Identification of Low Rank Vector Processes

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Abstract

We study modeling and identification of stationary processes with a spectral density matrix of low rank. Equivalently, we consider processes having an innovation of reduced dimension for which Prediction Error Methods (PEM) algorithms are not directly applicable. We show that these processes admit a special feedback structure with a deterministic feedback channel which can be used to split the identification in two steps, one of which can be based on standard algorithms while the other is based on a deterministic least squares fit. Identifiability of the feedback system is analyzed and a unique identifiable structure is characterized. Simulations show that the proposed procedure works well in some simple examples.

Key words: Multivariable system identification, low-rank process identification, feedback representation, rank-reduced output noise.

1 Introduction

Quite often in the identification of large-scale time series one has to deal with *low rank* signals which have a rank deficient spectral density. Such low rank time series may arise in diverse areas such as control systems, economics, networked systems, biology and other fields.

Suppose we want to identify an $(m+p)$-dimensional vector time series y by modeling it as a weakly stationary zero-mean purely non deterministic (p.n.d.) process $y \equiv \{y(t)\,;\, t \in \mathbb{Z}\}$, having a rank deficient rational spectral density $\Phi(z)$ of rank m. This spectral density can always be written in factorized form

$$\Phi(e^{i\theta}) = W(e^{i\theta})W(e^{-i\theta})^\top, \tag{1}$$

with W an $(m+p)\times m$ full rank stable rational spectral factor. It is well-known that there are in general many such factors only one of which has the property of being *minimum phase*, see the appendix B for a definition. This factor is *essentially unique*, that is unique modulo right multiplication by an arbitrary $(m \times m)$ constant orthogonal matrix.

The rank deficiency of the spectrum Φ and consequently of the process y appears in models used in a variety of applications and is discussed in the literature from different points of view.

Singular autoregressive (AR) or autoregressive moving average (ARMA) models are discussed in [31], [23], [12]. These models make contact with dynamic factor analysis representations; see [24], [14] where an essential role is played by a rank-deficient component driven by the common factors. They occur in biological networks reconstruction as discussed in [4], [38]. Low rank processes are also encountered in graphical models which are common in social networks, see [6], [11], [7], [39]. Specific engineering examples where identification of rank-deficient processes is involved are discussed in [20],[26].

The identification of singular processes has recently been addressed in [1], [13], [16], [31], [17], [32], and [15]. Some of these papers, like [13], [17], propose an ingenious adaptation of the Prediction Error Method (PEM) identification and are of special interest. We shall briefly comment on their approach later in this paper. For a recent survey of the literature see [25].

Let the process y be partitioned as

$$y(t) := \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \tag{2}$$

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where $y_1(t)$, $y_2(t)$ are jointly stationary of dimension m and p. By properly rearranging the components of y, we can assume that y_1 is a process of full rank m. The spectral density can then be partitioned as

$$\Phi(z) = \begin{bmatrix} \Phi_{11}(z) & \Phi_{12}(z) \\ \Phi_{21}(z) & \Phi_{22}(z) \end{bmatrix}, \tag{3}$$

where $\Phi_{11}(z)$ is full rank.
It is well-known that the PEM identification procedure requires that there must be a unique representation of the predictor in terms of past y. This is not the case unless the minimum phase spectral factor $W(z)$ is square full rank. In fact, let $e(t)$ be the m-dimensional normalized innovation of y and let us expand the innovation representation $y(t) = W(z)e(t)$ as

$$y(t) = [W(\infty) + z^{-1}\hat{W}(z)]\, e(t) = W(\infty)e(t) + \hat{W}(z)e(t-1),$$

where the last term is a causal function of the strict past innovations and is therefore (by the well-known causal equivalence of a process and its innovation) must be the predictor, although expressed as *a function of e*. Now since $W(z)$ is not invertible, there is no unique expression of $e(t)$ as a function of past y and therefore there is no unique expression of the predictor as a function of past y. Although $W(z)$ is full column rank, its left inverse is not unique, and one could end up with many expressions for the predictor. This difficulty is exacerbated when one is working with (parametric) estimates of the transfer function. Therefore a direct application of the PEM principle seems to be forbidden due to the reduced-rank noise. However in [19], [17] the authors essentially show that the past of the first component y_1 acts as a **sufficient statistic** for the predictor so that there is a unique expression of the joint predictor which is a function only of the past of y_1. This remarkable representation unfortunately requires a crucial minimum phase condition which is not always satisfied.

In this paper we follow a different approach based on ideas first presented in [36], [37] and especially in [15]. In the early paper [33] it was shown that there must exist a, in general non-causal, deterministic relation between the components of a singular vector process y. In [36], [37] and in [15] the existence and structure of such deterministic relations is elucidated and specified as a component of a special feedback model for the joint process. We should advise the reader that in the setting of this paper, the deterministic relation between the variables $y_1(t)$ and $y_2(t)$, is in a sense "dual" of that introduced in [36] and also studied in [37]. This relation is described by a rational transfer function which can be identified quite easily by a least squares algorithm.

The structure of this paper is as follows. In Section 2 we introduce the feedback model representation of low-rank processes and prove the existence of a deterministic dynamical relation which reveals the special structure of these processes. In Section 3 we exploit the special feedback structure for identification of the deterministic relation and of the transfer functions of the two stochastic components driven by white noise. In Section 4 we study the identifiability of the transfer functions of a feedback representation. The feedback structure is in general not identifiable and a characterization of all equivalent forward loop transfer functions is provided based on classical result of stabilization theory in robust control. Even under the constraint of stability of the forward loop, yet there are infinitely many equivalent (stable) forward transfer functions which realize the same transfer function of the feedback model. The existence of a canonical (unique) pair of transfer functions of the feedback loop is discussed in Subsection 4.2. This canonical structure is a causal Wiener filter plus an orthogonal error term. The identification of this canonical feedback structure is discussed in Section 5. The canonical model has an output-error representation where the additive error is not necessarily white. Two possible approaches to the identification of this model are briefly discussed.
From Section 3 to Section 5 we discuss the identification of low rank time series. The identification of processes with an external measurable input is considered in Section 6, where we also make a brief comparisons with the work of [19], [17]. Several simulation examples are reported in Section 7. Finally, in Section 8 we come to some conclusions.

Notation: All random processes in this paper are discrete-time ($t \in \mathbb{Z}$), wide sense stationary with zero mean and finite variance. Most notations comply with those used in the book [5] and should be quite standard in the system identification literature. In particular, multiplication by z is the one step ahead shift operator acting as: $zy(t) = y(t+1)$ and $y(t) = W(z)u(t)$ designates the response of a linear system with transfer function $W(z)$ to an input function $u \equiv \{u(t); t \in \mathbb{Z}\}$. A rational vector or matrix function is called *stable* if all of its poles belong to the interior of the unit disk. The strictly proper stable rational vector functions written as n-dimensional column vectors form a distinguished subspace of the vector Hardy space H_n^2 which, with some abuse of notation, in this paper will be denoted by the same symbol. $\bar{H}_n^2$ will denote the direct sum of H_n^2 plus the constants. This space contains the causal rational functions which are finite for $z \to \infty$ (but are not necessarily strictly causal). The notation $[\cdot]_+$ stands for the orthogonal projection operator onto H_n^2. It should be remembered that it maps rational functions into *proper* stable rational vector functions.

2 Feedback models of stationary processes

In this section, inspired by classical references such as [28] [29], [27] and [5, Sect. 17.1], we review the definition

and some properties of general feedback models which have been also used in our recent papers [36], [15] and [37] in the context of rank-deficient vector processes. Then we derive a special feedback model for low-rank processes and prove the existence of a deterministic relation between $y_1(t)$ and $y_2(t)$.

Definition 1 (Feedback Model) *A* Feedback model *of the process* $y(t) := \begin{bmatrix} y_1(t)^\top & y_2(t)^\top \end{bmatrix}^\top$ *of dimension* $m+p$*, is a pair of equations*

$$y_1(t) = F(z)y_2(t) + v(t), \tag{4a}$$
$$y_2(t) = H(z)y_1(t) + r(t), \quad t \in \mathbb{Z} \tag{4b}$$

satisfying the following conditions:

- v *and* r *are jointly stationary uncorrelated processes called the* modeling error *and the* input noise*;*
- $F(z)$ *and* $H(z)$ *are* $m\times p$, $p\times m$ *causal transfer function matrices, one of which is strictly causal, i.e., has at least one delay;*
- *the closed loop system mapping* $\begin{bmatrix} v \\ r \end{bmatrix} \to \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *is well-posed and* internally stable *;*

The block diagram illustrating a feedback representation is shown in Fig. 1. Note that the transfer functions $F(z)$ and $H(z)$ are in general not stable, but the overall feedback configuration needs to be internally stable [9, Chap. 3.2]. In the sequel, we shall often suppress the argument z whenever there is no risk of misunderstanding. The following construction shows that feedback representations of p.n.d. jointly stationary processes always exist. Let $\mathbf{H}_t^-(y_1)$ be the closed span of the past components $\{y_{11}(\tau), \ldots, y_{1m}(\tau)\} \mid \tau \le t\}$ of the vector process y_1 in an ambient Hilbert space of second order zero-mean random variables [5] and let $\mathbf{H}_t^-(y_2)$ be defined likewise in terms of $\{y_{21}(\tau), y_{22}(\tau), \ldots, y_{2p}(\tau) \mid \tau \le t\}$. A representation similar to (4) may be gotten from the formulas for causal Wiener filters expressing both $y_1(t)$ and $y_2(t)$ as the sum of the best linear estimate based on the past of the other process plus an error term

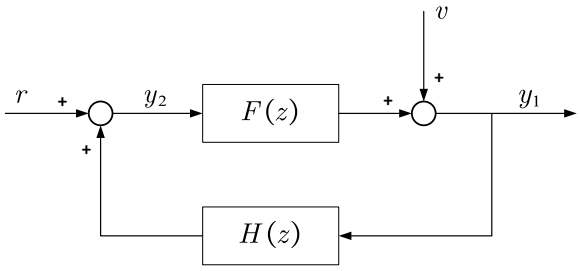

Fig. 1. Block diagram illustrating a feedback model

$$y_1(t) = \mathbb{E}\{y_1(t) \mid \mathbf{H}_{t-1}^-(y_2)\} + v(t), \tag{5a}$$
$$y_2(t) = \mathbb{E}\{y_2(t) \mid \mathbf{H}_t^-(y_1)\} + r(t). \tag{5b}$$

For a processes with a rational spectral density the Wiener predictors can be expressed in terms of causal rational transfer functions $F(z)$ and $H(z)$ as in Fig 1. Here we choose $F(z)$ to be strictly causal. An alternative representation with $H(z)$ strictly causal can also be given, to guarantee well-posedness of the feedback system. Although the errors v and r obtained by the procedure (5) may be correlated, in Appendix A we will show that there exist feedback model representations where they are uncorrelated. The following theorem describes basic properties of feedback representations of stationary processes. It has been proven in [36], [15] and is also reported in the companion paper [37], therefore its proof is omitted.

