Chapter 1

Fundamentals on digital signal processing

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1.1 Discrete-Time Signal and Systems

Signals play an important role in our daily life. Examples of signals that we encounter frequently are speech, music, picture and video signals. A signal is a function of independent variables such as time, distance, position, temperature and pressure. For examples, speech and music signals represent air pressure as a function of time at a point in space.

Most signals we encounter are generated by natural means. However, a signal can also generated synthetically or by computer simulation. Later we will see how to generate simple signal using the simulation environment *MATLAB*.

In this chapter we will focus our attention on a particulary class of signals: The so called *discrete-time signals*. This class of signals is the most important way to describe/model the sound signals with the aid of the calculator.

1.1.1 Characterization and Classification of Signals

Depending on the nature of the independent variables and the value of the function defining the signal, various type of signals can be defined. For example, independent variables can be continuous or discrete. Likewise, the signal can either be a continuous or a discrete function of the independent variables. Moreover, the signal can be either a real-valued function or a complex-valued function.

If we denote a function as follow:

$$x(t) : t \in \mathrm{D} \rightarrow x(t) \in \mathrm{C} \tag{1.1}$$

where D is the set of value of the independent variable t and C is the set of value of the function defining the signal, $x(t)$, is possible to classify the signal with respect the nature of the sets D and C. On the basis of nature of D we have:

- D= $\mathbb{R}$: *continuous-time* signal $x(t)$, $t \in \mathbb{R}$

- D= $\mathbb{I}$: *discrete-time* signal $x(t)$, $t \in \mathbb{I}$ where $\mathbb{I}$ is a countable set $\{\ldots, t_{-1}, t_0, t_1, \ldots\}$ The most common and important example is when $t_n = nT$, therefore $t_n \in \mathbb{Z}(T)$.

On the basis of the nature of C we have:

- C= $\mathbb{R}$: *continuous-amplitude* signal
- C= $\mathbb{I}$: *discrete-amplitude* signal. Commonly $\mathbb{I}$ is a countable and finite set of value $\{x_1, x_2, \ldots, x_M\}$. The most common examples are the quantized samples with uniform quantization law $x = kq$, with q the quantization step and k integer.

Finally combining the various domains we obtains the following class of signals, depicted if Fig.1.1:

1. D= $\mathbb{R}$, C= $\mathbb{R}$: "*analog*" signal.
2. D= $\mathbb{R}$, C= $\mathbb{I}$: "*quantized analog*" signal.
3. D= $\mathbb{I}$, C= $\mathbb{R}$: "*sampled*" signal or "*discrete-time*" signal.
4. D= $\mathbb{I}$, C= $\mathbb{I}$: "*numerical*" signal or "*digital*" signal. This is the common kind of signal analyzed with the aid of the calculator.

