Small-Signal Analysis of DC-DC Converters with Sliding Mode Control

P. Mattavelli*, L. Rossetto*, G. Spiazzi**

*Department of Electrical Engineering

**Department of Electronics and Informatics

University of Padova

Via Gradenigo 6/a, 35131 Padova - ITALY

Phone: +39-49-828.7500 Fax: +39-49-828.7599/7699

Abstract. **The paper deals with small-signal analysis of dc-dc converters with sliding mode control. A suitable small-signal model is developed, which allows selection of control coefficients, analysis of parameter variation effects and characterization of the closed-loop behavior in terms of audiosusceptibility, output and input impedances and reference-to-output transfer function.**

Unlike previous analyses, the model includes effects of the filters used to evaluate state variable errors.

Simulated and experimental results demonstrate model potentialities.

I. INTRODUCTION

A control technique suitable for dc-dc converters must cope with their intrinsic nonlinearity and wide input voltage and load variations, ensuring stability in any operating condition while providing fast transient response. Among the various control techniques proposed in the literature, sliding mode control [1,2] offers several advantages, namely, large-signal stability, robustness, good dynamic response and simple implementation [3-4].

In spite of these positive aspects, sliding mode control is not yet popular, probably because its theoretical complexity, which can make difficult selection of controller parameters. In fact, these parameters must be chosen so as to satisfy existence, hitting and stability conditions, the analysis being easily carried out only for second-order converters, which permit a phase-plane description of the system.

Another limitation is that sliding mode control requires, in theory, sensing of all state variables and generation of suitable references for each of them.

In practice, generation of reference signals for all state variables is not needed in dc-dc converters. In fact, since only error signals are required for the control, they can be achieved by high-pass filtering the state variable signals, in the assumption that their dc component is automatically adjusted by the converter according to input-to-output power balance condition. Of course, these high-pass filters, not considered in previous analyses, increase the system order and can heavily affect the converter dynamic.

Moreover, converter control can effectively be done by sensing only one inductor current in addition to the output voltage [4] even for high-order converters (e.g. Cuk and Sepic), resulting in a control complexity similar to that of standard current-mode controllers. However, a comparison between the two solutions (full-order control and reduced-order control) is not easy to obtain.

In summary, there is a lack of models able to describe the effects of circuit and controller parameter variations and to allow a comparison between the sliding mode approach and other popular control techniques, like current-mode control, PWM control, etc., in terms of converter transfer functions (audiosusceptibility, output and input impedances, reference-to-output transfer ratio).

This paper presents a small-signal model of sliding-mode controlled dc-dc converters operating in continuous conduction mode, which also includes the effects of the filters used to determine state variable references.

Simulated results of a Sepic converter are reported, which show the model potentialities.

Experimental results of a Boost converter are also given.

II. BASIC SLIDING MODE CONTROLLER

Fig. 1 shows the general sliding mode control scheme of dc-dc converters. Although non used in practice, this scheme emphasizes the properties and operation mechanism of this control. U_g and u_{CN} are input and output voltage, respectively, while i_{Li} and u_{Cj} (i=1÷r, j=r+1÷N-1) are the internal state variables of the converter (r inductor currents and N-r-1 capacitor voltages) and N is the system order.

For the sake of generality the state variables will be indicated as components of state vector x.

According to the general sliding mode control theory, all state variables are sensed, and the corresponding errors (defined by difference to the reference values x^*) are multiplied by proper gains $K=[k_1, k_2,...k_N]^T$ and added together to form sliding function ψ. Then, hysteretic block HC keeps this function near to zero by gating on and off the power switch S. We can therefore assume:

$$\psi = K^T(x - x^*) \cong 0 \quad (1)$$

This means that the control forces the system to evolve on the hyperplane (sliding surface) defined by (1).