Theorem 2 *The transfer function matrix* $T(z)$ *from* $\begin{bmatrix} v \\ r \end{bmatrix}$ *to* $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *of the feedback model is given by*

$$T(z) = \begin{bmatrix} P(z) & P(z)F(z) \\ Q(z)H(z) & Q(z) \end{bmatrix}, \tag{6a}$$

with

$$\begin{aligned} P(z) &= (I - F(z)H(z))^{-1}, \\ Q(z) &= (I - H(z)F(z))^{-1} \end{aligned} \tag{6b}$$

where the inverses exist. Moreover, $T(z)$ *is a full rank (invertible a.e.) and (strictly) stable function which yields*

$$\Phi(z) = T(z) \begin{bmatrix} \Phi_v(z) & 0 \\ 0 & \Phi_r(z) \end{bmatrix} T(z)^*, \tag{7}$$

where $\Phi_v(z)$ *and* $\Phi_r(z)$ *are the spectral densities of* v *and* r*, respectively, and* * *denotes transpose conjugate.*

Since $T(e^{i\theta})$ has full rank a.e., Φ is rank deficient if and only if at least one of Φ_v or Φ_r is. Thus the rank of Φ is equal to the sum of the ranks of Φ_v and Φ_r. The next lemma will play a crucial role in this paper. Although it can be seen as a dual of a continuous-time result in [36], for the benefit of the reader we shall provide a proof anyway.

Lemma 3 *Suppose* $(F\Phi_r F^* + \Phi_v)$ *is positive definite a.e. on the unit circle. Then*

$$H = \Phi_{21}\Phi_{11}^{-1} - \Phi_r F^*(\Phi_v + F\Phi_r F^*)^{-1}(I - FH), \tag{8}$$

that is

$$H = \Phi_{21}\Phi_{11}^{-1} \tag{9}$$

if and only if $\Phi_r \equiv 0$.

PROOF. From (6) and (7), we have

$$\Phi_{21} = Q(H\Phi_v + \Phi_r F^*)P^* = QH\Phi_v P^* + Q\Phi_r F^* P^*,$$
$$\Phi_{11} = P(\Phi_v + F\Phi_r F^*)P^*,$$

and using the easily verified relations

$$PF = FQ, \quad HP = QH.$$

we get

$$\Phi_{21} = HP\Phi_v P^* + Q\Phi_r F^* P^*.$$

Adding and subtracting the term $HPF\Phi_r F^* P^*$ we end up with

$$\begin{aligned}\Phi_{21} &= H\Phi_{11} + (Q - QHF)\Phi_r F^* P^* \\ &= H\Phi_{11} + \Phi_r F^* P^*\end{aligned}$$

since $Q - QHF = I$. Then (9) follows if and only if $\Phi_r = 0$ since P is invertible and F times a spectral density can be identically zero only if the spectral density is zero as otherwise this would imply that the output process of a filter with stochastic input would have to be orthogonal to the input. □

In the following we specialize to feedback models of rank deficient processes. We shall show that there are feedback model representations where the feedback channel is described by a *deterministic relation* between y_1 and y_2.

Theorem 4 *Let y be an $(m+p)$-dimensional process of rank m. Any full rank m-dimensional subvector process y_1 of y can be represented by a feedback scheme of the form*

$$y_1 = F(z)y_2 + v, \tag{10a}$$
$$y_2 = H(z)y_1. \tag{10b}$$

where the transfer functions $F(z)$ and $H(z)$ satisfy the conditions of Definition 1 and the input noise v is of full rank m.

PROOF. Recall that n-tuples of real rational functions form a vector space $\mathbb{R}^n(z)$ where the rank of a rational matrix is the rank almost everywhere.

The claim is equivalent to the two statements
1. If we have the structure (10), i.e. $\Phi_r \equiv 0$; then y_1 is of full rank $m = \text{rank}(\Phi)$.
2. Conversely if y_1 is of full rank $m = \text{rank}(\Phi)$ then $\Phi_r \equiv 0$.

Part 1 follows from Lemma 3 since because of (7) then Φ_v must have rank $m(= \text{rank}(\Phi))$.
Part 2 is not so immediate. One way to show it could be as follows.

Since $\Phi(z)$ has rank m a.e. there must be a full rank $p\times(m+p)$ rational matrix which we write in partitioned form, such that

$$\begin{aligned}[A(z)\ B(z)]\Phi(z) = 0 &\Leftrightarrow [A(z)\ B(z)]\begin{bmatrix}\Phi_{11}(z)\\ \Phi_{21}(z)\end{bmatrix} = 0 \\ &\Leftrightarrow [A(z)\ B(z)]\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = 0\end{aligned} \tag{11}$$

where A, B are $p\times m$, $p\times p$ matrices and the last formula has the usual interpretation.

We claim that $B(z)$ must be of full rank p. One can prove this using the invertibility of $\Phi_{11}(z)$. For, suppose $B(z)$ is singular, then pick a p-dimensional non-zero row vector $a(z)$ in the left null space of $B(z)$ and multiply from the left the second relation by $a(z)$. This would imply that also $a(z)A(z)\Phi_{11}(z) = 0$ which in turn implies $a(z)A(z) = 0$ since Φ_{11} is full rank. However $a(z)[A(z)\ B(z)]$ cannot be zero for the matrix $[A(z)\ B(z)]$ is full rank p and hence $a(z)$ must be zero. So $B(z)$ must be full rank.
Now take any nonsingular $p\times p$ rational matrix $M(z)$ and consider instead $M(z)[A(z)\ B(z)]$, which provides an equivalent relation to (11). By choosing $M(z) = B(z)^{-1}$ we can reduce $B(z)$ to the identity to get

$$[-H(z)\ \ I\,]\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = 0$$

where $H(z)$ is a rational matrix function, so that one gets the deterministic dynamical relation

$$y_2(t) = H(z)y_1(t)\,.$$

Substituting in the general feedback model one concludes that $y_2(t)$ must then be a functional of only the noise v since $y_1(t)$ is such. Therefore by the uncorrelation of v and r one must conclude that in the second equation of (4) r must be the zero process i.e. $\Phi_r = 0$. Hence a representation like (10) must hold. □

3 Identification of low rank processes

Suppose we want to identify by a PEM method a model of an $(m+p)$-dimensional time series y of rank m. To this

purpose, the model class should be selected to guarantee identifiablility (i.e. uniqueness) and it is specific of the PEM method that it should actually be an *innovation representation* of y which is well known to be essentially unique. This representation involves a *minimum phase* spectral factor $W(z)$ satisying (1) whereby

$$y(t) = W(z)e(t), \tag{12}$$

where $e(t)$ is the m-dimensional normalized innovation process of y, a white noise of covariance I_m.
Consider then the model (12) block-partitioned as in (2),

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} := \begin{bmatrix} W_1(z) \\ W_2(z) \end{bmatrix} e(t), \tag{13}$$

where y_1 and y_2 are described by the special feedback model (10). From the defining property of y_1 and y_2 in our partition, $W_1(z)$ must be square $m \times m$, stable, causal and non singular (invertible a.e.) and $W_2(z)$ stable and causal.

Proposition 5 *The transfer function of the feedback channel in model* (10) *is given by the expression*

$$H(z) = W_2(z)W_1(z)^{-1} \tag{14}$$

and is unique. In fact, it depends only on the joint spectrum (3)*. Stability of H holds if and only if W_1 is minimum phase.*

PROOF. The formula follows from the partition (13) since both components are driven by the same full rank process $e(t)$. Formula (9) in Lemma 3, provides the alternative expression $H(z) = \Phi_{21}(z)\Phi_{11}(z)^{-1}$ which must obviously coincide with (14) since $\Phi_{2,1}(z) = W_2(z)W_1(z)^*$ and $\Phi_1(z) = W_1(z)W_1(z)^*$. It is then clear that $H(z)$ depends only on the joint spectrum (3) and must therefore be *unique* for a given partition of the vector process y. That stability of H holds if and only if W_1 is minimum phase follows since there cannot be cancellations in forming the quotient (14). It is shown in Appendix B that if $W(z)$ is minimum phase then $W_2(z)$ and $W_1(z)$ cannot have common unstable zeros which could cancel in forming the product (14). □

Remark: Proposition 5 is in agreement with [37], where it was shown that $H(z)$, (called F in [24]) is unique but in general not stable by a counterexample provided in Section V-A. (Also see the conference version [36]). Incidentally this answered a question by Manfred Deistler in the negative. On the contrary we shall see that there are in general infinitely many transfer functions $F(z)$ generating y by means of the model (10).

