dot{\bar{\mathbf{d}}}(\mathbf{t}) = A_{\sigma(t)}\bar{\mathbf{d}}(t) + B_{\sigma(t)}\mathbf{1}_m, \tag{4.46}$$

with $\bar{\mathbf{d}}(t_k) = \boldsymbol{\xi}_{\sigma(t_{k+1})} = \boldsymbol{\xi}_j$. For every $t \in [t_k, t_{k+1})$ one finds:

$$\frac{d}{dt}(\bar{\mathbf{d}}(t) - \boldsymbol{\xi}_i) = A_i\bar{\mathbf{d}}(t) + B_i\mathbf{1}_m \ll A_i(\bar{\mathbf{d}}(t) - \boldsymbol{\xi}_i).$$

By integrating the previous equation and by making use of $\bar{\mathbf{d}}(t_k) = \boldsymbol{\xi}_j$, one gets for every $t \in [t_k, t_{k+1})$

$$\bar{\mathbf{d}}(t) \ll \boldsymbol{\xi}_i + e^{A_i(t-t_k)}(\boldsymbol{\xi}_j - \boldsymbol{\xi}_i). \tag{4.47}$$

Moreover, from (4.43) and (4.45), it follows

$$\begin{aligned}
\bar{\mathbf{d}}(t_{k+1}^-) &= e^{A_i(t_{k+1}-t_k)}\bar{\mathbf{d}}(t_k) + \int_{t_k}^{t_{k+1}} e^{A_i(t_{k+1}-\tau)}B_i\mathbf{1}_m d\tau \\
&= e^{A_i(t_{k+1}-t_k-T)}e^{A_iT}\boldsymbol{\xi}_j + \int_{t_k}^{t_{k+1}} e^{A_i(t_{k+1}-\tau)}B_i\mathbf{1}_m d\tau \\
&\ll e^{A_i(t_{k+1}-t_k-T)}\boldsymbol{\xi}_i - \int_0^T e^{A_i(t_{k+1}-t_k-T+\tau)}B_i\mathbf{1}_m d\tau \\
&+ \int_{t_k}^{t_{k+1}} e^{A_i(t_{k+1}-\tau)}B_i\mathbf{1}_m d\tau \\
&\ll \boldsymbol{\xi}_i - \int_0^{t_{k+1}-t_k-T} e^{A_i\tau}B_i\mathbf{1}_m d\tau - \int_0^T e^{A_i(t_{k+1}-t_k-T+\tau)}B_i\mathbf{1}_m d\tau \\
&+ \int_{t_k}^{t_{k+1}} e^{A_i(t_{k+1}-\tau)}B_i\mathbf{1}_m d\tau \\
&= \boldsymbol{\xi}_i.
\end{aligned} \tag{4.48}$$

Now, set $V(\mathbf{x}(t),t) := \max_r \frac{[\mathbf{x}(t)]_r}{[\bar{\mathbf{d}}(t)]_r}$, and $\bar{w} := \sup_{t\geq 0}\max_r[\mathbf{w}(t)]_r$. If t is a specific time in $[t_k, t_{k+1})$, and we let $s \in \{1,2,\dots,n\}$ be such that $V(\mathbf{x}(t),t) = \frac{[\mathbf{x}(t)]_s}{[\bar{\mathbf{d}}(t)]_s}$. From the equation of the system dynamics, (4.46) and (4.48), it follows that

$$\begin{aligned}
\frac{d}{dt}\left(\bar{w} - V(\mathbf{x}(t),t)\right) &> -\sum_{k=1}^{n}[B_i]_{sk}\left(\bar{w} - V(\mathbf{x}(t),t)\right), \\
\bar{w} - V(\mathbf{x}(t_{k+1},t_{k+1}) &> \bar{w} - V(\mathbf{x}(t_{k+1},t_{k+1}^-).
\end{aligned}$$

Therefore, $\bar{w} > V(\mathbf{x}(t),t)$, $\forall t \geq 0$. Finally, from (4.44) and (4.47), for any $i \in \{1, 2, \ldots, p\}$, we have

$$\begin{aligned}
[\mathbf{z}(t)]_i &= \sum_{j=1}^{n} [C_{\sigma(t)}]_{ij} [\mathbf{x}(t)]_j + \sum_{j=1}^{n} [D_{\sigma(t)}]_{ij} [\mathbf{w}(t)]_j \\
&\leq \sum_{j=1}^{n} [C_{\sigma(t)}]_{ij} [\bar{\mathbf{d}}(t)]_j V(\mathbf{x}(t), t) + \sum_{j=1}^{n} [D_{\sigma(t)}]_{ij} \bar{w} \\
&< \sum_{j=1}^{n} ([D_{\sigma(t)}]_{ij} - \gamma)(\bar{w} - V(\mathbf{x}(t), t)) + \gamma \bar{w} \\
&< \gamma \bar{w}.
\end{aligned}$$

In conclusion, for any positive $\mathbf{w} \in \mathcal{L}_\infty$ we have that $\sup_{t \geq 0} \max_i [\mathbf{z}(t)]_i < \gamma \bar{w}$ and the proof is concluded. □

A necessary and sufficient condition for $J^{ind}_{\mathcal{L}_\infty, 0} < \gamma$ to hold can be stated in the particular case where matrices A_i and B_i are not affected by the switching signal, i.e. $A_i = A$, $B_i = B$, $i = 1, 2, \ldots, M$. In this case, when only *sensor switching* occurs, one may want to have a certain performance level even in the worst case. Again, as for the $\mathcal{L}_1$ induced norm, the following result can be proven. The proof follows immediately from Theorem 4.10 (suffciency) and Lemma 4.9 (necessity), and it is therefore omitted.

Theorem 4.12. Assume that $A_i = A$, $B_i = B$, $i = 1, 2, \ldots, M$. Then $J^{ind}_{\mathcal{L}_1, 0} < \gamma$ if and only if there exists a strictly positive vector $\boldsymbol{\xi} \in \mathbb{R}^n_+$ such that

$$A\boldsymbol{\xi} + B\mathbf{1}_m \ll 0, \tag{4.49}$$

$$C_i\boldsymbol{\xi} + D_i\mathbf{1}_m \ll \gamma \mathbf{1}_m, \tag{4.50}$$

hold for any $i = 1, 2, \ldots, M$. In this case, the switched system (4.1)-(4.2) is exponentially stable,

$$J^{ind}_{\mathcal{L}_\infty, 0} = \max_{i \in \{1, 2, \ldots, M\}} \|G_i(0)\|_\infty$$

and $\sigma(t) = \arg\max_{i \in \{1, 2, \ldots, M\}} \|G_i(0)\|_\infty$, $t \geq 0$ is the worst switching signal.

4.4 Guaranteed induced $\mathcal{L}_2$ norm

In this section we consider the worst $\mathcal{L}_2$ induced norm of the positive switched system (4.1)-(4.2) for switching signals belonging to $\mathcal{D}_0$, defined as

$$J_{\mathcal{L}_2}^{ind} := \sup_{\sigma \in \mathcal{D}_0} \sup_{\substack{\mathbf{w} \in \mathcal{L}_2, \mathbf{w} \neq 0 \\ \mathbf{w}(t) \geq 0, \forall t \geq 0}} \frac{\int_0^{+\infty} \mathbf{z}(t)^\top \mathbf{z}(t) dt}{\int_0^{+\infty} \mathbf{w}(t)^\top \mathbf{w}(t) dt}. \tag{4.51}$$

Theorem 4.13. If there exist strictly positive vectors $\mathbf{v} \in \mathbb{R}_+^n$, $\boldsymbol{\xi} \in \mathbb{R}_+^n$, $\mathbf{h} \in \mathbb{R}_+^p$ and $\mathbf{g} \in \mathbb{R}_+^m$ such that

$$A_i^\top \mathbf{v} + C_i^\top \mathbf{h} \ll 0 \tag{4.52}$$

$$B_i^\top \mathbf{v} + D_i^\top \mathbf{h} \ll \gamma \mathbf{g} \tag{4.53}$$

$$A_i \boldsymbol{\xi} + B_i \mathbf{g} \ll 0 \tag{4.54}$$

$$C_i \boldsymbol{\xi} + D_i \mathbf{g} \ll \gamma \mathbf{h} \tag{4.55}$$

hold for any $i = 1, 2, \dots, M$, then the switched system (4.1)-(4.2) is exponentially stable and

$$\max_{i \in \{1,2,\dots,M\}} \|G_i(0)\|_2^2 \leq J_{\mathcal{L}_2,0}^{ind} < \gamma^2.$$

Proof. Exponential stability follows upon noticing that (4.52) implies $\mathbf{v}^\top A_i \ll 0$, $\forall i$, so that $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ is a linear copositive Lyapunov function for the system. In order to prove the first inequality, notice that $J_{\mathcal{L}_2,0}^{ind} \geq \|G_i(s)\|_\infty^2$, where $\|G_i(s)\|_\infty$ is the $\mathcal{H}_\infty$ norm of the time-invariant system with transfer function $G_i(s)$ obtained letting $\sigma(t) = i$, $t \geq 0$. Moreover, take $\sigma(t) = i$, $t \geq 0$, and observe that the $\mathcal{H}_\infty$ norm of the time invariant system associated with the ith mode is

$$\|G_i(s)\|_\infty^2 := \sup_\omega \|G_i(j\omega)\|_2^2 = \sup_{\substack{\mathbf{w} \in \mathcal{L}_2, \mathbf{w} \neq 0 \\ \mathbf{w}(t) \geq 0, \forall t \geq 0}} \frac{\int_0^{+\infty} \mathbf{z}(t)^\top \mathbf{z}(t) dt}{\int_0^{+\infty} \mathbf{w}(t)^\top \mathbf{w}(t) dt}.$$

From the definition it follows immediately that $\|G_i(s)\|_\infty \geq \|G_i(0)\|_2$. Moreover, for any m-dimensional complex vector $\mathbf{y}$

$$|G(j\omega)\mathbf{y}| = |\int_0^{+\infty} g_i(t) e^{j\omega t} \mathbf{y} dt| \leq \int_0^{+\infty} g_i(t) dt |\mathbf{y}| = G_i(0)|\mathbf{y}|$$

where $g_i(t)$ is the impulse response of the system associated with the ith mode and, for any vector $\mathbf{v}$, $|\mathbf{v}|$ is the vector whose ith entry is $|[\mathbf{v}]_i|$. Therefore

$$\mathbf{y}^{\sim} G(-j\omega)^{\top} G(j\omega)\mathbf{y} \leq |\mathbf{y}|^{\top} G_i(0)^{\top} G_i(0)|\mathbf{y}|,$$

where $\mathbf{y}^{\sim}$ is the complex conjugate transpose of $\mathbf{y}$. Since $\mathbf{y}$ is a generic vector it results that $\|G_i(j\omega)\| \leq \|G_i(0)\|_2$, for any ω, thus concluding

$$\|G_i(s)\|_{\infty} = \|G_i(0)\|_2.$$

The first inequality in the statement is then proven. Now, take the positive diagonal matrix P, with $[P]_{ii} = \gamma \frac{[\mathbf{v}]_i}{[\boldsymbol{\xi}]_i}$, $i = 1, 2, \ldots, n$, so that $P\boldsymbol{\xi} = \gamma \mathbf{v}$. After summing (4.52) with (4.54) pre-multiplied by P/γ, and using (4.55) in order to eliminate $\mathbf{h}$, we get

$$(A_i^{\top} P + PA_i + C_i^{\top} C_i)\boldsymbol{\xi} + (PB_i + C_i^{\top} D_i)\mathbf{g} \ll 0.$$

Again, if we pre-multiply (4.55) by $D_i^{\top}$ and eliminate $\mathbf{h}$ making use of (4.53) pre-multiplied by γ, we get

$$(B_i^{\top} P + D_i^{\top} C_i)\boldsymbol{\xi} + (D_i^{\top} D_i - \gamma^2 I_m)\mathbf{g} \ll 0,$$

and therefore the matrices

$$S_i = \begin{bmatrix} A_i^{\top} P + PA_i + C_i^{\top} C_i & PB_i + C_i^{\top} D_i \\ B_i^{\top} P + D_i^{\top} C_i & D_i^{\top} D_i - \gamma^2 I_m \end{bmatrix},$$

$i = 1, 2, \ldots, M$, satisfy the inequalities

$$S_i \begin{bmatrix} \boldsymbol{\xi} \\ \mathbf{g} \end{bmatrix} \ll 0.$$

Notice that matrices S_i, $i = 1, 2, \ldots, M$, are Metzler and symmetric. As such, they are all Hurwitz and negative definite, i.e. $S_i \prec 0$, $i = 1, 2, \ldots, M$. Now, compute the derivative of $V(\mathbf{x}) = \mathbf{x}^{\top} P \mathbf{x}$ along the trajectories of system (4.1). It follows

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \begin{bmatrix} \mathbf{x}^{\top} & \mathbf{w}^{\top} \end{bmatrix} \begin{bmatrix} A_{\sigma}^{\top} P + PA_{\sigma} & PB_{\sigma} \\ B_{\sigma}^{\top} P & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{x}^{\top} & \mathbf{w}^{\top} \end{bmatrix} S_{\sigma} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} - \mathbf{z}^{\top} \mathbf{z} + \gamma^2 \mathbf{w}^{\top} \mathbf{w}. \end{aligned}$$

Since $S_i \prec 0$ for every i, the thesis follows after integration from 0 to $+\infty$. $\square$

It is important to stress that (4.52)-(4.55) are linear inequalities whose feasibility can be easily checked by standard linear programs. In the time-invariant case, they provide a necessary and sufficient condition for $\|G_i(s)\|_\infty < \gamma$ to hold.

5

Stabilization

In this section we consider, again, the continuous-time positive switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \tag{5.1}$$

where $\sigma(t) \in \{1, 2, \ldots, M\}, \forall t \geq 0$, and for every value i taken by the switching signal σ the matrix A_i is Metzler. Differently from the previous sections, however, we assume that σ is not a completely arbitrary switching signal, but it represents a control input that we can suitably choose. In particular, our interest is in choosing σ in such a way that, for any positive initial condition $\mathbf{x}(0)$, the state $\mathbf{x}(t)$ converges to zero (*standard stabilization problem*) or it converges to zero with a guaranteed convergence speed (a problem that can be reduced to the standard stabilization one for a modified system, as pointed out in Proposition 3.6). In Sections 2.2, 2.3, 2.5 and 2.6, we provided several examples where these problems naturally arise.

In order to achieve this goal, namely to stabilize system (5.1), we can either resort to *open loop stabilization*, where σ is a function of the time variable t, or we can look for *feedback (equivalently, closed-loop) stabilization*, i.e., the switching signal is a (possibly time-varying) function of the state variable, and hence takes the form

$$\sigma(t) = \Psi(\mathbf{x}(t), t). \tag{5.2}$$

An interesting property of positive switched systems, that make them differ from standard switched systems (see Sun and Ge [2005]), is that open loop stabilization and feedback stabilization are equivalent problems, according to the following result (see Fornasini and Valcher [2012] and Blanchini et al. [2012] for details).

Theorem 5.1. For a positive switched system (5.1) the following conditions are equivalent:

i) There exists $\bar{\mathbf{x}}_0 \gg 0$ and a switching signal $\bar{\sigma}(t), t \in \mathbb{R}_+$, such that the trajectory $\bar{\mathbf{x}}(t), t \in \mathbb{R}_+$, generated corresponding to $\bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0$ and $\bar{\sigma}(t), t \in \mathbb{R}_+$, exponentially converges to zero.

ii) The switched system is *feedback stabilizable*, i.e., there exists a feedback law $\sigma(t) = \Psi(\mathbf{x}(t), t)$ such that the trajectory starting from any $\mathbf{x}(0) > 0$ exponentially converges to zero.

iii) The switched system is *consistently stabilizable*, i.e., there exists a switching signal $\sigma(t), t \in \mathbb{R}_+$, that drives $\mathbf{x}(t)$ to zero exponentially, independently of the positive [1] initial condition $\mathbf{x}(0) > 0$.

Proof. It is quite obvious that iii) $\Rightarrow$ ii) $\Rightarrow$ i). We want to show that i) $\Rightarrow$ iii) (see Fornasini and Valcher [2012]).

Let $\mathbf{x}(0)$ be any positive initial condition. Then there exists $\alpha > 0$ such that $\mathbf{x}(0) < \alpha\bar{\mathbf{x}}_0$. Let $\mathbf{x}(t)$ be the state trajectory generated corresponding to the initial condition $\mathbf{x}(0)$ and to the switching signal $\bar{\sigma}(t)$. Set $\mathbf{z}(t) := \alpha\bar{\mathbf{x}}(t) - \mathbf{x}(t)$. Then $\mathbf{z}(t)$ is a solution of the time-varying system

$$\dot{\mathbf{z}}(t) = A_{\bar{\sigma}(t)}\mathbf{z}(t).$$

Since $A_{\bar{\sigma}(t)}$ is Metzler at every time $t \geq 0$, and $\mathbf{z}(0) = \alpha\bar{\mathbf{x}}_0 - \mathbf{x}(0)$ is positive, then $\mathbf{z}(t) > 0$ for every $t \geq 0$, which means that $\alpha\bar{\mathbf{x}}(t) > \mathbf{x}(t) \geq 0$ for every $t \geq 0$. As the system is linear, the trajectory $\alpha\bar{\mathbf{x}}(t)$ asymptotically converges to zero, and so does $\mathbf{x}(t)$. □

[1] As a matter of fact, such a switching signal would work with any $\mathbf{x}(0) \in \mathbb{R}^n$, not necessarily positive.

Remark 5.1. It is worth noticing that condition i) in Theorem 5.1 can be restated by simply saying that there exists a switching signal $\bar{\sigma}(t), t \in \mathbb{R}_+$, that drives to zero the trajectory $\bar{\mathbf{x}}(t), t \in \mathbb{R}_+$, generated corresponding to $\bar{\mathbf{x}}(0) = \mathbf{1}_n$. Even more, it is easy to see (see, also, Corollary 3.12 in Sun and Ge [2011]) that if a system is consistently stabilizable then it can be stabilized by resorting to a periodic switching signal $\bar{\sigma}(t), t \geq 0$. This is interesting because, differently from stability under arbitrary switching (recall Remark 3.7), it is quite easy to verify that open loop stabilization, if achievable, can always be obtained by resorting to a periodic switching signal. □

In the following we will refer to a positive switched system satisfying any of the equivalent conditions of Theorem 5.1 simply as *stabilizable.*

5.1 Feedback stabilization based on Lyapunov functions

Even if, as we have just seen, stabilization can be equivalently obtained by means of open loop or feedback switching strategies, nonetheless a feedback solution is in general preferable for various reasons, first of all its robustness. To explore this problem we introduce the concept of copositive control Lyapunov function.