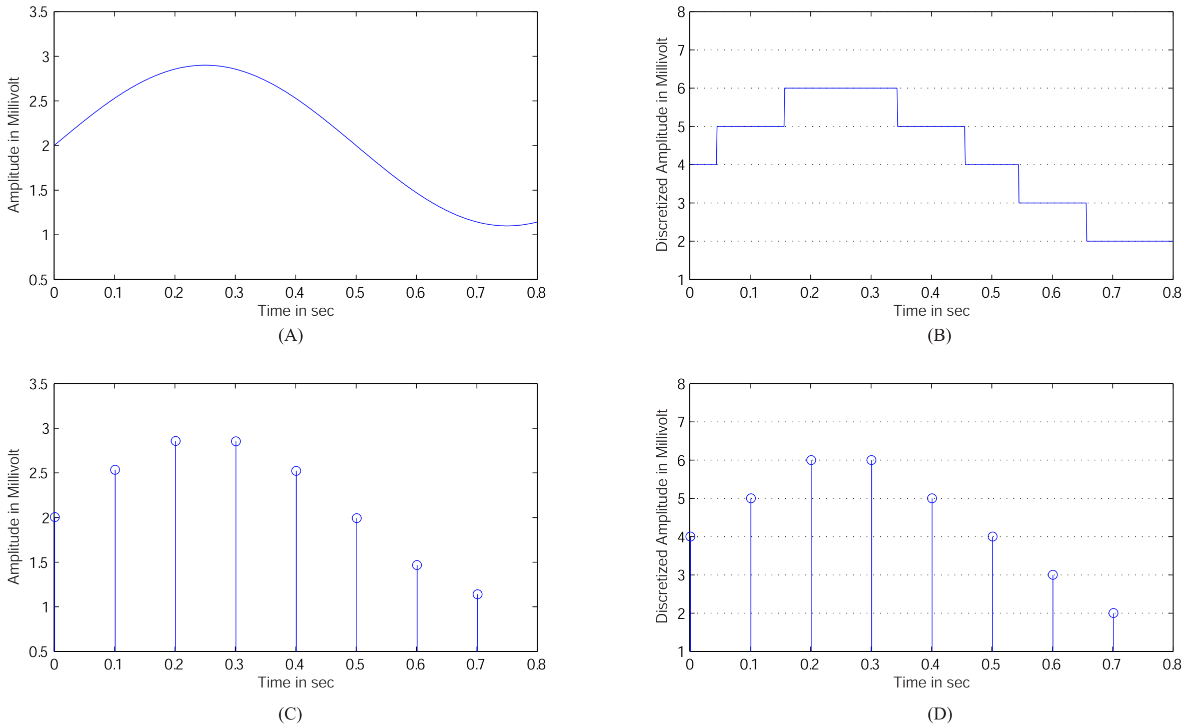

Figure 1.1: (a) Analog signal, (b) Quantized analog signal, (c) discrete-time signal, (d) numerical signal.

1.1.2 Discrete-Time signals: Sequences

In this section we will present the mathematical representation of discrete-time signals, also it will be introduced the mathematical formalism used in the rest of this book.

Discrete-Time signals are represented mathematically as sequences of numbers. A sequence of numbers x, in witch the *n*th number in the sequence is denoted x[n], is formally written as

$$x = \{x[n]\}, \quad -\infty < n < \infty, \tag{1.2}$$

where n is an integer. The graphical representation of a sequence $\{x[n]\}$ with real-valued samples is illustrated in Fig.1.2.

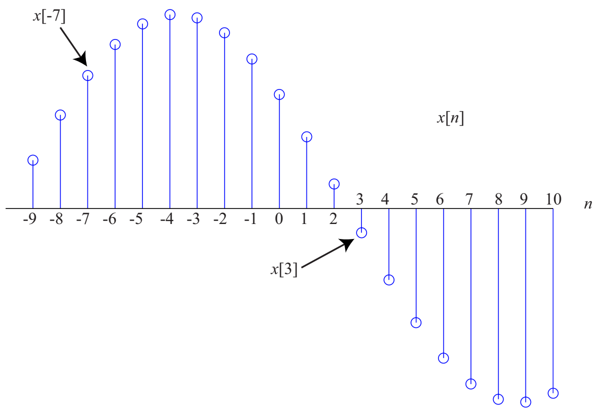

Figure 1.2: Graphical representation of a discrete-time sequence $\{x[n]\}$.

In a practical setting, such sequences can arise from **periodic sampling** of an analog signal. In this case, the numeric value of the *n*th number in the sequence is equal to the value of the analog signal $x_a(t)$ at time nT; i.e.,

$$x[n] = x_a(nT) \quad -\infty < n < \infty, \tag{1.3}$$

as illustrated in Fig.1.3. The quantity T is called *sampling period* and its reciprocal is the *sampling frequency*.

1.1.3 Operation on sequences

A single-input, single output discrete-time system operate on a sequence, called the *input sequence*, according to some prescribed rules and develops another sequence, called the *output sequence*, usually with more desiderable properties. In most cases, the operation defining a particular discrete-time system is composed of some basic operation that we describe next.

Product

Let $x[n]$ and $y[n]$ be two known sequences. By forming the *product* of the sample values of these two sequences at each instant, we form a sequence $w_1[n]$:

$$w[n] = x[n]y[n]. \tag{1.4}$$

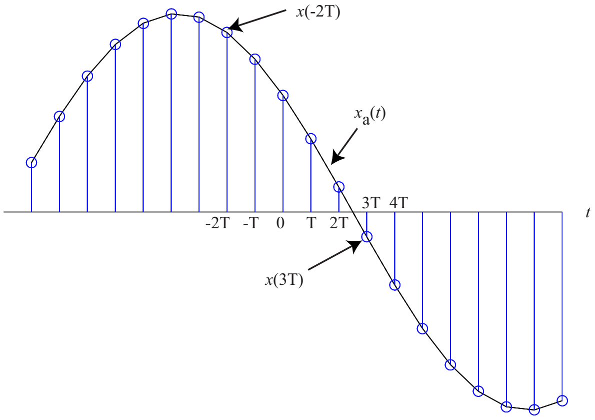

Figure 1.3: Sequence generated by sampling a continuous-time signal $x_a(t)$.

This operation is also know as *modulation*. Furthermore this operation is very useful when we want to obtain a finite-length sequence from an infinite-length sequence. This operation is performed by the product of the infinite-length sequence with a finite-length sequence called *window sequence*. This process is called *windowing*.

Time shifting

Another important operation is the *time shifting* or the *translation*:

$$w[n] = x[n - N], \tag{1.5}$$

with N integer. When $N > 0$, it is a *delaying* operation and if $N < 0$ it is an *advancing* operation.

Time reversal

The *time-reversal* operation is another useful scheme to develop a new sequence. An example is:

$$w[n] = x[-n], \tag{1.6}$$

which is the time-reversed version of the sequence $x[n]$.

