# GENERATION OF GAUSSIAN PROCESSES AND LINEAR CHAOS 

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#### Abstract

Any stationary Gaussian process with a rational spectral density can be represented as the output of a linear infinite-dimensional Hamiltonian system in "thermal equilibrium". The Hamiltonian system must have continuous spectrum of Lebesgue type. We show that on an extended phase space supporting invariant probability measures for the system, the Hamiltonian flow is $H y$ percyclic i.e. there is a vector generating a dense orbit. This is a well known topological condition for chaos. In fact, the hamiltonian flow is chaotic according to many standard definitions of the term.


## 1 Introduction

In this paper we continue previous investigations initiated in [15], [16], [17], to characterize "complicated" linear deterministic systems which, through a suitable randomization procedure, may admit a "simple" statistical description. It may happen that the external behaviour of certain linear deterministic systems is exactly described by a stochastic dynamic model so that the latter is equivalent to the original system, at least from the point of view of an observer having access only to the input-output terminals. The above phenomenon of exact model reduction by randomization, or "stochastic aggregation" as we have named it, can only occur when the time evoution of the external variables of the system is strongly irregular. In fact, as we shall argue in this paper, stochastic aggregability can essentially occur only when the underlying linear system is chaotic.

The word "chaotic" in the literature almost invariably refers to the non-linear/ hyperbolic setup. The dynamical systems studied in the literature either evolve in a compact phase space or, in any case, the interesting invariant sets on which irregular "stochastic" behaviour occurs, are "small", tipically of zero Lebesgue measure. By observing these systems one can generate effectively "random" processes but only of the finite-state type, e.g. Bernoulli i.i.d. processes. We want instead to generate stationary processes with continuous state space, say Gaussian processes taking values in $\mathbf{R}$. Strangely enough, the possibility of "chaos" in linear systems and its nature has not been reputed an interesting object of study in the mathematical literature until very recently and there are
very few known characterizations of linear chaos available (see [6], [4]).
In the present context, linear chaotic systems are very natural. In fact, an arbitrary purely-non-deterministic stationary zero-mean Gaussian random process $\{y(t)\}$, with a prescribed rational spectral density, can be represented (or "realized") as a linear observable of a linear "chaotic" system. This general result follows in part from previous work [18] and from the analysis presented here. The system will have to be infinite-dimensional. In fact, a natural choice (suggested by certain similarities with Statistical Mechanics,) leads to infinite dimensional autonomuos linear Hamiltonian systems $\Sigma_{o}$, evolving on a real Hilbert space $\mathbf{H}$. Such systems have the general form

$$
\Sigma_{o}:\left\{\begin{array}{l}
\dot{z}(t)=F z(t)  \tag{1}\\
y_{k}(t)=<h_{k}, z(t)>, k=1, \ldots, m
\end{array}\right.
$$

where $F$ is a densely defined linear skew-symmetric operator generating a group of continuous linear operators $\{\Phi(t) ; t \in \mathbf{R}\}$ on $\mathbf{H}$. The variable $z(t) \in \mathbf{H}, t \in \mathbf{R}$ is the microscopic state, and $y_{k}(t)$ the observables or outputs of the system at time $t$.
It may seem that a deterministic evolution equation of the type (1) has little to do with chaotic dynamics. Yet irregular behaviour must be present since a Gaussian purely non deterministic stationary output process $\{y(t)\}$, must have very irregular sample paths. Note that irregular sample path are a fact on a countably additive probability space only. This may seem trivial to point out but is really the key to explain the apparent paradox above.

Chaos really comes into play only at the point of introducing randomization. A probabilistic setup is introduced by equipping the phase space $\mathbf{H}$ with a "thermal equilibrium" invariant measure $\mu$ for the evolution group $\{\Phi(t)\}$, thus defining a condition of statistical steadystate of the system. Unfortunately the natural invariant Gaussian measures for the group are not $\sigma$-additive on the Hilbert space $\mathbf{H}$. This is a well-known difficulty with Gaussian measures on infinite-dimensional spaces [5], [7]. To be able to represent almost all sample paths of the process, with respect to a countably additive probability measuere $\mu$, a suitable extension of the system $\Sigma_{o}$ to a larger space than the "natural" phase space $\mathbf{H}$, is necessary. It will be seen that it is precisely this extension of
the original dynamics (1) which shows chaotic behaviour.