Definition 5.1. We say that a copositive function $V(\mathbf{x})$ is a *control Lyapunov function* for the system (5.1), if it is decreasing along the system trajectories, provided that a certain feedback switching strategy is applied. If we let $D_{\mathbf{w}}V(\mathbf{x})$ denote the derivative of the function $V(\mathbf{x})$ along the direction $\mathbf{w}$, this amounts to requiring that for every $\mathbf{x} \in \mathbb{R}^n_+$ and for the feedback switching law $\sigma(t) = \Psi(\mathbf{x}(t), t)$, we have

$$D_{A_\sigma \mathbf{x}}V(\mathbf{x}) := \lim_{h \to 0^+} \frac{V(\mathbf{x} + hA_\sigma \mathbf{x}) - V(\mathbf{x})}{h} < 0.$$

We underline that our definition of control Lyapunov function is different, yet equivalent, to the classical one used in the literature for systems of the form $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$, see Freeman and Kokotović [1996]. In the classical case it is required, under differentiability assumptions, that for every $\mathbf{x} \neq 0$, there exists $\mathbf{u}$ such that $\nabla V(\mathbf{x}) f(\mathbf{x}, \mathbf{u}) < 0$, and the feedback is proved to exists under suitable assumptions (Arstein's theorem, Artstein [1983]). Here we explicitly assume the existence of a feedback law Ψ such that $\mathbf{u} = \Psi(\mathbf{x}(t), t)$.

In general, for standard switched systems, convex control Lyapunov functions may not exist (see Blanchini and Savorgnan [2008]), even if the system is stabilizable. A natural question is whether there exists a class of Lyapunov functions that is universal for the stabilization problem in the case of positive switched systems. The following theorem provides an answer, see Hernandez-Vargas et al. [2011], and tells us that, surprisingly, for stabilizable positive switched systems, as long as we remain in the positive orthant, we can always find concave copositive control Lyapunov functions.

Theorem 5.2. If the positive switched system (5.1) is stabilizable, then it admits a concave copositive control Lyapunov function $V(\mathbf{x})$, positively homogeneous of order one (i.e. $V(\alpha\mathbf{x}) = \alpha V(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ and every $\alpha > 0$).

Proof. Let β be an arbitrarily small positive number and consider the positive switched system obtained by perturbing (5.1):

$$\dot{\mathbf{x}}(t) = [\beta I_n + A_{\sigma(t)}]\mathbf{x}(t) =: A_{\beta,\sigma(t)}\mathbf{x}(t).$$

Denote by $\mathbf{x}_\beta(t; \mathbf{x}_0, \sigma)$ and $\mathbf{x}(t; \mathbf{x}_0, \sigma)$ the solutions of the perturbed and of the unperturbed system, respectively, corresponding to the initial condition $\mathbf{x}_0$ and to the switching signal σ. We recall (see Proposition 3.6) that they are related as follows:

$$\mathbf{x}_\beta(t; \mathbf{x}_0, \sigma) = e^{\beta t}\mathbf{x}(t; \mathbf{x}_0, \sigma), \qquad \forall\, t \geq 0.$$

Also, if the original system is stabilizable, then for $\beta > 0$ sufficiently small the system remains stabilizable.

Introduce now, for the perturbed system, the following function:

$$V_\beta(\mathbf{x}_0) := \inf_{\sigma \in \mathcal{D}_0} \int_0^{+\infty} \mathbf{1}_n^\top \mathbf{x}_\beta(t; \mathbf{x}_0, \sigma)dt,$$

that is well defined since we assumed stabilizability. Clearly, $V_\beta(\mathbf{x}_0)$ is copositive and positively homogeneous of order one. To prove its concavity, assume $\mathbf{x}_0 = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$, with $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$. By the system linearity, for any switching signal σ we have

$$\begin{aligned}\int_0^{+\infty} \mathbf{1}_n^\top \mathbf{x}_\beta(t;\mathbf{x}_0,\sigma)dt &= \alpha_1 \int_0^{+\infty} \mathbf{1}_n^\top \mathbf{x}_\beta(t;\mathbf{x}_1,\sigma)dt \\ &\quad +\alpha_2 \int_0^{+\infty} \mathbf{1}_n^\top \mathbf{x}_\beta(t;\mathbf{x}_2,\sigma)dt,\end{aligned}$$

and hence

$$\begin{aligned}V_\beta(\mathbf{x}_0) &= \inf_{\sigma\in\mathcal{D}_0} \int_0^{+\infty} \mathbf{1}_n^\top \mathbf{x}_\beta(t;\mathbf{x}_0,\sigma) \\ &= \inf_{\sigma\in\mathcal{D}_0} \left[\alpha_1 \int_0^{+\infty} \mathbf{1}_n^\top \mathbf{x}_\beta(t;\mathbf{x}_1,\sigma)dt + \alpha_2 \int_0^{+\infty} \mathbf{1}_n^\top \mathbf{x}_\beta(t;\mathbf{x}_2\sigma)dt\right] \\ &\geq \alpha_1 \left[\inf_{\sigma\in\mathcal{D}_0} \int_0^{+\infty} \mathbf{1}_n^\top \mathbf{x}_\beta(t;\mathbf{x}_1,\sigma)dt\right] \\ &+ \alpha_2 \left[\inf_{\sigma\in\mathcal{D}_0} \int_0^{+\infty} \mathbf{1}_n^\top \mathbf{x}_\beta(t;\mathbf{x}_2,\sigma)dt\right] \\ &= \alpha_1 V_\beta(\mathbf{x}_1) + \alpha_2 V_\beta(\mathbf{x}_2).\end{aligned}$$

This proves that $V_\beta(\mathbf{x}_0)$ is concave. The derivative of V_β in $\mathbf{x}_0 \in \mathbb{R}^n_+$, along the direction of the perturbed system, is

$$D_{[\beta I_n + A_\sigma]\mathbf{x}} V_\beta(\mathbf{x}_0) := \lim_{h\to 0^+} \frac{V_\beta(\mathbf{x}_0 + h[\beta I_n + A_\sigma]\mathbf{x}_0) - V_\beta(\mathbf{x}_0)}{h}.$$

Consider any interval $[0,\tau]$, $\tau > 0$. By applying dynamic programming considerations (see for instance Luenberger [1979], Chapter 11.7) we deduce the identity

$$V_\beta(\mathbf{x}_0) = \inf_{\sigma\in\mathcal{D}_0} \left[\int_0^{\tau} \mathbf{1}_n^\top \mathbf{x}_\beta(t;\mathbf{x}_0,\sigma)dt + V_\beta(\mathbf{x}_\beta(\tau;\mathbf{x}_0,\sigma))\right],$$

and hence for ever $\tau > 0$

$$V_\beta(\mathbf{x}_0) > V_\beta(\mathbf{x}_\beta(\tau;\mathbf{x}_0,\sigma)).$$

As a consequence we have that

$$D_{[\beta I_n + A_\sigma]\mathbf{x}_0} V_\beta(\mathbf{x}_0) \leq 0.$$

Set $\eta := h/(1-\beta h)$ (so that $h \to 0$ implies $\eta \to 0$), and now, bearing in mind that $V_\beta(\lambda x) = \lambda V_\beta(x)$ for any real positive λ, consider the derivative of the function for the nominal system (and hence along the direction $A_\sigma \mathbf{x}_0$)

$$
\begin{aligned}
& D_{A_\sigma \mathbf{x}_0} V_\beta(\mathbf{x}_0) = \lim_{h \to 0^+} \frac{V_\beta(\mathbf{x}_0 + hA_\sigma \mathbf{x}_0) - V_\beta(\mathbf{x}_0)}{h} \\
= & \lim_{h \to 0^+} \frac{V_\beta(\mathbf{x}_0 - \beta h \mathbf{x}_0) - V_\beta(\mathbf{x}_0)}{h} \\
+ & \lim_{h \to 0^+} \frac{V_\beta(\mathbf{x}_0 + hA_\sigma \mathbf{x}_0 + \beta h \mathbf{x}_0 - \beta h \mathbf{x}_0) - V_\beta(\mathbf{x}_0 - h\beta \mathbf{x}_0)}{h/(1-\beta h)} \frac{1}{(1-\beta h)} \\
= & \lim_{h \to 0^+} \frac{(1-\beta h) - 1}{h} V_\beta(\mathbf{x}_0) \\
+ & \lim_{h \to 0^+} \frac{V_\beta((1-\beta h)\mathbf{x}_0 + h[A_\sigma + \beta I_n]\mathbf{x}_0) - (1-\beta h)V_\beta(\mathbf{x}_0)}{h/(1-\beta h)} \frac{1}{(1-\beta h)} \\
= & -\beta V_\beta(\mathbf{x}_0) + \lim_{\eta \to 0^+} \frac{V_\beta(\mathbf{x}_0 + \eta[\beta I_n + A_\sigma]\mathbf{x}_0) - V_\beta(\mathbf{x}_0)}{\eta} \\
= & -\beta V_\beta(\mathbf{x}_0) + D_{[\beta I_n + A_\sigma]\mathbf{x}} V_\beta(\mathbf{x}_0) \le -\beta V_\beta(\mathbf{x}_0).
\end{aligned}
$$

This ensures that V_β is a control Lyapunov function for the positive switched system (5.1), and the proof is completed. □

Remark 5.2. Note that the function V_β used within the proof of the previous theorem has a meaning. Consider again the fluid example and the integral cost function (2.10). By proceeding as in the proof, it can be shown that the cost-to-go function associated with (2.10), namely the function that associates with any $\mathbf{x}_0 > 0$ the optimal value of the cost function corresponding to the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, is a concave function. An algorithm for computing this type of functions that makes use of discretization has been proposed in Hernandez-Vargas et al. [2011]. We will reconsider this fact in the next section. □

As we have seen before (Theorem 3.1) a linear system (not necessarily positive) that is stable under arbitrary switching always admits a smooth convex Lyapunov function. So, one would be tempted to make the following:

Conjecture: for a positive switched system, stabilizability implies the existence of a smooth concave copositive (positively homogeneous of order 1) control Lyapunov function. □

Unfortunately, the above conjecture is false, except for the second order case, i.e. $n = 2$. To be precise, we can claim the following.

Theorem 5.3. Given the positive switched system (5.1), the following statements are equivalent.

i) The system is stabilizable and it admits a copositive, positively homogeneous of order 1, smooth control Lyapunov function such that
$$\min_i \nabla V(\mathbf{x}) A_i \mathbf{x} \le -\beta^* V(\mathbf{x}), \qquad \forall x \ge 0, \tag{5.3}$$
for some $\beta^* > 0$ [2].

ii) There exists $\alpha \in \mathcal{A}_M$ such that $A(\alpha)$ is Hurwitz.

iii) The system admits a linear copositive control Lyapunov function $V_L(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, with $\mathbf{v} \gg 0$.

Proof. The proof is carried on in the case where the Metzler matrices A_i $i = 1, 2, \dots, M$, are irreducible. The standard case may be obtained from the irreducible one, by first considering the irreducible matrices $A_i + \varepsilon \mathbf{1}_n \mathbf{1}_n^\top, \varepsilon > 0$, and then considering the limit for $\varepsilon \to 0^+$.
ii) $\Rightarrow$ iii) Assume that there exists a Hurwitz convex combination
$$A(\alpha) = \sum_{i=1}^{M} \alpha_i A_i, \quad \alpha_i \ge 0, \quad \sum_{i=1}^{M} \alpha_i = 1,$$
and let $\mathbf{v} \gg 0$ be the strictly positive left eigenvector of $A(\alpha)$ associated with the Frobenius eigenvalue $\lambda_F < 0$. Note that, by the irreducibility assumption on the matrices A_i, all the matrices $A(\alpha)$ are irreducible, in turn. Consider the linear copositive function $V_L(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$. Then, by linearity, as we have seen in the simple example in Section 2.5, for every $\mathbf{x} > 0$ we have
$$\min_{i=1,2,\dots,M} \mathbf{v}^\top A_i \mathbf{x} \le \mathbf{v}^\top A(\alpha) \mathbf{x} = \lambda_F \mathbf{v}^\top \mathbf{x} = \lambda_F V_L(\mathbf{x}) < 0.$$

[2] This condition ensures that $V(\mathbf{x}(t))$ is decreasing along the system trajectories, if we apply the switching strategy $\sigma(\mathbf{x}(t)) \in \arg\min_i \nabla V(\mathbf{x}) A_i \mathbf{x}$.

Therefore, the strategy $\sigma(t) \in \arg\min_i \mathbf{v}^\top A_i \mathbf{x}(t)$ is stabilizing. This shows that $V_L(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ is a linear copositive control Lyapunov function.

iii) $\Rightarrow$ i) Obvious, since a linear copositive function is smooth and positively homogeneous of order 1, and it satisfies (5.3).

i) $\Rightarrow$ ii) We give only a sketch of the proof here. For the complete proof see Blanchini et al. [2012, 2013]. Assume that there exists a copositive, positively homogeneous of order 1, smooth control Lyapunov function for which

$$\min_i \quad \nabla V(\mathbf{x}) A_i \mathbf{x} \leq -\beta^* V(\mathbf{x}),$$

for some positive $\beta^* > 0$. For any $\beta < \beta^*$, consider the following "relaxed condition":

$$\min_{\alpha \in \mathcal{A}_M} \quad \nabla V(\mathbf{x}) A(\alpha) \mathbf{x} \leq -\beta V(\mathbf{x}). \tag{5.4}$$

For each $\mathbf{x} > 0$ define the convex set-valued map, see Aubin [1991]:

$$\Omega(\mathbf{x}) = \{\alpha \in \mathcal{A} : \quad \nabla V(\mathbf{x}) A(\alpha) \mathbf{x} \leq -\beta V(\mathbf{x})\}.$$

The set valued map Ω has a non-empty relative interior in $\mathcal{A}_M$ for any $\mathbf{x} > 0$, because we chose $\beta < \beta^*$, and by the continuity of the gradient, it is a convex continuous set-valued map. Therefore there exists a continuous function $\bar{\alpha}(\mathbf{x})$, Freeman and Kokotović [1996], such that $\bar{\alpha}(\mathbf{x}) \in \Omega(\mathbf{x})$, for all $\mathbf{x} \gg 0$, Aubin [1991]. [3]

Now take the following two functions, the former from $\mathcal{A}_n$ to $\mathcal{A}_M$ and the latter from $\mathcal{A}_M$ to $\mathcal{A}_n$:

$$\mathbf{x} \to \bar{\alpha}(\mathbf{x})$$

and

$$\alpha \to \mathbf{v}_F(\alpha),$$

where $\mathbf{v}_F(\alpha)$ is the strictly positive Frobenius eigenvector associated with $A(\alpha)$, normalized in such a way that $\mathbf{1}_n^\top \mathbf{v}_F(\alpha) = 1$. Note also that

[3]For instance the choice of α with the smallest Euclidean norm (which is unique): $\bar{\alpha}(x) \doteq \arg\min_{\alpha \in \Omega(\mathbf{x})} \ \|\alpha\|$.

$\mathbf{v}_F(\alpha)$ is a positive and continuous function of α. The composed map from $\mathcal{A}_n$ to $\mathcal{A}_n$

$$\mathbf{x} \to \bar{\alpha}(\mathbf{x}) \to \mathbf{v}_F(\bar{\alpha}(\mathbf{x}))$$

is continuous and admits a fixed point $\hat{\mathbf{x}}$, because $\mathcal{A}_n$ is convex and compact. For such $\hat{\mathbf{x}}$ we have

$$\nabla V(\hat{\mathbf{x}})A(\hat{\alpha})\bar{\mathbf{x}} = \hat{\lambda}_F \nabla V(\hat{\mathbf{x}})\bar{\mathbf{x}} \le -\beta V(\hat{\mathbf{x}}),$$

where $\hat{\alpha} := \bar{\alpha}(\hat{\mathbf{x}})$ and $\hat{\lambda}_F$ is the Frobenius eigenvalue associated with $A(\hat{\alpha})$. On the other hand, any positively homogeneous smooth function is such that

$$V(\mathbf{x}) = \nabla V(\mathbf{x})\mathbf{x}.$$

This implies that the last inequality can be rewritten as

$$\lambda_F V(\hat{\mathbf{x}}) \le -\beta V(\hat{\mathbf{x}})$$

and hence $\lambda_F < 0$. This ensures that $A(\hat{\alpha})$ is Hurwitz. □

Remark 5.3. Condition ii) in the previous theorem is easily shown to be equivalent to the existence of a diagonal matrix D, with positive diagonal entries, and nonnegative parameters $\alpha_i, i = 1, 2, \ldots, M$, with $\sum_{i=1}^{M} \alpha_i = 1$, such that

$$\sum_{i=1}^{M} \alpha_i \mathbf{x}^\top [A_i^\top D + DA_i]\mathbf{x} < 0, \qquad \forall \mathbf{x} > 0.$$

□

As previously mentioned, stabilizability does not ensure the existence of a smooth copositive control Lyapunov functions, positively homogeneous of order 1. The proof of the following result can be found in Blanchini et al. [2012, 2013].

Proposition 5.1. There exist stabilizable positive switched systems (5.1), for which no copositive control Lyapunov function, positively homogeneous of order 1, which is continuously differentiable can be found.

Interestingly enough a counterexample is just the traffic system proposed in Section 2.6. Since positive switched systems are a special case of linear switched systems, the existence of a positively homogeneous smooth Lyapunov function would ensure the existence of a copositive one if we restrict our attention to the positive orthant. Hence, in general, the stabilizability of a positive switched system does not imply the existence of a smooth positively homogeneous control Lyapunov function. More details can be found in Blanchini et al. [2012, 2013].

Second order systems, however, represent an exception, as stated in the result below, whose proof can be found in Blanchini et al. [2012].

Proposition 5.2. A second order continuous-time positive switched system (5.1) is stabilizable if and only if there exists $\alpha \in \mathcal{A}_M$ such that $A(\alpha)$ is Hurwitz. $\square$

We stress that the existence of a Hurwitz convex combination is a sufficient condition for stabilizability even for nonpositive switched linear systems (see Wicks et al. [1998]). In the case of positive systems, the existence of such a Hurwitz combination leads to the important technique based on the left Frobenius eigenvector as explained by means of the fluid example reported in Section 2.5.

The previous discussion suggests that we have essentially two alternatives (for brevity, we consider the case when all matrices A_i, $i = 1, 2, \ldots, M$, are irreducible).

Lucky case. Find, if possible, a Hurwitz convex combination of the system matrices $A_i, i = 1, 2, \ldots, M$. This is not a simple problem in general, but if M is small, then gridding solutions are possible. If such a Hurwitz convex combination is found, then a control Lyapunov function can be inferred from its left Frobenius eigenvector.

Unlucky case. If there are no Hurwitz convex combinations, such as in the case of the traffic example in Section 2.6, then any attempt at finding a smooth, positively homogeneous of order 1, control Lyapunov function is hopeless. Necessarily, one has to find more general classes of functions such as the minimum of copositive linear functions, or approach the problem by seeking open loop switching signals, for instance of periodic type.