1.1.4 Properties of discrete-time signals

In this section we will see some basic properties of the discrete-time signals.

Periodicity

A sequence $x[n]$ satisfying

$$x[n] = x[n + kN] \quad -\infty < n < \infty \tag{1.7}$$

is called a *periodic* sequence with a *period* N where N is a positive integer and k is any integer. The *fundamental period* N_f of a periodic signal is the smallest value of N for wich Eq.(1.7) holds.

Energy

The total *energy* of a sequence $x[n]$ is defined by:

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2. \tag{1.8}$$

Note that an infinite-length sequence with finite sample values may or not have finite energy. The *average power* of an aperiodic-sequence $x[n]$ is defined by

$$\mathcal{P}_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2. \tag{1.9}$$

Finally is possible to define the average power of a periodic-sequence $x[n]$ with a period N by means

$$\mathcal{P}_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2. \tag{1.10}$$

Other type of Classification

A sequence $x[n]$ is said to be *bounded* if each of its samples is of magnitude less than or equal to a finite positive number $\mathcal{B}_x$,i.e.,

$$|x[n]| \leq \mathcal{B}_x > \infty \tag{1.11}$$

1.1.5 Some Basic Sequences

The most common basic sequences are the unit sample sequence, the unit step sequence, the sinusoid sequence and the exponential sequence. These sequences are defined next.

Unit Sample Sequence

The simplest and the most useful sequence is the *unit sample sequence*, often called *unit impulse*, as shown in Fig.1.4(a). It is denoted by $\delta[n]$ and defined by

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{1.12}$$

The unit sample sequence plays the same role for the discrete-time signals and systems that the impulse function (Dirac Delta function) does for continuous-time signal and systems.
One important aspects of this sequence is that an arbitrary sequence can be represented as a sum of scaled (linear combination), delayed impulses as expressed by:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{1.13}$$

Unit Step Sequence

A second basic sequence is the *unit step sequence* shown in Fig.1.4(b). It is denoted by $u[n]$ and is defined by

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{1.14}$$

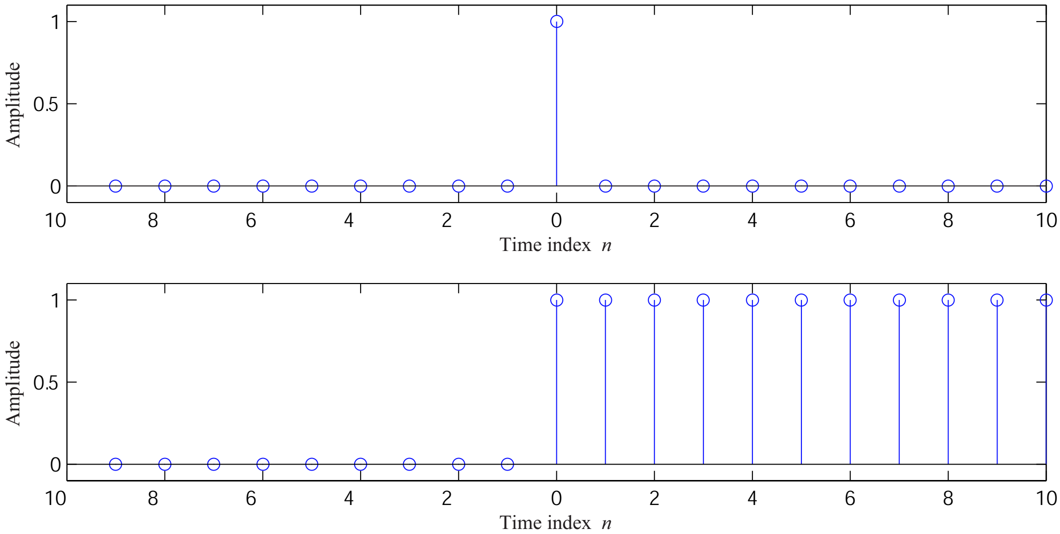

Figure 1.4: (a)The unite sample sequence $\delta[n]$, (b) The unit step sequence $u[n]$.

An alternative representation of the unit step in terms of the impulse is obtained by interpreting the unit step in Fig.1.4(b) in terms of a sum od delayed impulses. This is expressed as

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{1.15}$$

Conversely, the impulse sequence can be expressed as the *first backward difference* of the unit step sequence,i.e.,

$$\delta[n] = u[n] - u[n-1]. \tag{1.16}$$

Sinusoidal and Exponential Sequence

Exponential and sinusoidal sequences are extremely important in representing and analyzing linear discrete-time systems.

The general form of the *real sinusoidal sequence* with constant amplitude is

$$x[n] = A\cos(\omega_0 n + \phi), \quad -\infty < n < \infty, \tag{1.17}$$

where A, ω_0 and ϕ are real numbers. Different types of sinusoidal sequences are depicted in Fig.1.5.