## 2 Stochastic aggregation

From classical work in Statististical Mechanics, e.g. [3],[9], it is known that in certain situations it is possible to generate the output trajectories $t \rightarrow y(t)$ of an infinite-dimensional linear Hamiltonian system (like $\Sigma_{o}$ ) in thermal equilibrium, by means of a generalized Langevin equation, which is nothing else but a finite dimensional stochastic linear system of the type

$$
\Sigma:\left\{\begin{array}{l}
d x(t)=A x(t) d t+K d v(t)  \tag{2}\\
y(t)=C x(t)
\end{array}\right.
$$

where $x(t)$ is a finite dimensional Gauss-Markov process taking values say in $\mathbf{R}^{n}, K \in \mathbf{R}^{n \times p}$ and $v$ is a $p$-dimensional Wiener process. The problem of obtaining representations of the output of $\Sigma_{o}$ of the form (2) is in fact the stochastic aggregation problem in the present linear setting (the problem will actually be reformulated a bit more precisely below). It has been studied in a systematic way in [15] [17],etc, using ideas from Stochastic Realization Theory.

Besides being of intrinsic interest in itself, the "aggregation problem" mentioned above is believed to be relevant also in understanding stochastic modeling of engineering systems. In particular one of the main motivations in this paper is the desire of understanding (obviously, in a mathematically idealized context) how to treat unmodelled dynamics in engineering problems, especially in estimation and control. As a motivation one should think e.g. of the common engineering practice of describing complicated physical systems by stochastic low order models in which additive white noise terms are added to the state equations just to account for "modeling errors". This is for instance a very common practice in building state models for Kalman filtering.

In engineering terminology, we may say that a "large" system $\Sigma_{o}$ is aggregable if the state vector $z$ can be decomposed into a "low frequency" and a "high frequency" (or parasytic) component, the high frequency part being, for an external observer having access only to output measurements, equivalently "lumpable" as an additive white noise input acting on the low frequency dynamics. The basic definition in this respect is the following (the need to distinguish between "forward" and "backward" will be clear in the following).

DEFINITION 2.1 If, for all $t_{0} \in \mathbf{R}$ and for each initial condition $z\left(t_{o}\right)=z_{o} \in \mathbf{H}$ there are:

- $x_{o} \in \mathbf{R}^{n}$ and a driving input trajectory $v$ of $a$ Wiener process for the system (2), such that the corresponding output trajectories $y_{z_{0}}$ of $\Sigma_{o}$ and $y_{x_{o}, v}$ of $\Sigma$, respectively, coincide for all $t \geq t_{o}$.
- Conversely, if every output trajectory of (2) coincides for $t \geq t_{o}$ with some output trajectory of (1)
we shall say that the system $\Sigma_{o}$ is aggregable and $\Sigma$ is a forward aggregation of $\Sigma_{o}$.

A backward aggregation of $\Sigma_{o}$ is instead a finite dimensional stochastic system (2) for which the same happens for all $t \leq t_{o}$ for any initial time $t_{o}$.

Definition 2.1 is just formalizing the idea that the system $\Sigma_{o}$ in (1) is indistinguishable from a forward (resp. backward) aggregation for an observer having access only to future (resp. past) trajectories of the output function $y$.

In Statistical Mechanics the microscopic dynamics is always conservative i.e. governed by a energy-preserving group. This means a Hamiltonian type of microscopic time evolution. For applications to engineering systems this type of Hamiltonian structure may look restrictive. It can however be obtained, at least abstractly, by imbedding the actual physical (dissipative) model at hand, into a lossless one by means of a dilation procedure [2].

## 3 Linear Hamiltonian systems and stationary Gaussian processes

In this section we shall show that observables of linear Hamiltonian systems in thermal equilibrium, generate stationary Gaussian processes in a natural way.

For an Hamiltonian system with total energy $H(z)$, the solution of the canonical equations determines the phase of the system (say configurations and momenta $\left.z(t):=[q(t) p(t)]^{\prime}\right)$ at each time $t$, uniquely in terms of the initial value $z(0)$. The correspondence defines a flow $z(t)=\Phi(t) z(0)$ on the phase space which leaves invariant the total energy, $H(z(t))$, of the system.

In condition of "thermal equilibrium" the phase of the system is statistically distributed according to a probability distribution on the phase space which is invariant for the Hamiltonian flow $\Phi(t)$. In particular the distribution of the phase variable $z$ at time zero remains the same for all times. It is also well known known that in a finite-dimensional space, any absolutely continuous $\Phi(t)$-invariant probability measure admits a density $\rho(z)$ of the Gibbs type, i.e. equal to a normalization constant times $\exp \left[-\frac{1}{2 \beta} H(z)\right], \beta>0$.

It follows that in thermal equilibrium any measurable function $f$ on phase space can be regarded as defining a stationary stochastic process. In particular, a fixed $m$ dimensional family of observables $\left\{h_{1}, \ldots, h_{m}\right\}$ generates a vector-valued $m$-dimensional stationary process $\{y(t)\}$ with components

$$
\begin{equation*}
y_{k}(t, z(0)):=h(\Phi(t) z(0)) \tag{3}
\end{equation*}
$$

$z(0)$ being interpreted as the random elementary event and $\Phi(t)$ as the measure preserving group of transformations on the underlying probability (phase) space.