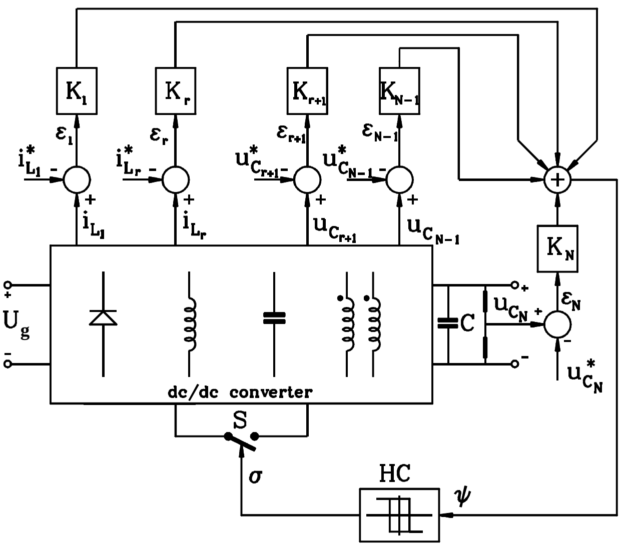


Fig.1 - Basic scheme of a sliding mode controller for dc-dc converters

As discussed in [4], selection of coefficients K must be done in order to satisfy some basic requirements: first, the state trajectories starting from points near the hyperplane must be directed toward the sliding surface (1) for both possible states of the converter switch (*existence condition*); second, the system trajectories must encounter the sliding surface irrespective of their starting point in the phase space (*hitting condition*); third, the system motion on the sliding surface must reach a stable point corresponding to the desired voltages and currents (*stability condition*).

Although the feasibility of a sliding mode controller of reduced order (where only one inductor current is sensed in addition to the output voltage) has been already demonstrated [4], in the following we will consider the case of Fig.1 for the sake of generality. Results for reduced-order controllers are easily obtainable by setting to zero some sliding coefficients k_i.

III. SMALL-SIGNAL MODEL DERIVATION

A generic dc-dc converter, working in continuous conduction mode, can be characterized by the following two sets of equations:

$$\dot{x} = A_1 x + B_1 u \quad \text{switch on} \quad (2)$$

$$\dot{x} = A_0 x + B_0 u \quad \text{switch off} \quad (3)$$

where x is the vector of state variables and u is the vector of the input variables (input voltage u_g and possibly load current i_o). According to the *state space averaging method* [5], the average system behavior is described by the equation:

$$\dot{x} = A\,x + B\,u \quad (4.a)$$

where:

$$A = A_1\delta + A_0(1-\delta)$$
$$B = B_1\delta + B_0(1-\delta) \quad (4.b)$$

δ being the converter duty-cycle in the steady state. After linearization (small-signal assumption), we can derive the following small-signal equation, in which the hat means perturbation from a steady-state working point (X, U):

$$\dot{\hat{x}} = A\,\hat{x} + B\,\hat{u} + C\,\hat{\delta} \quad (5)$$

Matrices A and B are the same as (4.b) and matrix C is given by:

$$C = (A_1 - A_0)X + (B_1 - B_0)U \quad (6)$$

In general, only some of the reference values x^* indicated in (1) are fed from the external control (often, only output voltage reference is assigned). The other references are derived internally to the controller, normally by low-pass filtering the corresponding state variables. For these latter references we can define M additional state variables x^*_{int} whose dynamic is described by:

$$\dot{x}^*_{i_{int}} = -\frac{1}{\tau_i} x^*_{i_{int}} + \frac{1}{\tau_i} x_i, \qquad i = 1 \div M \quad (7)$$

where τ_i are the corresponding filter time constants.

Thus, system order increases by M. The remaining N-M state variable references are additional inputs.

Note that linear equation (7) holds also for perturbed variables.

We call $\hat{x}'$ the vector of all state variables, including the additional state variables of the filters:

$$\hat{x}' = \begin{bmatrix} \hat{x} \\ \hat{x}^*_{int} \end{bmatrix} \quad (8)$$

Assuming that the converter operates in the sliding mode, constraint (1) also implies $\dot{\psi} = 0$, so that, from (5) and (7), we can express the converter duty-cycle as a function of complete state vector $\hat{x}'$, derivative of external references $\dot{\hat{x}}^*_{ext}$, and input variables:

$$\hat{\delta} = D_1 \hat{x}' + D_2 \hat{u} + D_3 \dot{\hat{x}}^*_{ext} \tag{9}$$

where matrices D_i (i=1÷3) are given in Appendix I. Substituting (9) in (5) and using (7) we obtain the linear system equation:

$$\dot{\hat{x}}' = A_c\, \hat{x}' + B_c\, \hat{u} + D_c\, \dot{\hat{x}}^*_{ext} \tag{10}$$

