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A new approach to circulant band extension

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***Abstract*— The circulant band-extension problem has been object of intense study in recent years which have led to a solution in terms of optimization of an Entropy-like functional. It is shown here that the problem can also be solved in terms of a special kind of matrix spectral factorization. The extension can be computed via a circulant analog of the matrix Levinson-Whittle algorithm and by solving a two point boundary value problem.**

I. Stationary periodic processes and block-circulant matrices

All random variables in this paper have zero mean and finite variance. Stationarity is understood in the weak sense. Random elements are denoted by lower case boldface letters while upper case boldface symbols are for block-matrices. Normally the blocks are of dimension $m \times m$ so an $N \times N$ block matrix will be actually of dimension $Nm \times Nm$. In particular, a $N \times N$ *block-circulant* matrix has a block-Toeplitz structure of the following kind

$$\mathbf{C} = \begin{bmatrix} C_0 & C_{N-1} & \dots & \dots & C_1 \\ C_1 & C_0 & C_{N-1} & \dots & \dots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & C_{N-1} \\ C_{N-1} & C_{N-2} & \dots & C_1 & C_0 \end{bmatrix}.$$

where $C_k \in \mathbb{R}^{m\times m}$. A block-circulant matrix $\mathbf{C}$ is fully specified by its first block-row (or column). It will be denoted by

$$\mathbf{C} = \mathrm{Circ}\{C_0, C_1, \dots, C_{N-1}\}. \tag{I.1}$$

A vector-valued random process $\mathbf{y}$ defined on a finite interval, say $[\,t_1, t_2\,]$ of the integer line $\mathbb{Z}$, is written as a column vector by listing its components with older variables in descending order; i.e. as $\mathbf{y}^\top = \begin{bmatrix}\mathbf{y}(t_2)^\top & \dots & \mathbf{y}(t_1)^\top\end{bmatrix}$. It has been shown [Carli et al.(2011)Carli, Ferrante, Pavon, and Picci] that stationary processes defined on a finite interval can be extended to a larger interval of finite length say $[-N+1,\, N\,]$ where they admit a description as periodic processes or as skew-periodic processes [Carli et al.(2011)Carli, Ferrante, Pavon, and Picci], [Levy et al.(1990)Levy, Frezza, and Krener]. In this paper we shall just deal with the periodic extension. It is important to keep in mind that the covariance matrix of a periodic process needs to have a special structure. We quote from [Carli et al.(2011)Carli, Ferrante, Pavon, and Picci] the following basic fact.

Theorem 1: An m-dimensional vector stochastic process $\mathbf{y}$ on a finite interval $[-N+1,\, N\,]$ is the restriction to the same interval of a wide-sense stationary, periodic process $\tilde{\mathbf{y}}$ of period $2N$ defined on $\mathbb{Z}$, if and only if its $2N \times 2N$ block covariance matrix $\mathbf{\Sigma}$ is symmetric block-circulant.

Any periodic process of period $2N$ can be imagined as being defined on the finite group $\mathbb{Z}_{2N}$ of the integers mod $2N$ and all the analysis of these objects can be carried on in this finite time setting.

Examples of periodic processes are the reciprocal processes of finite order n introduced in [Carli et al.(2011)Carli, Ferrante, Pavon, and Picci] [1], also called *n-reciprocal* for short, which generalize the ordinary reciprocal processes which are just of order 1. White noise on a finite interval is a reciprocal process of order zero with covariance $\mathbf{I}_{2N}$ which is trivially (block-) circulant. Let $\mathbf{\Sigma}$ be the $2N \times 2N$ positive definite covariance matrix of a vector process $\mathbf{x}$ which is reciprocal of order n. Then $\mathbf{\Sigma}$ must have a block-banded inverse of badwidth n

$$\mathbf{M} := \mathbf{\Sigma}^{-1} = \mathrm{Circ}\{M_0, \dots, M_n,\, 0, \dots, 0,\, M_{-n}, \dots, M_{-1}\} \tag{I.2}$$

where by symmetry

$$M_{-k} = M_k^\top\,, \qquad k = 1, \dots, n\,. \tag{I.3}$$

These matrices provide a dynamical model of $\mathbf{x}$ which is a bilateral autoregression (AR) of order n

$$\sum_{k=-n}^{n} M_k\, \mathbf{x}(t-k) \,=\, \mathbf{e}(t)\,, \qquad t \in \mathbb{Z}_{2N} \tag{I.4}$$

The solution of this difference equation is completely specified by assigning $2n$ boundary conditions. Because of periodicity we shall impose cyclic boundary conditions at the $2n$ endpoints of the interval. This is automatic in the circulant matrix formalism which will be used throughout. Introducing column vectors with $2N$ blocks, the model (I.4) together with the cyclic boundary conditions can be written compactly as

$$\mathbf{M}\,\mathbf{x} = \mathbf{e} \tag{I.5}$$

where $\mathbf{M}$ is the block-circulant matrix (I.2) and the *normalized conjugate process* $\mathbf{e} = \{\mathbf{e}(t)\}$ (originally called *double sided innovation* by [Masani(1960)]) satisfies the orthogonality relation

$$\mathbb{E}\,\mathbf{x}\,\mathbf{e}^\top \,=\, \mathbf{I}_{2N}\,. \tag{I.6}$$

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[1] The *order* of a vector reciprocal process is the number of lags appearing in its bilateral difference equation model.

It is easy to show that $\mathbf{M}$ is actually the covariance matrix of the normalized conjugate process $\mathbf{e}$. For multiplying (I.6) from the right by $\mathbf{e}^\top$ and taking expectations, one gets $\mathbb{E}\{\mathbf{e}\mathbf{e}^\top\} = \mathbf{M}\,\mathbb{E}\{\mathbf{x}\mathbf{e}^\top\}$ which obviously yields

$$\operatorname{Var}\{\mathbf{e}\} = \mathbf{M} \tag{I.7}$$

and, in particular, $\operatorname{Var}\{\mathbf{e}(t)\} = M_0$.

II. Preliminaries on finite harmonic analysis

Let $\zeta_1 := e^{i\Delta}$ be the primitive $2N$-th root of unity; i.e., $\Delta = \pi/N$, and define the discrete variable ζ taking the $2N$ values $\zeta_k \equiv \zeta_1^k = e^{i\Delta k}$; $k = -N+1, \dots, 0, \dots, N$ running counterclockwise on the discrete unit circle $\mathbb{T}_{2N}$. In particular, we have $\zeta_{-k} = \overline{\zeta_k}$ (complex conjugate).