It is worth noticing that the existence of a linear copositive control Lyapunov function can be checked with linear programming techniques. Indeed, if $V_L(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$ is a linear copositive function, with $\mathbf{v}$ a strictly positive vector, such a function is a control Lyapunov function for the system if (and only if) for any vector $\mathbf{x}$ in the closed positive simplex $\mathcal{A}_n$ there is at least one choice of the matrix A_i such that the derivative of V_L along the ith subsystem is negative, or, equivalently,

$$\min_{i=1,2,\dots,M} \mathbf{v}^\top A_i \mathbf{x} < 0, \quad \forall \mathbf{x} \in \mathcal{A}_n.$$

The previous condition is not verified if and only if the sets

$$\mathcal{P}_i := \{\mathbf{x} \in \mathcal{A}_n : \quad \mathbf{v}^\top A_i x \geq 0\}, \qquad i = 1, 2, \dots, M,$$

have a non-empty intersection in the positive orthant. Since these sets are polytopes, determining if they have a non-empty intersection is a linear programming problem.

Example 5.1. Let us reconsider the traffic problem presented in Section 2.6, with the following data:

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We have seen that there are no Hurwitz matrices in the convex hull $\{A(\alpha) = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 : \alpha \in \mathcal{A}_3\}$. Therefore for this positive switched system there cannot be smooth, positively homogeneous of order 1, copositive control Lyapunov functions.

For the sake of completeness, we show directly that no linear copositive control Lyapunov function can be found, since the three sets $\mathcal{P}_i, i = 1, 2, 3$, defined above have a nonempty intersection, no matter how $\mathbf{v} \gg 0$ is chosen. Set $\mathbf{v}^\top := [\; v_1 \; v_2 \; v_3 \;]$, with $v_i > 0$. The three sets $\mathcal{P}_i, i = 1, 2, 3$, are characterized by the following inequalities

$$\begin{aligned} \mathcal{P}_1 &= \{\mathbf{x} \in \mathcal{A}_3 : -v_1 x_1 + v_2 x_3 \geq 0\} \\ \mathcal{P}_2 &= \{\mathbf{x} \in \mathcal{A}_3 : -v_2 x_2 + v_3 x_1 \geq 0\} \\ \mathcal{P}_1 &= \{\mathbf{x} \in \mathcal{A}_3 : -v_3 x_3 + v_1 x_2 \geq 0\}. \end{aligned}$$

The above inequalities are indeed simultaneously satisfied as equalities by

$$x_1 = \frac{v_2}{v_1 + v_2 + v_3}, \quad x_2 = \frac{v_3}{v_1 + v_2 + v_3}, \quad x_3 = \frac{v_1}{v_1 + v_2 + v_3}$$

The conclusion is that no linear control Lyapunov functions exists.

We approach the stabilization of this system by considering an open loop periodic switching signal. As we have stated in Section 2.6, there exists a product of exponential matrices that is Schur, i.e. all its eigenvalues belong to the open unit disc. We want to analytically prove this statement.

The exponential matrices $e^{A_1 T}$, $e^{A_2 T}$ and $e^{A_3 T}$, corresponding to A_1, A_2 and A_3 are:

$$\begin{bmatrix} e^{-T} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & T & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & T \\ 0 & e^{-T} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ T & 1 & 0 \\ 0 & 0 & e^{-T} \end{bmatrix}.$$

Consider the product

$$\Phi_{123}(T) = e^{A_1 T} e^{A_2 T} e^{A_3 T} = e^{-T} \begin{bmatrix} 1 & 0 & Te^{-T} \\ T & 1 & 0 \\ T^2 & T & 1 \end{bmatrix}.$$

For $T \to +\infty$, matrix $\Phi_{123}(T)$ converges to 0. Hence, for T sufficiently large the product is Schur. This means that every periodic switching signal with T "sufficiently large" is stabilizing. We will propose a periodic strategy later (see (5.19)).

To illustrate why large periods are not necessarily the best solution, we investigate this traffic problem with an external scalar input, representing an incoming flow in the system:

$$\dot{\mathbf{x}}(t) = A(\alpha)\mathbf{x}(t) + \mathbf{1}_3 u,$$

with $u = const$ (for simplicity we assume that the input-to-state matrix has all identical components, set to 1 for simplicity). This model can be "augmented" in order to "become linear" by adding the equation

$$\dot{u}(t) = 0.$$

The resulting augmented matrix is

$$A_{aug}(\alpha) = \begin{bmatrix} -\alpha_1 & 0 & \alpha_2 & 1 \\ \alpha_3 & -\alpha_2 & 0 & 1 \\ 0 & \alpha_1 & -\alpha_3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

First note that, for any fixed $\alpha \in \mathcal{A}_3$ and $u > 0$, the state variables diverge. Consider the function $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}_{aug}$, with $\mathbf{x}_{aug} := [x_1\ x_2\ x_3\ u]$, where

$$\mathbf{v}^\top = \left[\frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} \quad \frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3} \quad \frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_3} \quad 1 \right].$$

Then

$$\dot{V}(\mathbf{x}) = \mathbf{v}^\top A_{aug}(\alpha)\mathbf{x}_{aug} = u > 0,$$

(since u is constant and positive). So

$$V(\mathbf{x}(t)) = V(\mathbf{x}(0)) + ut,$$

which means that the buffers diverges. This facts reveals an interesting property of the traffic system. In practice, a constant value of α can be realized by a fast switching, keeping the period of the red-green light sequence small. This is not efficient at all. On the other hand, a large period can create a mismatch among queues at different times. For instance, if we activate $\sigma = 1$ for a too long time, the buffers 2 and 3 can become very large, while we are focusing on emptying buffer 1.

So, in practice, a trade-off should be adopted. The simulations proposed in Section 2.6 are performed after such a trade-off has been considered (see Blanchini et al. [2012] for details).

A completely different case is illustrated in the next example, where fast switching is an efficient strategy.

Example 5.2. Reconsider the well-emptying problem described in Section 2.5. We applied the strategy of considering the convex combination of the system matrices for which the Frobenius eigenvalue is minimal and we denoted this value by $\bar{\lambda}_F$. We proved that in this way we can ensure a convergence speed equal to $-\bar{\lambda}_F > 0$, by considering the linear copositive control Lyapunov function associated with the left eigenvector corresponding to $\bar{\lambda}_F$. We can now prove that we cannot do any better. Indeed, the convergence speed β can be ensured for this second order system if and only if (see Proposition 3.6)

$$\dot{\mathbf{x}} = [\beta I_2 + A_\sigma]\mathbf{x}(t)$$

is stabilizable. But for any $\beta > -\bar{\lambda}_F$ the matrix $[\beta I_2 + A(\alpha)]$ is not Hurwitz for any choice of $\alpha \in \mathcal{A}_2$

As we have seen, a stabilizable positive switched system does not necessarily admit a smooth copositive control Lyapunov function. As clarified in Theorem 5.3, this is the case if and only if a linear copositive control Lyapunov function can be found. In this context, it is worth exploring, as we did for stability, what is the relation between the existence of linear copositive, positive definite and quadratic copositive Lyapunov functions. All these functions are smooth, and the condition that the functions are decreasing along the system trajectories can be simply checked by verifying that for every $\mathbf{x} > 0$

$$\min_{i=1,2\ldots,M} \nabla V(\mathbf{x}) A_i \mathbf{x} < 0.$$

We have the following result.

Theorem 5.4. If there exists a linear copositive control Lyapunov function for (5.1) then there exists a quadratic positive definite control Lyapunov function for (5.1) and this, in turn, implies the existence of a quadratic copositive control Lyapunov function for (5.1).

Proof. The proof is similar to the proof of Theorem 3.3 for stability. Assume that there exists a linear copositive control Lyapunov function $V_L(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, with $\mathbf{v} \gg 0$, such that

$$\min_{i=1,2\ldots,M} \mathbf{v}^\top A_i \mathbf{x} < 0, \qquad \forall \mathbf{x} \geq 0.$$

For every $\varepsilon > 0$, the function $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, where $P := \mathbf{v}\mathbf{v}^\top + \varepsilon I_n$, is positive definite. We want to show that, for a suitable choice of ε, it is a control Lyapunov function for the system, namely

$$\min_{i=1,2\ldots,M} \mathbf{x}^\top [A_i^\top P + P A_i]\mathbf{x} < 0, \qquad \forall \mathbf{x} \geq 0.$$

Introduce the compact set $\mathcal{K} := \{\mathbf{x} \in \mathbb{R}^n_+ : \|\mathbf{x}\| = 1\}$, and note that

$$\min_{i=1,2,\dots,M} \mathbf{x}^\top (A_i^\top P + PA_i)\mathbf{x} = \min_{i=12,\dots,M} 2[(\mathbf{v}^\top \mathbf{x})(\mathbf{v}^\top A_i \mathbf{x}) + \varepsilon(\mathbf{x}^\top A_i \mathbf{x})].$$

$\mathcal{K}$ is a compact set and hence, by Weierstrass's theorem, there exist

$$-\alpha := \max_{\mathbf{x}\in\mathcal{K}} \min_{i=1,2,\dots,M} (\mathbf{v}^\top \mathbf{x})(\mathbf{v}^\top A_i \mathbf{x}) < 0$$

and

$$\beta := \max_{\mathbf{x}\in\mathcal{K}} \max_{i=1,2,\dots,M} |\mathbf{x}^\top A_i \mathbf{x}| \geq 0.$$

If $\varepsilon \in (0, \alpha/\beta)$, then for every $\mathbf{x} \in \mathcal{K}$

$$\begin{aligned}
\min_{i=1,2,\dots,M} \mathbf{x}^\top (A_i^\top P + PA_i)\mathbf{x} &\leq \min_{i=1,2,\dots,M} 2[(\mathbf{v}^\top \mathbf{x})(\mathbf{v}^\top A_i \mathbf{x}) + \varepsilon|\mathbf{x}^\top A_i \mathbf{x}|] \\
&\leq \max_{\mathbf{x}\in\mathcal{K}} \min_{i=1,2,\dots,M} 2[(\mathbf{v}^\top \mathbf{x})(\mathbf{v}^\top A_i \mathbf{x}) + \varepsilon|\mathbf{x}^\top A_i \mathbf{x}|] \\
&\leq 2 \max_{\mathbf{x}\in\mathcal{K}} \min_{i=12,\dots,M} (\mathbf{v}^\top \mathbf{x})(\mathbf{v}^\top A_i \mathbf{x}) \\
&+ 2\varepsilon \max_{\mathbf{x}\in\mathcal{K}} \max_{i=12,\dots,M} |\mathbf{x}^\top A_i \mathbf{x}|] \\
&\leq -2\alpha + 2\varepsilon\beta < 0.
\end{aligned}$$

The extension to any $\mathbf{x} \in \mathbb{R}^n_+$ follows again the same lines as the proof of Theorem 3.3.

The fact that the existence of a quadratic positive definite control Lyapunov function implies the existence of a quadratic copositive control Lyapunov function is obvious. □

5.2 Other feedback stabilization techniques

The aim of this section is to provide additional stabilization techniques for positive switched systems with respect to the ones we investigated in the previous section, in order to drive the state to zero starting from any positive initial condition. We first consider the feedback stabilization problem in $\mathbb{R}^n_+$, and propose a stabilizing strategy solution, based on the so-called Lyapunov Metzler inequalities. Feasibility of such inequalities offers a sufficient condition for stabilization, that is shown to be not necessary through the counterexample provided in Example

5.1. Another sufficient condition based on a mixed open-closed loop strategy with dwell time is also provided in the same example. A third strategy is proposed next, based on a piecewise linear copositive control Lyapunov function. Finally, by referring again to the traffic model investigated in Example 5.1, we propose feedback techniques that drive the state trajectory from any initial condition to the positive orthant (or, equivalently, to the negative orthant), so that the previous techniques may be applied.

5.2.1 Lyapunov Metzler inequalities

In the previous section, we have focused our attention on linear and quadratic control Lyapunov functions. Clearly, the existence of any such function is extremely restrictive. A different approach to the stabilization problem consists in searching for sufficient conditions for the existence of control Lyapunov functions taking the form

$$V(\mathbf{x}) = \min_{i=1,2,\dots,M} V_i(\mathbf{x}), \tag{5.5}$$

where $V_i(\mathbf{x})$, $i \in \{1,2,\dots,M\}$, are suitable smooth functions, such that for every $\mathbf{x} \in \mathbb{R}^n_+$ there exists $i \in \{1,2,\dots,M\}$ such that $\dot{V}_i(\mathbf{x}) < 0$.

The use of Lyapunov inequalities parametrized by the entries of a Metzler matrix was first proposed in Geromel and Colaneri [2006], and the associated quadratic matrix inequalities were henceforth referred to as *Lyapunov Metzler inequalities*. For positive switched systems, one can resort to piecewise linear copositive Lyapunov functions, which amounts to saying that the V_i take the form $V_i(\mathbf{x}) = \mathbf{v}_i^\top \mathbf{x}$, for suitable strictly positive vectors $\mathbf{v}_i, i \in \{1,2,\dots,M\}$. A sufficient condition can be worked out by considering particular Metzler matrices arising in the study of stochastic stability, namely $M \times M$ Metzler matrices Λ satisfying $\Lambda \mathbf{1}_M = 0$. We denote such a set by the symbol $\mathcal{P}_M$. We have the following result (see also Blanchini et al. [2013]).

Theorem 5.5. If there exist strictly positive vectors $\mathbf{v}_i \in \mathbb{R}^n_+$, $i = 1,2,\dots,M$, and $M(M-1)$ nonnegative parameters λ_{ij}, $i,j = 1,2,\dots,M$, $i \neq j$, such that the following Lyapunov Metzler inequalities are satisfied:

$$A_i^\top \mathbf{v}_i + \sum_{\substack{j=1 \\ j\neq i}}^{M} \lambda_{ij} \left(\mathbf{v}_j - \mathbf{v}_i\right) \ll 0, \qquad i = 1, 2, \ldots, M, \tag{5.6}$$

then the positive switched system (5.1) is stabilizable. If this is the case, a stabilizing state feedback law is given by

$$\sigma(t) = \arg \min_{k=1,2,\ldots,M} \mathbf{v}_k^\top \mathbf{x}(t).$$

Proof. Consider the candidate piecewise linear copositive control Lyapunov function $V(\mathbf{x}) := \min_{i=1,2,\ldots,M} \mathbf{v}_i^\top \mathbf{x}$. Let $\mathcal{I}(\mathbf{x})$ be the set of all indices k such that $\mathbf{v}_k^\top \mathbf{x} \leq \mathbf{v}_r^\top \mathbf{x}$ (equivalently, $\left(\mathbf{v}_k - \mathbf{v}_r\right)^\top \mathbf{x} \leq 0$) for every $r \neq k$. If i is the active mode at time t, i.e. $\sigma(t) = i$, computing the Dini derivative of $V(\mathbf{x})$ leads to

$$D^+ V(\mathbf{x}) = \min_{r\in\mathcal{I}(\mathbf{x})} \mathbf{v}_r^\top A_i \mathbf{x} \leq \mathbf{v}_i^\top A_i \mathbf{x},$$

and hence

$$D^+ V(\mathbf{x}) < \sum_{\substack{j=1 \\ j\neq i}}^{M} \lambda_{ij} \left(\mathbf{v}_i - \mathbf{v}_j\right)^\top \mathbf{x} \leq 0.$$

□

The previous result deserves a few comments.

(i) Once the index $i \in \{1, 2, \ldots, M\}$ is fixed, the coefficients λ_{ij}, $j = 1, 2, \ldots, M$ and $j \neq i$, can be complemented with $\lambda_{ii} = -\sum_{j\neq i} \lambda_{ij}$, so that the matrix Λ with entries $[\Lambda]_{ij} = \lambda_{ij}$, $i, j = 1, 2, \ldots, M$, belongs to $\mathcal{P}_M$. As such, Λ corresponds to the infinitesimal transition matrix of a Markov chain, with associated Kolmogorov equation $\dot{\boldsymbol{\pi}} = \Lambda^\top \boldsymbol{\pi}$. Vector $\boldsymbol{\pi}(t)$ represents the probability vector at time t, and the stationary probability $\bar{\boldsymbol{\pi}}$ is the positive left Frobenius eigenvector of Λ corresponding to the zero eigenvalue, whose entries sum up to 1. System (5.1) where $\sigma(t)$ is a form process taking values from a Markov chain is known in the literature as a positive Markov jump linear system (PMJLS). Such systems are stable in the *mean* sense if the expectation of

$\mathbf{x}(t)$ tends to zero asymptotically for any initial state and any initial probability vector, see Bolzern et al. [2014]. It turns out that a PMJLS, with Λ as infinitesimal transition matrix of the Markov chain, is *mean stable* if and only if there exist strictly positive vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, M$, satisfying (5.6), and this is in turn equivalent to the Hurwitz stability of the Metzler matrix

$$\tilde{A} = \begin{bmatrix} A_1 + \lambda_{11} I_n & \lambda_{21} I_n & \ldots & \lambda_{M1} I_n \\ \lambda_{21} I_n & A_2 + \lambda_{22} I_n & \ldots & \lambda_{M2} I_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1M} I_n & \lambda_{2M} I_n & \ldots & A_M + \lambda_{MM} I_n \end{bmatrix}. \quad (5.7)$$

Indeed, it is easy to verify that the inequalities (5.6) can be equivalently rewritten in compact form as $\mathbf{v}^\top \tilde{A} \ll 0$, where $\mathbf{v}$ is an nM-dimensional vector whose ith block is $\mathbf{v}_i$, $i = 1, 2, \ldots, M$.

(ii) The inequalities (5.6) are linear once the Metzler matrix Λ is fixed. The search for $\Lambda \in \mathcal{P}_M$ such that the inequalities are satisfied, or equivalently, the associated augmented matrix $\tilde{A}$ in (5.7) is Hurwitz, can be simplified, at the cost of increased conservatism, if we assume for Λ a special structure, with just a few free parameters. For instance, if $\lambda_{ij} = \alpha > 0$, for $i \neq j$, checking the feasibility of the inequalities reduces to a linear problem along with a line search over α.