Taking now into account constraint (1), which holds even for perturbed variables, the system order is reduced by one. The final small-signal model has order N+M-1 and is given by:

$$\dot{\hat{x}}'' = A_T\, \hat{x}'' + B_T\, \hat{u} + D_{T1}\, \hat{x}^*_{ext} + D_{T2}\, \dot{\hat{x}}^*_{ext} \tag{11}$$

Expressions for matrices A_c, B_c, D_c, A_T, B_T, D_{T1}, D_{T2} are given in Appendix I.

From (11) we are now able to compute all closed-loop transfer functions. In particular, the input to output voltage transfer function (audiosusceptibility), the external reference-to-output voltage and the input admittance are directly derived from (11). Instead, derivation of converter output impedance, requires definition of the load current as an external input in (5).

It is noticeable that the above model derivation involves the same approximations of state-space-averaged modeling.

IV. MODEL VERIFICATION

In order to test the validity of the model, a Sepic converter operating at about 50kHz with sliding mode control was analyzed. According to [6], only two state variables are sensed: one is the output voltage and the other is the input inductor current, as shown in Fig.2.

Output impedance and audiosusceptibility were computed from the above small-signal model and by circuit simulation. The corresponding results are shown in Fig.3 a) and b), respectively (continuous line - model simulation; dotted line - converter simulation). As we can see, the maximum error is about 1dB in the whole range of operating frequencies.

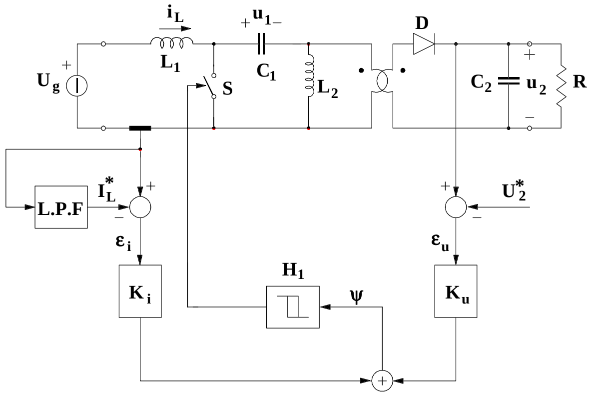
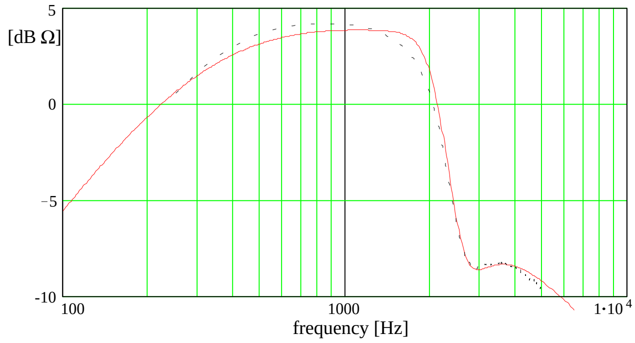
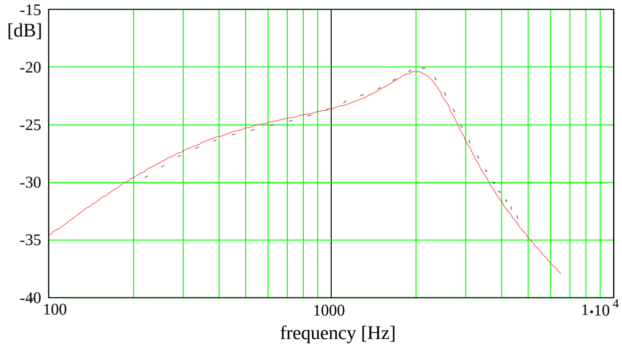


Fig.2 - Sepic converter with sliding mode control



a)



b)

Fig.3 - a) Output impedance, b) Audiosusceptibility
(continuous line - model simulation; dotted line - circuit simulation)

As already explained, the proposed small-signal model can be used both for full- and reduced-order control, thus showing how much this latter influences the system performances.