The discrete Fourier transform $\mathcal{F}$ maps a finite signal $g = \{g_k;\, k = -N+1, \dots, N\}$, into a sequence of complex numbers

$$\hat{g}(\zeta_j) := \sum_{k=-N+1}^{N} g_k \zeta_j^{-k}\,, \qquad j = -N+1, -N+2, \dots, N. \tag{II.1}$$

and the signal g can be recovered from its DFT $\hat{g}$ by the formula

$$g_k = \sum_{j=-N+1}^{N} \zeta_j^k \hat{g}(\zeta_j) \frac{\Delta}{2\pi}\,, \quad k = -N+1, -N+2, \dots, N, \tag{II.2}$$

where $\frac{\Delta}{2\pi} = \frac{1}{2N}$ plays the role of a uniform discrete measure $d\nu$ with total mass one on the discrete unit circle $\mathbb{T}_{2N}$. The map $\mathcal{F}$ is in fact unitary. If $\hat{f}$, $\hat{g}$ are the DFT of $\{f_k\}$, $\{g_k\}$, then

$$\begin{aligned}\sum_{k=-N+1}^{N} f_k g_k &= \sum_{k=-N+1}^{N} \hat{f}(\zeta_k)\hat{g}(\zeta_{-k}) \frac{1}{2N} \\ &= \int_{-\pi}^{\pi} \hat{f}(e^{i\theta})\hat{g}(e^{i\theta})^* d\nu(\theta)\end{aligned} \tag{II.3}$$

which is *Plancherel's Theorem* for DFT.

A. Spectral representation of periodic stationary stochastic processes

Let $\mathbf{y}$ be a zero-mean stationary process defined on a finite interval $[-N+1,\, N]$ of the integer line $\mathbb{Z}$ and extended to all of $\mathbb{Z}$ as a periodic stationary process with period $2N$. Let $\Sigma_{-N+1}, \Sigma_{-N+2}, \dots, \Sigma_N$ be the covariance lags $\Sigma_k := \mathbb{E}\{\mathbf{y}(t+k)\mathbf{y}(t)^\top\}$, so that the discrete Fourier transformation of Σ,

$$\Phi(\zeta_j) := \sum_{k=-N+1}^{N} \Sigma_k\, \zeta_j^{-k}\,, \qquad j = -N+1, \dots, N, \tag{II.4}$$

is a real-valued positive function of ζ, called by analogy, the *spectral density* of the process $\mathbf{y}$. Then, as seen from (II.2) and (II.2),

$$\Sigma_k = \sum_{j=-N+1}^{N} \zeta_j^k \Phi(\zeta_j) \frac{\Delta}{2\pi} = \int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta}) d\nu(\theta)\,, \tag{II.5}$$

for $k = -N+1, \dots, N$. In fact, let

$$\hat{\mathbf{y}}(\zeta_k) := \sum_{t=-N+1}^{N} \mathbf{y}(t)\zeta_k^{-t}, \quad k = -N+1, \dots, N, \tag{II.6}$$

be the discrete Fourier transformation of the process $\mathbf{y}$. The random variables (II.6) turn out to be uncorrelated, and

$$\frac{1}{2N}\mathbb{E}\{\hat{\mathbf{y}}(\zeta_k)\hat{\mathbf{y}}(\zeta_\ell)^*\} = \Phi(\zeta_k)\delta_{k\ell} \tag{II.7}$$

which leads to a spectral representation of $\mathbf{y}$ analogous to the usual one for stationary processes on $\mathbb{Z}$, namely

$$\mathbf{y}(t) = \sum_{k=-N+1}^{N} \zeta_k^t\, \hat{\mathbf{y}}(\zeta_k) \frac{1}{2N} = \int_{-\pi}^{\pi} e^{it\theta} d\hat{\mathbf{y}}(\theta), \tag{II.8}$$

where $d\hat{\mathbf{y}}(\theta) := \hat{\mathbf{y}}(e^{i\theta})d\nu(\theta)$.

Any block circulant matrix $\mathbf{M}$ can be represented in the form,

$$\mathbf{M} = \sum_{k=-N+1}^{N} M_k \mathbf{S}^{-k}, \tag{II.9}$$

where $\mathbf{S}$ is the nonsingular $2N \times 2N$ cyclic shift matrix,

$$\mathbf{S} := \begin{bmatrix} 0 & I & 0 & 0 & \dots & 0 \\ 0 & 0 & I & 0 & \dots & 0 \\ 0 & 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{II.10}$$

and the matrix products in (II.9) are Kronecker products.

Now, from (II.1) we get, for an arbitrary $\zeta_k \in \mathbb{T}_{2N}$,

$$\zeta_k \hat{f}(\zeta_k) = \sum_{\tau=-N+1}^{N} f(\tau)\zeta_k^{-\tau+1} = \sum_{\tau=-N+1}^{N} [Sf](\tau)\zeta_k^{-\tau}$$

so that the operator of multiplication by ζ on the Fourier transform of sequences f of length $2N$ corresponds to the action of the cyclic left shift on the vectorized signal f; i.e.

$$\zeta\, \hat{f}(\zeta) \,=\, \mathfrak{F}(Sf)(\zeta)\,, \qquad \zeta \in \mathbb{T}_{2N}\,. \tag{II.11}$$

Any block-circulant matrix can be represented as a polynomial in the shift whereby the action of $\mathbf{M}$ on a vectorized signal f is a combination of shifted versions of the signal

$$\mathbf{M}f = \sum_{k=-N+1}^{N} M_k\,(S^{-k}\, f)$$

and in the Fourier domain one gets,

$$\mathfrak{F}(\mathbf{M}f)(\zeta) = \left(\sum_{k=-N+1}^{N} M_k\, \zeta^{-k}\right) \hat{f}(\zeta)\,. \tag{II.12}$$

Hence, multiplication of a (vectorized) signal by a block-circulant matrix corresponds to pointwise multiplication of its Fourier transform by a matrix-valued polynomial in ζ. The polynomial matrix

$$M(\zeta) = \sum_{k=-N+1}^{N} M_k\, \zeta^{-k} \tag{II.13}$$

is called the *symbol of the circulant*. For example, the symbol of $\mathbf{S}$ is just $I\zeta$ where I is the $m\times m$ identity matrix. Note that the word "polynomial" here is used for linear combination of both positive and negative powers of ζ. When $\mathbf{M}$ is symmetric $M_{-k} = M_k^\top$ the polynomial is also symmetric, namely

$$M(\zeta^{-1}) = M(\zeta)^\top\,. \tag{II.14}$$

For example, the right shift S^{-3} has a block-circulant representative where all M_k's are zero except for M_3 which is equal to the identity. Its symbol is therefore $I\zeta^{-3}$.

A matrix polynomial of degree n involving only *negative powers* of ζ

$$M(\zeta) = \sum_{k=0}^{n} M_k\,\zeta^{-k}, \qquad n < N/2 \tag{II.15}$$

is the symbol of a *block-upper banded* circulant matrix of bandwidth n and an analogous characterization holds for polynomials involving only positive powers of ζ as representatives of *block-lower banded* circulant matrices of bandwidth n.

Theorem 2 (Circulant convolution theorem): Let $\mathbf{A}$, $\mathbf{B}$ be $N\times N$-block circulants of the same size with blocks of dimension $m\times m$. Then the sum and the product $\mathbf{C} := \mathbf{A}\,\mathbf{B}$ are also block circulant and the symbol of $\mathbf{C}$ is the product of the symbols of $\mathbf{A}$ and $\mathbf{B}$. In fact, the DFT is an *algebra homomorphism* of the set of block-circulant matrices with N blocks onto the $m\times m$ matrix polynomials of degree $N-1$ in the variable $\zeta\in\mathbb{T}_N$.