(iii) The inequalities (5.6) encompass the possibility of stabilization via chattering. As a matter of fact, assume that such inequalities are feasible for every matrix $\Lambda(\alpha) = \alpha\bar{\Lambda}$, where $\bar{\Lambda} \in \mathcal{P}_M$ is a given matrix and α satisfies $\alpha > \bar{\alpha} > 0$. Denote by $\boldsymbol{\pi}$ the left Frobenius eigenvector of $\bar{\Lambda}$ (and hence of $\Lambda(\alpha)$), whose entries sum up to 1. Finally, denote by $\mathbf{v}_i(\alpha)$, $i = 1, 2, \ldots, M$, the solutions of the inequalities for $\Lambda = \Lambda(\alpha)$. By dividing the inequalities by α and letting $\alpha \to +\infty$, one obtains that

$$\lim_{\alpha \to +\infty} \sum_{j \neq i} \bar{\lambda}_{ij} \left(\mathbf{v}_j(\alpha) - \mathbf{v}_i(\alpha) \right) = 0, \quad \forall i = 1, 2, \ldots, M.$$

If $\bar{\Lambda}$ is irreducible it is clear that the only possibility for the above inequalities to be verified is that all vectors $\mathbf{v}_i(\alpha)$ tend to the same vector, say $\bar{\mathbf{v}}$, i.e.

$$\lim_{\alpha \to +\infty} \mathbf{v}_i(\alpha) = \bar{\mathbf{v}}.$$

By multiplying by $[\boldsymbol{\pi}]_i$ the ith inequality of (5.6) and summing up, we get

$$\sum_{i=1}^{M} [\boldsymbol{\pi}]_i \mathbf{v}_i(\alpha)^\top A_i \ll 0,$$

so that, when $\alpha \to +\infty$, it turns out that

$$\bar{\mathbf{v}}^\top \left(\sum_{i=1}^{M} [\boldsymbol{\pi}]_i A_i \right) \ll 0.$$

This means that the matrices $A_1, A_2, \ldots, A_M$ of the system (5.1) admit a Hurwitz convex combination and hence the system is stabilizable, see Theorem 5.3.

It can be shown that the inequalities (5.6) are not satisfied for any choice of $\mathbf{v}_i \gg 0$ and $\Lambda \in \mathcal{P}_M$, for the Example 5.1. However we know that the system is stabilizable via a periodic switching strategy. Therefore, it is clear that system (5.1) in Example 5.1 can be stabilized, by means of a feedback switching law based on some piecewise linear copositive Lyapunov function described as in (5.5), only by imposing a nonzero *dwell-time* between two consecutive switching instants. Stabilization of switched systems under dwell-time constraints is an important problem, justified by practical/technological reasons. Inspired by Theorem 1 in Allherand and Shaked [2011], we state the following result, whose proof is omitted as it is follows the same lines as the one of the aforementioned Theorem 1.

Proposition 5.3. The positive switched linear system (5.1) is stabilizable if there exist $\Lambda \in \mathcal{P}_M$, vectors $\mathbf{v}_i \gg 0$, $i = 1, 2, \ldots, M$, and a scalar $T > 0$, such that

$$\mathbf{v}_i^\top A_i + \sum_{j \neq i}^{M} \lambda_{ij} (\mathbf{v}_j^\top e^{A_j T} - \mathbf{v}_i^\top) \ll 0, \qquad i = 1, 2, \ldots, M. \tag{5.8}$$

The switching law obtained from the previous result is a mixed open-loop/closed loop law. Let t_k, $k = 0, 1, \ldots,$ denote the switching instants with $t_0 = 0$. Then one can easily prove that the system is stabilizable by choosing

$$\sigma(t) = \begin{cases} \sigma(t^-), & \begin{array}{l}\text{if either } \; t - t_k \leq T \\ \text{or } \; \mathbf{v}_i^\top x \leq \mathbf{v}_j e^{A_j T}\mathbf{x}, \, \forall j \neq i;\end{array} \\ \arg\min_i \mathbf{v}_i^\top e^{A_i T}\mathbf{x}, & \text{otherwise.} \end{cases}$$

By referring to Example 5.1, with $T = 1$ and $\varepsilon = 0.1$, one can choose, for instance,

$$\begin{aligned} \Lambda &= 100 \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \\ \mathbf{v}_1^\top &= \begin{bmatrix} 0.8448 & 0.4126 & 0.3404 \end{bmatrix} && (5.9) \\ \mathbf{v}_2^\top &= \begin{bmatrix} 0.3404 & 0.8448 & 0.4126 \end{bmatrix} && (5.10) \\ \mathbf{v}_3^\top &= \begin{bmatrix} 0.4126 & 0.3404 & 0.8448 \end{bmatrix}. && (5.11) \end{aligned}$$

It is important to find the minimum T that ensures the feasibility of (5.8). This can be done by gridding the space of the unknown parameters and by checking the feasibility of a sequence of LMIs. In our counterexample (with $T = 1$ and $\varepsilon = 0.1$) it turns out that $T_{\min} = 0.23$ s, whereas the minimum T for which $e^{A_1 T} e^{A_2 T} e^{A_3 T}$ is Schur is 0.21. Of course the gap can also be justified by the impossibility of checking all the free parameters of $\Lambda \in \mathcal{P}_M$. This conclusion leads us to conjecture that Proposition 5.3 states also a necessary condition for stabilizability. This issue is however left to future investigations.

We end this section on Lyapunov Metzler inequalities to present a result on stabilization based on dual Lyapunov Metzler inequalities.

Theorem 5.6. If there exist strictly positive vectors $\boldsymbol{\xi}_i \in \mathbb{R}_+^n$, $i = 1, 2, \ldots, M$, and $M(M-1)$ nonnegative parameters $\lambda_{i,j}$, $i, j = 1, 2, \ldots, M, i \neq j$, such that the following inequalities are satisfied:

$$A_i \boldsymbol{\xi}_i + \sum_{j \neq i}^{M} \lambda_{ji} (\boldsymbol{\xi}_j - \boldsymbol{\xi}_i) \ll 0, \qquad i = 1, 2, \ldots, M, \tag{5.12}$$

then system (5.1) is exponentially stabilizable. When so, a stabilizing switching law is given by

$$\sigma(t) = \arg \max_{i=1,2,\ldots,M} \left(\max_r \frac{[\mathbf{x}]_r}{[\boldsymbol{\xi}_i]_r} \right).$$

Proof. Let $V(\mathbf{x}) = \max_i \max_r \frac{[\mathbf{x}]_r}{[\boldsymbol{\xi}_i]_r}$ and notice that this function can be also written as

$$V(\mathbf{x}) = \max_k \frac{[\mathbf{x}]_k}{[\boldsymbol{\eta}]_k}, \quad [\boldsymbol{\eta}]_k = \min_i [\boldsymbol{\xi}_i]_k$$

Therefore the Dini derivative at time t^+ is

$$D^+V(\mathbf{x}(t^+)) = \max_{r \in \mathcal{I}(\mathbf{x})} \frac{1}{[\boldsymbol{\eta}]_r} \sum_{s=1}^{n} [A_\sigma]_{rs} [\mathbf{x}(t)]_s$$

where $\mathcal{I}(\mathbf{x}) = \{r : \frac{[\mathbf{x}]_r}{[\boldsymbol{\eta}]_r} \geq \frac{[\mathbf{x}]_k, \forall k}{[\boldsymbol{\eta}]_k}\}$. Therefore, assuming that $\sigma(t) = i$, $k = \arg\max_r \frac{[\mathbf{x}(t)]_r}{[\boldsymbol{\xi}_i]_r}$ and that at time t^+ there is a jump $i \to j$ and $k \to r$, namely $\sigma(t^+) = j$, $\arg\max_r \frac{[\mathbf{x}(t^+)]_r}{[\boldsymbol{\xi}_i]_r} = r$, we can write

$$\begin{aligned} D^+V(\mathbf{x}(t^+)) &= \frac{1}{[\boldsymbol{\xi}_j]_r} \sum_{s=1}^{n} [A_i]_{rs} [\mathbf{x}(t)]_s \\ &= \frac{1}{[\boldsymbol{\xi}_j]_r} \left(\sum_{s \neq r}^{n} [A_i]_{rs} [\mathbf{x}(t)]_s + [A_i]_{rr} [\mathbf{x}(t)]_r \right). \end{aligned}$$

Since at time t we have

$$[\mathbf{x}(t)]_s \leq \frac{[\mathbf{x}(t)]_k [\boldsymbol{\xi}_i]_s}{[\boldsymbol{\xi}_i]_k}$$

and $[A_i]_{rs} \geq 0$, $r \neq s$, we have

$$D^+V(\mathbf{x}(t^+)) \leq \frac{1}{[\boldsymbol{\xi}_j]_r} \left(\frac{[\mathbf{x}(t)]_k}{[\boldsymbol{\xi}_i]_k} \sum_{s \neq r}^{n} [A_i]_{rs} [\boldsymbol{\xi}_i]_s + [A_i]_{rr} [\mathbf{x}(t)]_r \right).$$

Notice, from (5.12), that

$$\sum_{s \neq r}^{n} [A_i]_{rs} [\boldsymbol{\xi}_i]_s < -[A_i]_{rr} [\boldsymbol{\xi}_i]_r - \sum_{p \neq i}^{M} \lambda_{pi} \left([\boldsymbol{\xi}_p]_r - [\boldsymbol{\xi}_i]_r \right)$$

and therefore

$$\begin{aligned} D^+V(\mathbf{x}(t^+)) &< \frac{1}{[\boldsymbol{\xi}_j]_r}[A_i]_{rr}\left(\frac{[\mathbf{x}(t)]_r}{[\boldsymbol{\xi}_i]_r} - \frac{[\mathbf{x}(t)]_k}{[\boldsymbol{\xi}_i]_k}\right) \\ &\quad - \frac{[\mathbf{x}(t)]_k}{[\boldsymbol{\xi}_j]_r[\boldsymbol{\xi}_i]_k}\sum_{p\neq i}^{M}\lambda_{pi}\left([\boldsymbol{\xi}_p]_r - [\boldsymbol{\xi}_i]_r\right) \end{aligned}$$

Notice now that at time t, $\frac{[\mathbf{x}(t)]_r}{[\boldsymbol{\xi}_i]_r} = \frac{[\mathbf{x}(t)]_k}{[\boldsymbol{\xi}_i]_k}$. Therefore

$$D^+V(x(t^+)) < -\frac{[\mathbf{x}(t)]_k}{[\boldsymbol{\xi}_j]_r[\boldsymbol{\xi}_i]_k}\sum_{p\neq i}^{M}\lambda_{pi}\left([\boldsymbol{\xi}_p]_r - [\boldsymbol{\xi}_i]_r\right).$$

Finally $\frac{[\mathbf{x}(t)]_r}{[\boldsymbol{\xi}_i]_r} \geq \frac{[\mathbf{x}(t)]_r}{[\boldsymbol{\xi}_p]_r}$ so that $[\boldsymbol{\xi}_p]_r \geq [\boldsymbol{\xi}_i]_r$. Therefore $D^+V(\mathbf{x}(t^+)) < 0$ and the proof is concluded. □

Remark 5.4. It is clear that the existence of strictly positive vectors $\boldsymbol{\xi}_i \in \mathbb{R}^n_+$ satisfying the inequalities (5.12) is a sufficient condition for the stabilization of the dual system (3.15), and hence, by Theorem 3.4, also a sufficient condition for the stabilization of system (5.1). Therefore an alternative proof could have been derived by providing a control Lyapunov function for the dual system: $W(\mathbf{z}) = \min_i \boldsymbol{\xi}_i^\top \mathbf{z}$, where $\mathbf{z}$ is the state of the dual system. As we have seen in Theorem 5.6, the control Lyapunov function for system (5.1) is

$$V(\mathbf{x}) = \max_k \frac{[\mathbf{x}]_k}{[\boldsymbol{\eta}]_k}, \quad [\boldsymbol{\eta}]_k = \min_i [\boldsymbol{\xi}_i]_k$$

or also

$$V(\mathbf{x}) = \min_i [\boldsymbol{\xi}_i]_{k(x,i)}, \quad k(x,i) = \max_k \frac{[\mathbf{x}]_k}{[\boldsymbol{\xi}_i]_k}$$

Interestingly, the inequalities (5.12) are strictly related to the propagation of the *mean* of the state variable $\mathbf{x}(t)$ when $\sigma(t)$ is supposed to be generated by a Markov process with infinitesimal transition matrix Λ, the matrix whose entries are the scalars λ_{ij}, $i \neq j$, appearing in (5.12) and $\lambda_{ii} = [\Lambda]_{ii} = -\sum_{j=1}^{n}\lambda_{ij}$. □

Example 5.3. Consider system with

$$A_1 = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & 2.3 \\ 2.3 & -1 \end{bmatrix}$$

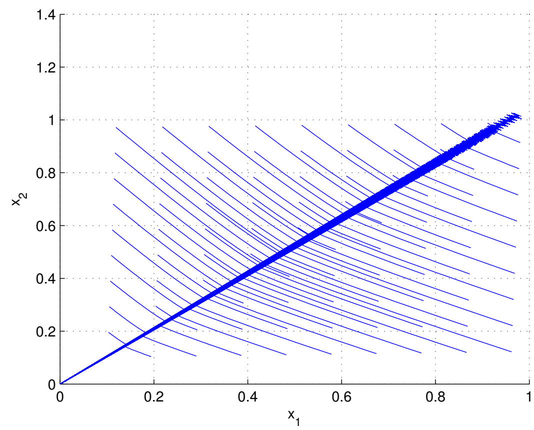

Figure 5.1: Phase portrait obtained by the switching law associated to the Lyapunov function $V(x) = \max_{i=1,2} \max_{k=1,2} \frac{[\mathbf{x}]k}{[\boldsymbol{\xi}_i]_k}$.

and take

$$\Lambda = \begin{bmatrix} -10 & 10 \\ 10 & -10 \end{bmatrix}$$

It follows that inequalities (5.12) in Theorem 5.6 are feasible with

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 0.4483 \\ 0.5001 \end{bmatrix}, \quad \boldsymbol{\xi}_2 = \begin{bmatrix} 0.447 \\ 0.5493 \end{bmatrix}$$

Then, the switching law $\sigma(t) = \arg\max_{i=1,2} \left(\max_{r=1,2} \frac{[\mathbf{x}]_r}{[\boldsymbol{\xi}_i]_r} \right)$ is stabilizing. It has been applied for 100 initial conditions. The corresponding phase portrait is reported in Figure 5.1. It is apparent that the state variables reaches the sliding line $[\boldsymbol{\eta}]_2[\mathbf{x}]_1 = [\boldsymbol{\eta}]_1[\mathbf{x}]_2$, where $[\boldsymbol{\eta}]_k = \min_{i=1,2}[\boldsymbol{\xi}_i]_k$, $k = 1, 2$, and then go along the sliding line.

5.2.2 Feedback techniques based on piecewise linear copositive functions

We now provide a necessary and sufficient condition for positive exponential stabilizability, whose proof hinges on the theory of periodic systems (the interested Reader is referred to Bittanti and Colaneri [2009] for details).

Proposition 5.4. A positive switched system (5.1) is exponentially stabilizable if and only if it there exist a positive integer N and strictly positive vectors $\mathbf{v}_i, i \in \{1, 2, \ldots, N\}$, such that

$$V(\mathbf{x}) = \min_{i \in \{1,2,\ldots,N\}} \mathbf{v}_i^\top \mathbf{x}, \tag{5.13}$$

is a control Lyapunov function.

Proof. We need to prove the necessity part only. Assume that the system is exponentially stabilizable, and hence consistently stabilizable. Without loss of generality (see Remark 5.1), the stabilizing switching law $\sigma(t), t \in \mathbb{R}_+$, can be chosen to be periodic. Denote by $T > 0$ the period of $\sigma(\cdot)$, and let t_k, $k = 1, 2, \ldots, N-1$, be the switching instants within the period $[0, T)$, with $t_0 = 0$. Clearly, $t_N = T$. It entails no loss of generality assuming that the instants t_k, $k = 1, 2, \ldots, N-1$, are rational numbers (if not, this can be achieved by means of a slight perturbation that does not affect the time-varying system stability). Therefore the T-periodic positive system

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) = A_{\sigma(t)}\mathbf{x}(t),$$

with $A(\cdot)$ a piecewise constant (right continuous) matrix function, is exponentially stable. The associated *monodromy matrix*

$$\Phi_A(T, 0) = e^{A_{\sigma(t_{N-1})}(T - t_{N-1})} \ldots e^{A_{\sigma(t_1)}(t_2 - t_1)} e^{A_{\sigma(t_0)} t_1}$$

is a Schur positive matrix. As such, there exists a strictly positive vector $\mathbf{v}$ such that $\mathbf{v}^\top \Phi_A(T, 0) \ll \mathbf{v}^\top$, and hence $\mathbf{v}^\top \Phi_A(T, 0) \ll \gamma \mathbf{v}^\top$, for some $0 < \gamma < 1$. Set, now, $\delta := \ln \gamma / T$, note that $\delta < 0$ and define for $k = 0, 1, \ldots, N-1$, the vector function

$$\mathbf{v}(t)^\top := \mathbf{v}(t_k)^\top e^{(A_{\sigma(t_k)} - \delta I)(t_{k+1} - t)}, \quad t \in [t_k, t_{k+1}),$$

with $\mathbf{v}(t_0) = \mathbf{v}$. It turns out that $\mathbf{v}(\cdot)$ satisfies

$$\dot{\mathbf{v}}(t)^\top + \mathbf{v}(t)^\top A_{\sigma(t)} \;=\; \delta \mathbf{v}(t)^\top \ll 0. \tag{5.14}$$

$\mathbf{v}(t), t \in \mathbb{R}$, is a strictly positive T-periodic vector function, everywhere differentiable over the period, except at the points $t \neq t_k$. However at these points the vector function $\mathbf{v}(t)$ is right differentiable. Note that $V(\mathbf{x},t) = \mathbf{v}(t)^\top \mathbf{x}(t)$ is a (time-varying) linear copositive Lyapunov function for the T-periodic system. By the rationality assumption on the points t_k, we can always select an arbitrarily small positive rational number h such that $t_{k+1} - t_k = N_k h$, with N_k integer, $k = 0, 1, \dots, N-1$ (with $t_N = T$). Hence, we can approximate arbitrarily well the right derivative $\dot{\mathbf{v}}(t)$ in $t = kh$ with $[\mathbf{v}((k+1)h) - \mathbf{v}(kh)]/h$. So, upon setting $N := \sum_{k=0}^{N-1} N_k$, $\mathbf{v}_i := \mathbf{v}((i-1)h)$, and $B_i := A_{\sigma((i-1)h)}$, for $i = 1, 2, \dots, N$, (and assuming, by the periodicity, that $\mathbf{v}_{N+1} = \mathbf{v}_1$), we have

$$h^{-1}\mathbf{v}_{i+1}^\top - h^{-1}\mathbf{v}_i^\top + \mathbf{v}_i^\top B_i \ll 0, \quad i = 1, 2, \dots, N.$$

These inequalities can be written as those in Theorem 5.5, i.e.

$$\mathbf{v}_i^\top B_i + \sum_{j \neq i}^{N} \lambda_{i,j} (\mathbf{v}_j - \mathbf{v}_i)^\top \ll 0, \qquad i = 1, 2, \dots, N,$$

with $\lambda_{i,i+1} = h^{-1}$, $i = 1, 2, \dots, N-1$, $\lambda_{N,1} = h^{-1}$ and $\lambda_{i,j} = 0$ otherwise. Therefore $V(\mathbf{x}) := \min_{i \in \{1,2,\dots,N\}} \mathbf{v}_i^\top \mathbf{x}$ is a control Lyapunov function for the switched system $\dot{\mathbf{x}}(t) = B_{\eta(t)}\mathbf{x}(t)$, with $\eta(t) \in \{1, 2, \dots, N\}$.