Another set of basic sequences is formed by taking the *n*th sample value to be the *n*th power of a real or complex constant. Such sequences are termed *exponential sequences* and their general form is

$$x[n] = A\alpha^n, \quad -\infty < n < \infty, \tag{1.18}$$

where A and α are real or complex constant.
The exponential sequence $A\alpha^n$ with complex α has real and imaginary part that are exponentially weighted sinusoid. Specifically, if $\alpha = |\alpha|e^{j\omega_0}$ and $A = |A|e^{j\phi}$,the sequence $A\alpha^n$ can be expressed as

$$\begin{aligned} x[n] &= |A||\alpha|^n e^{j(\omega_0 n + \phi)} \\ &= |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi) \end{aligned} \tag{1.19}$$

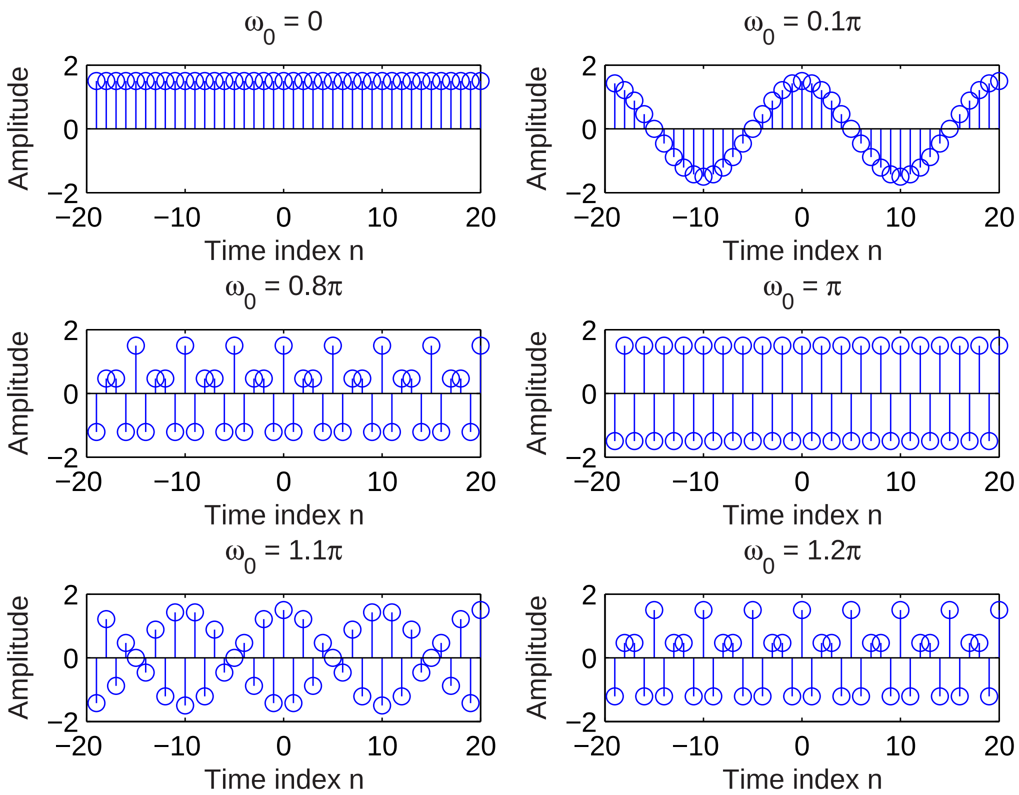

Figure 1.5: A family os sinusoidal sequences given by $x[n] = 1.5\cos(\omega_0 n)$: (a) $\omega_0 = 0$, (b) $\omega_0 = 0.1\pi$, (c) $\omega_0 = 0.8\pi$, (d) $\omega_0 = \pi$, (e) $\omega_0 = 1.1\pi$ and (f) $\omega_0 = 1.2\pi$.

If we write $x[n] = x_{re}[n] + jx_{im}[n]$, then from Eq.1.19:

$$x_{re}[n] = |A||\alpha|^n \cos(\omega_0 n + \phi), \quad (1.20)$$
$$x_{im}[n] = |A||\alpha|^n \sin(\omega_0 n + \phi). \quad (1.21)$$

These sequences oscillates with an exponential growing envelope if $|\alpha| > 1$ or with exponentially decay envelope if $|\alpha| < 1$.

When $|\alpha| = 1$, the sequence is referred to as a *complex exponential sequence* and has the form

$$x[n] = |A|e^{j\omega_0 n+\phi} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi), \quad (1.22)$$

where now the real and imaginary parts are real sinusoidal sequences with constant amplitude.