3.1 Estimation of $H(z)$

Since the relation between y_2 and y_1 is completely deterministic we can identify $H(z)$ by imposing a deterministic transfer function model to the observed data. The model can be written as $A(z^{-1})y_2(t) - B(z^{-1})y_1(t) = 0$, $t = 1, \dots, N$ (the minus sign is for convenience) where $A(z^{-1})$ and $B(z^{-1})$ are matrix polynomials in the delay variable z^{-1}, of dimension $p \times p$ and $p \times m$ such that

$$H(z) = A(z^{-1})^{-1}B(z^{-1})\,.$$

One can always choose $A(z^{-1})$ monic and parametrize the matrix polynomial $B(z^{-1})$ so that the transfer function corresponds to the difference equation

$$y_2(t) = -\sum_{k=1}^{q} A_k y_2(t-k) + \sum_{k=0}^{r} B_k y_1(t-k), \quad t = 1, \dots, N, \tag{15}$$

where we have written $A(z^{-1}) = I + \sum_{k=1}^{q} A_k z^{-k}$ and $B(z^{-1}) = \sum_{k=0}^{r} B_k z^{-k}$. The above equation involves delayed components of the observed trajectory data of y. The coefficients can then be estimated by solving a deterministic overdetermined linear system by least squares and a strongly consistent and unbiased result can be obtained whether the system is stable or not, assuming we know the true degrees of A and B. See the example in subsection 7.2.
Then, once W_1 is identified, the transfer function W_2 can be calculated using the relation

$$W_2(z) = H(z)W_1(z). \tag{16}$$

This procedure however may fail if the true $W_1(z)$ in (12) is not minimum phase and the identification is done by a time-recursive least squares algorithm. In fact if $W_1(z)$ has unstable zeros then $H(z)$ is unstable and in this case the noise superimposed to the data may tend to excite the unstable modes of the system (33) and cause divergence. To bypass the constraint of minimum phase of the true system one should rely on algorithms processing the whole data batch in one shot.

3.2 Identification of W_1

Next, since y_1 (and W_1) is full rank, it seems that one could easily identify, say an ARMA innovation model for y_1 based only on observations of $y_1(t)$ on some large enough time interval. By this procedure we would ideally identify an innovation representation for y_1, say $y_1(t) = G_1(z)e_1(t)$ where however the minimum phase transfer function $G_1(z)$ does not necessarily coincide with the upper block of the joint innovation representation of y. This would be true only if the upper block $W_1(z)$ of the minimum phase $W(z)$ was also minimum phase, *which in general may not be true* (the same clearly holding also for the lower block). See Appendix B for a discussion

of this point. In other words, the partitioned innovation representation of the full process y may not necessarily coincide with the separate innovation representations of the two components y_1 and y_2.
Therefore a PEM method applied to measurements of y_1 may not lead to a consistent estimate of the upper block $W_1(z)$ of the model (13) since there may be a nontrivial inner function $Q_1(z)$ such that

$$W_1(z) = G_1(z)Q_1(z) \tag{17}$$

One may then wonder if the identification problem we are after is well-posed and if there actually is a procedure to recover a non-minimum phase $W_1(z)$ from the data. To this end we shall first show that estimating G_1 can nevertheless lead to a consistent estimate of the joint spectrum.

Proposition 6 *Assume that the transfer function $H(z)$ is estimated as described in the previous subsection, that is using the data (y_1, y_2) and asymptotically satisfying the relation* (16)*. Then, even if the upper block $W_1(z)$ of the joint (minimum phase) transfer function $W(z)$ is not minimum phase, a consistent estimate of the minimum phase transfer function $G_1(z)$ does nevertheless produce a consistent estimate of the joint spectral density of the (joint) process y.*

PROOF. The statement is obviously true for the auto spectral density $\Phi_{11}(z)$. Then just recall that the cross spectral density of y_2 and y_1 can be expressed as

$$\Phi_{21}(z) = H(z)\Phi_{11}(z) = H(z)G_1(z)G_1(z)^*.$$

Using the estimate $\hat{G}_1(z)$ in place of $W_1(z)$ in formula (16) to compute the estimate $\hat{W}_2(z)$, although $\hat{W}_2(z) := \hat{H}(z)\hat{G}_1(z)$ may be a non-consistent estimate of $W_2(z)$, it does result in a consistent estimate of the cross spectrum $\Phi_{21}(z)$. A similar argument can be used for Φ_{22}. □

Hence a consistent estimate of the minimum phase transfer function $G_1(z)$ does produce a consistent estimate of the the joint (minimum phase) transfer function $W(z)$ of the (joint) process y and therefore also of its $m \times m$ upper block.

3.3 Procedure to recover W_1 and W_2 from consistent estimates of G_1 and H. (Equivalently, recovering the missing inner factor Q_1 in the outer-inner factorization (17)*).*

From the expression $H(z) = W_2(z)Q_1^*(z)G_1(z)^{-1}$, that is from

$$H(z)G_1(z) = W_2(z)Q_1^*(z) = W_2(z)Q_1(z)^{-1}. \tag{18}$$

One can get estimates of W_2 and Q_1 by performing a right-coprime factorization in the rational H^∞ space (see e.g. [21, sect. 5.4]), of the estimated product $\hat{H}(z)\hat{G}_1(z)$ imposing that Q_1 should be inner (see e.g. [10]). This guarantees uniqueness, see again [21, p. 368]. The conjugate inner function Q_1^* must contain exactly all the unstable poles of the left member.
In this way we are in principle able to obtain a consistent estimate of the full minimum phase model W even when W_1 is not minimum phase. The calculations are easy when W_1 is scalar but may be quite involved in the matrix case where one should need to use coprime factorization algorithms in terms of state-space realizations which we shall not dwell into.

4 Identification of the feedback model

The procedure described so far does not take into account the possibility of modeling the system by the structure (10), in particular by the "*internal*" feedback description of y_1 involving the transfer functions F, K and H. Assume that the model (13) is in innovation form, with $e(t)$ the innovation of the joint process $y(t)$ and let

$$y_1 = F(z)y_2 + K(z)e, \tag{19a}$$
$$y_2 = H(z)y_1. \tag{19b}$$

be the corresponding feedback representation with $K(z)$ a square spectral factor such that $v(t) := K(z)e(t)$, which we assume minimum phase for identifiability. From (6) we have

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = T \begin{bmatrix} K \\ 0 \end{bmatrix} = \begin{bmatrix} PK \\ QHK \end{bmatrix} = \begin{bmatrix} PK \\ HPK \end{bmatrix}, \tag{20}$$

with both $P(z)K(z)$ and $H(z)P(z)K(z)$ submatrices of a minimum phase transfer function.
One may ask how one could recover the direct transfer function $F(z)$ from the identified $W_1(z)$ and $H(z)$. This would amount to solving for F the relation $W_1 = (I - FH)^{-1}K$ which, assuming H is given, contains *two unknowns*. Hence $F(z)$ and $K(z)$ are *not identifiable* as they do not correspond uniquely to the minimum phase representation (13) and hence do not correspond uniquely to the joint spectral density of $y(t)$. In other words, there are in general infinitely many pairs $(F(z), K(z))$ realizing in feedback form the innovation representation (13). This actually agrees with the well-known identifiability analysis of feedback systems which dates back to [18], see the example in Sect. VI.

4.1 On equivalent feedback structures

In our setting the causal transfer function $H(z)$ of the feedback channel is uniquely determined by the two components of the process y, once the partition is fixed and

known, while there are in general a multitude of pairs (F, K) yielding the same transfer function $W_1(z)$. Note that each such pair should make W_1 stable. In particular, once H is given, each F should make the feedback configuration (10) internally stable. In this subsection we shall characterize the set of such equivalent F's. This problem can be regarded as the "dual" of a stabilization problem in control, which is also discussed in our companion paper [37] on modeling of low rank vector processes. Here we have a more limited scope than in [37] as we only want to analyze the identifiability of the system by explicitly describing all pairs of transfer functions (F, K) which realize the same stable W_1.

Since the feedback system must be internally stable the sensitivity function $P(z)$ defined in (6b) needs to be analytic in the complement of the open unit disk, without unstable pole-zero cancellation between $F(z)$ and $H(z)$. Assuming for the moment that $H(z)$ is a proper stable rational function, there is a whole class of proper rational functions $F(z)$ which accomplish this job. In the scalar case they are all described by the formula [9, Chapter 5.1],

$$F(z) = \frac{S(z)}{1 + S(z)H(z)} \tag{21}$$

where $S(z)$ is an arbitrary proper stable rational function. The corresponding sensitivity function is given by

$$P(z) = 1 + S(z)H(z)$$

linearly parameterized by an arbitrary such $S(z)$. All corresponding $K(z)$ are then obtained from the relation (19a), that is

$$K(z) = P(z)^{-1}W_1(z)$$

so that all such (F, K) yield the same transfer function $W_1(z)$.
When $W_1(z)$ is not minimum phase and $H(z) = W_2(z)W_1(z)^{-1}$ fails to be stable, closed-loop stability can still be characterized by using a coprime stable proper-rational factorization of $H(z)$ yielding a more general parametrization of all F's as described in [9, Sect. 5.4] (involving the so-called Youla parametrization).

In the matrix case, still assuming a stable H, there is a parametrization formula similar to (21), see e.g. reference [21]. But for the unstable case one needs to use matrix coprime factorizations to obtain the stabilizing F. This issue is fully discussed in the dual context of our companion paper [37].

4.2 A canonical feedback model

As seen from (21), there are infinitely many possible transfer functions $F(z)$ (and also companion $K(z)$) realizing the same closed loop transfer function W_1. In this subsection we shall ask the following natural question: If one restricts F to be stable and causal, does there exists a *unique* feedback representation (19)? Since the identifiability analysis of the previous section involves also the transfer function $K(z)$, it is quite evident that the answer should be negative. The following example provides in fact a few different pairs (F, K), all with a strictly causal stable F, which realize the same transfer function $W(z)$.