The associated switching rule comes from the Dini derivative $D^+V(\mathbf{x})$ of $V(\mathbf{x})$. To be precise, if $\mathcal{I}(\mathbf{x}) = \{i : \ V(\mathbf{x}) = \mathbf{v}_i^\top \mathbf{x}\}$, then $D^+V(\mathbf{x}) = \min_{j \in \mathcal{I}(\mathbf{x})} \mathbf{v}_j^\top B_i \mathbf{x}$, with $i \in \mathcal{I}(\mathbf{x})$, and hence $\eta(t) = \arg\min_{j \in \mathcal{I}(\mathbf{x}(t))} \mathbf{v}_j^\top B_i \mathbf{x}(t)$. Notice that as $\{B_i, i \in \{1, 2, \dots, N\}\} \subseteq \{A_i, i \in \{1, 2, \dots, M\}\}$, $V(\mathbf{x})$ is a control Lyapunov function also for the original switched system. Indeed, at every time instant it is sufficient to impose that $\sigma(t)$ takes values in the set of indices that correspond to the matrices A_i that appear in $\{B_i, i \in \{1, 2, \dots, N\}\}$. Specifically, once we define the map

$$\begin{aligned} \psi \ &: \ \{1, 2, \dots, N\} \to \{1, 2, \dots, M\} \\ &: \ i \mapsto \sigma((i-1)h), \end{aligned}$$

the stabilizing switching law for the original switched system $\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t)$ is

$$\sigma(t) = u(\mathbf{x}(t)) = \psi\left(\arg \min_{j \in \mathcal{I}(\mathbf{x}(t))} \mathbf{v}_j^\top B_i \mathbf{x}(t)\right), \tag{5.15}$$

with $i \in \mathcal{I}(\mathbf{x}(t))$. $\square$

Remark 5.5. The switching law (5.15) used within the proof works nicely to prove that the given function $V(\mathbf{x})$ is a control Lyapunov function for the positive switched system, but it requires the evaluation of the periodic switching signal $\sigma(\cdot)$ as well as of the map $\psi(\cdot)$ in (5.15). In practice, however, the previous switching law can be equivalently rewritten as

$$u(\mathbf{x}) = \arg\min_{j=1,2,\ldots,M} \mathbf{v}_{i(x)}^\top A_j \mathbf{x}, \tag{5.16}$$

where

$$i(\mathbf{x}) := \arg\min_{i=1,2,\ldots,N} \mathbf{v}_i^\top \mathbf{x}.$$

Moreover, we can define the set $\mathcal{T}(\mathbf{x}) := \{t(\mathbf{x}) : \mathbf{v}(t(\mathbf{x}))^\top \mathbf{x} \leq \mathbf{v}(\tau)^\top \mathbf{x}, \forall \tau \in [0, T)\}$, and introduce the new control Lyapunov function $\bar{V}(\mathbf{x}) = \mathbf{c}(t(\mathbf{x}))^\top \mathbf{x}$. As a consequence of (5.14), for each $t(\mathbf{x}) \in \mathcal{T}(\mathbf{x})$ there exists $i \in \{1, 2, \ldots, M\}$ such that $\mathbf{v}(t(\mathbf{x}))^\top A_i \mathbf{x} < 0$, so that, when $N \to +\infty$, (5.16) tends to the stabilizing switching law $\sigma(t) = u(\mathbf{x}(t))$ with

$$u(\mathbf{x}) = \arg\min_{j=1,2,\ldots,M} \mathbf{c}(t(\mathbf{x}))^\top A_j \mathbf{x}. \tag{5.17}$$

□

We now discuss the above stabilizing strategy by referring to Example 5.1. Let $T > 0$ be such that $e^{A_1 T} e^{A_2 T} e^{A_3 T}$ is Schur stable. Define the periodic system

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t), \tag{5.18}$$

where $A(t) = A_{\sigma(t)}$ and

$$\sigma(t) = \begin{cases} 3, & t \in [3Tk, 3Tk + T), k \in \mathbb{Z}_+; \\ 2, & t \in [3Tk + T, 3Tk + 2T), k \in \mathbb{Z}_+; \\ 1, & t \in [3Tk + 2T, 3Tk + 3T), k \in \mathbb{Z}_+. \end{cases} \tag{5.19}$$

System (5.18) is a positive periodic system with period $\bar{T} = 3T$. Now let $\mathbf{v} \gg 0$ be the left Frobenius eigenvector of the irreducible monodromy matrix $e^{A_1 T} e^{A_2 T} e^{A_3 T}$ associated with $A(\cdot)$, i.e.:

$$\mathbf{v}^\top e^{A_1 T} e^{A_2 T} e^{A_3 T} = \lambda_F \mathbf{v}^\top,$$

with $0 \leq \lambda_F < 1$. Upon introducing the *characteristic exponent* $\mu_F := \frac{1}{3T}\ln(\lambda_F) < 0$, one can find the unique $3T$-periodic solution $\mathbf{v}(t)$ of the differential equation

$$\dot{\mathbf{v}}(t)^\top + \mathbf{v}(t)^\top A(t) = \mu_F \mathbf{v}(t)^\top \ll 0 \tag{5.20}$$

with initial condition $\mathbf{v}(0) = \mathbf{v}$. For $T = 1$, one finds $\mu_F = -0.0917$,

$$\begin{aligned}
\mathbf{v}(0)^\top &= \begin{bmatrix} 0.8448 & 0.4126 & 3404 \end{bmatrix} \\
\mathbf{v}(1)^\top &= \begin{bmatrix} 0.4126 & 0.3406 & 0.8448 \end{bmatrix} \\
\mathbf{v}(2)^\top &= \begin{bmatrix} 0.3406 & 0.8448 & 4126 \end{bmatrix}.
\end{aligned}$$

Notice that the above vectors $\mathbf{v}(0)$, $\mathbf{v}(1)$, and $\mathbf{v}(2)$ coincide with $\mathbf{v}_1$, $\mathbf{v}_3$ and $\mathbf{v}_2$ in (5.9)-(5.11), respectively. The solution of (5.20) is strictly positive for each $t \in \mathbb{R}_+$, and $V(\mathbf{x}, t) = \mathbf{v}(t)^\top \mathbf{x}(t)$ is a (time-varying) linear copositive Lyapunov function for the periodic system (5.18), since $\dot{V}(\mathbf{x}, t) = \mu_F V(\mathbf{x}, t) < 0$. As shown in the proof of Proposition 5.4, $V(\mathbf{x}, t)$ directly induces a control Lyapunov function for the switched system and the stabilizing feedback control law (5.17). In Figure 5.2 the time evolutions of the three state variables under the switching law (5.17) are plotted for $\mathbf{x}(0) = [1 \;\; 5 \;\; 10]^\top$. In Figure 5.3 the trajectories in the positive orthant are plotted by taking 20 randomly generated initial conditions with entries in the range $[0, 5]$.

We now show how it is possible to find a feedback switching law capable of driving any state trajectory to the positive orthant (or to the negative orthant), so that, by combining this law with any of the laws we previously discussed, it is possible to construct a feedback law driving any initial state to zero (*global stabilization*).

Consider the Lyapunov-like function "distance from the bisector of the positive (negative) orthant", whose direction is identified by the vector $\mathbf{r} = \mathbf{1}_3/\sqrt{3}$. Elementary geometric reasonings lead to deduce that such a function is defined as

$$V_{dist}(\mathbf{x}) := \|\mathbf{x}\|^2 - (\mathbf{r}^\top \mathbf{x})^2 = \frac{1}{3}\mathbf{x}^\top [3I_3 - \mathbf{1}_3 \mathbf{1}_3^\top]\mathbf{x}. \tag{5.21}$$

This function is positive semi-definite, and it is zero if and only $\mathbf{x} = \rho \mathbf{r}$, for some $\rho \in \mathbb{R}$, namely $\mathbf{x}$ has all identical entries.

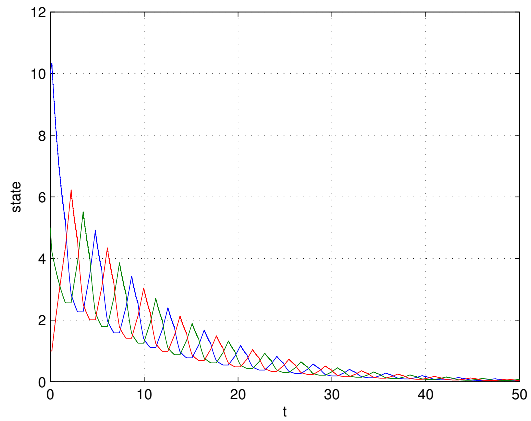

Figure 5.2: State variables.

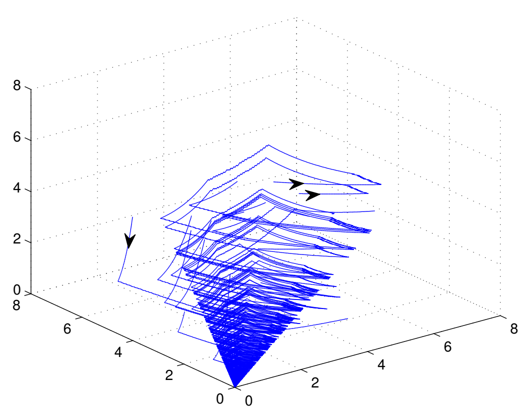

Figure 5.3: Phase portrait.

We prove that the Lyapunov derivative of this function is negative except on the line $\mathbf{x} = \rho\mathbf{r}, \rho \in \mathbb{R}$. Consider the *average system*

$$\dot{\mathbf{x}}(t) = A_c\mathbf{x}(t),$$

where

$$A_c = \frac{1}{3}\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} + \frac{1}{3}\varepsilon I_3.$$

After some tedious computations one gets

$$\dot{V}_{dist}(\mathbf{x}) = -\left(1 - \frac{2}{3}\varepsilon\right) V_{dist}(\mathbf{x}).$$

The switching law

$$\sigma(t) = u(\mathbf{x}(t)) \in \arg\min_{\alpha\in\mathcal{A}} \nabla V_{dist} A(\alpha)\mathbf{x}(t)$$

drives $\mathbf{x}(t)$ to the bisector, generated by $\mathbf{r}$, and hence $\mathbf{x}(t)$ asymptotically reaches either the positive or the negative orthant.

6

Performances optimization

6.1 Optimal control

Optimal control of switched and hybrid systems has been widely studied, see Cassandras et al. [2001] and Dmitruk and Kaganovich [2008, 2011]. The problem is closely related to the variational approach to the stability of switched systems previous developed by Rapoport [1996], Boscain [2002] and Margaliot [2006]. The simplest optimal control problem for a positive switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \tag{6.1}$$

can be defined by introducing the linear cost functional

$$J(\mathbf{x}_0, \sigma) := \mathbf{c}^\top \mathbf{x}(t_f), \tag{6.2}$$

where $\mathbf{c} \in \mathbb{R}^n_+, \mathbf{c} \gg 0$, is the cost vector, $t_f > 0$ is a given terminal time instant, $\sigma \in \mathcal{D}_0$ is a switching signal, and $\mathbf{x}_0 \in \mathbb{R}^n_+$ is a given initial state in the positive orthant. The cost (6.2) has to be minimized with respect to $\sigma \in \mathcal{D}_0$. This is a so-called *Mayer problem*, i.e. a problem where the cost is a function of the final state only.

Despite the simplicity of its definition, this optimal control problem does not admit, in general, a solution $\sigma \in \mathcal{D}_0$. Indeed, when dealing with switched systems it is possible to encounter sliding trajectories, i.e. σ exhibits infinite frequency switching. In order to include sliding trajectories, we embed again the switched system in the bilinear system described by

$$\dot{\mathbf{x}}(t) = A(\mathbf{u}(t))\mathbf{x}(t), \tag{6.3}$$

where the matrix $A(\mathbf{u})$ is defined as

$$A(\mathbf{u}) := \sum_{i=1}^{M} A_i[\mathbf{u}]_i, \tag{6.4}$$

and $\mathbf{u}(t) \in \mathcal{A}_M$, for every $t \in [0, t_f]$, is the control vector. The cost functional, with a little abuse of notation, can be written as

$$J(\mathbf{x}_0, \mathbf{u}) := \mathbf{c}^\top \mathbf{x}(t_f) \tag{6.5}$$

and has to be minimized with respect to $\mathbf{u}$.

This approach was used in recent papers dealing with optimal control, including Bengea and DeCarlo [2005], Bai and Yang [2007]. By extending the concept of valid switching signals to sliding modes based on the appropriate differential inclusions, we consider the optimal control of the system (6.3) with cost functional (6.5). For further details on the so-called *viscosity* solutions of non-smooth differential equations and their relation with the optimal control of differential inclusions, see Bardi and Capuzzo-Dolcetta [2008] and Brandi and Salvadori [1998], respectively. The role of sliding modes (singular control) in optimization problems in terms of finite time convergence to the sliding surface is emphasized in McDonald [2008].

Remark 6.1. Note that the optimal control of system (6.3)-(6.4) with cost functional (6.5) always exists. Indeed, a sufficient condition for its existence is that the sets of velocities $F(\mathbf{x}, \mathbf{u}) := \{A(\mathbf{u})\mathbf{x}; \mathbf{u} \in \mathcal{A}_M\}$ are convex and that the vector field is bounded by an affine function of the norm of the state variable, i.e. there exists some positive scalar α such that $\|A(\mathbf{u})\mathbf{x}\| \leq \alpha(1 + \|\mathbf{x}\|)$, for all $\mathbf{x} \in \mathbb{R}^n_+$ and $\mathbf{u}(t) \in \mathcal{A}_M$, $t \geq 0$, see e.g. Theorem 5.1.1 in Bressan and Piccoli [2007]. These conditions are satisfied for our problem and therefore the optimal control exists. $\square$

In the literature on optimal control, great relevance has been given to the analysis of necessary conditions for optimality. In most cases these necessary conditions are the starting point to find the optimal solutions, since direct sufficient conditions (for instance associated with Hamilton-Jacobi-Bellman equations) are often unpractical. For our control problem, necessary conditions can be easily found by writing the Hamiltonian function $H(\mathbf{x}, \mathbf{u}, \boldsymbol{\pi}) =: \boldsymbol{\pi}^\top A(\mathbf{u})\mathbf{x}$, where $\boldsymbol{\pi}$ is the so-called *co-state*, and using a minor extension of the Pontryagin principle to cope with the input-affine form of the Hamiltonian function. As a result we now introduce the definition of a Pontragyin solution, namely a candidate optimal solution satisfying the necessary conditions. For further details see, e.g., Bressan and Piccoli [2007].

Definition 6.1. A triple $(\mathbf{u}^o(t), \mathbf{x}^o(t), \boldsymbol{\pi}^o(t))$ that satisfies (for almost every $t \geq 0$) the system of equations:

$$
\begin{aligned}
\dot{\mathbf{x}}^o(t) &= \left(\sum_{i=1}^{M} [\mathbf{u}^o(t)]_i A_i\right) \mathbf{x}^o(t), && (6.6)\\
-\dot{\boldsymbol{\pi}}^o(t) &= \left(\sum_{i=1}^{m} [\mathbf{u}^o(t)]_i A_i^\top\right) \boldsymbol{\pi}^o(t), && (6.7)\\
\mathbf{u}^o(t) &\in \arg\min_{\mathbf{u(t)}\in\mathcal{A}_M} \{\boldsymbol{\pi}^\top(t) \left(\sum_{i=1}^{M} [\mathbf{u}(t)]_i A_i\right) \mathbf{x}^o(t)\}, && (6.8)
\end{aligned}
$$

with the boundary conditions $\mathbf{x}^o(0) = \mathbf{x}_0$ and $\boldsymbol{\pi}^o(t_f) = \mathbf{c}$, is called a *Pontryagin solution* for the optimal control problem:

$$
\min_{\mathbf{u}} J(\mathbf{x}_0, \mathbf{u}). \tag{6.9}
$$

As noted earlier, in general a Pontryagin solution need not be optimal, since the conditions expressed by Definition 6.1 are only necessary for optimality. We know that for linear systems and (for instance) quadratic cost, the Pontryagin solution is also optimal and can be found through backward integration of a Riccati differential equation. Two classes of optimal control problems for which any Pontryagin solution is necessarily optimal are discussed in the following remark.

Remark 6.2. In some cases, the necessary conditions satisfied by the Pontryagin solutions are also sufficient to guarantee optimality. One is trivially the case when the Pontryagin solution is unique. The second case is when the cost functional is convex with respect to the control variable. The following result can be stated, see Theorem 7.2.1 in Bressan and Piccoli [2007]. Let $\mathcal{U}_{li}^M$ be the set of measurable and locally integrable functions taking values in $\mathcal{A}_M$. If the functional $\mathbf{u} \to \mathbf{c}^\top \mathbf{x}(t_f)$ from $\mathcal{U}_{li}^M$ into $\mathbb{R}_+$ is convex, then any Pontryagin solution gives an optimal input trajectory and state trajectory pair $(\mathbf{u}^o, \mathbf{x}^o)$. $\square$

In general, even for positive switched systems and linear costs, the functional $J(\mathbf{x}_0, \mathbf{u}) : \mathcal{U}_{li}^M \longrightarrow \mathbb{R}_+$ is not convex. There is however a special class of such systems that enjoys this important property, as specified in the assumption below and formalized in Theorem 6.1.

Assumption 1. The off-diagonal entries of each matrix A_i, $i = 1, 2, \ldots, M$, do not depend on i, i.e.

$$A_i = D_i + \Pi^\top, \qquad i = 1, 2, \ldots, M,$$

where Π is a Metzler matrix and $D_i, i = 1, 2, \ldots, M$, are (not necessarily positive) diagonal matrices. When so, without any loss of generality, matrix Π can be selected so that $\Pi \mathbf{1}_n = 0$.

Remark 6.3. The class of positive switched systems satisfying Assumption 1 is relevant to several applications. One arises when trying to approximate the dynamics of HIV mitigation under therapy switching. It is assumed that each therapy only affects the diagonal entries of the matrices, and this simplifying assumption seems to be good enough to represent the behaviour in a particular transient phase of the disease growth, see Hernandez-Vargas et al. [2013]. This class also encompasses some epidemiology models (see for example Ait Rami et al. [2014], Moreno et al. [2002], Blanchini et al. [2014]). Under some additional assumptions on the interactions, and considering the initial phase of the infection, when trying to slow down the spread of a disease, the model above is appropriate. $\square$

Positive switched systems, described in their embedded form (6.3), whose matrices A_i, $i = 1, 2, \ldots, M$, enjoy Assumption 1, are characterized by an important convexity result that helps in the derivation of an optimal control $\mathbf{u}(t)$ minimizing a convex cost. We now state our main result. For the proof the Reader is referred to Colaneri et al. [2014] (see also Blanchini et al. [2014] and Rantzer and Bernhardsson [2014]).