1.2 Discrete-Time Systems

A discrete-time system is defined mathematically as a transformation that maps an input sequence with value$x[n]$ into an output sequence with values $y[n]$ and can be denoted by

$$y[n] = \mathcal{T}\{x[n]\} \tag{1.23}$$

and is showed if Fig.1.6. Classes of systems are defined by placing constraints on the properties of the transformation $\mathcal{T}\{\cdot\}$. Doing so often leads to very general mathematical representation, as we will see.

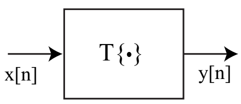

Figure 1.6: Representation of a discrete-time system,i.e., a transformation that maps an input sequence $x[n]$ into a unique output sequence $y[n]$.

1.2.1 Linear Systems

The class of *linear systems* is defined by the principle of superposition. If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is linear if and only if

$$\mathcal{T}\{x_1[n] + x_2[n]\} = \mathcal{T}\{x_1[n]\} + \mathcal{T}\{x_2[n]\} = y_1[n] + y_2[n] \tag{1.24}$$

and

$$\mathcal{T}\{ax[n]\} = a\mathcal{T}\{x[n]\} = ay[n]. \tag{1.25}$$

where a is an arbitrary constants. The two properties can be combined into the *principle of superposition*, stated as

$$\mathcal{T}\{a_1x_1[n] + a_2x_2[n]\} = a_1\mathcal{T}\{x_1[n]\} + a_2\mathcal{T}\{x_2[n]\} \tag{1.26}$$

for an arbitrary constants a_1 and a_2.

1.2.2 Time-Invariant Systems

A time-invariant system is one for which a time shift or delay of the input sequence causes a corresponding shift in the output sequence. Specifically, suppose that a system transform the input sequence with values $x[n]$ into the output sequence $y[n]$. The system is said to be time-invariant if for all n_0 the input sequence with values

$$x_1[n] = x[n - n_0]$$

produces the output sequences with values

$$y_1[n] = y[n - n_0].$$

This relation between the input and the output must hold for any arbitrary input sequence and its corresponding output.

1.2.3 Causal Systems

A system is causal if for every choice of n_0 the output sequence value at index $n = n_0$ depends only on the input sequence values for $n \leq n_0$ and does non depend on input samples for $n > n_0$. That is the system is *not anticipative*. Thus if $y_1[n]$ and $y_2[n]$ are the responses os a causal discrete-time system to the inputs $x_1[n]$ and $x_2[n]$, respectively, then

$$x_1[n] = x_2[n] \quad \text{for} \quad n < N$$

implies also that

$$y_1[n] = y_2[n] \quad \text{for} \quad n < N$$

1.2.4 Stable Systems

A system is stable in the bounded-input bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence. The input $x[n]$ is bounded if there exist a fixed positive value B_x such that

$$|x[n]| \leq B_x < \infty \quad \text{foralln}.$$

Stability requires that for every bounded input there exists a fixed positive finite value B_y such that

$$|y[n]| \leq B_y < \infty \quad \text{foralln}.$$

1.3 Linear Time-Invariant Systems (LTI)

A *linear-time invariant* (LTI) discrete-time system satisfies both the linearity and the time-invariance properties. Such systems are mathematically easy to analyze, and characterize.
If the linearity property is combined with the representation of a general sequence as a linear combination of delayed impulses as in Eq.(1.13), it follow that a linear system can be completely characterized by its *impulse response*. Specifically, let $h_k[n]$ be the response of the system to $\delta[n-k]$, an impulse occurring at $n = k$. Then from Eq.(1.13),

$$y[n] = \mathcal{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}. \tag{1.27}$$

From the principle of superposition in Eq.(1.26), is possible to write

$$y[n] = \sum_{k=-\infty}^{\infty} x[n]\mathcal{T}\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[n]h_k[n]. \tag{1.28}$$

According to this last equation, the system response to any input cam be expressed in terms of the response of the system to $\delta[n-k]$.

The property of time invariance implies that if $h[n]$ is the response to $\delta[n]$, then the response to $\delta[n-k]$ is $h[n-k]$. With this constraint, Eq.(1.28) becomes

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{1.29}$$

As a consequence of Eq.(1.29), an LTI system is completely characterized by its impulse response $h[n]$, in the sense that, it is possible to use Eq.(1.29) to compute the output $y[n]$ due to *any* input $x[n]$. The above sum in Eq.(1.29) is called *convolution sum* of the sequence $x[n]$ and $h[n]$, and represented by

$$y[n] = x[n] \circledast h[n], \tag{1.30}$$

where the convolution sum is represented by $\circledast$.
Is important to state that the convolutional sum is a linear operator that satisfies the *commutative*, *associative* and *distributive* properties.