The above "randomization" of the phase space is introduced in Statistical Mechanics in the hope that it leads
to a description of the process $\{y(t)\}$ of (3) by some sort of statistical model of a simpler structure than the microscopic Hamiltonian description. However it is trivial to check that for linear Hamiltonian systems with a finite dimensional phase space, the stochastic processes $\{y(t)\}$ generated by linear observables of a system in thermal equilibrium are purely deterministic quasi-periodic processes. These are quite uninteresting from the point of view of aggregation. In fact, it is easy to see (and has been stressed very early by Lewis and Thomas [9]) that only infinite-dimensional linear Hamiltonian systems can admit aggregation. Moreover, for aggregable systems the infinitesimal generator of the Hamiltonian group must have continuous Lebesgue spectrum.
We shall then consider only infinite-dimensional linear Hamiltonian systems with phase space a real Hilbert space $\mathbf{H}$. We shall further assume that the Hamiltonian function can be normalized to the squared norm, $H(z)=1 / 2\|z\|^{2}$, of the phase variable (the so-called energy norm) so that the Hamiltonian flow $\Phi(t)$ becomes norm preserving, i.e. an orthogonal group of linear operators on $\mathbf{H}$.
The observables (state-output maps) will be taken to be described by $m$ linear functionals $h_{k}^{*}: \mathbf{H} \rightarrow \mathbf{R}$ represented by $m$ vectors $h_{k}$ in $\mathbf{H}$, so that the relative observation processes $y_{k}(t), k=1, \ldots, m$, of the system started in the initial phase $z(0)$, are described by

$$
\begin{equation*}
y_{k}(t, z(0)):=<h_{k}, \Phi(t) z(0)>\quad z(0) \in \mathbf{H} \tag{4}
\end{equation*}
$$

where $\langle\ldots$, $\rangle$ denotes inner product in $\mathbf{H}$.
This setup is quite general and can be shown to accomodate many classical linear models of Statistical Mechanics like the Brownian particle in a heat bath, the Lamb's model [9], [10] etc..
Linear Hamiltonian system (1), will hereafter be assumed irreducible, in the sense that the smallest $\{\Phi(t)\}$ invariant subspace $\mathbf{H}_{o}$ containing the vectors $\left\{h_{k} ; k=\right.$ $1, \ldots, m\}$, and given by

$$
\begin{equation*}
\mathbf{H}_{o}:=\overline{\operatorname{sp} a \bar{n}}\left\{\Phi(t) h_{k} ; k=1, \ldots, m, t \in \mathbf{R}\right\} \tag{5}
\end{equation*}
$$

will be assumed to coincide with $\mathbf{H}$. This is a natural condition of non-redundancy of the model. For all phase trajectories generated by initial phases in the orthogonal complement $\mathbf{H}_{o}^{\perp}$ in $\mathbf{H}$, will live there forever and therefore will automatically also be in the nullspace of the observables of the system. Hence they will be completely invisible to an external observer.
Note that for irreducible systems the Hamiltonian group $\{\Phi(t)\}$ has finite multiplicity $(\leq m)$.

A technical nuisance especially associated with continuous-spectrum Hamiltonian systems in a infinite dimensional Hilbert spaces, is that the natural invariant probability measures for the orthogonal group $\Phi(t)$ are families of Gaussian cylinder measures $\mu_{\beta}$ with covariance operator $\beta I, \beta>0$, which are not countably additive on $\mathbf{H}[7]$. The standard way out to this difficulty is to enlarge the "natural" probability space, $\mathbf{H}$, to a larger

Banach space on which the invariant Gaussian measures can be supported. The large space will in general contain "nonphysical" phases.

As explained in [7], [5], the "large" Banach space (which can sometimes be taken to be Hilbert), is usually seen as the dual, $\mathcal{S}^{\prime}$, of a "smooth" space $\mathcal{S} \subset \mathbf{H}$, densely imbedded in $\mathbf{H}$. The crucial request for extendability of $\mu_{\beta}$ to $\mathcal{S}^{\prime}$ is that the imbedding map $\mathcal{S} \rightarrow \mathbf{H}$ be a nuclear operator, or, which amounts to the same, that the dual imbedding $i^{\prime}: \mathbf{H} \rightarrow \mathcal{S}^{\prime}$ be nuclear. This implies that $\mathcal{S}^{\prime}$ is a separable Banach space. Note that there may be many different choices of such Gelfan'd triples

$$
\begin{equation*}
\mathcal{S} \subset \mathbf{H} \subset \mathcal{S}^{\prime} \tag{6}
\end{equation*}
$$