Theorem 6.1. Consider the system (6.3), the cost (6.5), and let Assumption 1 be verified. Then the cost functional is convex with respect to $\mathbf{u} \in \mathcal{U}_{li}^M$, the optimal control problem admits at least one Pontryagin solution $(\mathbf{u}^o, \mathbf{x}^o, \boldsymbol{\pi}^o)$ and $\mathbf{u}^o(t)$ is a global optimal control input corresponding to $\mathbf{x}_0$. Moreover, the value of the optimal cost functional

$$J^o(\mathbf{x}_0) = \min_{\mathbf{u} \in \mathcal{U}_{li}^M} J(\mathbf{x}_0, \mathbf{u}) \tag{6.10}$$

is $J^o(\mathbf{x}_0) = \boldsymbol{\pi}^{\boldsymbol{o}}(0)^\top \mathbf{x}_0$.

6.1.1 Extensions

The proof of convexity in the previous theorem is carried out by considering the Mayer problem for positive switched systems. The same result can be proved, under the same assumptions on the positive switched system, for the more general cost function

$$J(\mathbf{x}_0, \mathbf{u}) = \mathbf{c}^\top \mathbf{x}(t_f) + \int_0^{t_f} \mathbf{d}^\top \mathbf{x}(t) dt, \tag{6.11}$$

where $\mathbf{d} \in \mathbb{R}_+^n, \mathbf{d} \gg 0$. Indeed, the optimal control problem for system (6.3) and cost (6.11) can be transformed into the optimal control problem for system (6.3) and cost (6.5) by introducing the augmented system

$$\dot{\boldsymbol{\xi}} = \bar{A}(\mathbf{u})\boldsymbol{\xi}, \quad \bar{A}(\mathbf{u}) := \begin{bmatrix} A(\mathbf{u}) & 0 \\ \mathbf{d}^\top & 0 \end{bmatrix}, \quad \boldsymbol{\xi}(0) = \begin{bmatrix} \mathbf{x}_0 \\ 0 \end{bmatrix}$$

and the corresponding augmented cost

$$J(\mathbf{x}_0, \mathbf{u}) = \bar{\mathbf{c}}^\top \boldsymbol{\xi}(t_f), \quad \bar{\mathbf{c}}^\top := \begin{bmatrix} \mathbf{c}^\top & 1 \end{bmatrix}.$$

Notice that the assumption of diagonal switching on $A(\mathbf{u})$, i.e. Assumption 1, is inherited by $\bar{A}(\mathbf{u})$ as well, since $\mathbf{d}$ is independent of $\mathbf{u}$.

Following a similar rationale, we can also establish an extension of the present theory to systems affected by a constant input, i.e.

$$\dot{\mathbf{x}} = A(\mathbf{u})\mathbf{x} + \mathbf{b},$$

where $\mathbf{b} > 0$. This system can be rewritten as

$$\dot{\boldsymbol{\xi}} = \bar{A}(\mathbf{u})\boldsymbol{\xi}, \quad \bar{A}(\mathbf{u}) := \begin{bmatrix} A(\mathbf{u}) & \mathbf{b} \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{\xi}(0) = \begin{bmatrix} \mathbf{x}_0 \\ 0 \end{bmatrix}$$

and the related cost as

$$J(\mathbf{x}_0, \mathbf{u}) = \bar{\mathbf{c}}^\top \boldsymbol{\xi}(t_f), \quad \bar{\mathbf{c}}^\top := \begin{bmatrix} \mathbf{c}^\top & 0 \end{bmatrix}.$$

Again the new $\bar{A}(\mathbf{u})$ satisfies Assumption 1 if $A(\mathbf{u})$ does.

An important result concerns the concavity of the optimal cost

$$J^o(\mathbf{x}_0) = \min_{\mathbf{u} \in \mathcal{U}_{li}^M} J(\mathbf{x}_0, \mathbf{u})$$

with respect to the initial state (see also Theorem 5.2).

Lemma 6.2. For any $t_f > 0$, the function $J^o(\mathbf{x}_0)$ is concave and positively homogeneous of order 1, as a function of $\mathbf{x}_0$.

Proof. The fact that $J(\mathbf{x}_0)$ is positively homogeneous of order 1 is obvious from the fact that $J^o(\mathbf{x}_0) = \boldsymbol{\pi}^{\boldsymbol{o}}(0)^\top \mathbf{x}_0$. To prove concavity, consider two initial states $\mathbf{x}_A$ and $\mathbf{x}_B$ and take any convex combination $\mathbf{x}_0 = \alpha \mathbf{x}_A + \beta \mathbf{x}_B$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Let $\mathbf{u}^o(t)$ be the optimal control input associated with the initial condition $\mathbf{x}_0$, achieving the optimal cost $J^o(\mathbf{x}_0)$. Let $\mathbf{x}_A(t)$ and $\mathbf{x}_B(t)$ be the state trajectories corresponding to $\mathbf{u}^o(t)$ and to the initial states $\mathbf{x}_A(0) = \mathbf{x}_A$ and $\mathbf{x}_B(0) = \mathbf{x}_B$, respectively. By the system linearity, we have

$$\mathbf{x}(t_f) = \alpha \mathbf{x}_A(t_f) + \beta \mathbf{x}_B(t_f).$$

Therefore

$$J^o(\mathbf{x}_0) = \alpha J(\mathbf{x}_A, \mathbf{u}^o) + \beta J(\mathbf{x}_B, \mathbf{u}^o) \geq \alpha J^o(\mathbf{x}_A) + \beta J^o(\mathbf{x}_B)$$

This proves the concavity of $J^o(\mathbf{x}_0)$. $\square$

The previous lemma has several implications including the fact that given any convex combination (in a general polytope) of initial conditions, the optimal cost function $J^o(\mathbf{x}_0)$ reaches its minimum on a vertex.

6.1.2 Algorithm

By Theorem 6.1, under Assumpion 1, the cost functional is convex with respect to $\mathbf{u}$, and this allows to use different types of algorithms to find the solution of

$$\min_{\mathbf{u}\in\mathcal{U}_{li}^{M}} J(\mathbf{x}_0,\mathbf{u}), \quad J(\mathbf{x}_0,\mathbf{u}) = \mathbf{c}^\top \mathbf{x}(t_f). \tag{6.12}$$

Computations can be cast in discrete-time, by taking a subdivision of the interval $[0,t_f]$ into N intervals ol lengths T_1, T_2, …, T_N, and by approximating the control function with a piecewise constant function, i.e. by assuming

$$\mathbf{u}(t) = \begin{cases} \bar{\mathbf{u}}_1, & t\in[0,T_1); \\ \bar{\mathbf{u}}_2, & t\in[T_1,T_1+T_2); \\ \vdots & \vdots \\ \bar{\mathbf{u}}_N, & t\in[\sum_{i=1}^{N-1}T_i, t_f). \end{cases}$$

The discretized control, denoted by $\bar{\mathbf{u}} = \begin{bmatrix} \bar{\mathbf{u}}_1^\top & \bar{\mathbf{u}}_2^\top & \dots & \bar{\mathbf{u}}_N \end{bmatrix}^\top$, takes values in the Cartesian products of $\mathcal{A}_M$, denoted by $\mathcal{A}_{NM}$. Hence the problem becomes to find

$$J^o(\mathbf{x}_0) = \min_{\bar{\mathbf{u}}\in\mathcal{A}_{NM}} J(\mathbf{x}_0,\bar{\mathbf{u}}), \qquad J(\mathbf{x}_0,\bar{\mathbf{u}}) := \mathbf{c}^\top \prod_{i=N}^{1} e^{(\Pi^\top + D_i\bar{\mathbf{u}}_i)T_i}\mathbf{x}_0.$$

Notice that $\mathcal{A}_{NM}$ is a convex set and that, by Assumption 1, $J(\mathbf{x}_0,\bar{\mathbf{u}})$ is a convex function of $\bar{\mathbf{u}}$. Therefore the constrained optimization problem can be solved using the standard Matlab function `fmincon.m` or an *ad hoc* algorithm based on a projected (sub)gradient method, e.g.

$$\bar{\mathbf{u}}^{[k+1]} = \mathrm{Proj}_{\mathcal{A}_{NM}}\left(\bar{\mathbf{u}}^{[k]} - \alpha\mathbf{g}^{[k]}\right), \tag{6.13}$$

where α is a speed factor (possibly varying with k), $\mathrm{Proj}_{\mathcal{A}_{NM}}$ is the projection on $\mathcal{A}_{NM}$, $\mathbf{g}^{[k]}$ is the gradient of $J(\mathbf{x}_0,\bar{\mathbf{u}})$ evaluated at $\bar{\mathbf{u}} = \bar{\mathbf{u}}^{[k]}$ and k indicates the iteration index. The gradient of $J(\mathbf{x}_0,\bar{\mathbf{u}})$ is an NM-dimensional row vector and can be computed in a simple way from the expression of $J(\mathbf{x}_0,\bar{\mathbf{u}})$. Notice also that numerical algorithms might be further enhanced by explicit computation of the Hessian matrix using similar techniques.

As already pointed out in Lemma 6.2, the optimal cost is a concave function of $\mathbf{x}_0$. Then taking $\mathbf{x}_0 \in \mathcal{A}_n$, it may also be of interest to find a saddle point solution of the min-max problem

$$\min_{\mathbf{u}\in\mathcal{U}_{li}^M} \max_{\mathbf{x}_0\in\mathcal{A}_n} J(\mathbf{x}_0, \mathbf{u}),$$

i.e. a solution pair $(\mathbf{u}^*, \mathbf{x}_0^*)$ such that $J(\mathbf{x}_0, \mathbf{u}^*) \leq J(\mathbf{x}_0^*, \mathbf{u}^*) \leq J(\mathbf{x}_0^*, \mathbf{u})$ for any $\mathbf{x}_0 \in \mathcal{A}_n$ and any $\mathbf{u} \in \mathcal{U}_{li}^M$. In this respect, by assuming again the above discretization of $\mathbf{u}$, we are able to write the computational scheme:

$$\bar{\mathbf{u}}^{[k+1]} = \mathrm{Proj}_{\mathcal{A}_{NM}} \left(\bar{\mathbf{u}}^{[k]} - \alpha \mathbf{g}^{[k]}\right), \tag{6.14}$$

$$\bar{\mathbf{x}}_0^{[k+1]} = \mathrm{Proj}_{\mathcal{A}_{NM}} \left(\bar{\mathbf{x}}_0^{[k]} + \alpha \mathbf{h}^{[k]}\right), \tag{6.15}$$

where $\mathbf{h}^{[k]}$ is the gradient of $J(\mathbf{x}_0, \mathbf{u})$ with respect to $\mathbf{x}_0$ at the kth iteration. The vector $\mathbf{h}^{[k]}$ can be easily computed by the linearity of $J(\mathbf{x}_0, \mathbf{u})$ with respect to $\mathbf{x}_0$.

6.2 Suboptimal control via linear programming

In general, finding the optimal control in the form of a state-feedback law is a rather complicated problem, due to the inherent nonlinearity of the problem itself and the difficulty in finding a closed-form formula for the cost-to-go function $J^o(\mathbf{x})$, that, as it is well-known, satisfies a non-smooth version of the classical Hamilton-Jacobi equation. This problem also arises for the class of systems satisfying Assumption 1 (that ensures the convexity of the cost functional). Therefore, easy-to-compute subotimal state-feedback strategies can be devised instead, for instance based on appropriate piecewise linear control Lyapunov functions. As a first step, let us associate with system (6.1) the integral cost with final state penalty

$$J(\mathbf{x}_0, \sigma) = \mathbf{c}^\top \mathbf{x}(t_f) + \int_0^{t_f} \mathbf{d}_{\sigma(\tau)}^\top \mathbf{x}(\tau) d\tau, \tag{6.16}$$

with $\mathbf{d}_i > 0$, $i = 1, 2, \ldots, M$, and $\mathbf{c} > 0$. The following result can be proven.

Theorem 6.3. Consider system (6.1), cost (6.16), with $\mathbf{c} > 0$, $\mathbf{d}_i > 0$, $i = 1, 2, \ldots, M$, and the solution $\mathbf{v}_i(t)$ of the coupled linear differential equations

$$-\dot{\mathbf{v}}_i(t) = A_i^\top \mathbf{v}_i(t) + \sum_{j=1, j\neq i}^{M} \lambda_{ij} \left(\mathbf{v}_j(t) - \mathbf{v}_i(t)\right) + \mathbf{d}_i, \quad \mathbf{v}_i(t_f) = \mathbf{c}, \tag{6.17}$$

where λ_{ij}, $i \neq j$, $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, M$, are given nonnegative scalars. Then, the switching law

$$\sigma(t) = \arg\min_i \mathbf{v}_i(t)^\top \mathbf{x}(t) \tag{6.18}$$

is such that $J^o(\mathbf{x}_0) \leq \min_i \mathbf{v}_i(0)^\top \mathbf{x}_0$.

Proof. Define the Lyapunov function

$$V(\mathbf{x}, t) := \min_i \mathbf{v}_i(t)^\top \mathbf{x},$$

where $\mathbf{v}_i(t)$ satisfies (6.17), for every $i = 1, 2, \ldots, M$, and denote by $\mathcal{I}(\mathbf{x}, t)$ the set of all indices i such that $\mathbf{v}_i(t)^\top \mathbf{x} \leq \mathbf{v}_j(t)^\top \mathbf{x}$, for $j \neq i$, at the given time instant t. $V(\mathbf{x}, t)$ is not differentiable with respect to $\mathbf{x}$ and hence we have to compute the right-upper Dini derivative $D^+(V(\mathbf{x}, t))$. Letting i be the minimizing index at time t, it follows that, see Ladson [1970]

$$\begin{aligned}
D^+(V(\mathbf{x}, t)) &= \min_{j \in \mathcal{I}(\mathbf{x}, t)} \left[\mathbf{v}_j(t)^\top A_i \mathbf{x}\right] + \dot{\mathbf{v}}_i(t)^\top \mathbf{x} \\
&\leq \left(\mathbf{v}_i(t)^\top A_i + \dot{\mathbf{v}}_i(t)^\top\right) \mathbf{x} \\
&= -\sum_{j=1, j\neq i}^{M} \lambda_{ij} \left(\mathbf{v}_j(t)^\top \mathbf{x} - \mathbf{v}_i(t)^\top \mathbf{x}\right) - \mathbf{d}_i^\top \mathbf{x}.
\end{aligned}$$

Therefore, since $\lambda_{ij} \geq 0$ for $i \neq j$ and $\mathbf{v}_j(t)^\top \mathbf{x} \geq \mathbf{v}_i(t)^\top \mathbf{x}$ since i is the minimizing index at time t, we conclude that $D^+(V(\mathbf{x}, t)) < -\mathbf{d}_i^\top \mathbf{x}$ for $t \geq 0$. By recalling the terminal condition $V(\mathbf{x}, t_f) = \mathbf{c}^\top \mathbf{x}(t_f)$ and by integrating from 0 to t_f, it results that

$$\mathbf{c}^\top \mathbf{x}(t_f) + \int_0^t \mathbf{d}_{\sigma(t)}^\top \mathbf{x}(t) dt < \min_i \mathbf{v}_i(0)^\top \mathbf{x}_0$$

and the thesis is proved. $\square$

Remark 6.4. The differential equations (6.17) can be written in a compact form by stacking the vectors $\mathbf{v}_i(t)$ and $\mathbf{d}_i$, $i = 1, 2, \ldots, M$, in vectors $\mathrm{vec}\{\mathbf{v}_i(t)\}$ and $\mathrm{vec}\{\mathbf{d}_i\}$, respectively. Indeed, one finds

$$-\mathrm{vec}\{\dot{\mathbf{v}}_i(t)\} = \Big(\mathrm{diag}\{A_i^\top\} + \Lambda \otimes I_n\Big)\,\mathrm{vec}\{\mathbf{v}_i(t)\} + \mathrm{vec}\{\mathbf{d}_i\}, \quad (6.19)$$

$$\mathrm{vec}\{\mathbf{v}_i(t_f)\} = \mathbf{1}_m \otimes \mathbf{c}, \quad (6.20)$$

where $[\Lambda]_{i,j}$, the (i,j)th entry of the matrix Λ is, λ_{ij}, for $i \neq j$, and $[\Lambda]_{ij} = \lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$. Notice that the matrix Λ, thus constructed, is Metzler and such that $\Lambda \mathbf{1}_M = 0$. We previously denoted the set of all such Metzler matrices of dimensions $M \times M$ by $\mathcal{P}_M$. If Λ is irreducible, then 0 is its Frobenius eigenvalue. As such, (recall comment (i) after Theorem 5.5), Λ can be regarded as the infinitesimal transition matrix of a Markov chain, governing the switching $\sigma(t)$, and equations (6.17) induce the stochastic Lyapunov function $V(\mathbf{x}, \sigma, t) = \mathbf{v}_\sigma(t)^\top \mathbf{x}$ for the cost $E\left[\mathbf{c}^\top \mathbf{x}(t_f) + \int_0^{t_f} \mathbf{d}_{\sigma(\tau)}^\top \mathbf{x}(\tau) d\tau\right]$, where expectation is operated with respect to the probability measure induced by $\Lambda \in \mathcal{P}_M$. □

The solutions $\mathbf{v}_i(t)$, $i = 1, 2, \ldots, M$, of equations (6.17) always exist for $t \in [0, t_f]$ and for any $\Lambda \in \mathcal{P}_M$, and are bounded nonnegative vector functions. Letting $\Lambda = \alpha \bar{\Lambda}$, for some $\bar{\Lambda} \in \mathcal{P}_M$ and $\alpha > 0$, and denoting by $\mathbf{v}_i(\alpha, t)$, $i = 1, 2, \ldots, M$, the solutions of (6.17), we can study the limit of such solutions as $\alpha \to +\infty$. It follows that (the easy check is left to the Reader):

$$\lim_{\alpha \to +\infty} \bar{\Lambda} \cdot \mathrm{vec}\{\mathbf{v}_i(\alpha, t)\} = 0,$$

so that, if $\bar{\Lambda}$ is irreducible (and henceforth its right Frobenius eigenvector is $\mathbf{1}_M$),

$$\lim_{\alpha \to +\infty} (\mathbf{v}_i(\alpha, t) - \mathbf{v}_j(\alpha, t)) = 0, \quad \forall i, j.$$

Therefore it is possible to define:

$$\mathbf{v}(t) = \lim_{\alpha \to +\infty} \mathbf{v}_i(\alpha, t), \quad \forall i.$$

On the other hand, let $\bar{\boldsymbol{\pi}}$ be the left Frobenius eigenvector of $\bar{\Lambda}$ associated with the zero eigenvalue, i.e. $\bar{\boldsymbol{\pi}}^\top \bar{\Lambda} = 0$, and consider again the solutions of (6.17) associated with $\Lambda = \alpha\bar{\Lambda}$. By multiplying each equation by $[\bar{\boldsymbol{\pi}}]_i$, letting α go to $+\infty$, and summing up, we have

$$-\dot{\mathbf{v}}(t) = \left(\sum_{i=1}^{M} A_i^\top [\bar{\boldsymbol{\pi}}]_i\right) \mathbf{v}(t) + \sum_{i=1}^{M} \mathbf{d}_i[\bar{\boldsymbol{\pi}}]_i, \quad \mathbf{v}(t_f) = \mathbf{c}, \tag{6.21}$$

so that $J^o(\mathbf{x}_0) \leq \mathbf{v}(0)^\top \mathbf{x}_0$ under the action of $\mathbf{u} = \bar{\boldsymbol{\pi}}$ for the associated bilinear system $\dot{\mathbf{x}} = \left(\sum_{i=1}^{M} A_i[\mathbf{u}]_i\right) \mathbf{x}$.