Example *Let a 2×1 transfer function $W(z)$ be partitioned by two scalar blocks of respective transfer functions*

$$W_1(z) = \frac{z^3}{(z-0.5)(z+0.5)(z-0.2)}, \tag{22a}$$

$$W_2(z) = \frac{z^3}{(z-0.5)(z-0.2)(z+0.1)}. \tag{22b}$$

the corresponding transfer function H being (from (14))

$$H(z) = \frac{z+0.5}{z+0.1}.$$

We can provide three different pairs F, K realizing the system, all three with a stable strictly causal F. The first being

$$F_1 = \frac{-0.4}{z+0.5}, \tag{23a}$$

$$K_1 = \frac{z^3}{(z-0.5)(z-0.2)(z+0.1)}. \tag{23b}$$

the second,

$$F_2 = \frac{0.4}{z+0.5},\ K_2 = \frac{z^3(z-0.3)}{(z+0.5)(z-0.5)(z-0.2)(z+0.1)}. \tag{24a}$$

and finally

$$F_3 = \frac{(0.2z^2 + 0.25z - 0.5)(z+0.1)}{(z+0.5)z^3}, \tag{25a}$$

$$K_3 = 1. \tag{25b}$$

To check that all three pairs realize the minimum phase W_1 in the example, just calculate the noise transfer functions K_i from $K_i = (I - F_iH)W_1$, yielding all K_i to be minimum phase, and the corresponding $P_i = (I - F_iH)^{-1} = W_1K_i^{-1}$ being stable.

However, the last example offers a hint leading to the characterization of uniqueness: one can choose a particular function $F(z)$ which, besides being stable with at least one unit delay, acts as the transfer function of the Wiener predictor of $y_1(t)$ based on the (strict) past of y_2.

Then one should have a representation like (19a) where $v(t)$ is the *prediction error*, uncorrelated with (i.e. orthogonal to) the past space $\mathbf{H}_{t-1}^{-}(y_2)$.
The proof of uniqueness of such a representation is just based on the uniqueness of the orthogonal decomposition of $y_1(t)$ as a linear causal functional of the strict past of y_2 plus an error part orthogonal to the past space $\mathbf{H}_{t-1}^{-}(y_2)$. By the orthogonal projection lemma [5, p. 27], given such a decomposition, the linear causal functional of the strict past of y_2 must then be the (unique) orthogonal projection $\mathbb{E}[y_1(t) \mid \mathbf{H}_{t-1}^{-}(y_2)]$ onto $\mathbf{H}_{t-1}^{-}(y_2)$, i.e. the Wiener predictor.

In particular, when $K(z)$ is a constant matrix as in the third example, the noise $K_3 e(t)$ is automatically orthogonal to the strict past space of y_2 and we automatically get the remarkable interpretation of $F(z)$ as the transfer function of the Wiener predictor. Indeed, below we shall show that this will surely happen when W_2 is minimum phase.

Theorem 7 *Assume that W_2 is minimum phase; then there is a representation* (19) *where F is stable and strictly causal, that is $F(z) = z^{-1}\bar{F}(z)$ with $\bar{F}(z)$ causal and stable (analytic in $\{|z| \geq 1\}$) and $K(z)$ is a constant matrix K_+. In fact, this $\bar{F}(z)$ coincides with the transfer function $F_+(z)$ of the one-step ahead Wiener predictor based on the strict past of y_2, that is*

$$F_+(z)y_2(t-1) = \mathbb{E}\{y_1(t) \mid \mathbf{H}_{t-1}^{-}(y_2)\} \tag{26}$$

and the prediction error $\tilde{y}_1(t) := y_1(t) - F_+(z)y_2(t-1)$ can be written $K_+ e(t)$ where $e(t)$ is the innovation of the joint process y. The representation

$$y_1(t) = F_+(z)y_2(t-1) + K_+ e(t) \tag{27}$$

is the unique feedback representation of $y_1(t)$ in which $v(t)$ is uncorrelated with the strict past of y_2.

PROOF. Let $W_2(z) = G_2(z)Q_2(z)$ with $Q_2(z)$ the inner factor of $W_2(z)$; it is a standard fact explained for example in [5, Chap. 3] that the Fourier representative of $\mathbf{H}_{t-1}^{-}(y_2)$ is the subspace $Q_2 H_m^2$ of H_m^2. Denoting by $P^{Q_2 H_m^2}$ the orthogonal projection operator onto $Q_2 H_m^2$, we can write the formal representative of the error process $\tilde{y}_1(t) := y_1(t) - \mathbb{E}[y_1(t) \mid \mathbf{H}_{t-1}^{-}(y_2)]$ as

$$\tilde{y}_1 := W_1 e - [P^{Q_2 H_m^2} W_1]\, e$$

so that

$$K(z) := W_1(z) - [P^{Q_2 H_m^2} W_1](z) \tag{28}$$

is the transfer function of the error process $v(t) := K(z)e(t)$ which by construction is uncorrelated with the strict past $\mathbf{H}_{t-1}^{-}(y_2)$. In other words,

$$K(z) \perp Q_2 H_m^2 \tag{29}$$

the orthogonality being understood as holding columnwise in the L^2 space of vector functions on the unit circle. Now if (and only if) $Q_2(z) = I_m$ then $K(z) \perp H_m^2$ which means that $K(z)$ (in fact its column functions) belong to the orthogonal complement $(H_m^2)^{\perp}$. But since $K(z)$ is analytic, this can happen only when $K(z)$ is a constant matrix. □

Naturally, for a general representation (19a) with a strictly causal F, the error process $v(t)$, given by $v(t) = [W_1(z) - F(z)W_2(z)]e(t) := K(z)e(t)$ may not necessarily be orthogonal to the past of y_2.

5 Structure and estimation of the predictor

Denoting for convenience the one-step ahead predictor $\mathbb{E}\{y_1(t) \mid \mathbf{H}_{t-1}^{-}(y_2)\}$ by the symbol $\hat{y}_1(t)$, we may calculate F_+ by the Wiener predictor formula. see e.g. [5, p. 105]. Introducing the cross spectral density of the processes $y_1(t)$ and $e_2(t-1) \equiv z^{-1}e_2(t)$, one has

$$F_+ = [\Phi_{\hat{y}_1, z^{-1}e_2}]_+ G_2^{-L} = [zW_1 Q_2^*]_+ G_2^{-L}\,. \tag{30}$$

where $[\cdot]_+$ denotes the (causal) orthogonal projection of a function onto the complete $\bar{H}_m^2$ space, $G_2(z)$ is the minimum phase factor of $W_2(z)$, G_2^{-L} its (Moore-Penrose) left inverse and $Q_2(z)$ the inner factor of W_2 so that the innovation of y_2 is $e_2(t) = Q_2(z)e(t)$. Hence, if W_2 is minimum phase the above simplifies to

$$F_+(z) = [zW_1(z)]_+ W_2^{-L}(z)\,. \tag{31}$$

and one gets $\hat{y}_1(t+1) = F_+(z)y_2(t) = [zW_1(z)]_+ e(t)$ and so, when W_1 is also minimum phase, $e(t) = W_1(z)^{-1}y_1(t)$ and

$$F_+(z)y_2(t) = z[W_1(z) - W_1(\infty)]W_1(z)^{-1}y_1(t) \tag{32}$$

is exactly the one-step Wiener predictor of $y_1(t+1)$ given its own past. This agrees with the sufficient statistic role of the past of y_1 in the predictor formulas (12) and (13) of [19].

5.1 Estimation of $F_+(z)$

A conceptually simple way to estimate $F_+(z)$ is to resort to estimates of the transfer functions $H(z)$ and the minimum phase factor $G_1(z)$. From consistent estimates of these functions one can perform the coprime factorization (18) to obtain estimates of $W_2(z)$ and of the inner factor $Q_1(z)$. To estimate $G_2(z)$ and compute the inner

factor $Q_2(z)$ one can then perform an outer-inner factorization on the estimate of $W_2(z)$, i.e.,

$$W_2(z) = G_2(z)Q_2(z).$$

With these data one may in principle compute $F_+(z)$ by formula (30) and the companion noise transfer function $K(z)$ by implementing the formula (28) or by $K = (I - z^{-1}F_+H)W_1$. Although this may look like a rather complicated indirect procedure, for scalar transfer functions it can be implemented quite easily, see Example 2 in Section 7.2.

One may instead attempt to estimate the transfer function $F_+(z)$ directly from the data. For simplicity we shall restrict to the case of scalar processes, the generalization to the vector case being relatively straightforward. We assume a rational structure, say

$$F(z) = D(z^{-1})^{-1}N(z^{-1})$$

where $D(z^{-1})$ and $N(z^{-1})$ are polynomials in the delay variable z^{-1}, of degree n and m. Choosing $D(z^{-1})$ monic and the numerator polynomial $N(z^{-1})$ with a zero constant term, the transfer function corresponds to the difference equation

$$\hat{y}_1(t) = -\sum_{k=1}^{n} D_k\hat{y}_1(t-k)+\sum_{k=1}^{r} N_k y_2(t-k) \quad t = 1,\dots,N, \tag{33}$$

involving delayed components of the unobserved trajectory of the predictor $\hat{y}_1$ and of the " input" time series y_2. Assuming we know the true orders, this could act as a parametric representation of the predictor transfer function. Of course $\hat{y}_1$ is not observed and the identification problem needs to be formulated in an *output-error* setting. Introducing the prediction error

$$v(t) := y_1(t) - \hat{y}_1(t)$$

and letting

$\varphi(t-1) := \begin{bmatrix} y_1(t-1) & \dots & y_1(t-n) & y_2(t-1) & \dots & y_2(t-r) \end{bmatrix}^\top$

$= \begin{bmatrix} \mathbf{y}_1(t-1) \\ \mathbf{y}_2(t-1) \end{bmatrix}$, where the boldface symbols $\mathbf{y}_1(t-1)$, $\mathbf{y}_2(t-1)$ represent arrays made of n- and r-dimensional delayed variables $y_1(t-k)$ and $y_2(t-k)$ as specified by the model (33), the representation (19a) can be written as a "constrained" pseudo-linear structure

$$y_1(t) = \varphi(t-1)^\top\theta + \varepsilon(t)\,. \tag{34}$$

where θ is the $(n+r)$-dimensional vector of unknown parameters and $\varepsilon(t) := D(z^{-1})v(t)$ still dependent on the parameter θ. From what we have seen previously, in general $v(t)$ and hence $\varepsilon(t)$ may be far from being white so attempts to use ARX identification may lead to badly biased estimates. In addition, for $F_+(z)y_2(t-1)$ to be the Wiener predictor, $v(t)$ must be orthogonal to the strict past of y_2 which should be added as a further constraint to the model.