for the same system. The enlarged phase (=probability) space of the system, $\mathcal{S}^{\prime}$ will be denoted by $\Omega$ hereafter. The dual, $\mathcal{S}$, is to play the role of ambient space for "smooth" linear functionals (i.e. observables of the system) on the extended space $\Omega$.
In this setup the orginal observables (4) are only defined on the dense subset $\mathbf{H}$ of the probability space $\Omega$. They can however be extended by isometry as $L^{2}\left(\Omega, \mu_{\beta}\right)$-limits, to the whole space (see Hida's book for details). So, eventually we end up with a bona-fide $m$-dimensional Gaussian stationary stochastic process $\{y(t)\}$ whose sample paths, at least for "physical" initial phases, represent the time evolution of the measured observables (4).
It should be stressed that the finite-dimensional distributions of $\{y(t)\}$ do not depend on the extension of the probability measure but are completely determined by the linear observable maps (4) on $\mathbf{H}$ and by $\mu_{\beta}$ as a finitely additive (cylinder) Gaussian measure on $\mathbf{H}$. However the extended space and the choice of a "best" $\mathcal{S}^{\prime}$ is crucial for the representation problem we consider here. As it may appear from the discussion above the possible choices of $\mathcal{S}^{\prime}$ have for example a lot to do with the degree of smoothness of the sample paths of $\{y(t)\}$ ). We shall come back to this point later.

The Spectral Distribution Matrix of the process can be written explicitely in terms of the spectral data of the Hamiltonian group ([15]). Let

$$
\begin{equation*}
\Phi(t) z=\int e^{i \lambda t} d \hat{E}(i \lambda) z \tag{7}
\end{equation*}
$$

be the spectral representation of the unitary group $\{\Phi(t)\}, \hat{E}($.$) denoting the spectral measure mapping$ Borel subsets of the imaginary axis into orthogonal projection operators on $\mathbf{H}$. Define the random measures $\hat{y}_{k}$, by setting

$$
\begin{equation*}
\hat{y}_{k}(\Delta, \omega):=<\hat{E}(\Delta) h_{k}, \omega>, k=1, \ldots, m \tag{8}
\end{equation*}
$$

where $\Delta$ is a Borel set of the imaginary axis and initially $\omega=z(0)$ is taken in $\mathbf{H} \subset \Omega$. Extend then the linear functionals (8) to the whole of $\Omega$ by isometric extension in the $L^{2}\left(\Omega, \mu_{\beta}\right)$-sense.

It descends from the defining relation (4) and formula (8) that $\hat{y}_{k}$ are the components of the random spectral
measure (i.e. the Fourier transform ) of the stationary process $\{y(t)\}$ (see [19]). In other words $\{y(t)\}$ admits the spectral representation

$$
\begin{equation*}
y_{k}(t)=\int_{-\infty}^{+\infty} e^{i \lambda t} d \hat{y}_{k}(i \lambda), k=1, \ldots, m \tag{9}
\end{equation*}
$$

and hence its spectral distribution is an $m \times m$ matrix measure $F$ with entries

$$
\begin{equation*}
F_{k, j}(\Delta)=E \hat{y}_{k}(\Delta) \hat{y}_{j}(\Delta)^{*}=\beta<\hat{E}(\Delta) h_{k}, h_{j}>k, j=1, \ldots, m \tag{10}
\end{equation*}
$$

It is seen that the spectrum, and hence the probability law of the observation process is completely described (modulo an arbitrary "temperature" parameter $\beta>0$ of the invariant measure,) by known data of the Hamiltonian system.

At this point the problem of stochastic aggregation of a linear Hamiltonian system, with respect to a given family of linear observables $h_{k}^{*}$, has been seen to be identical to representing a stationary Gaussian process $\{y(t)\}$ given in Eq. 4 above, as a function of a finite dimensional p.n.d. Markov process. Of course a very natural requirement on the representation is that the Gaussianness and stationarity of $\{y(t)\}$ should be inherited by the Markov process. This in turn calls for linear time invariant representations of the type

$$
\begin{equation*}
y(t)=C x(t) \tag{11}
\end{equation*}
$$

where $C$ is a linear map from the state space of the Markov process $\{x(t)\}$ into $\mathbf{R}^{m}$.

## 4 Constructing the state of a stationary Gaussian process

In this section we shall briefly review the characterization of stationary Gaussian processes $\{y(t)\}$ representable by finite-dimensional Markov processes and the main steps of a procedure for constructing such Markovian representations. The details can be found for example in the paper [11].
THEOREM 4.1 There are finite-dimensional Markovian representations of $\{y(t)\}$ if and only if the spectral distribution matrix $F$ of the process is absolutely continuous with a rational spectral density $\hat{\Phi}(i \lambda)$,

$$
\begin{equation*}
\hat{\Phi}(i \lambda)=\frac{d}{d \lambda} F(i \lambda) \tag{12}
\end{equation*}
$$

Finite dimensional Markovian representations of $y(t)$ are computed via the following algorithm