Proof: The first block column of $\mathbf{C}$ is just the circulant convolution of the first block column of $\mathbf{A}$ and the first block column of $\mathbf{B}$. Hence $C(\zeta) = A(\zeta)\,B(\zeta)$. ■

In particular, if $\mathbf{A}$ is non singular; i.e. there is a matrix, $\mathbf{B}$, necessarily block-circulant, such that

$$\mathbf{A}\,\mathbf{B} = \mathbf{I}$$

then, since the symbol of $\mathbf{I}$ is the matrix function identically equal to the $m\times m$ identity matrix, by the circulant convolution theorem we have

$$A(\zeta)\,B(\zeta) = I\,, \qquad \zeta\in\mathbb{T}_N$$

so that

Corollary 3: The symbol of the inverse $\mathbf{A}^{-1}$ is equal to the inverse of the symbol of $\mathbf{A}$.

In other words, if $A(\zeta) = \sum_{-N+1}^{N} A_k\,\zeta^{-k}$ is the symbol of $\mathbf{A}$, then the matrix polynomial $B(\zeta)$ with values

$$B(\zeta_k) := \left[\sum_{-N+1}^{N} A_h\,\zeta_k^{-h}\right]^{-1}$$

is the symbol of $\mathbf{A}^{-1}$. The polynomial with these values is unique and is their Lagrange interpolating polynomial. The coefficients of $B(\zeta)$ can therefore be computed by Lagrange interpolation.

The corollary 3 is an instance of a more general *spectral mapping theorem* [Dunford and Schwartz(1958), p. 557] valid in much wider generality (*but not valid for Toeplitz matrices*). The theorem below is a matrix generalization of a well-known result on diagonalization of scalar-circulant matrices by the Fourier map [Tee(2005)].

Theorem 4 (Spectral decomposition of block-circulants): The spectrum of a block-circulant $\mathbf{M}$ is the union of the spectra of the DFT symbol $M(\zeta)$; i.e. is the union of the spectra of the $m\times m$ matrices $\{M(\zeta_k)\,;\, k = -N+1,\dots,-1,0,1,\dots,N\}$. If $\mathbf{M}$ is symmetric then $M(1/\zeta_k)^\top = M(\zeta_k)$; i.e. each $M(\zeta_k)$ is Hermitian. Hence its spectrum and the spectrum of $\mathbf{M}$ are real.

III. Factorization and unilateral AR representations

The problem of representing periodic processes by unilateral recursions leads to the problem of *causal* or *anticausal* factorization of a (block-) circulant matrix. This problem is discussed next.

A N-blocks *upper banded block-circulant* matrix of bandwidth n, $\mathbf{U} = \mathrm{Circ}\{U_0, 0\,,\dots,0\,,U_n\,U_{n-1}\,\dots,U_1\}$ has the following structure

$$\begin{bmatrix} U_0 & U_1 & \dots & U_{n-1} & U_n & 0 & \dots & 0\\ 0 & U_0 & U_1 & \ddots & U_{n-1} & U_n & \dots & 0\\ \vdots & & \ddots & \ddots & & \ddots & \ddots & \vdots\\ 0 & 0 & \dots & U_0 & U_1 & \dots & U_{n-1} & U_n\\ U_n & 0 & \dots & 0 & U_0 & & \dots & U_{n-1}\\ U_{n-1} & U_n & \dots & \dots & 0 & U_0 & & \dots\\ \vdots & & \ddots & & & \ddots & \ddots & U_1\\ U_1 & U_2 & \dots & U_n & \dots & \dots & 0 & U_0 \end{bmatrix} \tag{III.1}$$

where the $n+1$-st block U_n in the list is nonzero Dually, a N-blocks *lower banded block-circulant* matrix of bandwidth n, has the representation

$$\mathbf{L} = \mathrm{Circ}\{L_0, 0,\dots,0, L_n, L_{n-1},\dots,L_1\}.$$

Note that upper or lower banded block-circulant matrices of bandwidth $N-1$; i.e. with $U_{N-1}\neq 0$, are just full block-circulant matrices. In this case the concept of upper or lower banded block-circulant degenerates. $\mathbf{U}$ or $\mathbf{L}$ are said to be *normalized* when $U_0 = I$ or $L_0 = I$. Later in this paper the number of blocks will actually be $2N$ according to the conventions established in the previous section. The modifications to be introduced to adapt to this situation are obvious.

Note that while the inverse of a upper or lower triangular matrix is still upper or lower triangular, the inverse of a upper (lower) banded block-circulant matrix is in general neither upper nor lower banded.

Consider a stochastic process $\mathbf{y}$ on $\mathbb{Z}_{2N}$ described by the unilateral AR model

$$\sum_{k=0}^{n} A_k\mathbf{y}(t-k) = \mathbf{w}(t)\,, \qquad t\in\mathbb{Z}_{2N}\,. \tag{III.2}$$

where $\mathbf{w}$ is stationary white noise on $\mathbb{Z}_{2N}$; that is

$$\mathbb{E}\,\{\mathbf{w}\mathbf{w}^\top\} = \mathrm{diag}\{D,\dots,D\}\,; \quad D>0\,.$$

Lemma 5: If the recursion (III.2) is associated to the cyclic boundary values

$$\mathbf{y}(-n) = \mathbf{y}(N-n)\,, \ldots, \mathbf{y}(0) = \mathbf{y}(N) \tag{III.3}$$

then the process $\mathbf{y}$ is reciprocal of order n.
The covariance matrix of a reciprocal process of order n has a symbol whose inverse is a bilateral polynomial matrix of degree n. Therefore computing unilateral AR representations leads to a particular kind of polynomial spectral factorization of the symbol. Before discussing this point we shall address spectral factorization in general terms.

Since the frequency samples $\Phi(\zeta_k)$ of a spectral density matrix (II.4) are Hermitian positive definite, they admit a factorization

$$\Phi(\zeta_k) = \hat{W}(\zeta_k)\hat{W}(\zeta_k)^*\,, \qquad k = -N+1, \ldots, N \tag{III.4}$$

where $\hat{W}(\zeta_k)$ can be chosen square and invertible whenever $\Phi(\zeta_k)$ is such. Since $\Phi(\zeta_{-k}) = \Phi(\zeta_k)^\top$, we may also choose $\hat{W}(\zeta_{-k})$ to be the complex conjugate of $\hat{W}(\zeta_k)$. The inverse Fourier transform of the sequence $\{\hat{W}(\zeta_k)\}$,

$$W_k := \sum_{j=-N+1}^{N} \hat{W}(\zeta_j)\,\zeta_j^k\,\frac{\Delta}{2\pi}\,,$$

is therefore a sequence of real matrices whose "Zeta-transform"

$$W(\zeta) := \sum_{k=-N+1}^{N} W_k\,\zeta^{-k} \tag{III.5}$$

takes by construction, the values $\hat{W}(\zeta_j)$ at the frequencies ζ_j; i.e.