The output-error model (34) could be identified by an instrumental-variable method see [22, p. 192-198]. In the standard procedure the unknown parameters should first be roughly estimated by minimizing the average squared prediction error $v(t)$ i.e. minimizing

$$J_N(\theta) := \frac{1}{N}\sum_{t=1}^{N} v(t)^2$$

by least squares pretending v is white, that is imposing orthogonality to the delayed data $\varphi(t-1)$, i.e.

$$\frac{1}{N}\sum_{t=1}^{N} \varphi(t-1)\,v(t) = 0\,. \tag{35}$$

which leads to the normal equations

$$\frac{1}{N}\sum_{t=1}^{N} \varphi(t-1)\varphi(t-1)^\top\,\theta = \frac{1}{N}\sum_{t=1}^{N} \varphi(t-1)y_1(t)\,.$$

In the limit for $N \to \infty$ we are led to solve an equation of the form

$$\mathbf{H}\,\theta = \mathbb{E}[\varphi(t)y_1(t)]\,. \tag{36}$$

where the matrix

$$\mathbf{H} = \begin{bmatrix} \Sigma_{\mathbf{y}_1} & \Sigma_{\mathbf{y}_1,\mathbf{y}_2} \\ \Sigma_{\mathbf{y}_2,\mathbf{y}_1} & \Sigma_{\mathbf{y}_2} \end{bmatrix}$$

is formed by obvious limit covariance matrices of the observed data. Due to the non-identifiability caused by the deterministic feedback $y_2(t) = H(z)y_1(t)$ the matrix $\mathbf{H}$ turns out to have a large nullspace and the minimization does not lead to a unique estimate. Most standard software can however compute a solution via the Moore Penrose pseudoinverse. A constraint which should be satisfied in order to get a consistent estimate of the transfer function $F(z) = z^{-1}F_+(z)$ is the stability of the estimated $D(z)$ polynomial. This condition can be imposed by implementing a spectral factorization procedure by which the estimated parameters D_k are substituted by a spectrally equivalent stable set via a fast Cholesky spectral factorization algorithm due to [8],[30].
The estimated $\hat{D}_k$ (and $\hat{D}(z)$) can then be used to filter the prediction error to improve the output error estimate obtained as $\hat{\varepsilon}(t) := \hat{D}(z^{-1})v(t)$ and thereby implement an iteratively refined parameter estimation algorithm by solving a sequence of *weighted* least squares problems. We shall however leave the analysis of this procedure to a future publication.

As a simpler alternative, assuming W_2 minimum phase, one may revert to the simpler model (27) which is unique and hence identifiable and therefore a Prediction Error method should be able to identify the transfer function directly from observed data, [22, p. 203]. One may attempt a simple least squares estimation method by using a rational (or matrix-fraction) descriptions and transforming (27) to a constrained output-error model with a white output error. It is well-known that this model leads however to a predictor which is a *nonlinear function of the parameters of the denominator* and the estimation procedure needs to be carried on iteratively. Moreover the estimate is still *constrained by the stability condition on* $F_+(z)$. The naive least squares method can be consistent only if $F_+(z)$ is a FIR-type transfer function, that is the denominator of $F_+(z)$ is a constant (see again [22, Sect. 7.3]). As a first approximation one may use models of this kind. With this proviso, in spite of feedback, a suitably constrained PEM method may work anyway [35, p. 416], [22].

6 Identification of a low rank model with an external input

Suppose we want to identify a multidimensional system with an external input $u(t)$, say

$$y(t) = F(z)u(t) + K(z)e(t) \tag{37}$$

where e is a white noise process. The input u is assumed to be completely uncorrelated with e (no feedback) and persistently exciting of an appropriate order. When $\dim e = \dim y$ and $K(z)$ is square invertible, one could attack the problem by a standard PEM method. The method however runs into difficulties when the noise is of smaller dimension than y since, exactly for the same reasons explained in Sect. 1, the predictor and the prediction error are not well-defined.
When the dimension of e is strictly smaller than the dimension of y the model (37) is also called *low-rank*. This low-rank problem is actually the one discussed in [13], [16] and [17] where the authors propose an approximate solution depending on a regularization parameter. In this section we shall propose a two-stage scheme to compute estimates of F and K which in principle does not use approximations.

Referring to the general feedback model for the joint process we can always assume F causal and $K(\infty)$ full rank and normalized in some way. Consider then the prediction error of $y(t)$ given the past history of u. We have

$$\tilde{y}(t) := y(t) - \mathbb{E}[y(t) \mid \mathbf{H}_t^-(u)] = K(z)e(t) \tag{38}$$

since, by causality of $F(z)$, the Wiener predictor is exactly $F(z)u(t)$. Hence $\tilde{y}$ is a low rank time series in the sense described in the previous sections (now with the current $K(z)$ playing the same role of $W(z)$). In principle we could then use the procedure described above for time series as we could preliminarily estimate $F(z)$ by solving a deterministic regression of $y(t)$ on the past of u and hence get $\tilde{y}(t)$. If we choose linear least square methods, we will obtain a consistent estimation. Then a standard ARMA identification can be applied to estimate the minimum phase $K(z)$ in terms of the preprocessed data $\tilde{y}(t)$.

Compared with the approach in [13], [16] and [17], we use a composition of basic least squares and ARMA identification methods which avoids the approximations, and the possible complex computations of a regularized optimization problem with a tuning parameter.

7 Simulation Examples

7.1 Example 1 [Both W_1 and W_2 minimum phase]

As a first simulation example consider a two-dimensional process of rank 1 described by

$$y(t) = \begin{bmatrix} W_1(z) \\ W_2(z) \end{bmatrix} e(t) \tag{39}$$

where both $W_1(z)$ and $W_2(z)$ **are minimum phase rational transfer functions** and e is a scalar Gaussian white noise of zero mean and variance λ^2. By simulation we produce a sample of two-dimensional output data of the system (39). With these data we shall:

- Identify W_1 and W_2 by two separate AR models.
- Identify a transfer function model for y_1 and estimate $H(z)$ according to the first procedure described in Section 3.2. And then do the same for the other component.
- Estimate $F_+(z)$ and $K_+(z)$ in (27) using the estimated value of $W_1(z)$ and $H(z)$.

We choose W_1 and W_2 as in (22) and e a scalar zero mean white noise of variance $\lambda^2 = 1$. The process $y(t)$ has rank 1. The two transfer functions functions W_1 and W_2 are normalized at infinity and minimum phase rational transfer functions. Note that in this particular example both y_1 and y_2 are full rank so that our procedure would work for both.
We have generated 100 samples of the two-dimensional time series with $N = 500$ data points $\{y_i(t); t = 1, \dots, N,\ i = 1, 2\}$ and used Monte-Carlo simulations in MATLAB. The results are condensed in Box plots.

Assume the orders of W_1 and W_2 are known. Since the two AR models of y_1 and y_2 are of order 3, we just implement two AR identification in MATLAB for models

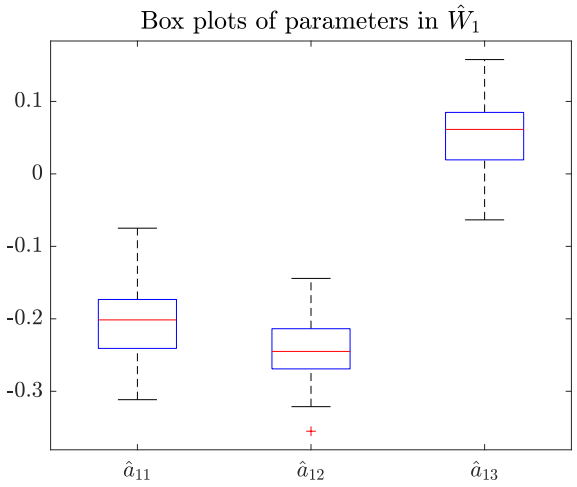

Fig. 2. Box plots of $\hat{a}_{1k}$ for $k = 1, 2, 3$ in example 1, where the true values are $a_{11} = -0.2,\ a_{12} = -0.25,\ a_{13} = 0.05$.

