On the State Space and Dynamics Selection in Linear Stochastic Models: A Spectral Factorization Approach

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***Abstract*—Matrix spectral factorization is traditionally described as finding spectral factors having a fixed analytic pole configuration. The classification of spectral factors then involves studying the solutions of a certain algebraic Riccati equation, which parametrizes their zero structure. The pole structure of the spectral factors can also be parametrized in terms of solutions of another Riccati equation. We study these two Riccati equations and describe how they can be combined for the construction of general spectral factors, which involve both zero and pole flipping on an arbitrary reference spectral factor.**

***Index Terms*—Causality, modeling, Riccati equations, spectral factorization, state space, stochastic realization.**

I. Introduction

An important and widely used class of models in control engineering and signal processing describes an m-dimensional observed random signal $\{y(t)\}$ as the output of a linear system driven by white noise

$$\begin{cases} x(t+1) = Ax(t) + Bw(t) \\ y(t) = Cx(t) + Dw(t) \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and w is a normalized white noise. The n-dimensional signal x is the state vector. The basic steps for the constructions of models of the form (1) from observations of $\{y(t)\}$ lead to the following three problems which in various forms permeate linear systems and control theory:

1) Estimate the spectral density $\Phi_y(z)$ of y (see [8], [13], [14], [24]–[26], and references therein).
2) Compute a stochastically minimal[1] spectral factor of $\Phi_y(z)$, i.e., a matrix transfer function $W(z)$ of minimal McMillan degree such that
$$\Phi_y(z) = W(z)W^\top(z^{-1}) \tag{2}$$
(see [3], [4], and references therein).
3) Fix a minimal realization $W(z) = C(zI - A)^{-1}B + D$ to provide a parametrization of model (1).

The literature on these topics being enormous, we have chosen to quote only a few recent papers in which one can find a more extensive bibliography. The study of models (1) of the signal y without *a priori* constraints of causality or analyticity is motivated by the fact that a stochastic process can be seen as a flow of trajectories, which has no privileged direction of time built in and hence must admit representations that are neither causal nor anticausal. This general point of view is discussed in the recent book [18, pp. 637–639].

The objective of this paper is to continue the analysis and study in more depth the relations among different models (1), which are in a sense *equivalent* as they serve to represent the same process but may have different system-theoretic structures and properties. Indeed, representations (1) have several degrees of freedom. The most obvious (and least interesting) one is the choice of basis in the input and in the state space. In particular, the matrices A, B, C, and D in step 3) are determined up to a transformation of the form $T^{-1}AT, T^{-1}BU, CT$, and DU, where T is an arbitrary invertible matrix and U is an arbitrary orthogonal matrix. Once these degrees of freedom are factored out, we are left with two more interesting objects:

1) The state space as a coordinate free representative of model (1);
2) The (dynamical) causality structure (related in particular to the choice of direction of the time arrow) of equivalent models.

One of the key results of the stochastic realization theory (see [18]) is that these two choices correspond, respectively, to the *choice of zeros and poles* of the spectral factor $W(z)$ in (2). Each pole configuration of the spectral factor corresponds to a certain causality structure so that, once this configuration is fixed, one is left with the choice of the zero structure of the spectral factor, which just means choosing a (minimal) state space of the realization.

Matrix spectral factorization is traditionally described as finding spectral factors having a fixed analytic (also called *Schur stable*) pole configuration so that all corresponding models are causal, and classifying different models corresponds to parametrizing all possible zero structures of W. However, a zero structure fixes, independently of causality, a possible minimal state space[2] for y. Hence, once a minimal state space (i.e., the zero structure of W) is fixed, there is a whole family of possible causality structures, which can be parametrized by the allowed pole locations of a spectral factor W.

If some minimal *reference* spectral factor is fixed, minimal spectral factorization can be seen as a *zero- or pole-flipping* transformation performed on the reference factor. In this paper, we analyze the interplay between these two operations in relation to the solution sets of two families of algebraic Riccati equations (AREs). We derive closed-form formulas that allow to compute the model corresponding to a given causality structure and state space. This may be viewed as the completion of an endeavor first undertaken in [20] in continuous time but not pushed to the final consequences. Here we shall address the discrete-time situation and give a complete solution.