To conclude, for a certain choice of $\Lambda \in \mathcal{P}_M$, a switching law guaranteeing an upper bound on the cost can be easily defined as in (6.18) and optimized with respect to $\Lambda \in \mathcal{P}_M$. Constant controls $\mathbf{u}$ are recovered taking $\Lambda = \alpha\bar{\Lambda}$ with $\alpha \to +\infty$ and $\bar{\Lambda}\mathbf{u} = 0$. This corresponds to a sliding control law in which $\sigma(t)$ switches at infinite frequency, with a frequency pattern of the modes (the values of σ) in accordance with the entries of $\mathbf{u}$.

Now, we consider the infinite horizon version of the cost (6.16), namely

$$J(\mathbf{x}_0, \sigma) = \lim_{t_f \to +\infty} \mathbf{c}^\top \mathbf{x}(t_f) + \int_0^{t_f} \mathbf{d}_{\sigma(\tau)}^\top \mathbf{x}(\tau) d\tau. \tag{6.22}$$

It is clear that an optimal control exists if and only if the system is *stabilizable*, see Chapter 5, and in this case

$$\inf_{\sigma \in \mathcal{D}_0} J(\mathbf{x}_0, \sigma) = \min_{\mathbf{u} \in \mathcal{U}_{li}^M} J(\mathbf{x}_0, \mathbf{u}), \tag{6.23}$$

where, with the usual abuse of notation, we have set

$$J(\mathbf{x}_0, \mathbf{u}) = \int_0^{+\infty} \sum_{i=1}^{M} [\mathbf{u}(t)]_i \mathbf{d}_i^\top \mathbf{x}(\tau) d\tau \tag{6.24}$$

to be minimized for $\mathbf{u} \in \mathcal{U}_{li}^M$ under the dynamic constraint given by the system

$$\dot{\mathbf{x}}(t) = \left(\sum_{i=1}^{M} A_i[\mathbf{u}(t)]_i\right) \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{6.25}$$

It is of course true that the set of input functions can be restricted to the ones driving the state vector to zero, so that the term $\mathbf{c}^\top \mathbf{x}(t_f)$ does not play any role in the cost function $J(\mathbf{x}_0, \mathbf{u})$. Looking at the differential equations in Theorem 6.3, see also (6.19) and (6.20), we can state the following result, whose proof can be straightforwardly derived from the one given for Theorem 6.3, for $t_f \to +\infty$, and therefore is omitted.

Theorem 6.4. Consider system (6.1), cost (6.22), with $\mathbf{c} > 0$ and $\mathbf{d}_i > 0$, $i = 1, 2, \ldots, M$, and assume that there exist the solutions $\bar{\mathbf{v}}_i > 0$, $i = 1, 2, \ldots, M$, of the coupled linear equations

$$A_i^\top \bar{\mathbf{v}}_i + \sum_{j=1, j \neq i}^{M} \lambda_{ij} \left(\bar{\mathbf{v}}_j - \bar{\mathbf{v}}_i \right) + \mathbf{d}_i = 0, \tag{6.26}$$

where λ_{ij}, $i \neq j$, $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, M$, are given nonnegative scalars. Then, the switching law

$$\sigma(t) = \arg\min_i \bar{\mathbf{v}}_i^\top \mathbf{x}(t) \tag{6.27}$$

is such that

$$J^o(\mathbf{x}_0) = \inf_{\mathbf{u} \in \mathcal{U}_{li}^M} J(\mathbf{x}_0, \mathbf{u}) \leq \min_i \bar{\mathbf{v}}_i^\top \mathbf{x}_0. \tag{6.28}$$

Following what was said in Remark 6.4, the coupled linear equations (6.26) can be written in compact form as

$$\left(\mathrm{diag}\{A_i^\top\} + \Lambda \otimes I_n \right) \mathrm{vec}\{\bar{\mathbf{v}}_i\} + \mathrm{vec}\{\mathbf{d}_i\} = 0. \tag{6.29}$$

Therefore $\mathrm{vec}\{\mathbf{v}_i\} > 0$ exists if and only if $\Lambda \in \mathcal{P}_M$ is such that

$$\tilde{A} = \begin{bmatrix} A_1 + \lambda_{11} I_n & \lambda_{21} I_n & \ldots & \lambda_{M1} I_n \\ \lambda_{12} I_n & A_2 + \lambda_{22} I_n & \ldots & \lambda_{M2} I_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1M} I_n & \ldots & \ldots & A_M + \lambda_{MM} I_n \end{bmatrix}$$

is Hurwitz.

Finally, if $\bar{\mathbf{u}} \in \mathcal{A}_M$ is such that $A(\bar{\mathbf{u}}) = \sum_{i=1}^M A_i [\bar{\mathbf{u}}]_i$ is Hurwitz, then $J^o(\mathbf{x}_0) \leq \bar{\mathbf{v}}^\top x_0$, where $\bar{\mathbf{v}} > 0$ satisfies

$$\left(\sum_{i=1}^{M} A_i [\bar{\mathbf{u}}]_i \right) \bar{\mathbf{v}} + \sum_{i=1}^{M} \mathbf{d}_i [\bar{\mathbf{u}}]_i = 0, \tag{6.30}$$

and the upper bound is achieved by the constant control $\mathbf{u} = \bar{\mathbf{u}}$.

Remark 6.5. It is worth noticing that equations (6.26) can be replaced by the linear inequalities

$$A_i^\top \mathbf{v}_i + \sum_{j=1, j\neq i}^{M} \lambda_{ij} (\mathbf{v}_j - \mathbf{v}_i) + \mathbf{d}_i \ll 0.$$

If $\mathbf{v}_i \gg 0$, $i = 1, 2, \dots, M$, exist, and $\Lambda \in \mathcal{P}_M$, then the switching control $\sigma(t) := \arg\min_i \mathbf{v}_i^\top \mathbf{x}(t)$ guarantees $J^o(\mathbf{x}_0) < \min_i \mathbf{v}_i^\top \mathbf{x}_0$. Consequently, also (6.29) and (6.30) can be replaced by the corresponding inequalities. □

It is well-known that the initial state $\mathbf{x}_0$ can be set to zero and its effect replaced by that of an impulsive input affecting the state derivative. In Section 4.2 we have studied the worst $\mathcal{L}_1$ performances for systems that are exponentially stable under arbitrary switching. Here we consider the minimization of the $\mathcal{L}_1$ performances. By making use of the same notations as in Section 4.2, consider the cost

$$J_{\mathcal{L}_1}(\sigma, h) := \int_0^{+\infty} \mathbf{1}_p^\top \mathbf{z}^{[h]}(t) dt,$$

and the minimization problem

$$\hat{J}_{\mathcal{L}_1,0} := \inf_{\sigma \in \mathcal{D}_0} \sum_{h=1}^{m} \int_0^{+\infty} \mathbf{1}_p^\top \mathbf{z}^{[h]}(t) dt.$$

In order to investigate this problem we can resort to Theorem 6.4, recall also Remark 6.5, upon noticing that the (forced) state evolution $\mathbf{x}^{[h]}(t)$ coincides with the free state response associated with the initial state $\mathbf{x}(0) = B_{\sigma(0)} \mathbf{e}_h$, and hence

$$\mathbf{z}^{[h]}(t) = C_{\sigma(t)} \mathbf{x}^{[h]}(t) + D_{\sigma(t)} \delta(t) \mathbf{e}_h, \qquad t \in \mathbb{R}_+.$$

This implies that for every switching signal σ we have

$$J_{\mathcal{L}_1}(\sigma, h) = \int_0^{+\infty} \mathbf{1}_p^\top C_{\sigma(t)} \mathbf{x}^{[h]}(t) dt + \mathbf{1}_p^\top D_{\sigma(0)} \mathbf{e}_h.$$

Theorem 6.5. Consider system (4.1)-(4.2) with $\mathbf{x}(0) = 0$. Assume that there exist strictly positive vectors $\mathbf{v}_i \in \mathbb{R}^n_+$, $i = 1, 2, \ldots, M$, such that:

$$A_i^\top \mathbf{v}_i + \sum_{j=1, j\neq i}^{M} \lambda_{ij} \left(\mathbf{v}_j - \mathbf{v}_i\right) + C_i^\top \mathbf{1}_p \ll 0. \tag{6.31}$$

Then, the switching control law

$$\sigma(t) = \begin{cases} \arg\min_i \mathbf{v}_i^\top \mathbf{x}(t), & t > 0, \\ \arg\min_j \min_i \left(\mathbf{v}_i^\top B_j + \mathbf{1}_p^\top D_j\right) \mathbf{1}_m, & t = 0 \end{cases} \tag{6.32}$$

is such that

$$\hat{J}_{\mathcal{L}1,0} < \min_{j\in\{1,2,\ldots,M\}} \min_{i\in\{1,2,\ldots,M\}} \left(\mathbf{v}_i^\top B_j + \mathbf{1}_p^\top D_j\right) \mathbf{1}_m. \tag{6.33}$$

Remark 6.6. Consider again inequalities (6.31) for $\Lambda = \alpha\bar{\Lambda}$, with $\bar{\Lambda} \in \mathcal{P}_M$, $\alpha > 0$, and assume that such inequalities are feasible for $\alpha > 0$ arbitrarily large. This is equivalent to the existence of a vector $\mathbf{v} \gg 0$ satisfying

$$\mathbf{v}^\top (\sum_{i=1}^{M} A_i [\bar{\boldsymbol{\pi}}]_i) + \sum_{i=1}^{M} \mathbf{1}_p^\top C_i [\bar{\boldsymbol{\pi}}]_i \ll 0,$$

where $\bar{\boldsymbol{\pi}}$ is the Frobenius left eigenvector associated with $\bar{\Lambda}$. The control law reduces to the constant control law $\mathbf{u}(t) = \bar{\boldsymbol{\pi}}$, $t > 0$, for the embedded system (4.16)-(4.17). □

Remark 6.7. Notice that in (6.33) the upper bound is given by the minimum over j of the right hand expression. The value of the minimizing j corresponds to the value of the minimizing $\sigma(0)$, or the minimizing vertex of $\mathbf{u}(0) \in \mathcal{A}_M$, respectively. □

6.2.1 Application to optimal therapy scheduling

Here we reconsider the model of optimal therapy scheduling for viral mitigation in HIV disease, briefly discussed in Section 2.2, and provide a few simulation results. For more details see Hernandez-Vargas et al. [2011], Hernandez-Vargas et al. [2013], Colaneri et al. [2014].

The viral dynamics is represented by equation (2.2). Under Assumption 1, the associated embedded system can be written as follows:

$$\dot{\mathbf{x}}(t) = (D_1[\mathbf{u}(t)]_1 + D_2[\mathbf{u}(t)]_2 + \Pi^\top)\mathbf{x}(t), \tag{6.34}$$

where $[\mathbf{u}(t)]_1 + [\mathbf{u}(t)]_2 = 1$, at every time $t \geq 0$, and

$$D_\sigma = \mathrm{diag}\{2\mu - \delta + \rho_{i\sigma},\, i = 1, 2, 3, 4\}, \quad \Pi = \mu \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

with virus clearance rate $\delta = 0.24$, $days^{-1}$, mutation rate $\mu = 10^{-4}$, and replication rates ρ_{ij}, $i = 1, 2, 3, 4$, $j = 1, 2$, as in Table 2.1. Notice that other authors (Huang et al. [2010], Luo et al. [2012], Putter et al. [2002]) suggest a faster clearance rate δ, approximately 1 per day. These numbers are of course idealized, however the general principles are realistic, see Hernandez-Vargas et al. [2011], Hernandez-Vargas and Middleton [2014].

Remark 6.8. Note that the final time penalty is motivated by the observation that frequently the final viral escape is at an exponential rate that is largely independent of the treatment selection. Thus, the terminal cost (6.5) is a surrogate for delaying the time to escape. □

The optimal control problem for system (6.34) and cost (6.12) with $\mu \neq 0$ and data as in Table 2.1 has been completely solved in Hernandez-Vargas et al. [2013], where a Pontryagin solution has been found as a function of $\mathbf{c}$ and $\mathbf{x}_0$. Optimality of the Pontragyn solution has been proved showing its uniqueness. All this was possible because of the *symmetric constraints* satisfied by the data in Table 2.1, namely

$$\rho_{21} > \delta, \rho_{22} < \delta, \rho_{31} < \delta, \rho_{32} > \delta,\ \rho_{21} - \rho_{22} + \rho_{31} - \rho_{32} = 0. \tag{6.35}$$

In Colaneri et al. [2014] we have compared the interior point method in the Matlab routine `fmincon.m`, an *ad hoc* algorithm based on the projected gradient method, with optimally varying speed, and the closed-form solutions given by the necessary conditions. As expected, the results using both algorithms and the Pontryagin solutions are identical apart from minor differences due to numerical issues. In the three simulations the final time is $t_f = 50$ *days*, the initial state is $\mathbf{x}_0 = [10^3 \quad 5 \quad 0 \quad 10^{-5}]^\top$, whereas the cost vector is $\mathbf{c} = \mathbf{1}_4$, $\mathbf{c} = [1 \quad 5 \quad 1 \quad 1]^\top$, and $\mathbf{c} = [1 \quad 1 \quad 5 \quad 1]^\top$, respectively. In Figure 6.1 the optimal control $\mathbf{u}$ is shown for the three different choices of terminal cost vector.

Finally, we have slightly perturbed one parameter of the system in order to violate (6.35). In particular, we have set $\rho_{21} = 0.31$. In this case the closed form solution worked out in Hernandez-Vargas et al. [2013] is no longer valid and the optimal solution, even in the long horizon case, cannot be easily computed. Figure 6.2 shows the results using the numerical algorithms of Section 6.1.2. In this figure there seem to be intervals of time in which the optimal control is constant (sliding mode optimal solution), however a numerical analysis reveals that the input is not really constant in those intervals. Therefore, differently from the symmetric case, the existence of sliding mode solutions in the asymmetric case remains an open problem.

As far as the min-max optimal control problem is concerned, algorithm (6.14)-(6.15) has been applied for several different choices of $\mathbf{c}$. As a result, the saddle point solution is found to be $(\mathbf{x}_0^*, \mathbf{u}^*)$, where $\mathbf{x}_0^* = [0\ 0\ 0\ 1]^\top$ and $\mathbf{u}^*$ coincides with the optimal control function associated with $\mathbf{x}_0^*$, which means $[\mathbf{u}(t)]_1 = [\mathbf{u}(t)]_2 = 0.5$ for $t \in [0, 42.69]$ and $[\mathbf{u}(t)]_1 = 2 - k$, $[\mathbf{u}(t)]_2 = |1 - k|$ for $t \in [42.69, 50]$, where $k = \arg\min\{[\mathbf{c}]_2, [\mathbf{c}]_3\}$.

Remark 6.9. Optimal trajectories are associated with chattering switching laws, that are of course not realistically applicable for HIV treatment. However, this theoretical result provides an important insight since it clarifies when the therapies have to be alternated more frequently in order to better control the viral load. Indeed, using a *switch on virological failure strategy,* see D'Amato et al. [1998], the therapy is changed after 9 months (when viral load $\geq$ 1000 copies/ml) and therefore the population of the resistant genotype is so large that can not be contained by the second therapy. On the contrary, proactive switching may reduce viral load to very low levels during the whole

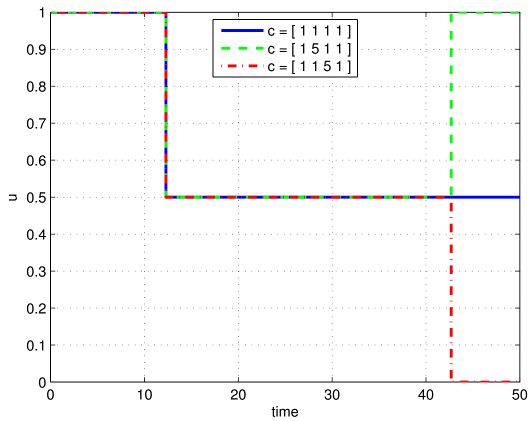

Figure 6.1: Optimal control variable - symmetric case.