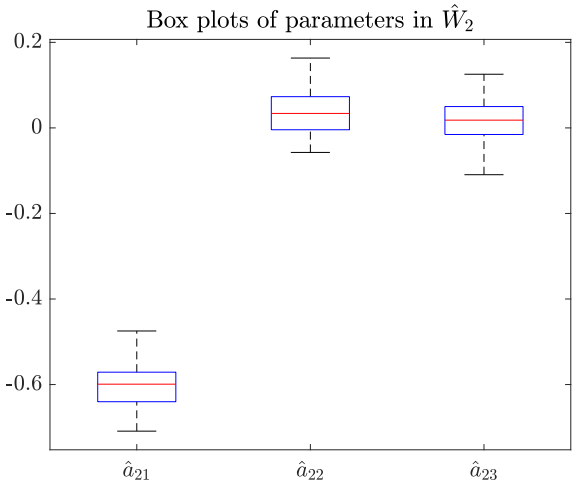

Fig. 3. Box plots of $\hat{a}_{2k}$ for $k = 1, 2, 3$ in example 1, where the true values are $a_{21} = -0.6,\ a_{22} = 0.03,\ a_{23} = 0.01$.

of the form

$$y_i(t) = -\sum_{k=1}^{3} a_{i,k} y_i(t-k) + e(t), \qquad t = 1, \dots N,$$

The box plots of the estimated parameters in $\hat{W}_1$ and $\hat{W}_2$ are shown in Fig. 2 and Fig. 3. [1] In the two box plots, all median estimated values are close to the real ones, with the ranges of estimation values acceptable and only one outlier for $\hat{a}_{12}$. We also use the average of 100 runs of Monte-Carlo simulation to estimate the asymptotic covariance of the estimated parameters which are of the order of magnitudes 10^{-4}, quite small compared with the magnitude of parameters. The box plots in Figure 2 and 3 show that our AR estimators work well.

Next we do least-squares estimation of the transfer function $H(z)$. Since H satisfies the identities

$$W_2(z) = H(z)W_1(z), \quad W_1(z) = \bar{H}(z)W_2(z),$$

we can use the following theoretical formula for H and $\bar{H}$:

$$H(z) = \frac{1 + 0.5z^{-1}}{1 + 0.1z^{-1}}, \quad \bar{H}(z) = \frac{1 + 0.1z^{-1}}{1 + 0.5z^{-1}}.$$

which is equivalent to the difference equation

$$(1 + 0.1z^{-1})y_2(t) = (1 + 0.5z^{-1})y_1(t),$$

[1] In all box plots, the red horizontal line is the median of the data, the blue box contains half of the data points, the horizontal lines are at 25% and 75% level. The black tails (black horizontal lines) are at the minimum and maximum values, except for the outliers that are indicated by a red '+' sign.

This is just a theoretical model which we keep for comparison.
Assuming now that we don't know the true degrees of the model polynomials in (33); then we first carry on an order estimation to choose the appropriate q and r in the model

$$y_2(t) - y_1(t) = -\sum_{k=1}^{q} a_k y_2(t-k) + \sum_{k=1}^{r} b_j y_1(t-j),$$

and then use least square to get estimates of the parameters of the model

$$\hat{H}(z) = \frac{1 + \sum_{k=1}^{r} \hat{b}_k z^{-k}}{1 + \sum_{k=1}^{q} \hat{a}_k z^{-k}}.$$

From a BIC table values we see that when $(q, r) = (1, 1)$ the BIC index reaches a minimum. So we do least squares estimation of a first order model

$$y_2(t) - y_1(t) = -a_1 y_2(t-1) + b_1 y_1(t-1).$$

All the parameter estimates turn out to be equal to the true values of the parameters $a_1 = 0.1, b_1 = 0.5$, affected by extremely small errors. In Monte-Carlo simulations, the calculated estimated variances are all smaller than 10^{-29}. We don't show box plots here. For estimating $\bar{H}(z)$, we obtain very similar results, which are therefore not presented. Here both H and $\bar{H}$ are stable functions. We shall check if our algorithm also works when H is not stable in the next example.

Next we shall use the previous estimates $\hat{W}_1$ and $\hat{H}$ to calculate estimates of F_+ and K_+. We choose one estimate from the previous Monte-Carlo simulations,

namely

$$\hat{W}_1 = \frac{1}{1 - 0.1627z^{-1} - 0.2256z^{-2} + 0.0505z^{-3}},$$
$$\hat{H} = \frac{1 + 0.5000z^{-1}}{1 + 0.1000z^{-1}}.$$

From Theorem 7, we know that there is one and only one pair (F_+, K_+) with F_+ the one-step Wiener predictor filter. In our case W_1, W_2 are both normalized and minimum phase, and from (32) we obtain the estimate of F_+ described by

$$\begin{aligned}\hat{F}_+ &= z(1 - \hat{W}_1^{-1})\hat{H}^{-1} \\ &= \frac{(0.1627 + 0.2256z^{-1} - 0.0505z^{-2})(1 + 0.1000z^{-1})}{1 + 0.5000z^{-1}}\end{aligned}$$

and $\hat{K}_+$ equal to the constant part of $\hat{W}_1$, i.e.,

$$\hat{K}_+ = \hat{W}_1(\infty) = 1.$$

The parameters of these functions are very close to the true values and hence appear to be consistent estimates of zF_3, K_3 in (25).
In fact, we get $\hat{K}_+ = 1$ each time in different simulations. What's more, since we are identifying with true orders in the previous Monte-Carlo simulations, we obtain a $\hat{F}$ with true orders as in (25), i.e.,

$$F_+ = zF_3 = \frac{0.2 + 0.27z^{-1} - 0.025z^{-2} - 0.005z^{-3}}{1 + 0.5z^{-1}}$$

The box plot of the estimated parameters in $\hat{F}_+$, represented as

$$\hat{F}_+ = \frac{\sum_{k=0}^{3} \hat{b}_k z^{-k}}{1 + \hat{a}_1 z^{-1}}.$$

are in Figure 4, showing that the estimate $\hat{F}_+$ obtained from $\hat{W}_1$ and the calculations in Section 5 is a good estimate of the true causal Wiener filter F_+.

7.2 Example 2 [Both W_1 and W_2 not minimum phase]

In this subsection, a simple simulation example will be presented to show that our method **can identify H well also when it is unstable, can recover the minimum phase factor G_1 when W_1 is not minimum phase** as discussed in subsection 3.2, and **can estimate the Wiener Filter F_+ when W_2 is not minimum phase** as explained in the beginning of subsection 5.1.

Consider a two-dimensional process $y(t)$ described by (13), where e is a zero mean white scalar noise of variance $\lambda^2 = 1$, and W has the two blocks with transfer functions

$$W_1 = \frac{z + 2}{z - 0.2}, \quad W_2 = \frac{z - 2}{z - 0.2}.$$

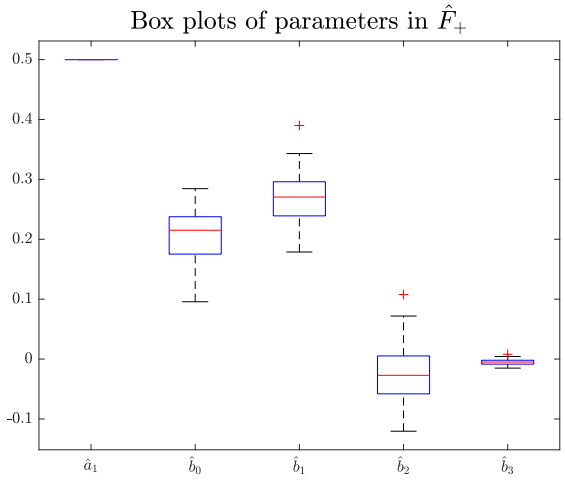

Fig. 4. Box plots of the parameters in $\hat{F}_+$ in example 1.

It is easy to obtain an outer-inner factorization of W_1 as in (17), where

$$G_1 = \frac{2z + 1}{z - 0.2} = \frac{2 + z^{-1}}{1 - 0.2z^{-1}}, \quad Q_1 = \frac{z + 2}{2z + 1}.$$

From these we get the transfer function

$$H = \frac{1 - 2z^{-1}}{1 + 2z^{-1}},$$

which is not stable.
Here for simplicity, we do not use Monte-Carlo simulations and order estimations. We just generate one group of data as in Example 1, with e scalar zero mean and of variance 1. Assume the orders of G_1 and H are known.

Though G_1 is not normalized at infinity, we may still implement an ARMA estimation in MATLAB and obtain an estimated model

$$y_1(t) - 0.1442y_1(t-1) = \hat{e}(t) + 0.5666\hat{e}(t-1),$$

where the variance of the innovation $\hat{e}$ is $\hat{\lambda}^2 = 4.3127$. Then calculate the corresponding estimate of G_1

$$\hat{G}_1 = \frac{\lambda(1 + 0.5666z^{-1})}{1 - 0.1442z^{-1}} = \frac{2.077z + 1.177}{z - 0.1442},$$

which is minimum phase.
Next we estimate H by least squares on the model

$$y_1(t) + a_1 y_1(t-1) = y_2(t) + b_1 y_2(t-1),$$

and obtain the estimate

$$\hat{H} = \frac{1 + \hat{b}_1 z^{-1}}{1 + \hat{a}_1 z^{-1}} = \frac{1 - 2.000z^{-1}}{1 + 2.000z^{-1}},$$

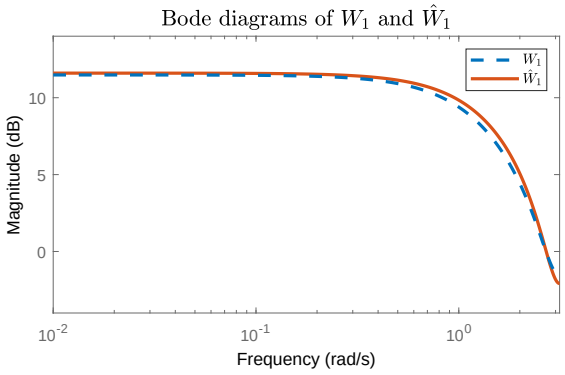

Fig. 5. Bode diagrams of W_1 and $\hat{W}_1$ in example 2.