$$W(\zeta_j) := \hat{W}(\zeta_j)\,, \qquad j = -N+1, \ldots, N\,. \tag{III.6}$$

Hence by (III.4) the function $W(\zeta)$ satisfies the Spectral Factorization equation

$$\Phi(\zeta) = W(\zeta)W(\zeta^{-1})^\top\,, \qquad \zeta \in \mathbb{T}_{2N}\,. \tag{III.7}$$

Since $\Phi(\zeta)$ is the symbol of the circulant covariance matrix $\mathbf{\Sigma}$ and in force of the isomorphism of Theorem 2, this equation is equivalent to the factorization of the circulant covariance matrix of the process as

$$\mathbf{\Sigma} = \mathbf{W}\mathbf{W}^\top \tag{III.8}$$

where $\mathbf{W}$ is the block circulant matrix with symbol $W(\zeta)$. Hence spectral factorization of the symbol $\Phi(\zeta)$ is *equivalent* to circulant covariance factorization. Note that this is generally *not true* for the Toeplitz covariance of stationary processes defined on the integer line.
Consider the matrix polynomial $W(z)$ obtained by substituting the discrete variable ζ in (III.5) by the complex variable z. Since for ζ running on the discrete torus $\mathbb{T}_{2N}$ the sample values $W(\zeta_j)$ determine uniquely the matrix coefficients W_k by means of the inverse finite Fourier transform, the matrix polynomial $W(z), z \in \mathbb{C}$ is uniquely determined by the discrete counterpart $W(\zeta)$, $\zeta \in \mathbb{T}_{2N}$ and there is an isomorphic continuous polynomial spectral density $\Psi(z); z \in \mathbb{C}$ such that

$$\Psi(z) = W(z)W(z^{-1})^\top\,, \qquad z \in \mathbb{C}\,. \tag{III.9}$$

which is in fact the spectrum of a stationary (non periodic) process of the Moving Average (MA) type. Hence it follows that the discrete spectral factorization problem (III.7) becomes isomorphic to the ordinary spectral factorization of a matrix polynomial spectrum $\Psi(z), z \in \mathbb{C}$. Even if the two processes have little to do with each other, once the problem (III.9) is solved the discrete factors $W(\zeta)$ can be obtained just by identifying the coefficients of the two matrix polynomials $W(z)$ and $W(\zeta)$. In fact the polynomial spectral factorization (III.9) can be solved by a number of known techniques, see e.g. [Rissanen(1973)]. Therefore we have the following discrete version of a well-known continuous spectral factorization result.

Theorem 6: Every nonsingular spectral density of a full rank reciprocal process on $\mathbb{T}_{2N}$ admits square spectral factors satisfying (III.7). Any such factor is determined modulo right multiplication by an arbitrary collection of (square) unitary matrices $U(\zeta_k)\,;\, k \in \mathbb{T}_{2N}$ such that

$$U(\zeta_k)U(\zeta_k^{-1})^\top = I\,, \qquad k = -N+1, \ldots, N\,.$$

Equivalently, every symmetric positive definite block circulant matrix $\mathbf{\Sigma}$ admits a block-circulant factorization

$$\mathbf{\Sigma} = \mathbf{W}\,\mathbf{W}^\top\,, \tag{III.10}$$

where each block-circulant factor $\mathbf{W}$ is unique modulo right multiplication by a *unitary block-circulant* matrix $\mathbf{U}$ which satisfies $\mathbf{U}\,\mathbf{U}^\top = \mathbf{I}$.
If the process $\mathbf{y}$ is full-rank, namely $\mathbf{\Sigma} > 0$, the square factor $\mathbf{W}$ in (III.10) must be invertible. Hence we may define the random process $\mathbf{w}$ on $\mathbb{Z}_{2N}$ by setting

$$\mathbf{w} := \mathbf{W}^{-1}\,\mathbf{y}$$

This process is (periodic) *white noise* since

$$\mathbb{E}\,\mathbf{w}\,\mathbf{w}^\top = \mathbf{W}^{-1}\mathbf{\Sigma}\mathbf{W}^{-\top} = \mathbf{I}$$

therefore whitening and spectral factorization are related exactly as in the standard theory of stationary processes on $\mathbb{Z}$. In particular, from the existence of block-circulant factors of the covariance $\mathbf{\Sigma}$ in Theorem 6, it follows that *every stationary periodic process can be whitened*; i.e. there always exist a sequence $\{W_k\,;\, k = -N+1, \ldots, N\}$ such that the process

$$\mathbf{w}(t) = \sum_{k=-N+1}^{N} W_k\,\mathbf{y}(t-k)\,, \qquad t \in \mathbb{Z}_{2N}$$

is white noise.

Matrix symbols whose coefficients of the positive powers of ζ are equal to zero are called *Analytic*. These factors are of the form

$$A(\zeta) = \sum_{k=0}^{N} A_k\,\zeta^{-k}$$

while *coanalytic* symbols have all coefficients of the negative powers equal to zero. They have an obvious dual expression. Analytic symbols correspond to block-upper banded circulant matrices and, dually, coanalytic symbols correspond to block-lower banded circulant matrices (naturally, "lower banded and upper banded" circulant matrices are not exactly lower banded and upper banded in the ordinary sense because of the circulant structure). A relevant question in this respect is the following.

Problem 7: Characterize the spectral densities which admit analytic (or coanalytic) spectral factors. In other words, under what conditions does a spectral density $\Phi(\zeta)$ admit matrix functions

$$A(\zeta) := \sum_{k=0}^{n} A_k \zeta^{-k}, \qquad B(\zeta) := \sum_{k=0}^{n} B_k \zeta^{k} \tag{III.11}$$

with $n < N$, such that

$$\Phi(\zeta) = A(\zeta)A(\zeta^{-1})^\top, \qquad \Phi(\zeta) = B(\zeta)B(\zeta^{-1})^\top. \tag{III.12}$$

Essentially the same argument used in the continuous case yields the following result.

Theorem 8: A positive definite Hermitian block-circulant matrix $\mathbf{M}$ admits lower or upper banded factors of bandwidth n if and only if it is bilaterally banded of bandwidth n.

Therefore a periodic process $\mathbf{y}$ with a block-circulant covariance matrix $\boldsymbol{\Sigma}$ admits causal or anticausal whitening filters. Define

$$\mathbf{w} := \mathbf{A}^{-1}\mathbf{y}, \qquad \bar{\mathbf{w}} := \mathbf{B}^{-1}\mathbf{y} \tag{III.13}$$

then both $\mathbf{w}$ and $\bar{\mathbf{w}}$ are white noise processes since their variances are block-diagonal; that is,

$$\mathbb{E}\,\mathbf{w}\mathbf{w}^\top = \mathbf{I}, \qquad \mathbb{E}\,\bar{\mathbf{w}}\bar{\mathbf{w}}^\top = \mathbf{I}$$

and hence any process $\mathbf{y}$ having a banded covariance admits the representations $\mathbf{y} = \mathbf{A}\mathbf{w}$ and $\mathbf{y} = \mathbf{B}\bar{\mathbf{w}}$ which can be written as causal and anticausal Moving Average (MA)-type models which may formally be written as $\mathbf{y}(t) = A(\zeta)\mathbf{w}(t)$ and $\mathbf{y}(t) = B(\zeta)\bar{\mathbf{w}}(t)$. As we shall see in a moment this result applies in particular to the conjugate of a reciprocal process. Below is the main result of this section. The statement appeared already in [Carli and Picci(2010)]. Due to length the proof is skipped; it can be found in the forthcoming paper [Picci(2015)].