1.4 Spectral Analysis of Discrete-Time signals

The spectral analysis is one of the powerful analysis tool in several fields of engineering. The fact that we can decompose complex signals with the superposition of other simplex signals, commonly sinusoid or complex exponentials, highlight some signal features that sometimes are very hard to discover with some other kind of analysis. For example, acoustical features such as *pitch* and *timbre*, are commonly obtained with algorithm that works in the frequency domain. Furthermore, the decomposition with simplex function is very useful when we want to modify a signal. In the frequency domain, the possibility to manipulate single spectral component give us the possibility to modify some fundamental feature of the sound, such as the timbre, that are hard, and sometimes impossible, to manipulate operating on the sound waveform.

A rigorous mathematical approach of the huge field of spectral analysis is out the scope of this book. In the next chapters we focus our attention on the most common and used spectral analysis tool: the **S**hort **T**ime **F**ourier **T**ransform (STFT).Sounds are time-varying signals in the real world and, indeed, all of their meaning is related to such time variability. Therefore, it is interesting to develop sound analysis techniques that allow to grasp at least some of the distinguished features of time-varying sounds, in order to ease the tasks of understanding, comparison, modification, and resynthesis. With STFT, often defined as the time-dependent Fourier Transform, we intend the joint analysis of the temporal and frequency features of the sound. In other word with this tool is possible to follow the temporal evolution of the spectral parameters of a sound.
The concept of STFT is based on the concept of the **D**iscrete-**T**ime **F**ourier **T**ransform, DTFT, that is the fundamental tool used to analyze a signal in the frequency domain. This is the discrete time version of the classical Fourier Transform commonly used for continuous-time signals. After a brief introduction on the DTFT, we will se how the DTFT on a periodic discrete-time signal specializes in the so called DFT that is at the bases of the STFT.

1.4.1 The Discrete-Time Fourier Transform: DTFT

First of all we will clarify the meaning of the variables which are commonly associated to the word "frequency" for signals defined in both the continuous and the discrete-time domain. The various symbols are collected in table 1.1, where the limits imposed by the Nyquist frequency are also indicated. With the term "digital frequencies" we indicate the frequencies of discrete-time signals.

Recalling that for a continuous-time signal $x(t)$ the Fourier Transform is defined as:

$$F(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \tag{1.31}$$

Nyquist Domain	Symbol	Unit	
$[-F_s/2 \quad \ldots \quad 0 \quad \ldots \quad F_s/2]$	f	[Hz] = [cycles/s]	
$[-1/2 \quad \ldots \quad 0 \quad \ldots \quad 1/2]$	f/F_s	[cycles/sample]	digital
$[-\pi \quad \ldots \quad 0 \quad \ldots \quad \pi]$	$\omega = 2\pi f/F_s$	[radians/sample]	freqs.
$[-\pi F_s \quad \ldots \quad 0 \quad \ldots \quad \pi F_s]$	$\Omega = 2\pi f$	[radians/s]	

Table 1.1: Frequency variables

where $\omega = 2\pi f$ is the continuous-frequency expressed in radians, we can try to re-express this expression in the case of a discrete-time signal $x[n]$.
If we think about a discrete-time signal as the sampled version of a continuous-time signal, $x(t)$, with a sampling interval $T = \frac{1}{F_s}$, $x[n] = x(nT)$, we can define the DTFT starting from the Eq.1.31 where the integral is substituted by a summation:

$$X(f) = \sum_{n=-\infty}^{+\infty} x(nT)e^{-j2\pi\frac{f}{F_s}n} \; . \tag{1.32}$$

As we will see later, $X(f)$ is a periodic function of the continuous-frequency variable f, with period F_s. Now if we use the variable $\omega = 2\pi\frac{f}{F_s}$, a more compact expression arise from the Eq:1.32 :

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(nT)e^{-j\omega n} \; . \tag{1.33}$$

where the variable ω is called normalized frequency,and is expressed in radians.Eq.1.33 is the general expression used to compute the DTFT.
In general $X(\omega)$ is a complex function of the real variable ω and can be written in rectangular form as

$$X(\omega) = X_{re}(\omega) + jX_{im}(\omega) \quad , \tag{1.34}$$

where $X_{re}(\omega)$ and $X_{im}(\omega)$ are, respectively, the real and imaginary parts of $X(\omega)$, and are real function of ω. $X(\omega)$ can alternately be expressed in the polar form as

$$X(\omega) = |X(\omega)|e^{\theta(\omega)} \tag{1.35}$$

and

$$\theta(\omega) = arg[X(\omega)] \tag{1.36}$$

The quantity $|X(\omega)|$ is called the *magnitude function* and the quantity $\theta(\omega)$ is called the *phase function* with both function again being real function of ω.
As can be seen from the definition Eq.1.33, the Fourier Transform $X(\omega)$ of a discrete-time sequence is a periodic function in ω with a period 2π. Note that the periodicity, with period 2π in the domain of the normalized-frequency $\omega = 2\pi\frac{f}{F_s}$, is equivalent to a periodicity of F_s in the domain of absolute-frequency f.
It therefore follows that Eq.1.33 represent the Fourier series representation of the periodic function $X(\omega)$. As a result, the Fourier coefficients $x[n]$ can be computed from $x(\omega)$ using the Fourier Integral given by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\mathrm{e}^{j\omega n}d\omega \quad , \tag{1.37}$$