which is practically equal to the true H, with an estimation error variance of 1.0607×10^{-29}. Incidentally; all of our simulation results show that the least squares method works well in identifying unstable H's.
Since in this example G_1 and W_2 are scalar, we do not need coprime factorization for obtaining Q_1. In this case, Q_1^* is the conjugate inner factor of H of formula (18), i.e., Q_1 is the greatest inner factor of H^{-1}. From

$$\hat{H}^{-1} = \frac{z+2.000}{z-2.000} = \frac{2.000z+1}{z-2.000} \cdot \frac{z+2.000}{2.000z+1},$$

we have the estimate,

$$\hat{Q}_1 = \frac{z+2.000}{2.000z+1}.$$

Hence the estimate of W_1 is

$$\hat{W}_1 = \hat{G}_1\hat{Q}_1 = \frac{1.039z^2+2.666z+1.177}{z^2+0.3558z-0.0721},$$

whose magnitude Bode graph is compared with the true W_1 in Fig. 5. The Bode diagrams show that we can obtain a consistent estimate of W_1 even if it is not minimum phase. The corresponding estimate of W_2 can be calculate from

$$\hat{W}_2 = \hat{H}\hat{W}_1 = \frac{1.039z^2-1.489z-1.177}{z^2+0.3558z-0.0721},$$

whose Bode diagram is close to that of the true W_2. We omit the graphs due to space limitations. It is easy to check $\hat{W} = [\hat{W}_1\ \hat{W}_2]^\top$ is minimum phase.
Next we perform an outer-inner factorization on $\hat{W}_2$, i.e., $\hat{W}_2 = \hat{G}_2\hat{Q}_2$, and obtain

$$\hat{G}_2 = \frac{2.077z^2+0.1385z-0.5885}{z^2+0.3558z-0.0721},\ \hat{Q}_2 = \frac{z-2.000}{2.000z-1}.$$

At last, the estimate of F_+ can be calculated by (30)

$$\hat{F}_+ = [z\hat{W}_1\hat{Q}_2^*]_+\hat{G}_2^{-1} = \frac{0.3915z(z+0.6023)}{z^2+0.0667z-0.2834},$$

and the companion noise transfer function $K(z)$ by implementing the formula (28),

$$\begin{aligned}\hat{K} &= \hat{W}_1 - z^{-1}[z\hat{W}_1\hat{Q}_2^*]_+\hat{Q}_2 \\ &= \frac{1.039z^3+1.7397z^2+0.4125z-0.0988}{z^3-0.1442z^2-0.2500z+0.0361}.\end{aligned}$$

It can be checked that $\hat{K}$ satisfies the equation $\hat{W}_1 = (1-z^{-1}\hat{F}_+\hat{H})^{-1}\hat{K}$. And we can see that when W_2 is not minimum phase, K is not a constant anymore as stated in Theorem 7.

7.3 Example 3 [With external input]

In this subsection we consider the identification of a two-dimensional process of rank 1 **subjected to an external input** u. We generate a scalar white noise u independent of e and identify a 2-dimensional process model (37) as described in the previous section 6.
In this example the true system is described by

$$\begin{aligned}F(z) =& z^{-1}\begin{bmatrix}0.3+0.7z^{-1}+0.3z^{-2}\\ 0.15+0.9z^{-1}-0.5z^{-2}\end{bmatrix},\\ K(z) =:& \begin{bmatrix}K_1(z)\\K_2(z)\end{bmatrix} = \begin{bmatrix}\frac{1+0.1z^{-1}+0.4z^{-2}}{1+0.3z^{-1}+0.4z^{-2}}\\ \frac{1+0.1z^{-1}+0.4z^{-2}}{1-0.2z^{-1}+0.1z^{-2}}\end{bmatrix}.\end{aligned} \tag{40}$$

where we have used the same F as in [13] (called $G(q)$ there). Since the K_2 of [13] is not normalized to 1, we use a different one. Both components of our $K(z)$ here are normalized and minimum-phase so the overall model is an innovation model. By calculation the deterministic relation from $K_1(z)$ to $K_2(z)$ is

$$H(z) = K_2(z)K_1(z)^{-1} = \frac{1+0.3z^{-1}+0.4z^{-2}}{1-0.2z^{-1}+0.1z^{-2}}.$$

For the model (40) we generate 100 groups of two-dimensional time series of $N = 500$ data points $\{y_i(t); t = 1,\dots,N,\ i = 1,2\}$. The Monte-Carlo simulations are run with u and e independent scalar white noises of variances 2 and 1. Of course here we also measure the input time series u. Suppose we do not know the orders of both F_i's for $i = i, 2$.

First, let $F_i(z) = z^{-1}A_i(z^{-1})^{-1}B_i(z^{-1})$ for $i = 1.2$, where the polynomials are parametrized as

$$A_1(z^{-1}) = 1+\sum_{k=1}^{q_1} a_{1,k}z^{-k}, \quad A_2(z^{-1}) = 1+\sum_{k=1}^{q_2} a_{2,k}z^{-k}.$$

$$B_1(z^{-1}) = \sum_{k=0}^{r_1} b_{1,k}z^{-k}, \quad B_2(z^{-1}) = \sum_{k=0}^{r_2} b_{2,k}z^{-k}$$

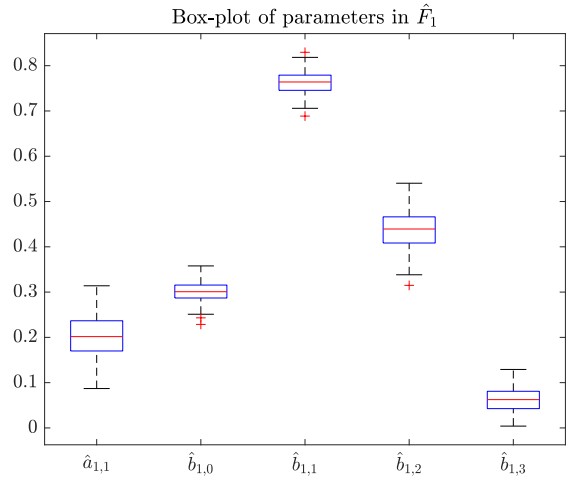

Fig. 6. Box plots of parameters of $\hat{F}_1(z)$ in example 3.

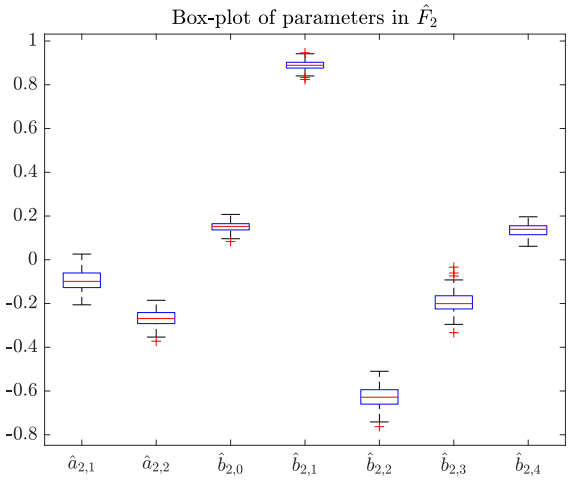

Fig. 7. Box plots of parameters in $\hat{F}_2(z)$ in example 3.

corresponding to the dynamic relations

$$A_i(z^{-1})y_i(t) = B_i(z^{-1})u(t-1) + \varepsilon_i(t), \quad t = 1, \ldots, N,$$

$i = 1, 2$ where we have added a small white noise error term. We do a standard least squares regression on these models, written in the form,

$$\hat{y}_i(t) = -\sum_{k=1}^{q_i} a_{i,k}y_i(t-k) + \sum_{k=0}^{r_i} b_{i,k}u(t-1-k), \quad (i = 1, 2). \tag{41}$$

where the orders are to be estimated. Order estimation by minimum BIC leads to choose $(q_1, r_1) = (1, 3)$ and $(q_2, r_2) = (2, 4)$. Although we don't get the right model structures, with these orders we get the reasonable box plots shown in Fig. 6 and Fig. 7, with very few outliers.

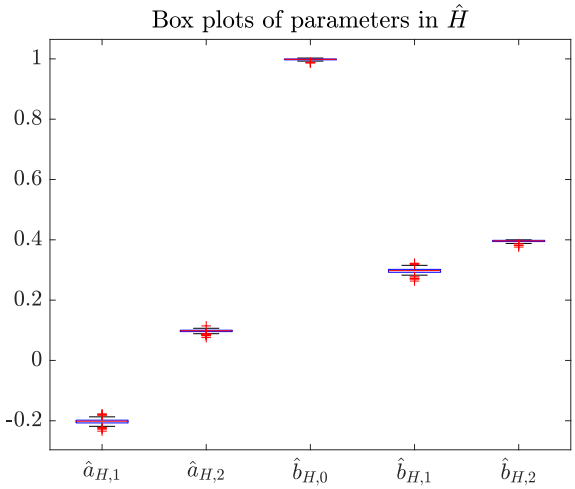

Fig. 8. Box plots of parameters in $\hat{H}(z)$ in example 3.

Next we compute the deviations (38) by

$$\tilde{y}_i(t) = y_i(t) - \hat{F}_i u(t), \qquad i = 1, 2$$

which are components of a 2-dimensional low rank process. With these data we estimate $K_1(z)$ and $K_2(z)$ by the procedure illustrated in Section 3. This time, to smooth the influence of the wrong model structure used in estimating F_1 and F_2, we assume that the true degrees of H and K_1 are known.