Theorem 9: Every full rank reciprocal process of order n can be described by a n-th order unilateral AR-type model (either causal or anticausal) with cyclic initial (or terminal) conditions. Conversely, every stationary full-rank process described by a unilateral AR-type model of order $n < N$, can also be described by a bilateral model of the same order. Hence the class of full-rank processes of order n described by unilateral AR-type models on $\mathbb{Z}_{2N}$ coincides with the class of reciprocal processes of order n.

This equivalence has been briefly addressed in [Levy et al.(1990)Levy, Frezza, and Krener, Sect. V] for non stationary reciprocal processes of order one and as a limit case for stationary processes on a doubly infinite interval, in the paper [Levy(1992)]. We have generalized these representation results to reciprocal processes of arbitrary order n.

IV. Circulant Band extension

In practice the data of an estimation problem are necessarily finite and may just consist of (estimates of) a finite sequence of $n+1$ covariance matrices $\{C_k\}$ which we shall collect in a block-Toeplitz matrix

$$T_n := \begin{bmatrix} C_0 & C_1 & \dots & C_n \\ C_1^\top & C_0 & C_1 & \dots \\ \dots & & & \dots \\ C_n^\top & \dots & C_1^\top & C_0 \end{bmatrix}, \tag{IV.1}$$

assumed to be positive definite. The problem we shall discuss in this section, is how to compute bilateral AR representations of finite order of the type (I.4), starting from the initial covariance data (IV.1) of the process. This realization problem can be phrased as a *moment problem with complexity constraints* and has been discussed in detail in [Carli et al.(2011)Carli, Ferrante, Pavon, and Picci]. Here we shall propose an approach which is not based on optimization but rather on spectral factorization.

Consider a unilateral n-th order AR model (III.2) associated with cyclic boundary conditions on $[-N+1, N]$, namely $\mathbf{y}(-N) = \mathbf{y}(N), \dots, \mathbf{y}(-N+n) = \mathbf{y}(N-n)$ which can be written in matrix form as $\mathbf{A}\mathbf{y} = \mathbf{w}$ where $\mathbf{A} = \mathrm{Circ}\{I, A_1, \dots, A_n, 0, \dots, 0\}$ is an invertible block-circulant matrix and $\mathbf{w}$ is white noise of covariance $\mathbf{D}$. The covariance of $\mathbf{y}$ must admit the factorization

$$\boldsymbol{\Sigma} = \mathbb{E}\,\mathbf{y}\mathbf{y}^\top = \mathbf{A}^{-1}\mathbf{D}\mathbf{A}^{-\top}$$

which, rewritten in terms of symbols reads

$$\Phi(\zeta) = A(\zeta)^{-1} D A(\zeta^{-1})^{-\top}. \tag{IV.2}$$

This formula is reminiscent of the well-known formula of the classical maximum entropy Toeplitz extension of the block-Toeplitz matrix of covariance data (IV.1) which we shall very briefly review in the following paragraphs.

Recall that the (Toeplitz) maximum entropy extension of T_n is the infinite covariance matrix of a process $\mathbf{y}$ admitting an autoregressive description of order n of the form $L_n(z^{-1})\mathbf{y}(t) = \mathbf{w}(t)$ where $L_n(z^{-1})$ is the n-th Levinson-Whittle polynomial associated to the data matrix (IV.1) and $\mathbf{w}$ is a white noise on $\mathbb{Z}$ having the same variance matrix of the forward innovation process of memory n, denoted $\mathbf{e}_n(t)$. In fact this extension is the unique covariance extension having a *banded inverse* of (symmetric) bandwidth n.

Note that the so-called Toeplitz extension is not really a Toeplitz matrix since its elements vary along the diagonals. In particular, in the main diagonal are listed the variance matrices of the sequance $\mathbf{e}_1(t), \dots, \mathbf{e}_n(t), ..$ of the finite memory innovation processes which are generally different. For more details see for example the survey [Carli and Picci(2010)] presented at the 2010 MTNS conference.

Instead of the classical banded Toeplitz extension it will be more convenient to consider the *banded Laurent extension problem* of T_n, as this avoids dealing with the "transient" phenomenon alluded at before.

Now a *block-Laurent* matrix $\mathbf{\Lambda}$ is a doubly infinite array of matrix blocks $[\,\Lambda_{i,j}\,]_{i,j\in\mathbb{Z}}$, having a Toeplitz structure that is $\Lambda_{i,j} = \Lambda_{j-i}$ for all $i, j \in \mathbb{Z}$. Every south-east principal corner of a Laurent matrix is Toeplitz. The symbol of $\mathbf{\Lambda}$ is

$$\Psi(z) := \sum_{k=-\infty}^{+\infty} \Lambda_k z^{-k}\,, \qquad z \in \mathbb{C}$$

which we shall assume to be well defined and invertible a.e.[2]. We shall say that $\mathbf{\Lambda}$ is positive definite if it is symmetric, the principal minors of any order are positive and convene to say that $\mathbf{\Lambda}$ has a *banded inverse of bandwidth* n if all elements of $\mathbf{\Lambda}^{-1}$ external to the two n-th diagonals, are zero.

Problem 10: Given a positive definite data matrix (IV.1) find an infinite positive definite symmetric block-Laurent matrix $\mathbf{\Lambda}$ satisfying the moment conditions

$$\Lambda_k = \int_{-\pi}^{\pi} \Psi(e^{j\theta}) e^{jk\theta} \frac{d\theta}{2\pi} = C_k \qquad k = 0, 1, \dots, n\,, \tag{IV.3}$$

and such that $\mathbf{\Lambda}^{-1}$ is banded of bandwidth n.

This problem can also be restated by saying that the inverse symbol $\Psi(z)^{-1}$ should be a finite symmetric Laurent (i.e. bilateral) polynomial matrix of degree n. It is then immediate that $\mathbf{\Lambda}^{-1}$ will be banded of bandwidth n if and only if $\Psi(z)^{-1}$ admits polynomial spectral factors of degree n. In particular analytic and co-analytic polynomial factors will correspond to doubly infinite upper and lower banded block-Laurent matrix factors.

The solution of Problem 10 is essentially the same as that of the well-known Toeplitz band extension problem of [Dym and Gohberg(1981)].

Theorem 11: There is a unique banded Laurent extension of T_n whose symbol is

$$\Psi_n(z) = L_n(z)^{-1} D_n L_n(z^{-1})^{-\top}\,, \qquad z \in \mathbb{C}\,, \tag{IV.4}$$

where $L_n(z)$ is the n-th Levinson-Whittle polynomial associated to the data matrix (IV.1) and D_n is the variance matrix of the forward, memory n innovation $\mathbf{e}_n(t)$, of the underlying stationary process $\mathbf{y}$.