called the *inverse discrete-time Fourier transform*. Equations 1.33 and 1.37 together form a Fourier representation for the sequence $x[n]$. Equation 1.37, the Inverse Fourier Transform, is a *synthesis* formula. That is, it represent $x[n]$ as a superposition of infinitesimally small complex sinusoid of the form

$$\frac{1}{2\pi}X(\omega)\mathrm{e}^{j\omega n}d\omega \tag{1.38}$$

with ω ranging in the interval of length 2π and with $X(\omega)$ determining the relative amount of each complex sinusoidal component. Although in writing 1.37 we have chosen the range of values for ω between $-\pi$ and π, any interval of length 2π can be used. Equation 1.33, the Fourier Transform, is an expression for computing $X(\omega)$ from the sequence $x[n]$, i.e., for *analyzing* the sequence $x[n]$ to determine how much of each component is required to synthesize $x[n]$ using Eq.1.37.

1.4.2 The Discrete Fourier Transform: DFT

In the case of a finite-length sequence $x[n], \quad 0 \leq n \leq N-1$, there is a simpler relation between the sequence and its discrete-time Fourier tranform $X(\omega)$. In fact, for a lenght-N sequence, only N values of $X(\omega)$, called *frequency samples*, at N distinct frequency points, $\omega =$ $omega_k, \quad 0 \leq k \leq N-1$, are sufficient to determine $x[n]$, and hence $X(\omega)$, uniquely. This lead to the concept of the discrete Fourier Transform, a second transform-domain representation that is applicable only to a finite-length sequence.
The DFT is at the heart of digital signal processing, because it is a **computable** transformation. Although the Fourier, Laplace and z-transform are the analytical tools of signal processing as well as many other disciplines, it is the DFT that we must use in a computer program such as Matlab.

1.4.2.1 Definition

The simplest relation between a finite-length sequence $x[n]$, defined for $0 \leq n \leq N-1$, and its DTFT $X(\omega)$ is obtained by uniformly sampling $X(\omega)$ on the ω-axis between $0 \leq \omega \leq 2\pi$ at $\omega_k = 2nk/N$, $0 \leq k \leq N-1$. From Eq.1.33,

$$X[k] = X(\omega)|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}, \quad 0 \leq k \leq N-1 \tag{1.39}$$

Note that $X[k]$ is also a finite-length sequence in the frequency domain and is of length N. The sequence $X[k]$ is called the *Discrete Fourier transform (DFT)* of the sequence $x[n]$. Using the commonly used notation

$$W_N = e^{-j\frac{2\pi}{N}} \tag{1.40}$$

we can rewrite Eq.1.39 as

$$X[k] = \sum_{k=0}^{N-1} x[n]W_N^{kn}, \quad 0 \leq k \leq \tag{1.41}$$

The *inverse discrete Fourier Transform*(IDFT) is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad 0 \leq n \leq . \tag{1.42}$$

1.5 The z-Transform

The discrete-time Fourier transform provides a frequency-domain representation of discrete-time signals and LTI systems. In this section, we consider a generalization of the Fourier transform referred to as the *z-transform*. This transformation for discrete-time signals is the counterpart of the Laplace transform for continuous-time signals. A principal motivation for introducing this generalization is that the Fourier Transform does not converge for all sequences and it is useful to have a generalization of the Fourier transform that encompasses a broader class of signals.

1.5.1 Definition

for a given sequence $g[n]$, its *z*-transform $G(z)$ is defined as:

$$G(z) = \mathcal{Z}\{g[n]\} = \sum_{n=-\infty}^{\infty} g[n] z^{-n}, \tag{1.43}$$

where *z* = Re(z)+Im(z) is a complex variable. If we let $z = r\mathrm{e}^{j\omega}$, then the right-hand side of the above expression reduces to

$$G(r\mathrm{e}^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} \mathrm{e}^{-j\omega n},$$

witch can be interpreted as the discrete-time Fourier transform of the modified sequence $\{g[n]r^{-n}\}$. For $r = 1$ (i.e., $|z| = 1$), the *z*-transform of $g[n]$ reduces to its discrete-time Fourier transform, provided that the latter exists.

Like the discrete-time Fourier transform, there are condition on the convergence of the infinite series of Eq.(1.43). For a given sequence, the set $\mathcal{R}$ of values of *z* for witch its *z*-transform converges is called the *regione of convergence* (ROC).