An important point, which needs to be stressed, is that once the spectral density $\Phi_y(z)$ is assigned, the selection of a state space, that is, the selection of a specific zero structure of the spectral factor, is *independent of the selection of the causality structure, i.e., of the poles*. Of course both need to be compatible with the zero–pole configuration

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[1] Stochastic minimality means that we are only interested in models of minimal complexity (McMillan degree).

[2] We stress that the choice of the state space must not be confused with the choice of basis in $\mathbb{R}^n$.

of $\Phi_y(z)$. In practice, once a reference spectral factor is fixed, the zero selection (that is the zero flipping) is operated by choosing a specific solution of an ARE whose coefficients are obtained from the matrices of the reference model. More generally, to obtain all the models with a fixed desired zero configuration starting from an arbitrary reference model, a family of AREs (each corresponding to a specific pole selection) needs to be considered. In other words, for any choice of poles, in principle we have to solve a different ARE in this family to select the desired zeros. Of course, to achieve the wanted fixed zero structure, a specific solution for each ARE of the family must be selected. On the other hand, the zeros can be chosen only based on the zero structure of $\Phi_y(z)$ and independently of the poles of a reference model so that the sets of solutions of the AREs of the family (parametrized by pole selection) must be in a one-to-one correspondence with each other. In this paper, we derive an explicit formula for such a correspondence, so that the problem can be solved by selecting suitable solutions of only two *independent* AREs instead of considering one ARE coupled with a family of (possibly infinitely many) other AREs.

Although our main motivation is stochastic modeling, our contribution can also be viewed as related to AREs and to spectral factorization. Both have important applications in several areas of control, signal processing, and system theory.

Some technical assumptions of this paper could probably be weakened, however, at the expense of clarity. For pedagogical reasons, we have decided to work in a setting, which reduces technicalities to a minimum.

II. Background on Spectral Factorization and AREs

Let $\Phi(z)$ be an $m \times m$, full-rank rational spectral density matrix of a *regular* stationary process, where regularity means that $\Phi(z)$ has no zeros at $z = 0$, nor at infinity, see [12] for more details on the concept of regularity. Let

$$W(z) := C(zI - A)^{-1}B + D \tag{3}$$

be a minimal realization of a minimal square spectral factor of $\Phi(z)$ so that $\Phi(z) = W(z)W(z)^*$, where $W(z)^* := W(z^{-1})^\top$ is the conjugate transpose. By regularity, the matrix D is nonsingular for any minimal square spectral factor $W(z)$ [12]; without loss of generality, the matrix D can be assumed to be symmetric and positive definite: this rules out the uninteresting degree of freedom corresponding to multiplying a spectral factor on the right side by a constant orthogonal matrix. By regularity, the *numerator matrix* $\Gamma := A - BD^{-1}C$ is nonsingular (see [18, Th. 6.8.2]).

Definition 2.1: Let $W_i(z)\,;\, i = 1, 2$ be minimal spectral factors of the same rational spectral density. We shall say that $W_1(z)$ and $W_2(z)$ have the same pole structure if they admit a state-space realization with the same state transition matrix. Likewise, we say that $W_1(z)$ and $W_2(z)$ have the same zero structure if they admit a state-space realization with the same numerator matrix.

In this paper, we consider a *reference* spectral factor $W_0(z) = C(zI - A)^{-1}B + D$. We hasten to stress that we do *not* assume $W_0(z)$ to be analytic outside of the unit disk. We only assume that both A and Γ are *unmixed*, i.e., they do not have reciprocal pairs $(\lambda, 1/\lambda)$ of eigenvalues. Hence, in particular, both A and Γ cannot have eigenvalues of modulus one.

Once the state matrix A is fixed, all minimal spectral factors having a fixed pole structure are classified in terms of their zero structure; equivalently, in terms of invariant subspaces of the transpose of the numerator matrix Γ. It is well known that this involves the study of an ARE. In continuous time, the connection between spectral factorization and solutions of the ARE has been known for a long time, see [2] and [22, Remark 27]; see, in particular, [17, p. 381] for the connection between solutions of the ARE and the zero location (i.e., the numerator matrix). These ideas are now a cornerstone of the linear system theory and an enormous literature has been produced discussing various applications to linear control and filtering theory. The discrete-time versions seem, to the best of our knowledge, to have been first collected in a systematic way in the monograph [6]. In particular, we quote here the following result, slightly different versions of which have appeared in several places of the literature.