We first use a least square method to estimate H based on the data $\tilde{y}_1$ and $\tilde{y}_2$, assuming true orders,

$$\hat{H}(z) = \frac{\hat{b}_{H,0} + \hat{b}_{H,1}z^{-1} + \hat{b}_{H,2}z^{-2}}{1 + \hat{a}_{H,1}z^{-1} + \hat{a}_{H,2}z^{-2}}.$$

Then let $K_1 = A_1^{-1}C_1$ so that

$$A_1(z^{-1})\tilde{y}_1(t) = C_1(z^{-1})e(t),$$

where

$$A_1(z^{-1}) = 1 + a_{1,1}z^{-1} + a_{1,2}z^{-2},$$
$$C_1(z^{-1}) = 1 + c_{1,1}z^{-1} + c_{1,2}z^{-2}.$$

The box plot of the Monte-Carlo simulations of the estimates of $H(z)$ are shown in Fig. 8. With the estimate of $H(z)$ we can calculate the estimate of K_2 by

$$\hat{K}_2 = \hat{H}\hat{K}_1.$$

Because of multiplication of estimates, $\hat{K}_2$ turns out to have a large number of parameters. In order to save space, we do not show their box plot. Instead of drawing box plots, we have compared the average of Monte-Carlo estimates with the true functions. Denote by $\bar{\hat{K}}_i$

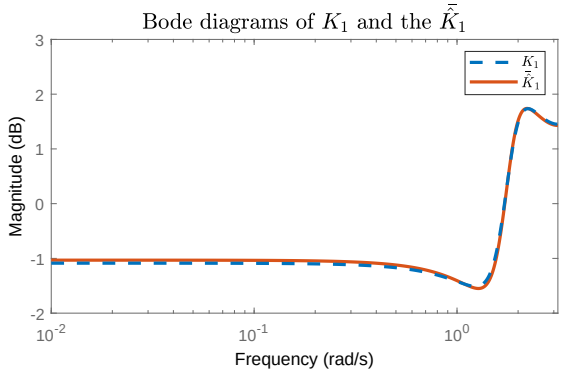

Fig. 9. Magnitude Bode plots of $\bar{\hat{K}}_1(z)$ of example 3.

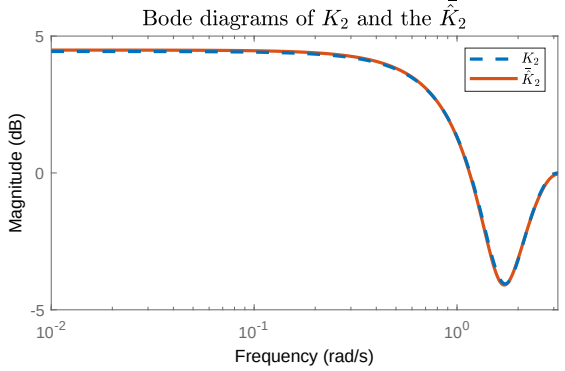

Fig. 10. Magnitude Bode plots of $\bar{\hat{K}}_2(z)$ of example 3.

$(i = 1, 2)$ the Monte-Carlo averages of the estimates $\hat{K}_i, i = 1, 2$; the Bode diagrams of the comparisons are shown in Fig. 9 and Fig. 10. Both average estimates have Bode diagrams quite close to those of the true ones. The results are nice even if we didn't guess the true model structures when estimating F.

8 Conclusions

In this paper we have shown that a rank-deficient process admits a special feedback representation with a deterministic feedback channel, which can be used to split the identification in two steps, one of which can be based on standard PEM algorithms while the other is based on a deterministic least squares fit. Identifiability of these feedback structures is not guaranteed and we show how to choose an identifiable representative. A consequent method of identifying low rank processes with an external input is also proposed. It is shown that standard identification algorithms can be easily applied to identify the transfer functions of low-rank models in diverse circumstances. Several simulations confirm the validity of the proposed approach.

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A Proof of the existence of models (4) with uncorrelated noises

Consider a feedback model like (5) where the input noises (r, v) may be correlated and let

$$\hat{r}(t) := \mathbb{E}[r(t) \mid v(s); s \in \mathbb{Z}]$$

be the *acausal* Winer estimate of $r(t)$ given the whole history of the process v [5, p. 105]. Since the joint spectral density is rational there is a rational transfer function say $S(z)$ by which we can represent $\hat{r}$ as $\hat{r}(t) = S(z)v(t)$ (with the usual convention on the symbols). Hence

$$r(t) = S(z)v(t) + w(t)$$

where $w(t)$ is a stationary process uncorrelated with the whole history of v. Now, after substituting into the first equation, the second equation of (4) can be written

$$y_2(t) = [H(z) + S(z)]y_1(t) - S(z)F(z)y_2(t) + w(t)$$

from which

$$\begin{aligned} y_2(t) = {} & [I + S(z)F(z)]^{-1}[H(z) + S(z)]y_1(t) \\ & + [I + S(z)F(z)]^{-1}w(t) \end{aligned} \tag{A.1}$$

which, after setting $\tilde{r}(t) := [I+S(z)F(z)]^{-1}w(t)$ may be written $y_2(t) = \tilde{H}(z)y_1(t)+\tilde{r}(t)$, of the same form of the second equation in (4) but now with v and $\tilde{r}$ completely uncorrelated. □

B On minimum phase matrix functions

Let $W(z)$ be an $(m+p) \times m$ full column rank stable matrix possibly a spectral factor of our $(m + p) \times (m + p)$ spectral density matrix $\Phi(z)$ of rank m. Minimum phase functions are called *outer* in the mathematical literature. Although our functions are rational it will be convenient to refer to the general definitions in Hardy spaces of the literature. For these we shall use the row-vector convention of the book [5]. The following is an intuitive definition which matches that for scalar functions [5, Theorem 4.6.11, p.137].

Definition 8 *A rational matrix function $W(z)$ is minimum-phase, i.e., outer , if and only if it has all its poles in the open unit disc and all its zeros in the closed unit disc.*

One should refer to the definition of (right) zeros [5, Definition 4.6.10, p.136] for full column rank matrix functions with rows in H_m^2. For example, α is a zero of a 2×1 matrix $W = [W_1, W_2]'$, if and only if it is a common zero of both W_1 and W_2. Equivalently there is a scalar inner function $q(z)$, a Blaschke product with a zero in α, such

that $W(z) = \hat{W}(z)q(z)$ with $\hat{W}(\alpha) \neq 0$. More generally, we want to consider a partition of $W(z)$

$$W(z) = \begin{bmatrix} W_1(z) \\ W_2(z) \end{bmatrix} \tag{B.1}$$

where $W_1(z)$, $W_2(z)$ are $m \times m$, $p \times m$ analytic matrix functions with rows in H^2_m. Next we recall the classical definition of an outer matrix function in the matrix Hardy space $H^2_{(p+m),m}$. The matrix function $W(z) \in H^2_{(p+m),m}$ is outer, if the row-span

$$\overline{\text{span}}\,\{\phi(z)W(z)\,;\,\phi \in H^\infty_{(p+m)}\}$$

is the whole space H^2_m. This is equivalent to saying that in the outer-inner factorization $W(z) = \hat{W}(z)Q(z)$, the inner (matrix) function Q must be a unitary constant, which we may identify with the the identity I_m.

Consider now the outer-inner factorizations

$$W_1(z) = \hat{W}_1(z)Q_1(z), \tag{B.2a}$$
$$W_2(z) = \hat{W}_2(z)Q_2(z), \tag{B.2b}$$

where $\hat{W}_1$, $\hat{W}_2$ are the outer (minimum-phase) factors and Q_1, Q_2 are inner (in fact matrix Blaschke products). The question we want to answer is: if W is outer, does it follow that any (or both) of the two components W_1, W_2 should also be outer? We shall see that the answer is in general negative.
Let us recall the definition of *greatest common right inner divisor* of two inner functions Q_1 and Q_2, see [3, p. 188 top] denoted $Q_1 \wedge_R Q_2$. This is the inner function representative of the closed vector sum $H^2_m Q_1 \vee H^2_m Q_2$.

Theorem 9 *Let a full column rank matrix function $W(z) \in H^2_{(p+m),m}$ be partitioned as in* (B.1)*. The W is outer if and only Q_1 and Q_2 are right-coprime, equivalently, the* greatest common right inner divisor *of Q_1 and Q_2 is the identity, i.e. $Q_1 \wedge_R Q_2 = I_m$.*

PROOF. Follows from the identity see [3, p. 188 top].

$$H^2_m Q_1 \vee H^2_m Q_2 = H^2_m(Q_1 \wedge Q_2)$$

Hence $W(z) \in H^2_{(p+m),m}$ can be outer even if none of the two submatrices W_1 and W_2 is. They just need to have no (unstable) zeros in common. On the other hand, when W_1 or W_2 have no unstable zeros, they are automatically outer.

C Stability of the Moore-Penrose Pseudo-Inverse

This section was contributed by Augusto Ferrante [2]. It deals with stability of the Moore-Penrose pseudo inverse of a minimum phase rational matrix function. We might only concentrate on stability of a left inverse which is what is needed in this paper but the result is more general.

Suppose we have a rational spectral factor $W(z)$ with n rows and $p \leq n$ columns and assume the minimum phase condition that $W(z)$ has full column rank for any $|z| \geq 1$. This means that the *Smith-McMillan form* [34, p.443-445] of $W(z)$ is

$$W(z) = U(z)G(z)V(z)$$

where:
1) $U(z)$ is a $n \times n$ unimodular polynomial matrix so that its inverse $U^{-1}(z)$ is polynomial.
2) $G(z)$ is a $n \times p$ rational matrix having the form $G(z) = \begin{bmatrix} D(z) \\ 0 \end{bmatrix}$ where $D(z)$ is a $p \times p$ diagonal matrix whose diagonal elements are non-zero rational functions having only zeros strictly inside the unit circle.
3) $V(z)$ is a square unimodular polynomial matrix with p rows and p columns so that its inverse $V^{-1}(z)$ is also polynomial. Thus if $W(z)$ is minimum phase the left inverse

$$W^{-L}(z) := V^{-1}(z)\,[D^{-1}(z) \mid 0]\,U^{-1}(z)$$

is clearly analytic. Since there is an algorithm to compute the Smith-McMillan form, $W^{-L}(z)$ as defined above can be effectively computed.