Although a full-order control generally results in better performances, this is not true in general. For example, the audiosusceptibility of the converter of Fig.2, is considerably better in terms of peak value and low frequency behavior.

V. EFFECTS OF CONTROL PARAMETER VARIATIONS

An important advantage of the derived small-signal model is the possibility to analyze the dynamic of the controlled system as a function of all control parameters, i.e. filter time constants and sliding coefficients k_i. This makes straightforward the design process, unlike previous analyses which disregard filter time constants and do not give information on dependence of the system dynamic on sliding coefficients.

Let us consider, for example, the boost converter with sliding mode control shown in Fig.4.

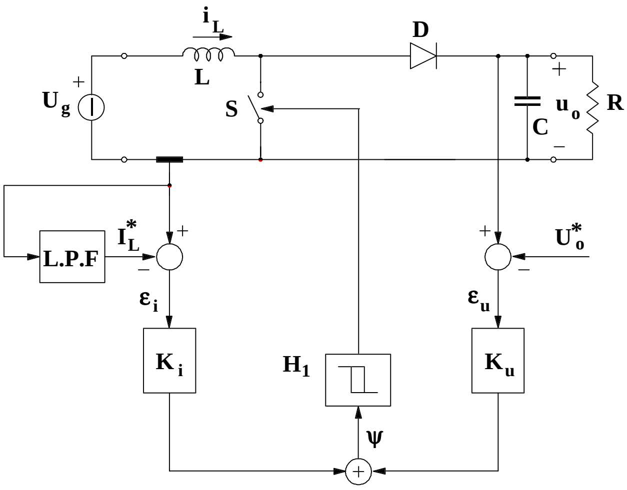


Fig.4 - Boost converter with sliding mode control

The order of the overall system is 2, because we have two state variables (N=2) and one low-pass filter (M=1). Following the procedure reported in Appendix I, we compute matrix A_T which is given by:

$$A_T = \begin{bmatrix} -\dfrac{\dfrac{2}{RC} + g\dfrac{\delta'}{C}\left(1 - \dfrac{L_{eq}}{R\tau}\right)}{1 - g\dfrac{L_{eq}}{RC}\delta'} & \dfrac{\dfrac{\delta'}{C}}{1 - g\dfrac{L_{eq}}{RC}\delta'} \\ -\dfrac{g}{\tau} & 0 \end{bmatrix} \qquad (12)$$

where $K^T=[k_i \;\; k_u]$, $g=k_u/k_i$, $\delta'=1-\delta$, τ is the filter time constant and $L_{eq}=L/\delta'^2$. Note that coefficients k_i and k_u were chosen in order to make adimensional the value of ψ, so that g results to be a conductance. The characteristic polynomial of matrix A_T is given by the determinant of matrix $(sI-A_T)$:

$$\Delta = s^2 + s\cdot\left[\frac{\dfrac{2}{RC} + g\cdot\dfrac{\delta'}{C}\left(1 - \dfrac{L_{eq}}{R\tau}\right)}{1 - g\dfrac{\delta'}{C}\dfrac{L_{eq}}{R}}\right] + \frac{g\delta'}{\tau C}\frac{1}{1 - g\dfrac{\delta'}{C}\dfrac{L_{eq}}{R}} \qquad (13)$$

From the analysis of the coefficients of polynomial Δ (see Appendix II), we recognize that if we choose

$$0 < g < g_{crit} = \frac{RC\delta'}{L} \qquad (14)$$

the system stability is ensured provided that:

$$\tau > \tau_{crit} = \frac{L_{eq}}{R + 2\dfrac{L_{eq}}{RC}} \qquad (15)$$

It can be demonstrated that condition (14), derived under small-signal time-averaging approximation, is the same as the existence condition of the sliding mode for a boost converter in the steady-state. This is easily derived from converter trajectory analysis (see Appendix III). The same result can be found for other basic converter topologies, for which values of τ_{crit} and g_{crit} are reported in Table I.