In general, the regione of convergence $\mathcal{R}$ of a *z*-transform of a sequence $g[n]$ is an annular region of the *z*-plane:

$$R_{g-} < |z| < R_{g+}$$

where $0 \le R_{g-} < R_{g+} \le \infty$.

Some commonly used *z*-transform pairs are listed in Table (1.2).

Sequence	***z*-Transform**	**ROC**
$\delta[n]$	1	All values of *z*
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos\omega_0 n)u[n]$	$\frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$(r^n \sin\omega_0 n)u[n]$	$\frac{(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Table 1.2: Some commonly used *z*-transform pairs.

1.5.2 Rational z-Transform

In the case of LTI discrete-time systems all pertinent *z*-transforms are rational function of z^{-1}, i.e., are ratios of two polynomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + +p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + +d_{M-1} z^{-(N-1)} + d_M z^{-N}}$$

where the *degree* of the numerator polynomial $P(z)$ is M and that of the denominator polynomial $D(z)$ is N.

The above equation can be alternately written in factored form as

$$G(z) = \frac{p_0}{d_0} \frac{\prod_{l=1}^{M} (1 - \xi_l z^{-1})}{\prod_{l=1}^{N} (1 - \lambda_l z^{-1})} = z^{N-M} \frac{p_0}{d_0} \frac{\prod_{l=1}^{M} (z - \xi_l)}{\prod_{l=1}^{N} (z - \lambda_l)}.$$

At a root $z = \xi_l$ of the numerator polynomial, $G(\xi_l) = 0$, and as a result, these values of z are known as the *zeros* of $G(z)$. Likewise, at a root $z = \lambda_l$ of the denominator polynomial, $G(\lambda_l) \to \infty$, and these points in the *z*-plane are called the *poles* of $G(z)$.

A physical interpretation of the concept of poles and zeros can be given by plotting the log-magnitude $20 \log_{10} |G(z)|$. This last expression is a two-dimensional function of Re(*z*) and Im(*z*). Hence its plot will describe a surface in the complex *z*-plane as illustrated in Figure 1.7 for the rational *z*-transform

$$G(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}.$$

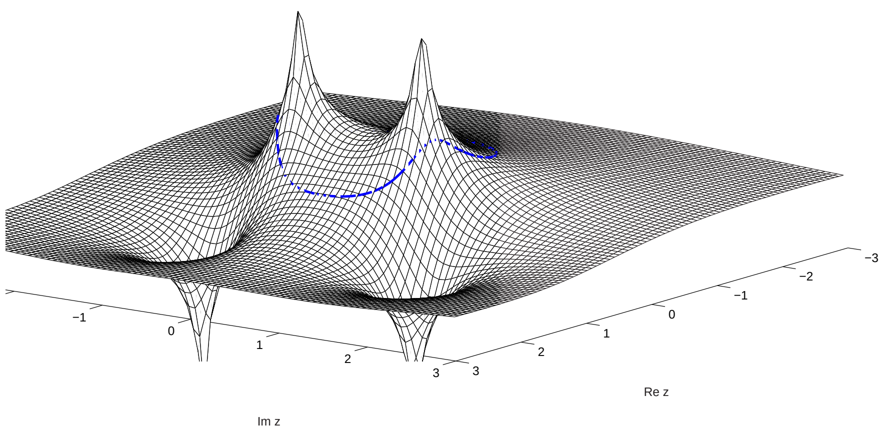

Figure 1.7: The 3-D plot of $20 \log_{10} |G(z)|$ as a function of Re(*z*) and Im(*z*)

1.5.3 Inverse z-Transform

The inverse *z*-transform relation is given by the contour integral

$$g[n] = \oint_C G(z) z^{n-1} dz, \tag{1.44}$$

where C is a counterclockwise closed contour in the region of convergence (ROC) of $G(z)$.

Equation (1.44) is the formal inverse *z*-transform expression. If the region of convergence includes the unit circle and if the contour of integration is taken to be the unit circle, then on this contour, $G(z)$ reduces to the Fourier transform and Eq.(1.44) reduces to the inverse Fourier transform expression

$$g[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega,$$

where we have used the fact that integrating in *z* counterclockwise arount the unit circle is equivalent to integrating in ω from $-\pi$ to π and that $dz = je^{j\omega} d\omega$.

1.5.4 z-Transform Properties

In Table 1.3 are summarized some specific properties of the *z*-transform.

It should be noted that the convolution property plays a particularly important role in the analysis of LTI systems. Specifically, as a consequence of this property, the *z*-transform of the output of an LTI system is the product of the *z*-transform of the input and the *z*-transform of the system impulse response.

Property	**Sequence**	***z*-Transform**	**ROC**
	$g[n]$	$G(z)$	$\mathcal{R}_g$
	$h[n]$	$H(z)$	$\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_0]$	$z^{-n_0} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$

Table 1.3: Some useful properties of the *z*-transform.