Proposition 2.1: Let $W_0(z) := C(zI - A)^{-1}B + D$ be a minimal realization of a square reference spectral factor.

1) There is a one-to-one correspondence between symmetric solutions of the homogeneous ARE

$$P = \Gamma P \Gamma^\top - \Gamma P C^\top (DD^\top + CPC^\top)^{-1} CP\Gamma^\top \tag{4}$$

and minimal spectral factors of $\Phi(z)$ having the same pole structure of $W_0(z)$. This correspondence is defined by the map assigning to each solution P the spectral factor

$$W_P(z) := C(zI - A)^{-1}B_P + D_P \tag{5}$$

where

$$\begin{aligned} B_P &:= (BD^\top + APC^\top)(DD^\top + CPC^\top)^{-1/2} \\ D_P &:= (DD^\top + CPC^\top)^{1/2}\,. \end{aligned} \tag{6}$$

2) There is a one-to-one correspondence between symmetric solutions of (4) and $\Gamma^\top$-invariant subspaces which is defined by the map assigning to each solution P the $\Gamma^\top$-invariant subspace $\ker(P)$.

For a proof, we shall just refer the reader to [18, Corollary 16.5.7 and Lemma 16.5.8] where the equation differs by an inessential change of sign. A similar Riccati equation although in a different context is studied in [23].[3]

We just recall that in a basis adapted to the orthogonal decomposition

$$\mathbb{R}^n = \ker P \oplus \operatorname{im} P$$

we have $P = \operatorname{diag}\{0,\ \hat{P}\}$, where $\hat{P}$ is nonsingular. Letting $\Gamma_P := A - B_P D_P^{-1} C$, the ARE (4) can be rewritten in the form

$$\Gamma^{-1} P = P \Gamma_P^\top$$

which implies $\hat{\Gamma}^{-1}\hat{P} = \hat{P}\hat{\Gamma}_P^\top$, where $\hat{\Gamma}$ and $\hat{\Gamma}_P$ are the restrictions of Γ and Γ_P to $\operatorname{im} P$. In particular, let P_+ be the unique nonsingular solution of (4), then the corresponding $\Gamma^\top$-invariant subspace $\ker P_+$ is trivial, $\Gamma_P^\top$ is similar to Γ^{-1}, and the zeros of $W_0(z)$ are all flipped to reciprocal positions. We shall denote the corresponding spectral factor by $W_{P_+}(z)$. Zero flipping can also be visualized as the right multiplication of $W_0(z)$ by a suitable square all-pass function so as to preserve minimality. The entailed factorization of $W_P(z)$ is in turn uniquely identified by the existence of a $\Gamma^\top$-invariant subspace [5].

On the other hand, we have the following fact that describes the *pole-flipping* relation among spectral factors keeping a *fixed zero structure*.

Proposition 2.2: Let $W_0(z) := C(zI - A)^{-1}B + D$ be a minimal realization of a square reference spectral factor.

1) There is a one-to-one correspondence between symmetric solutions of the ARE

$$Q = A^\top Q A - A^\top Q B\,(I + B^\top Q B)^{-1} B^\top Q A \tag{7}$$

[3]Any solution P can actually be seen as the difference say $X - X_0$ of two arbitrary solutions of an equivalent Riccati equation parametrizing the minimal spectral factors, which is defined directly in terms of a realization of Φ and does not involve a reference spectral factor, see [18, Sec. 16.5]. Here, X_0 is kept fixed as a reference solution and Γ describes the zero structure of the reference spectral factor W_0.

and minimal normalized spectral factors having the same zero structure of $W_0(z)$. This correspondence is defined by the map assigning to each solution Q the spectral factor

$$W_Q(z) := C_Q(zI - A_Q)^{-1}B_Q + D_Q \tag{8}$$

where

$$\begin{aligned}\Delta_Q &:= I + B^\top QB\\ C_Q &:= C - D\Delta_Q^{-1}B^\top QA\\ A_Q &:= A - B\Delta_Q^{-1}B^\top QA\\ B_Q &:= B\Delta_Q^{-1/2}U\\ D_Q &:= D\Delta_Q^{-1/2}U\end{aligned} \tag{9}$$

and U is the orthogonal matrix

$$U := (D\Delta_Q^{-1/2})^\top((D\Delta_Q^{-1/2})(D\Delta_Q^{-1/2})^\top)^{-1/2}$$

which is selected in such a way that D_Q is symmetric and positive definite.[4]

2) There is a one-to-one correspondence between symmetric solutions of (7) and A-invariant subspaces, which is defined by the map assigning to each solution Q the A-invariant subspace $\ker(Q)$.