TABLE I : Values of τ_{crit} and g_{crit} for basic converter topologies

	Buck	Boost	Buck-boost
g_{crit}	∞	$\dfrac{RC\delta'}{L}$	$\dfrac{RC\delta'}{L\delta}$
τ_{crit}	0	$\dfrac{L_{eq}}{R + 2\dfrac{L_{eq}}{RC}}$	$\dfrac{L_{eq}}{\dfrac{R}{\delta} + (2-\delta')\dfrac{L_{eq}}{RC}}$

For higher order systems, for which is not easy to derive simple conditions as (14) and (15), eq.(11) can be used directly to observe the effects of the controller parameters on the system dynamic. For example, Fig.5 shows the root locus resulting from different values of the filter time constant of the Sepic converter of Fig.2. Too low values cause real poles P_1 and P_2 to become complex or even to cross the imaginary axis.

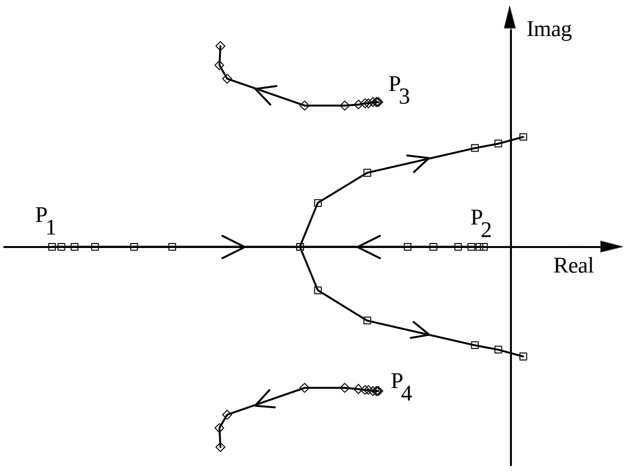


Fig.5 - Root locus of closed-loop system for variation of low-pass filter time constant of the Sepic converter of Fig.2

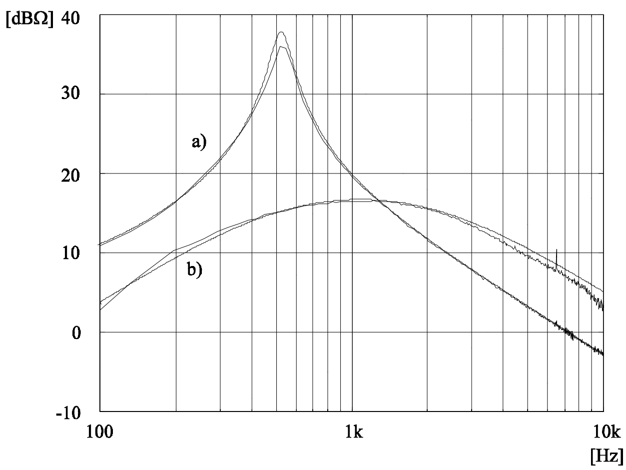


Fig.6 - Comparison between model forecast and experimental results: output impedance a) open loop, b) closed loop

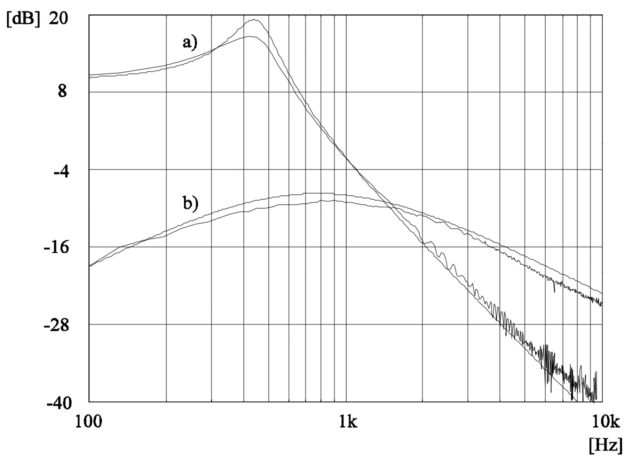


Fig.7 - Comparison between model forecast and experimental results: audiosusceptibility a) open loop, b) closed loop

VI. EXPERIMENTAL RESULTS

A boost converter prototype, according to the scheme of Fig.4, was used to test the validity of the above theoretical results. Converter parameters are listed in Table II.