1. Do spectral factorization of $\hat{\Phi}(i \lambda)$, i.e. find a rational $m \times r$ matrix functon $W$ satisfying

$$
\begin{equation*}
\hat{\Phi}(i \lambda)=W(i \lambda) W(i \lambda)^{*} \tag{13}
\end{equation*}
$$

and such that $W$ extends to an analytic matrix function on the left-half complex plane (such factors are called analytic). We restrict for simplicity to leftinvertible factors, in which case $r=\operatorname{rank}(\hat{\Phi})$, a.e.
2. For each analytic spectral factor $W$ define the Gaussian stationary-increments process $\{w(t)\}$ by assigning its spectral measure $d \hat{w}(i \lambda)$ as

$$
\begin{equation*}
d \hat{w}(i \lambda):=W^{-L}(i \lambda) d \hat{y}(i \lambda) \tag{14}
\end{equation*}
$$

the superscript $-L$ denoting left inverse. Then $\{w(t)\}$ is a $r$-dimensional vector Wiener process and $\{y(t)\}$ has the following spectral representation

$$
\begin{equation*}
y(t)=\int_{-\infty}^{+\infty} e^{i \lambda t} W(i \lambda) d \hat{w}(i \lambda) \tag{15}
\end{equation*}
$$

in terms of $\{w(t)\}$.
3. Find a minimal realization of $W(i \lambda)$, i.e. compute constant real matrices $\{A, B, C\}$ with $A$ square $n \times n$, $B$ of dimension $n \times r$ and $C$ of dimension $m \times n$ with $n$ as small as possible,such that

$$
\begin{equation*}
W(i \lambda)=C(i \lambda I-A)^{-1} B \tag{16}
\end{equation*}
$$

Then, corresponding to each spectral factor $W,\{y(t)\}$ admits a Markovian representation of the form

$$
\begin{align*}
d x(t) & =A x(t) d t+B d w(t)  \tag{17}\\
y(t) & =C x(t) \tag{18}
\end{align*}
$$

the representation corresponding to $W$ being unique modulo change of basis on the state space and $r \times r$ orthogonal transformations on the Wiener process $w$.

The Markovian Splitting Subspace associated to the representation described by Eqs. (17), (18), namely $\mathbf{X}:=\operatorname{span}\left\{x_{1}(0), \ldots, x_{n}(0)\right\}$ is not necessarily of minimal dimension. For this to happen the spectral factor $W$ must be of "smallest degree" [11], [12]. An intriguing feature of the Markovian representation problem is that even minimal representations are non unique. This nonuniqueness is already present in the spectral factorization problem as it is well known that there are in general many minimal degree analytic spectral factors of a generic spectral density matrix $\hat{\Phi}$. The question of minimality of Markovian Splitting Subspaces is examined in great detail in the two references cited above.

Note that for the Gauss-Markov process $x$ described by the stochastic equation (17), the past histories of the processes $x$ and $w$ coincide, i.e. we have $\mathbf{H}^{-}(x)=$ $\mathrm{H}^{-}(d w)$. This follows since the spectrum of $A$ lies in the left-half plane (which is in turn a consequence of analiticity of the spectral factor $W$ ). So, the subspace $\mathrm{H}^{-}(x)$ is, in the terminology of Lax-Phillips Scattering theory [8], an outgoing subspace for the orthogonal group $\Phi(t)$. It is shown in [11], [12] that outgoing subspaces are essentially in one-to-one correspondence with analytic spectral factors of the density matrix $\hat{\Phi}(i \lambda)$. The "incoming" part of the scattering picture associated to $\mathbf{X}$ comes from a dual construction. In fact we have,

PROPOSImION 4.1 Let X be a (finite dimensional) Markovian Splitting Subspace for the process $\{y(t)\}$.

Then there is a unique pair ( $W, \bar{W}$ ) of (rational) $m \times r$ spectral factors corresponding to $\mathbf{X}$, the first being analytic and the second coanalytic (i.e. analytic on the right half-plane), such that the $r \times r$ unitary matrix function $K(i \lambda)$ defined by

$$
\begin{equation*}
K(i \lambda):=\bar{W}^{-L}(i \lambda) W(i \lambda) \tag{19}
\end{equation*}
$$

is a (rational) inner function.
The random measures $d \hat{w}, d \hat{\bar{w}}$ defined by

$$
\begin{equation*}
d \hat{y}(i \lambda)=W(i \lambda) d \hat{w}=\bar{W}(i \lambda) d \hat{\bar{w}} \tag{20}
\end{equation*}
$$

and hence related by $d \hat{\bar{w}}(i \lambda)=K(i \lambda) d \hat{\omega}(i \lambda)$ are the Fourier transforms of two Wiener processes $w, \bar{w}$ generating, respectively, the past and future histories of the Markov process $x$. The inner function (19) is in fact the scattering matrix corresponding to the scattering pair $\left(\mathbf{H}^{-}(d w), \mathbf{H}^{+}(d \bar{w})\right.$ attached to $\mathbf{X}$.