Proof: Since the Levinson-Whittle polynomials $L_n(z) = I + \sum_{k=1}^{n} L_k z^{-k}$ associated to T_n are analytic, the infinite Laurent matrix $\mathfrak{L}$ with symbol $L_n(z)$ is upper triagular. An arbitrary principal submatrix of $\mathfrak{L}$ has the following structure

$$\mathfrak{L} = \begin{bmatrix} I & L_1 & \dots & L_n & 0 & & \dots & 0 & 0 \\ 0 & I & L_1 & \dots & L_n & 0 & 0 & \dots & \\ \dots & \ddots & \ddots & \ddots & & \ddots & \dots & \ddots & 0 \\ \dots & & 0 & I & L_1 & \dots & L_n & & 0 \\ 0 & 0 & & 0 & \dots & & \dots & \dots & \dots \end{bmatrix}. \tag{IV.5}$$

Each row of this matrix is a whitening filter which applied to the underlying stationary process $\mathbf{y}$, written as a semi-infinite column vector

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}(t)^\top & \mathbf{y}(t-1)^\top & \dots & \mathbf{y}(t-n)^\top & \dots \end{bmatrix}^\top$$

produces the memory n innovation at successive times $t, \dots, t-n, \dots$, so that

$$\begin{bmatrix} \mathbf{e}_n(t) \\ \dots \\ \mathbf{e}_n(t-n) \\ \dots \end{bmatrix} = \mathfrak{L}\mathbf{y}$$

Computing the covariance matrix of both members in this expression we find

$$\mathfrak{L}\mathbf{\Sigma}\mathfrak{L}^\top = D_n \mathbf{I}$$

where $\mathbf{\Sigma}$ is the Toeplitz covariance matrix of $\mathbf{y}$. Since the symbols of the Laurent extension $\mathbf{\Lambda}$ and of $\mathbf{\Sigma}$ are the same, this relation can also be written $L_n(z)\Psi_n(z)L_n(z^{-1})^\top = D_n$, which is equivalent to (IV.4). $\square$

Return now to the circulant band extension problem. It has been shown in [Carli et al.(2011)Carli, Ferrante, Pavon, and Picci] that, for N large enough the data (IV.1) have a unique extension to a symmetric *block circulant* covariance matrix of N blocks having a banded inverse. This unique extension can in fact be computed by solving a variational problem [Carli et al.(2011)Carli, Ferrante, Pavon, and Picci]. Hence, for N large enough and positive Toeplitz data matrix (IV.1), there is a unique process $\mathbf{y}$, reciprocal of order n, defined on $\mathbb{Z}_{2N} \equiv [-N+1,\, N\,]$, whose covariance is a block-circulant extension of T_n. This $N \times N$ blocks covariance has in fact a banded inverse of bandwidth n.

By Theorem 9 this process admits a description by normalized causal (or anticausal) AR models of order n on $\mathbb{Z}_{2N}$ of the form (III.2). The question is how to compute the coefficients. The following argument is inspired by the similarity of the formulas (IV.2) and (IV.4).

V. A CIRCULANT LEVINSON-WHITTLE ALGORITHM

We want to see if there is a circulant analog of the Levinson-Whittle algorithm.

Assume that N is large enough for T_n to admit a circulant extension. Then there is a stationary *n-reciprocal* periodic process $\mathbf{y}$ on $\mathbb{Z}_{2N}$, whose initial covariance lags are the entries of the matrix T_n. Now, inspired by the procedure explained in e.g. [Carli and Picci(2010)], we introduce the *innovation process of finite memory* n of $\mathbf{y}$, defined as

$$\mathbf{e}_n(t) := \mathbf{y}(t) - \hat{\mathbb{E}}\,[\mathbf{y}(t) \mid \mathbf{y}_{[t-n,t\,)}]\,, \tag{V.1}$$

where the time interval is $\mathbb{Z}_{2N}$ with modular arithmetic mod $2N$.

Proposition 12: The process $\mathbf{e}_n(t)$ is the output of a circulant convolution filter depending only on the previous $n+1$ variables of the process

$$\mathbf{e}_n(t) = \sum_{k=0}^{n} A(k)\,\mathbf{y}(t-k)\,, \qquad t \in \mathbb{Z}_{2N} \tag{V.2}$$

[2]Invertibility conditions of infinite Toplitz and Laurent matrices in terms of their symbol are discussed in many places in the literature. See [Hartman and Wintner(1950)] for the first original characterization.

where the coefficients $\{A(k)\}$ are independent of t and are uniquely determined by the initial data (IV.1). They coincide with the coefficients $L_n(k)$ of the n-th Levinson-Whittle polynomial $L_n(z)$ computed from the initial covariance data (IV.1).

Proof: Time invariance follows from the joint stationarity of $\mathbf{e}_n(t)$ and $\{\mathbf{y}(t),\, \mathbf{y}(t-1),\, \mathbf{y}(t-2)\, \ldots,\, \mathbf{y}(t-n)\}$ for all $t \in \mathbb{Z}_{2N}$ which implies that all cross covariances of these random elements do not depend on t. The coefficients $A(k)$ are determined by the orthogonality condition $\mathbf{e}_n(t) \perp \{\mathbf{y}(t-1),\, \mathbf{y}(t-2)\, \ldots,\, \mathbf{y}(t-n)\}$ which must hold for all $t \in \mathbb{Z}_{2N}$. To make formulas easier to read we shall provisionally shift the time axis from $[-N+1,\, N\,]$ to $[1,\, 2N\,]$ as an isomorphic representation of $\mathbb{Z}_{2N}$. Due to stationarity this shift does not change the cross covariances and since the coefficients $\{A(k)\}$ are determined by the cross covariances of the process intervening in the orthogonality condition, they will also remain unchanged. Without loss of generality, for this proof the variable t will run from 1 to $2N$ with arithmetics $\mod 2N$.

For $n < t$, (V.1) is just the same forward innovation of memory n introduced in the standard Toepltz case. Note however that, when $n \geq t$, the subinterval of $[t-n, t\,)$ with nonpositive times gets "folded around" by the arithmetics mod $2N$, so that we have[3]

$$\begin{aligned}
\mathbf{e}_n(1) &:= \mathbf{y}(1) - \hat{\mathbb{E}}\,[\mathbf{y}(1) \mid \mathbf{y}_{[1-n,0\,]}] \\
&= \mathbf{y}(1) - \hat{\mathbb{E}}\,[\mathbf{y}(1) \mid \mathbf{y}(0), \mathbf{y}(-1), \ldots, \mathbf{y}(-n+1)\,] \\
&= \mathbf{y}(1) - \hat{\mathbb{E}}\,[\mathbf{y}(1) \mid \mathbf{y}(2N), \mathbf{y}(2N-1), \ldots, \mathbf{y}(2N-n+1)] \\
\mathbf{e}_n(2) &:= \mathbf{y}(2) - \hat{\mathbb{E}}\,[\mathbf{y}(2) \mid \mathbf{y}_{[2-n,1\,]}] \\
&= \mathbf{y}(2) - \hat{\mathbb{E}}\,[\mathbf{y}(2) \mid \mathbf{y}(1), \mathbf{y}(0), \ldots, \mathbf{y}(-n+2)] \\
&= \mathbf{y}(2) - \hat{\mathbb{E}}\,[\mathbf{y}(2) \mid \mathbf{y}(1), \mathbf{y}(2N), \ldots, \mathbf{y}(2N-n+2)] \\
&\cdots\cdots \\
\mathbf{e}_n(n) &:= \mathbf{y}(n) - \hat{\mathbb{E}}\,[\mathbf{y}(n) \mid \mathbf{y}_{[0,n-1\,]}] \\
&= \mathbf{y}(n) - \hat{\mathbb{E}}\,[\mathbf{y}(n) \mid \mathbf{y}(0), \mathbf{y}(1), \ldots, \mathbf{y}(n-1)\,] \\
&= \mathbf{y}(n) - \hat{\mathbb{E}}\,[\mathbf{y}(n) \mid \mathbf{y}(2N), \mathbf{y}(1), \ldots, \mathbf{y}(n-1)\,]
\end{aligned}$$