TABLE II: Experimental prototype parameters

U_g = 24 V	U_o = 48 V	P_o = 50 W	f_s = 50 kHz
L = 570 μH	C = 22 μF	τ = 0.4 ms	g = 0.35

Converter audiosusceptibility and output impedance were measured and compared with those predicted by the model. For this purpose, the parameters used in the model were measured directly on the prototype and the inductor series resistance was taken into account.

In Fig.6, the open- and closed-loop output impedance is reported. As we can see, theoretical and experimental results agree pretty well. The sliding mode control reduces the peak in the output impedance diagram by almost 20dB. Fig.7 shows the audiosusceptibility behavior in the same conditions. Once again, simulated and experimental curves look very similar, while the improvement in the audiosusceptibility peak is about 30dB.

Note that control causes a worsening of the dynamic characteristics at high frequencies, as compared to the open-loop case. This behavior agrees with the multi-loop nature of sliding mode control, as explained in [7].

VII. CONCLUSIONS

A small-signal model of dc-dc converters with sliding mode control is derived. It allows evaluation of closed-loop performances like audiosusceptibility, output and input impedances and reference-to-output transfer function.

With this model, stability as well as effects of controller parameter variations can deeply be investigated and control parameters can be carefully selected.

Simulated results of a Sepic converter and experimental results of a Boost converter are reported, showing the validity of the approach.

APPENDIX I

In order to derive the expressions of matrices in (9), (10), (11) it is convenient to split the vector of state variable references between internal and external variables, as follows:

$$x^* = \begin{bmatrix} x^*_{int} \\ x^*_{ext} \end{bmatrix} \tag{A1.1}$$

From (1), it follows that also the derivative of sliding function ψ is zero. Thus, considering perturbed signals, we can write:

$$\dot{\hat{\psi}} = K^T\left(\dot{\hat{x}} - \dot{\hat{x}}^*\right) = K^T\dot{\hat{x}} - K^T_{int}\dot{\hat{x}}^*_{int} - K^T_{ext}\dot{\hat{x}}^*_{ext} = 0 \tag{A1.2}$$

in which the vector of sliding coefficients K is divided in two terms K_{int} and K_{ext}.

The dynamic description of the internal references is given by (7), which can be rewritten as:

$$\dot{\hat{x}}^*_{int} = T_1\,\hat{x}^*_{int} - T_2\,\hat{x} \tag{A1.3}$$

where T_1 is a diagonal M×M matrix, while T_2 is a M×N matrix given by

$$T_2 = [T_1 \quad 0] \tag{A1.4}$$

which holds in the hypothesis that the state variables in vector x are ordered as in vector x^*.

Substituting (A1.3) and (5) into (A1.2), and calculating the duty-cycle we obtain (9), whose matrices are given by:

$$D_1 = \begin{bmatrix} D_{11} & D_{22} \end{bmatrix} \tag{A1.5a}$$

$$D_2 = -\left(K^TC\right)^{-1} \cdot K^TB \tag{A1.5b}$$

$$D_3 = \left(K^TC\right)^{-1} \cdot K^T_{ext} \tag{A1.5c}$$

where

$$\begin{aligned} D_{11} &= -\left(K^TC\right)^{-1} \cdot \left(K^TA + K^T_{int}T_2\right) \\ D_{22} &= \left(K^TC\right)^{-1} \cdot K^T_{int}T_1 \end{aligned} \tag{A1.6}$$

Note that term K^TC is scalar.