An obvious corollary of the existence of incomingoutgoing subspaces for the group $\Phi(t)$ (which is in turn equivalent to factorizability conditions like (13) and its "conjugate analytic" cunterpart) is that $\Phi(t)$ is unitarily equivalent to the translation group on the Lebesgue space $L_{r}^{2}:=L^{2}\left(\mathbf{R} ; \mathbf{R}^{r}\right)$. This follows from the translation representation of [8].

## 5 Hypercyclic flows and chaotic Hamiltonian systems

A linear bounded operator $T$ on a separable Hilbert, or, more generally, Banach space, $\mathbf{H}$, is called Hypercyclic if there is a vector $v$ such that the orbit generated by $v,\left\{T^{k} v ; k \geq 0\right\}$ is dense in $\mathbf{H}$. Note that this is much stronger than just requiring the linear span of the orbit to be dense.
It is known that there are many such operators, [4]. Recall that a map (usually on a compact metric space) is called chaotic, see e.g. [1], if it has a dense orbit, a dense set of periodic points and has "sensitive dependence" on initial conditions. A certain type of sensitive dependence on initial conditions is present in all Hypercyclic operators, [4]. Thus a Hypercyclic operator defines a chaotic (discrete-time) dynamical system on $\mathbf{H}$ if it has a dense set of periodic points. D. Herrero, [6], has given a spectral charaterization of a class of Hypercyclic operators which have a dense set of periodic points and show a sort of hyperbolic structure very similar to the one leading to chaos in finite dimensions. We may therefore call such operators chaotic.
It can be proven that topologically mixing operators i.e. operators $T$ such that for every two open sets $U, V \subset \mathbf{H}$ and arbitrary $k_{o}$ there always exist $k \geq k_{0}$ such that $T^{k} U \cap V \neq \emptyset$ have dense orbits (i.e. are Hypercyclic).

Naturally all these definitions and properties have a direct counterpart in continuous-time. A strongly continuous group of bounded linear operators (i.e. a linear flow) $\{\Phi(t) ; t \in \mathbf{R}\}$ on $\mathbf{H}$ is called Hypercyclic if there are
vectors $v$ generating dense orbits $\{\Phi(t) v ; t \in \mathbf{R}\}$ in $\mathbf{H}$. (In this case it is most natural to define orbits relative to the whole time axis.)

It turns out that Hypercyclic flows arise naturally in connection with ergodic stationary Gaussian processes $\{y(t)\}$. In fact, under very mild technical conditions, the shift flow on the sample space of an ergodic Gaussian process is necessarily hypercyclic. In particular, Gaussian processes of the type discussed in section 3 above are purely non-deterministic and hence ergodic. Essentially, in the latter case, the following conditions are needed,

1. The countably additive extension, $\tilde{\mu}$ of the Gussian cylinder measure $\mu$ on $\mathbf{H}$ to the Banach space $\Omega$ has full support, i.e. $\operatorname{supp}(\tilde{\mu})=\Omega$. Since the extension $\tilde{\mu}$ is a Borel measure on $\Omega$ ([7], Thm 4.2), all open subsets of $\Omega$ have positive $\tilde{\mu}$-measure.
2. The extended flow $\tilde{\Phi}(t): \Omega \rightarrow \Omega$ is a strongly continuous group on $\Omega$.

The above guarantee that $\bar{\Phi}(t)$ is hypercyclic on $\Omega$. The proof of this statement (which will be given in full detail elsewhere) is based on the following observations.
For every open set $U$, the set $U_{t \in \mathbf{R}} \tilde{\Phi}(t) U$ is open and $\tilde{\Phi}(t)$-invariant. By ergodicity it must have measure zero or one. Being open, it has obviously measure one and hence it must be the whole space, i.e.

$$
\begin{equation*}
U_{t \in \mathbf{R}} \tilde{\Phi}(t) U=\Omega \tag{21}
\end{equation*}
$$

for all open $U \subset \Omega$. Now (21) in conjunction with strong continuity of the flow $\tilde{\Phi}(t)$ and separability of $\Omega$ permits to prove existence of a dense orbit in $\Omega$.
Recall further that the Hamiltonian flow $\Phi(t)$ of aggregable systems is unitarily equivalent to the translation group on $L_{r}^{2}$. In this case we may as well take $\mathbf{H}=L_{r}^{2}$ and $\Phi(t)$ equal to the translation group $(S(t) \omega)(\tau):=$ $\omega(t+\tau)$ acting on $r$-dimensional functions of time $\omega$. This leads to a variety of possible choices for the extended sample space $\Omega$, which will typically be spaces of continuous functions of time defined on $\mathbf{R}$, admitting a strongly continuous translation group and thus satisfying the two conditions required for hypercyclicity.