and so on, for all $t \in \mathbb{Z}_{2N}$. Rearranging the variables of the process $\mathbf{y}$ into a column vector with the endpoint variable $\mathbf{y}(2N)$ in the last block and $\mathbf{y}(1)$ at the top, the expressions above can be rewritten as a causal filter of the form (V.2) as

$$\begin{aligned}
\mathbf{e}_n(1) &:= \begin{bmatrix} I & 0 & \ldots & 0 & A(n) & A(n-1) & \ldots & A(1) \end{bmatrix}\mathbf{y} \\
\mathbf{e}_n(2) &:= \begin{bmatrix} A(1) & I & 0 & \ldots & 0 & A(n) & A(n-1) & \ldots & A(2) \end{bmatrix}\mathbf{y} \\
&\cdots\cdots \\
\mathbf{e}_n(n) &:= \begin{bmatrix} A(n-1) & A(n-2) & \ldots & I & 0 & \ldots & 0 & A(n) \end{bmatrix}\mathbf{y} \\
\mathbf{e}_n(n+1) &:= \begin{bmatrix} A(n) & A(n-1) & \ldots & A(1) & I & 0 & \ldots & 0 \end{bmatrix}\mathbf{y}\,.
\end{aligned} \tag{V.3}$$

which has a block-circulant structure. Now, enforcing the periodic boundary conditions on $\mathbf{y}$, the orthogonality condition

$$\mathbf{e}_n(1) \perp \{\mathbf{y}(0),\, \mathbf{y}(-1), \ldots, \mathbf{y}(-n+1)\}$$

is the same as $\mathbf{e}_n(1) \perp \{\mathbf{y}(2N),\, \mathbf{y}(2N-1)\, \ldots,\, \mathbf{y}(2N-n+1)\}$. Rearranging the first equation in (V.3), we obtain

$$\begin{bmatrix} I & A(1) & \ldots & A(n-1) & A(n) \end{bmatrix} \times \mathbb{E} \begin{bmatrix} \mathbf{y}(1) \\ \mathbf{y}(2N) \\ \mathbf{y}(2N-1) \\ \vdots \\ \mathbf{y}(2N-n+1) \end{bmatrix} \begin{bmatrix} \mathbf{y}(2N) \\ \mathbf{y}(2N-1) \\ \vdots \\ \mathbf{y}(2N-n+1) \end{bmatrix}^\top = 0 \tag{V.4}$$

On the other hand $\mathbb{E}\,\mathbf{e}_n(1)\mathbf{y}(1)^\top = \mathbb{E}\,\mathbf{e}_n(1)^2 = D_n$, which yields

$$\begin{bmatrix} I & A(1) & \ldots & A(n-1) & A(n) \end{bmatrix} \times \begin{bmatrix} \Sigma_0 & \Sigma_{2N-1}^\top & \Sigma_{2N-2}^\top & \ldots & \Sigma_{2N-n}^\top \\ \Sigma_{2N-1} & \Sigma_0 & \Sigma_1 & \ldots & \Sigma_{n-1} \\ \Sigma_{2N-2} & \Sigma_1^\top & \Sigma_0 & \Sigma_1 & \ldots \\ \ldots & \ddots & & \ddots & \ldots \\ \Sigma_{2N-n} & \ldots & & \Sigma_1^\top & \Sigma_0 \end{bmatrix} = \begin{bmatrix} D_n & 0 & \ldots & 0 \end{bmatrix}$$

Substituting the known boundary values $\Sigma_k = C_k\,;\, \Sigma_{N-k}^\top = C_k\,;\, k = 1, 2, \ldots, n$ imposed by circulant symmetry, one obtains the "circulant" Yule-Walker equation

$$\begin{bmatrix} I & A(1) & \ldots & A(n-1) & A(n) \end{bmatrix} \times \begin{bmatrix} C_0 & C_1 & C_2 & \ldots & C_n \\ C_1^\top & C_0 & C_1 & \ldots & C_{n-1} \\ C_2^\top & C_1^\top & C_0 & C_1 & \ldots \\ \ldots & & & \ldots & \\ C_n^\top & C_{n-1}^\top & \ldots & C_1^\top & C_0 \end{bmatrix} = \begin{bmatrix} D_n & 0 & \ldots & 0 \end{bmatrix} \tag{V.5}$$

which is exactly the same holding for the Toeplitz case see [Whittle(1963)] and has the unique solution

$$\begin{bmatrix} A(1) & \ldots & A(n-1) & A(n) \end{bmatrix} = -\begin{bmatrix} C_1 & C_2 & \ldots & C_n \end{bmatrix} T_{n-1}^{-1}\,;$$

$$D_n = C_0 + \sum_{k=1}^{n} A(k) C_k^\top \tag{V.6}$$

so that

$$A(1) = L_n(1), \ldots, A(n) = L_n(n)\,, \tag{V.7}$$

where $L_n(k),\, k = 1, 2, \ldots, n$ are the coefficients of the Levinson-Whittle polynomial of order n for the data matrix (IV.1). It can be checked that, imposing the orthogonality condition $\mathbf{e}_n(t) \perp \{\mathbf{y}(t-1), \mathbf{y}(t-2)\, \ldots, \mathbf{y}(t-n)\}$ and taking into account the periodic boundary values, all other model equations in (V.3) yield exactly the same Yule Walker equation (V.5). ■

Remarks

1. This result clearly agrees with the statement of Theorem 11. Since the circulant Yule Walker equation is exactly the same as that for the Toeplitz case then determining the polynomial $L_n(\zeta) = \sum_{k=0}^n L_n(k)\zeta^{-k}$ can be done by the same recursive algorithm used to compute the n-th order Levinson-Whittle polynomial in the Toeplitz case.

2. Let N be large enough and $\boldsymbol{\Sigma}$ be the *unique* circulant extension of T_n with a banded inverse, see

[3]The symbol $\hat{\mathbb{E}}\,[\cdot \mid \cdot]$ denotes wide-sense conditional expectation.