The complete description of the system is given by (10), which is obtained by substituting (9) in (5) and using (A1.3) and (8). Matrices A_c, B_c, D_c result:

$$A_c = \begin{bmatrix} A + CD_{11} & CD_{22} \\ -T_2 & T_1 \end{bmatrix} \tag{A1.7a}$$

$$B_c = \begin{bmatrix} B + CD_{22} \\ 0 \end{bmatrix} \tag{A1.7b}$$

$$D_c = \begin{bmatrix} CD_3 \\ 0 \end{bmatrix} \tag{A1.7c}$$

Lastly, in order to derive the matrices of the reduced order system (11), let us consider non zero the first element k_1 of vector K. Thus, from (1) and (A1.1) we can write:

$$K^T\left(\hat{x} - \hat{x}^*\right) = \left[K^T - K^T_{int}\right] \cdot \hat{x}' - K^T_{ext} \cdot \hat{x}^*_{ext} = 0 \tag{A1.8}$$

Solving for the first state variable and substituting in (10) we obtain equations (11), in which we have:

$$\hat{x}'' = \hat{x}'(2{:}N+M)$$

$$A_T = A_c(2{:}N+M, 2{:}N+M) + A_c(2{:}N+M, 1) \cdot F_1$$

$$B_T = B_c(2{:}N+M, :)$$

$$D_{T1} = A_c(2{:}N+M, 1) \cdot F_2$$

$$D_{T2} = D_c(2{:}N+M, :)$$

where

$$F_1 = -\frac{1}{k_1} \cdot \left[K^T(2{:}N) - K^T_{int}\right]$$

$$F_2 = \frac{1}{k_1} \cdot K^T_{ext}$$

and the notation A(*n*:*m* , *j*:*k*) means the submatrix obtained from matrix A taking rows from index *n* to *m* and columns from index *j* to *k*, while symbol *:* alone means all.

APPENDIX II

In order to derive conditions (14) and (15), which ensure system stability for a Boost converter with sliding mode control, we analyze the sign of coefficients of the characteristic polynomial Δ (13). Calling r_1 and r_2 the roots of Δ, we can write:

$$\Delta = s^2 - s \cdot (r_1 + r_2) + r_1 \cdot r_2 \tag{A2.1}$$

The system is stable if both r_1 and r_2 have negative real part, i.e. if

$$r_1 \cdot r_2 > 0 \tag{A2.2}$$

and

$$r_1 + r_2 < 0 \tag{A2.3}$$

From (13), (A2.2) and (A2.3) the following inequalities result:

$$\frac{g}{\tau}\frac{\delta'}{C}\frac{1}{1 - g\dfrac{\delta'}{C}\dfrac{L_{eq}}{R}} > 0 \tag{A2.4}$$

$$\frac{\dfrac{2}{RC} + g\dfrac{\delta'}{C}\left(1 - \dfrac{L_{eq}}{R\tau}\right)}{1 - g\dfrac{\delta'}{C}\dfrac{L_{eq}}{R}} > 0 \tag{A2.5}$$

From (A2.4) it results

$$0 < g < g_{crit} = \frac{RC}{\delta' L_{eq}} = \frac{RC\delta'}{L} \qquad (A2.6)$$

which is the condition (14).

From (A2.5), taking into account (A2.6), it results:

$$\tau > \frac{L_{eq}}{R} \frac{1}{1 + \frac{2}{R\delta' g}} \qquad (A2.7)$$

where, substituting the maximum value g_{crit} for g given by (A2.6), the minimum value τ_{crit} as given by (15) results.

APPENDIX III

As already mentioned, the existence condition requires that the phase trajectories are directed toward the sliding surface in a small volume around the surface itself. This statement translates into the conditions [1-2]:

$$\lim_{\psi \to 0^+} \frac{d\psi}{dt} < 0$$
$$\lim_{\psi \to 0^-} \frac{d\psi}{dt} > 0 \qquad (A3.1)$$

Writing the expression of $\dot{\psi} = K^T \dot{x}$ for the boost converter in the two situations corresponding to the different values of the switch status, the above inequalities give:

$$\frac{k_u}{k_i} < \frac{RC}{L} \cdot \frac{U_g}{u_C} \qquad (A3.2)$$

where u_C is the instantaneous voltage of the output filter capacitor. Now, if we consider a small volume around the operating point in the phase plane (small-signal approximation), we can use the nominal capacitor voltage in (A3.2), which is equal to the output voltage, thus obtaining condition (14).

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