## 6 A "canonical" example of Aggregable System

A class of linear distributed-parameter conservative systems admitting aggregate description will be described in this section. The underlying model is a generalization of the so-called Lamb model which was proposed in 1900 as a mechanical model for explaining radiation. See [9],[14],[10].
Consider a semi-infinite string tautly stretched at ten$\operatorname{sion} \tau$, connected at the left-end point, situated at $x=0$, to a lumped conservative mechanical load, for example composed of an arbitrary (but finite) number of point masses and linear springs connected together.

Let $\varphi(t, x)$ be the vertical deflection of the string at distance $x$ from the load and let

$$
\begin{equation*}
v(t, x)=\frac{\partial}{\partial t} \varphi(t, x), \quad f(t, x)=\tau \frac{\partial}{\partial x} \varphi(t, x) \tag{22}
\end{equation*}
$$

be the vertical components of the velocity and of the tension of the string at $x$.
At $x=0$ the pulling force $f_{o}(t):=f(t, 0)$ acts on the mechanical load inducing a constrained motion along the vertical axis with velocity $v_{o}(t):=v(t, 0)$, related to the acting force by the mechanial impedence of the load. Using Laplace transforms we can write

$$
\begin{equation*}
\hat{v}_{o}(s)=Z_{o}(s) \hat{f}_{o}(s) \tag{23}
\end{equation*}
$$

where $Z_{0}(s)$ is the mechanical impedence of the load seen from the connecting point with the string.
The system can be described by the following state equations

$$
\begin{align*}
{\left[\begin{array}{c}
\frac{\partial v}{\partial t} \\
\frac{\partial}{\partial t}
\end{array}\right] } & =\left[\begin{array}{cc}
0 & 1 / \rho \frac{\partial}{\partial x} \\
\tau \frac{\partial}{\partial x} & 0
\end{array}\right]\left[\begin{array}{l}
v \\
f
\end{array}\right]  \tag{24}\\
\dot{x} & =A x+b_{o} f_{o}  \tag{25}\\
v_{o} & =c_{o} x \tag{26}
\end{align*}
$$

where $\rho$ is the density of the string. The last two equations (25) and (26), can be thought as a realization of the mechanical impedence $Z_{o}(s)$ which will then be expressible as

$$
\begin{equation*}
Z_{o}(s)=c_{o}(s I-A)^{-1} b_{o} \tag{27}
\end{equation*}
$$

Note that $Z_{o}(s)$ is a Lossless impedence function. In particular there is no direct feedtrough term in the realization.
We want to model an (observed) output variable of the system which is formed as a linear combination of the state of the load $x$ with perhaps a direct feedtrough term from the pulling force of the string $f_{o}$

$$
\begin{equation*}
y=c x+d f_{o} \tag{28}
\end{equation*}
$$

The string equation (24) is just the wave equation written in vector form. We assume suitable units have been chosen to insure $\tau / \rho=1$ (so that the speed of propagation along the string is one). The evolution of the composite system (24),(25) can be seen as the evolution of a conservative Hamiltonian system $\dot{z}=F z$ with state (phase) vector

$$
z:=\left[\begin{array}{l}
x  \tag{29}\\
v \\
f
\end{array}\right]
$$

taking place in the phase space $\mathbf{H}:=\mathbf{R}^{2 n} \oplus L_{2}^{2}\left(\mathbf{R}_{+}\right)$. This space can be given a Hilbert space structure by introducing the energy norm

$$
\left\|\left[\begin{array}{l}
x  \tag{30}\\
v \\
f
\end{array}\right]\right\|^{2}=1 / 2 x^{T} \Omega x+1 / 2 \int_{0}^{+\infty}\left(\rho v^{2}+f^{2}\right) d x
$$

where $\Omega$ is a symmetric nonnegative matrix representing the total energy (hamiltonian) of the load, a
quadratic form in the state $x$. By choosing $x$ minimally we can always guarantee $\Omega>0$.

The $F$ operator operator corresponding to the dynamical equations (24),(25), (26) is skew-adjoint on its natural domain (of smooth functions satisfying the boundary conditions (23)) and generates an energy preserving (i.e. orthogonal) group on $\mathbf{H}$.

Since the string subsystem obeys the wave equation we can express the displacement in the well known "scattering" form

$$
\begin{equation*}
\varphi(t, x)=a(t+x)+b(t-x) \quad t \in \mathbf{R}, x \geq 0 \tag{31}
\end{equation*}
$$

where the functions $a$ and $b$ are called the incoming and outgoing waves respectively. In the present setup it is actually only the derivatives $a^{\prime}$ and $b^{\prime}$ which will enter the scattering representation of the state vector $(v, f)^{T}$ as determined by the initial data of (vertical) velocity and tension along the string at (say) time zero. In fact, putting $t=0$ we have from(22), (31)

$$
\begin{align*}
& v_{i}(x):=v(0, x)  \tag{32}\\
& f_{i}(x):=f(0, x)=a^{\prime}(x)+b^{\prime}(-x) \\
& a^{\prime}(x)-b^{\prime}(-x)
\end{align*}
$$

Note that this system of equations determines only $a^{\prime}(x)$ for $x \geq 0$ and $b^{\prime}(x)$ for $x \leq 0$. Moreover, for arbitrary initial data $\left(v_{i}, f_{i}\right)^{T}$ in $L_{2}^{2}\left(\mathbf{R}_{+}\right)$the restrictions $\left.a^{\prime}\right|_{x \geq 0}$ and $\left.b^{\prime}\right|_{x \leq 0}$ are essentially "free variables" in $L^{2}\left(\mathbf{R}_{+}\right)$and $L^{2}\left(\mathbf{R}_{-}\right)$, respectively.