[Carli et al.(2011)Carli, Ferrante, Pavon, and Picci]. We have just proved that the correspondence

$$T_n \leftrightarrow (\{L_n(k)\ k = 1, 2, \ldots, n\},\ D_n = \mathrm{Var}\{\mathbf{e}_n(t)\})$$

is one-to-one. This is so because the coefficients $L(k)$ and the innovation covariance D_n are uniquely determined by T_n as it follows from (V.6). On the other hand T_n is in turn uniquely determined by the n-th order Levinson-Whittle polynomial $L_n(z^{-1})$ and by the relative innovation variance D_n since equation (V.5) can be interpreted as a recursion in the C_k's, namely

$$C_n + \sum_{k=1}^{n} A(k)C_{n-k} = D_n\delta_{n,0}$$

which, of course, also follows from classical Toeplitz extension theory. Hence the circulant extension $\mathbf{\Sigma}$, must be uniquely determined by the parameters $L_n(k)$ and D_n. How this circulant extension is concretely implemented is the content of the next theorem.

Theorem 13: Assume N is large enough for (IV.1) to admit a positive circulant extension and form the upper triangular $N \times N$ block-circulant matrix

$$\mathbf{L} := \mathrm{Circ}\{I,\ L_n(1),\ L_n(2),\ \ldots, L_n(n),\ 0\ldots0\} \tag{V.8}$$

having symbol $L_n(\zeta)$. Let

$$\mathbf{M} := \mathbf{L}^\top\mathbf{D}^{-1}\mathbf{L} \tag{V.9}$$

where $\mathbf{D} := D_n\mathbf{I}$, D_n being the covariance matrix of $\mathbf{e}_n(t)$[4]. Then $\mathbf{M}$ is a non-singular block-circulant symmetric matrix which is banded of bandwidth n.

Then the covariance matrix $\mathbf{\Sigma}$, solution of the circulant band extension problem for the data (IV.1), satifies the equation

$$\mathbf{M\Sigma} = \mathbf{I} \tag{V.10}$$

which is equivalent to the sequence $\{\Sigma_k\,;\ k \in \mathbb{Z}_N\}$ being the solution of the two point boundary value problem:

$$\sum_{j=-n}^{n} M(j)\Sigma_{k-j} = I\delta_k\,; \qquad k \in \mathbb{Z}_N \tag{V.11}$$

$$\Sigma_k = C_k \quad \Sigma_{N-k} = C_k^\top\,; \quad k = 1, 2, \ldots, n\,. \tag{V.12}$$

Proof: Clearly (V.9) is a block-circulant positive definite banded symmetric matrix of bandwidth n as required for modes of reciprocal processes of order n, in (I.2). Then (V.10) implies that

$$\mathbf{\Sigma} = \mathbf{M}^{-1} = \mathbf{L}^{-1}\mathbf{D}\mathbf{L}^{-\top}$$

and it obvioous that $\mathbf{\Sigma}$ is a block-circulant matrix having a banded inverse of bandwidth n. We just need to show that it is an extension of T_n.

The last equation, once rewritten in terms of symbols shows that the symbol of $\mathbf{\Sigma}$ is isomorphic to that of the Laurent extension of T_n as stated in Theorem 11, namely we have $\Phi(\zeta) = \Psi(\zeta)$ as it follows from (IV.2) and (IV.4). Therefore, since their symbols are isomorphic matrix polynomials, $\mathbf{\Sigma}^{-1}$ turns out to be a circulant N-section of the inverse Laurent extension $\mathbf{\Lambda}^{-1}$ (see Definition 14 and Lemma 15 in the appendix). This implies that both $\mathbf{\Sigma}$ and $\mathbf{\Lambda}$ are banded extensions of the same T_n. ■

[4]Note that we do not need to assume that $\mathbf{D}$ is the covariance of $\mathbf{e}_n$; i.e. that $\mathbf{e}_n$ is a white process.

References

[1] F. Carli, A. Ferrante, M. Pavon, and G. Picci. A maximum entropy solution of the covariance extension problem for reciprocal processes. *IEEE Transactions Automatic Control*, 56:1999–2012, 2011.

[2] F. P. Carli and G. Picci. On the factorization approach to band extension of block-circulant matrices. In *Proc. of the19th Int. Symposium on the Mathematical Theory of Networks and Systems (MTNS 2010), Budapest, Hungary*, pages 907–914, 2010.

[3] N. Dunford and J.T. Schwartz. *Linear Operators, Part I: General Theory*, volume I. John Wiley & Sons, New York, 1958.

[4] H. Dym and I. Gohberg. Extension of band matrices with band inverses. *Linear Algebra and Applications*, 36:1–24, 1981.

[5] P. Hartman and A. Wintner. On the spectra of Toeplitz's matrices. *Amer. J. Math.*, 72:359–366, 1950.

[6] B. C. Levy. Regular and reciprocal multivariate stationary Gaussian reciprocal processes over **Z** are necessarily Markov. *J. Math. Systems, Estimation and Control*, 2:133–154, 1992.

[7] B. C. Levy, R. Frezza, and A.J. Krener. Modeling and estimation of discrete-time Gaussian reciprocal processes. *IEEE Trans. Automatic Control*, AC-35(9):1013–1023, 1990.

[8] P. Masani. The prediction theory of multivariate stochastic proceses, iii. *Acta Mathematica*, 104:141–162, 1960.

[9] G. Picci. A finite factorization approach to band extension of block-circulant matrices and application to filtering of periodic processes. Technical report, DEI, University of Padova, 2015.

[10] J. Rissanen. Algorithms for triangular decomposition of block-hankel and toeplitz matrices with applications to factoring positive matrix polynomials. *Math. Comp.*, 27:147–154, 1973.

[11] G. J. Tee. Eigenvectors of block circulant and alternating circulant matrice. *Res. Lett. Inf. Math. Sci.*, 8:123–142, 2005.

[12] P. Whittle. On the fitting of multivariate autoregressions and the approximate canonical factorization of a spectral density matrix. *Biometrica*, 50:129–134, 1963.

Appendix

Definition 14: Let $\mathbf{H} = [\,H_{i,j}\,]_{i,j\in\mathbb{Z}}$ be a banded symmetric block-Laurent (or Toeplitz) matrix of (not necessarily symmetric) bandwidth n and let $N > n$. A block-circulant N-section of $\mathbf{H}$ is a circulant completion of a finite principal submatrix of $N \times N$ blocks, namely

$$\mathbf{H}_N = \mathrm{Circ}\{H_0,\ H_{-1}\ldots H_{-n}\,,0\,,\ldots\,,0\,,H_n\,,\ldots\,,H_1\} \tag{.1}$$

We have the following immediate but important fact.

Lemma 15: Let $\mathbf{H}$ be a banded infinite block-Laurent matrix. Then $\mathbf{H}$ and $\mathbf{H}_N$ have isomorphic polynomial symbols $H(z)$ and $H(\zeta)$. Hence for every factorization $H(z) = A(z)B(z^{-1})^\top$ there corresponds a circulant factorization $\mathbf{H}_N = \mathbf{A}_N\mathbf{B}_N^\top$ where $\mathbf{A}_N$ has symbol $A(\zeta)$ and $\mathbf{B}_N$ has symbol $B(\zeta)$..

The lemma in particular applies to upper or lower triangular matrices.