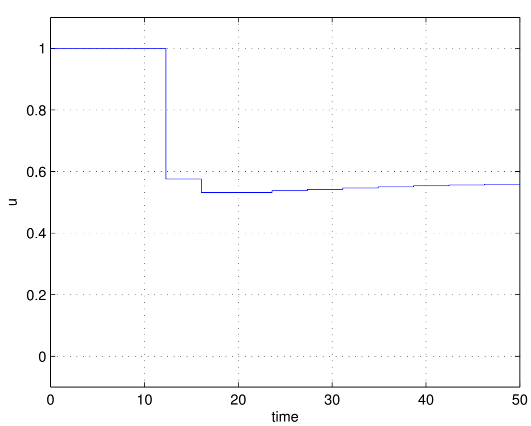

Figure 6.2: Optimal control variable - asymmetric case.

treatment (100 copies/ml), thus promoting a larger delay in the viral escape. This means that a periodic oscillating strategy may be effective in postponing viral escape without requiring a detailed model, high computational time and full state measurements. □

Remark 6.10. When $\mu = 0$, the system matrices commute, and existing results (e.g. see Agrachev and Liberzon [2001] and Margaliot [2007]) can be applied. In particular, it is possible to prove that if we let T_i, $i = 1, 2$, denote the total time in the interval $[0, t_f]$ in which the ith mode is active (clearly $T_1 + T_2 = t_f$), then all optimal controls are characterized by

$$T_1 = \frac{t_f}{2} + \frac{1}{0.44} ln \frac{[\mathbf{c}]_3[\mathbf{x}_0]_3}{[\mathbf{c}]_2[\mathbf{x}_0]_2}, \quad T_2 = \frac{t_f}{2} - \frac{1}{0.44} ln \frac{[\mathbf{c}]_3[\mathbf{x}_0]_3}{[\mathbf{c}]_2[\mathbf{x}_0]_2}.$$

Moreover, the optimal cost is

$$J^o(\mathbf{x}_0) = 2e^{-0.08t_f}\sqrt{[\mathbf{x}_0]_2[\mathbf{x}_0]_3[\mathbf{c}]_2[\mathbf{c}]_3} + [\mathbf{x}_0]_1[\mathbf{c}]_1 e^{-0.19t_f} + [\mathbf{x}_0]_4[\mathbf{c}]_4 e^{0.03t_f}.$$

The optimal solution is not unique and for initial/final conditions satisfying $[\mathbf{c}]_2 = [\mathbf{c}]_3$, $[\mathbf{x}_0]_2 = [\mathbf{x}_0]_3$, the sliding mode control, $\mathbf{u}(t) = [\alpha \quad 1-\alpha]^\top$, $t \geq 0$, with $\alpha = 0.5$, is also optimal. □

6.2.2 Optimal therapy scheduling: Epidemiological models

In this section, we follow up on the epidemiological model presented in Section 2.3, and refer to Blanchini et al. [2014] for the simulations. The linearized system (2.6) can be also written as in Assumption 1, for

$$\begin{aligned}
D_1 &= \text{diag}\{-1.6811, -0.9587, -3.0005, -0.1098\} \\
D_2 &= \text{diag}\{-0.0199, -1.4265, -2.1732, -2.2430\} \\
\Pi^\top &= \begin{bmatrix} -1.9462 & 0.6324 & 0.9575 & 0.9572 \\ 0.9058 & -1.4578 & 0.9649 & 0.4854 \\ 0.1270 & 0.2785 & -2.8930 & 0.8003 \\ 0.9134 & 0.5469 & 0.9706 & -2.2429 \end{bmatrix}.
\end{aligned}$$

The goal is to find the optimal control that minimizes the cost function

$$J(\mathbf{x}_0, \mathbf{u}) = \int_0^\infty \sum_{i=1}^4 [\mathbf{x}]_i(t)dt,$$

with the initial state given by $\mathbf{x}_0 = [0.05 \;\; 0.15 \;\; 0.25 \;\; 0.35]^\top$. Similarly to what we did in Section 6.1.1, we set $\bar{\mathbf{c}} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 1]^\top$,

$$\bar{A}(\mathbf{u}) = \begin{bmatrix} \Pi^\top + D(\mathbf{u}) & 0 \\ \bar{\mathbf{c}}^\top & 0 \end{bmatrix},$$

with $D(\mathbf{u}) = D_1[\mathbf{u}]_1 + D_2[\mathbf{u}]_2$, so that we can tackle the problem of minimizing $J(\mathbf{x}_0, \mathbf{u}) = \lim_{t_f \to +\infty} \mathbf{c}^\top \boldsymbol{\xi}(t_f)$, by assuming that the system dynamics is given by $\dot{\boldsymbol{\xi}} = \bar{A}(\mathbf{u})\boldsymbol{\xi}$.

Notice that both $A_1 = D_1 + \Pi^\top$ and $A_2 = D_2 + \Pi^\top$ are Hurwitz, so that the minimal value of the cost function one can obtain with $\mathbf{u}$ equal to either $\mathbf{e}_1$ or $\mathbf{e}_2$ can be easily found to be $\min_i \mathbf{q}_i^\top \mathbf{x}_0 = 0.622$, where $\mathbf{q}_i = -\mathbf{c}^\top (D_i + \Pi^\top)^{-1}$, $i = 1, 2$. Notice that by taking a constant $\mathbf{u}$ strictly inside the polytope, the best constant control is $[\mathbf{u}]_1 = 0.26$, $[\mathbf{u}]_2 = 0.74$, that corresponds to a sliding mode for the switched system. Corresponding to this control the cost function takes the value $J(\mathbf{x}_0) = 0.601$.

As a consequence of the cost convexity, following from Assumption 1, it is possible to numerically compute the optimal control. By assuming a time horizon of 20 time units, and 0.1 time units as discretization step, we obtain the optimal control shown in Figure 6.3.

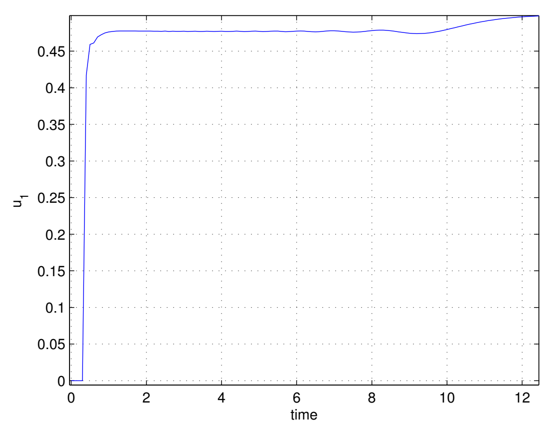

Figure 6.3: Optimal control variable.

For this example, the optimisation was computed with an analytical gradient, using the interior point algorithm in the Matlab function `fmincon`.

The optimal cost for the linearized system is $J(\mathbf{x}_0) = 0.582$. One can also optimize the values of the parameters λ_{ij} in order to minimize the upper bound on the cost given by the copositive Lyapunov Metzler inequalities (6.31). By taking $\lambda_{12} = 0.74$, $\lambda_{21} = 0.26$ one obtains $p_1 = [0.876 \quad 0.933 \quad 0.652 \quad 1.083]$, $p_2 = [1.229 \quad 0.833 \quad 0.723 \quad 0.725]$ and by applying the associated sub-optimal switching strategy one obtains a cost equal to 0.582. Therefore, in this particular case, the optimal cost for the linearized system is equal to the cost obtained by the sub-optimal switching strategy. The (state-feedback) control law generates a sort of periodic behaviour in that the control periodically switches from $[\mathbf{u}]_1 = 0$ to $[\mathbf{u}]_1 = 1$.

As for the concave-convex mixed strategy, see (6.14), (6.15), the results show that the worst initial state is $\mathbf{x}_0 = \alpha[0.206\,0.1631, 0.0043\,0.126]^\top$, where α is any positive scalar. Notice that since the system is linear the cost is also linear with respect to $\mathbf{x}_0$ and hence only the direction is important. The associated cost is $J(\mathbf{x}_0) = 0.4321\alpha$.

When applied to the nonlinear system (2.5), the best constant control constrained to belong to the vertices of the polytope corresponds to $[\mathbf{u}]_1 = 0$ and the associated cost is 0.411. The best constant control within the polytope is $[\mathbf{u}]_1 = 0.26$, $[\mathbf{u}]_2 = 0.74$ and the associated cost is 0.403. The switching strategy based on (6.31) provides a cost equal to 0.3922. For the nonlinear system, the cost associated with this optimal control is again equal (up to numerical errors) to the cost due to the switching strategy based on the copositive Lyapunov Metzler inequalities.

The associated transients of the state variables are depicted in Figure 6.4. If we compute the optimal control for the linearized system and then apply it to the nonlinear system we obtain the results shown in Figure 6.5, where the state-variables are plotted. Finally, the min-max optimal strategy corresponding to the initial state $\mathbf{x}_0 = [0.206\,0.1631, 0.0043\,0.126]^\top$ has been applied to the nonlinear system, giving a cost equal to 0.407. The transient of the state-variables and the optimal input function are illustrated in Figs. 6.6 and 6.7, respectively.

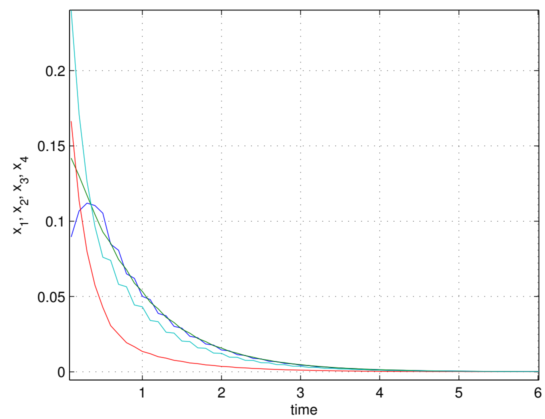

Figure 6.4: State variables of the nonlinear system under the action of the suboptimal control for the linearized system.

6.3 Input-output norms minimization

In Chapter 4 we studied the input-output norms of a linear positive switched system, for which the switching signal σ was considered as a (uncontrolled) disturbance. In particular we were interested in finding an upper bound on the worst induced norm between the input $\mathbf{w}$ and the output $\mathbf{z}$, upon assuming a zero initial condition for the state variable. In the present section the switching signal σ is considered as a control variable and the aim is to find an upper bound on the minimum achievable induced norm. Therefore, consider the $\mathcal{L}_1$ induced norm, i.e.

$$\hat{J}^{ind}_{\mathcal{L}_1,0} := \inf_{\sigma\in\mathcal{D}_0} \sup_{\substack{\mathbf{w}\in\mathcal{L}_1,\mathbf{w}\neq 0\\ \mathbf{w}(t)\geq 0,\forall t\geq 0}} \frac{\int_0^{+\infty} \mathbf{1}_p^\top \mathbf{z}(t)dt}{\int_0^{+\infty} \mathbf{1}_m^\top \mathbf{w}(t)dt}. \tag{6.36}$$

The next result provides an upper bound on this norm, by exploiting a Lyapunov function constructed as the minimum of *piecewise linear* Lyapunov functions.

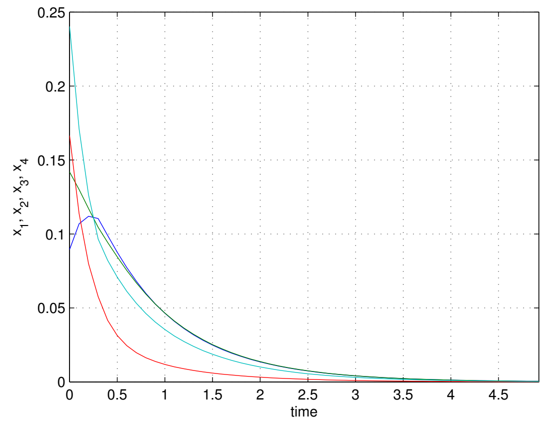

Figure 6.5: State variables of the nonlinear system under the action of the optimal control for the linearized system.

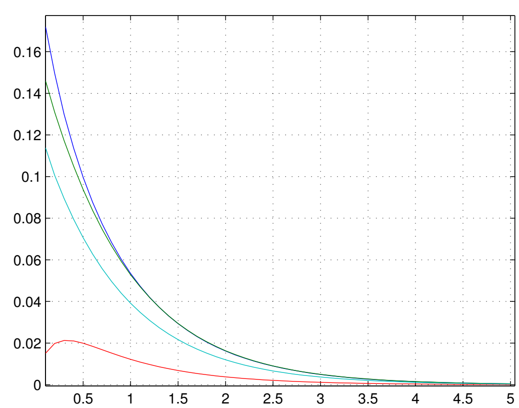

Figure 6.6: State variables of the nonlinear system under the action of the optimal mixed control for the linearized system.

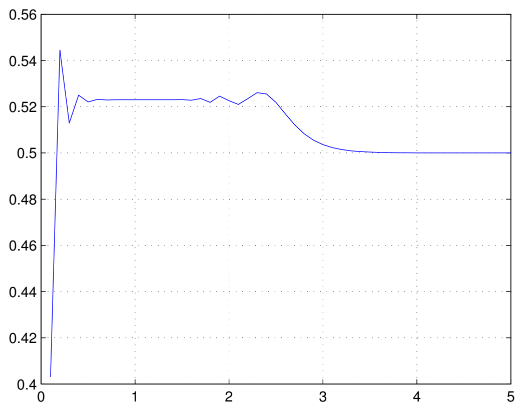

Figure 6.7: Optimal control variable associated with the worst initial state.

Theorem 6.6. Consider the positive switched system (4.1)-(4.2) and assume that there exist strictly positive vectors $\mathbf{v}_i \in \mathbb{R}^n_+$, $i = 1, 2, \dots, M$, and a Metzler matrix $\Lambda \in \mathcal{P}_M$, such that for every $i, j \in \{1, 2, \dots, M\}, i \neq j$,

$$\mathbf{v}_i^\top A_i + \sum_{j=1, j\neq i}^{M} \lambda_{ij}(\mathbf{v}_j^\top - \mathbf{v}_i^\top) + \mathbf{1}_p^\top C_i \ll 0. \tag{6.37}$$

Let γ be any positive number satisfying

$$\gamma > \min_{i=1,2,\dots M} \max_{j=1,2,\dots,m} \left(\mathbf{v}_i^\top B_i + \mathbf{1}_p^\top D_i\right) \mathbf{e}_j. \tag{6.38}$$

Then, under the switching law

$$\hat{\sigma}(t) = \arg \min_{i=1,2,\dots,M} \begin{cases} \mathbf{v}_i^\top \mathbf{x}(t), & \mathbf{x}(t) > 0; \\ \max_j \left(\mathbf{v}_i^\top B_i + \mathbf{1}_p^\top D_i\right) \mathbf{e}_j, & \mathbf{x}(t) = 0. \end{cases} \tag{6.39}$$

the closed loop system is exponentially stable and such that $\hat{J}^{ind}_{\mathcal{L}_1,0} < \gamma$.

Proof. The feasibility of the inequalities (6.37) implies the feasibility of inequalities (5.6), so that the system is stabilizable under the switching law (6.39). Set

$$V(\mathbf{x}) := \arg \min_i \mathbf{v}_i^\top \mathbf{x},$$

and let $\mathcal{I}(\mathbf{x})$ be the set of all indices i such that $\mathbf{v}_i^\top \mathbf{x} \leq \mathbf{v}_j^\top \mathbf{x}$ for every $j \neq i$. If i is the active mode at time t, i.e. $\hat{\sigma}(t) = i$, computing the Dini derivative of $V(\mathbf{x})$ leads to

$$\begin{aligned} D^+V(\mathbf{x}) &= \min_{j \in \mathcal{I}(\mathbf{x})} \mathbf{v}_i^\top (A_j\mathbf{x} + B_j\mathbf{w}) \leq \mathbf{v}_i^\top (A_i\mathbf{x} + B_i\mathbf{w}) \\ &< \mathbf{v}_i^\top B_i\mathbf{w} - \mathbf{1}_p^\top C_i\mathbf{x} - \sum_{j=1, j\neq i}^{M} \lambda_{ij}(\mathbf{v}_j^\top \mathbf{x} - \mathbf{v}_i^\top \mathbf{x}) \\ &\leq \mathbf{v}_i^\top B_i\mathbf{w} - \mathbf{1}_p^\top C_i\mathbf{x} \\ &= \left(\mathbf{v}_i^\top B_i + \mathbf{1}_p^\top D_i\right)\mathbf{w} - \mathbf{1}_p^\top \mathbf{z}. \end{aligned}$$

By the exponential stability and the zero initial condition, integration from 0 to $+\infty$ for any $\mathbf{w} \in \mathcal{L}_1$, $\mathbf{w}(t) \geq 0, \forall t \geq 0$, leads to

$$\int_0^{+\infty} \mathbf{1}_p^\top \mathbf{z}(t)dt < \int_0^{+\infty} \left(\mathbf{v}_{\hat{\sigma}(t)}^\top B_{\hat{\sigma}(t)} + \mathbf{1}_p^\top D_{\hat{\sigma}(t)}\right)\mathbf{w}(t)dt$$

and hence

$$\begin{aligned} \hat{J}_{\mathcal{L}_1,0}^{ind} &< \sup_{\substack{\mathbf{w}\in\mathcal{L}_1, \mathbf{w}\neq 0 \\ \mathbf{w}(t)\geq 0, \forall t\geq 0}} \frac{\int_0^{+\infty} \left(\mathbf{v}_{\hat{\sigma}(t)}^\top B_{\hat{\sigma}(t)} + \mathbf{1}_p^\top D_{\hat{\sigma}(t)}\right)\mathbf{w}(t)dt}{\int_0^{+\infty} \mathbf{1}_m^\top \mathbf{w}(t)dt} \\ &\leq \sup_{t\geq 0} \max_j \left(\mathbf{v}_{\hat{\sigma}(t)}^\top B_{\hat{\sigma}(\mathbf{t})} + \mathbf{1}_p^\top D_{\hat{\sigma}(t)}\right)\mathbf{e}_j \\ &= \left(\mathbf{v}_{\hat{\sigma}(\bar{t})}^\top B_{\hat{\sigma}(\bar{\mathbf{t}})} + \mathbf{1}_p^\top D_{\hat{\sigma}(\bar{t})}\right)\mathbf{e}_{\bar{j}}, \end{aligned}$$

where $\bar{t}$ and $\bar{j}$ are the time instant and the index maximizing $\left(\mathbf{v}_{\hat{\sigma}(t)}^\top B_{\hat{\sigma}(\mathbf{t})} + \mathbf{1}_p^\top D_{\hat{\sigma}(t)}\right)\mathbf{e}_j$. The worst case of the bound is obtained for (the $\mathcal{L}_1$ approximation of) $\mathbf{w}(\mathbf{t}) = \delta(t - \bar{t})\mathbf{e}_{\bar{j}}$, Hence, in $t = \bar{t}^-$, the state variable is identically zero, so that, according to (6.39), $\hat{\sigma}(\bar{t}) = \arg\min_i \left(\mathbf{v}_i^\top B_i + \mathbf{1}_p^\top D_i\right)\mathbf{e}_{\bar{j}}$, and therefore $\hat{J}_{\mathcal{L}_1}^{ind} < \gamma$, with γ satisfying (6.38). $\square$

It is worth noticing that the inequalities (6.37) do not depend on γ and their feasibility (namely the existence of $\Lambda \in \mathcal{P}_M$ such that the inequalities are verified for some strictly positive vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, M$, and $\Lambda \in \mathcal{P}_M$) is not guaranteed, not even for stabilizable systems. As a matter of fact, as shown in Chapter 5, this condition is only sufficient for stabilizability. Moreover, inequalities (6.37) are not linear inequalities for the presence of a product between unknowns (the coefficients of Λ and the coefficients of vectors $\mathbf{v}_i$). This latter problem can be circumvented by gridding the space of the $M(M-1)$ free parameters in Λ or by imposing (at the price of increased conservatism) a special structure to Λ with less free parameters. Of course, one can set up a minimization problem for the upper bound in the following way. Once the solutions $\mathbf{v}_i$, $i = 1, 2, \ldots, M$, satisfying (6.37), for a certain $\Lambda \in \mathcal{P}_M$, have been found, inequalities (6.38) are always feasible for $\gamma \geq \max_i \max_j \left(\mathbf{v}_i^\top B_i + \mathbf{1}_p^\top D_i\right) \mathbf{e}_j$. Therefore, one can minimize γ by optimizing this bound with respect to $\Lambda \in \mathcal{P}_M$.

Remark 6.11. Observe that in the case when only the actuators switch, and hence $A_i = A$, $C_i = C$, for each $i = 1, 2, \ldots, M$, the switched system must be exponentially stable, i.e. A must be Hurwitz (since it is not possible to stabilize it by switching), and the best switching signal $\sigma(t)$ for the minimization of the induced $\mathcal{L}_1$ norm is $\sigma(t) = i$, $t \geq 0$, where $i = \arg\min_j \|G_j(0)\|_1$, and $G_j(s)$ is the transfer matrix of the time-invariant system associated with the jth mode. $\square$

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