Now, we shall see that a reinterpretation of an idea of Scattering Theory, [8], leads to a mathematical descriptions of the boundary variables $v_{o}(t)$ and $f_{o}(t)$ by linearfinite dimensional models driven by free $L^{2}$-input variables. These linear models have similar structural properties to those of stochastic models driven by white noise processes.

The procedure starts with the identities

$$
\begin{align*}
v_{o}(t) & =a^{\prime}(t)+b^{\prime}(t)  \tag{33}\\
f_{o}(t) & =a^{\prime}(t)-b^{\prime}(t) \tag{34}
\end{align*}
$$

and then uses the "steady-state" boundary condition at $x=0$, relating $v_{o}(t)$ and $f_{o}(t)$ specified by (23). It is immediate to check that (34) and (34) are both interpretable as output feedback laws on the mechanical load system ( $A, b_{o}, c_{o}$ ) i.e. they can be written

$$
\begin{align*}
& f_{o}(t)=-v_{o}(t)+2 a^{\prime}(t)  \tag{35}\\
& f_{o}(t)=v_{o}(t)-2 b^{\prime}(t) \tag{36}
\end{align*}
$$

and it follows then that (36) and, respectively, (36), after substitution of (26) and combining with (28), yield the following pair of representations of the "output signal" $y^{1}$

$$
\begin{align*}
\dot{x}(t) & =\left(A-b_{o} c_{o}\right) x(t)+2 b_{o} a^{\prime}(t)  \tag{37}\\
y(t) & =\left[c-d c_{o}\right] x(t)+2 a^{\prime}(t) \tag{38}
\end{align*}
$$

[^0]and, respectively
\[

$$
\begin{align*}
\dot{x}(t) & =\left(A+b_{o} c_{o}\right) x(t)-2 b_{o} b^{\prime}(t)  \tag{39}\\
y(t) & =\left[c+d c_{o}\right] x(t)-2 b^{\prime}(t) \tag{40}
\end{align*}
$$
\]

The above calculation shows that any output $y$ of the system, of the type (28) admits a bona fide ForwardBackward pair of representations in the spirit of stochastic realization of stationary processes [11]. In fact,

THEOREM 6.1 Assume the realization (27) is minimal. Then the representations (37) and (39), respectively, are asymptotically stable and antistable, in fact

$$
\begin{equation*}
\Re \lambda\left(A-b_{o} c_{o}\right)<0, \quad \sigma\left(A+b_{o} c_{o}\right)=-\sigma\left(A-b_{o} c_{o}\right) \tag{41}
\end{equation*}
$$

the symbol $\sigma(A)$ denoting the spectrum of the matrix $A$.
Moreover the two representations are related by a "change of white noise input" formula of the type $\hat{b}^{\prime}=$ $K(s) \hat{a}^{\prime}$ with $K(s)$ an inner function. In fact $K(s)$ is precisely the scattering function associated to the boundary condition (23), i.e.

$$
\begin{equation*}
K(s)=\frac{Z_{0}(s)-1}{Z_{0}(s)+1} \tag{42}
\end{equation*}
$$

This generalizes a similar result presented in[18]. Moreover the construction can be reversed. Starting with a process $y$ of assigned rational spectral density $\Phi$, we can select an analytic-coanalytic pair of spectral factors and the relative scattering matrix $K$ as in (19) and then form the lossless impedence $Z_{o}$ solving (42). The connection of a lossless load of impedence $Z_{0}$ to a lossless infinitely long string will then generate a process $y$ with the given spectrum.

Note that, since the input function $a^{\prime}$ is determined by the initial conditions of the string only on positive half lines $\left\{t \geq t_{o}\right\}$, the Forward representation (37) can only represent the time evolution of $y(t)$ on positive half lines $\left\{t \geq t_{o}\right\}$ of the time axis. The evolution of $y(t)$ Backwards in time is governed by a different model (39) which does not correspond to the trivial change of direction of time transformation $t \rightarrow-t$ on (37). Hence the temporal evolution of the variable $y(t)$ is irreversible in time, a truly stochastic phenomenon which has no counterpart in finite dimensional deterministic systems.

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[^0]:    ${ }^{1}$ Of course we can obtain analogous representations also for the tension $f_{o}$ and velocity $v_{o}$