Proof: That the zero structures of $W_Q(z)$ and of $W_0(z)$ coincide is the content of [18, Th. 16.4.4]. The rest is readily checked. □

Notwithstanding pole flipping may be viewed as dual of the classical results on zero flipping, and indeed can be dealt with by considering zero flipping for the spectral factor $(W^*)^{-1}$ of the inverse spectral density Φ^{-1}, the literature on this topic is, to the best of our knowledge, very limited. As far as we know, the first contribution in this direction (in the continuous-time case) was proposed in [19], (see also [7], [9], [20]) while we do not know any reference earlier than [18] for the discrete-time case. In fact, while the idea of considering the spectral factors of Φ^{-1} leads to pole flipping with simple algebraic manipulations, the ideas proposed in [20] and developed in [18] are founded on much deeper system-theoretics grounds that allow for the treatment of a more general situation where the spectral factor can possibly be nonsquare. A different approach, where pole and zero flipping are treated without reference to state-space realization, is in [4].

III. Combining Pole and Zero Flipping

We want to understand the combination of zero and pole flipping leading to an arbitrary minimal square spectral factor $W(z)$. To this end, let us consider the spectral factor $W_Q(z)$ defined in (8) as a reference spectral factor and describe the zero-flipping process on $W_Q(z)$. By direct computation we easily find that the numerator matrix of $W_Q(z)$ is the same as that of the numerator matrix of $W_0(z)$, i.e., the matrix Γ. Hence, the Riccati equation (4) corresponding to the reference $W_Q(z)$ takes the form

$$P_Q = \Gamma P_Q\Gamma^\top - \Gamma P_Q C_Q^\top(D_Q D_Q^\top + C_Q P_Q C_Q^\top)^{-1}C_Q P_Q\Gamma^\top \tag{10}$$

where C_Q and D_Q are defined in (9). Notice that, since (4) and (10) involve the same matrix Γ and each symmetric solution of either equation is uniquely attached to a $\Gamma^\top$-invariat subspace [22], the map assigning to each solution P of (4) the solution P_Q of (10) such that $\ker(P) = \ker(P_Q)$ is a one-to-one correspondence between the set $\mathcal{P}$ of solutions of (4) and the set $\mathcal{P}_Q$ of solutions of (10).

[4]Of course if we do not require D_Q to be symmetric and positive definite, we can take U to be an arbitrary orthogonal matrix including the identity matrix.

Our main contribution is to analyze the relations between $\mathcal{P}$ and $\mathcal{P}_Q$ and to provide an explicit formula to compute the solution P_Q from a given pair P, Q. In this way, once we have parametrized the solutions of (4) and (7), we do not need to solve (10) and we have a closed-form formula for the spectral factor with assigned pole and zero structures, or equivalently for the model with the assigned state space and causality structure.

We may represent the situation by the following commutative diagram:

$$\begin{array}{ccc} W_0 & \xrightarrow{P} & W_P \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q_P} \\ W_Q & \xrightarrow[P_Q]{} & W \end{array} \tag{11}$$

Here, W_P has the same pole structure of the the reference spectral factor W_0 (but a different zero structure) and W_Q has the same zero structure of the the reference spectral factor W_0 (but a different pole structure). We want to compute the target spectral factor W (having the same pole structure of W_Q and the same zero structure of W_P) *directly*, i.e., without solving (10) for computing P_Q [or, dually, an equation similar to (7) to compute Q_P].

We shall proceed in two steps: First consider the following diagram corresponding to a *complete* flipping of zeros:

$$\begin{array}{ccc} W_0 & \xrightarrow{P_+} & W_{P_+} \\ {\scriptstyle Q}\downarrow & & \downarrow \\ W_Q & \xrightarrow[P_{Q,+}]{} & W_{Q,+} \end{array} \tag{12}$$

where P_+ is the unique invertible solution of ARE (4). This solution leads to flipping *all* the zeros of $W_0(z)$. Similarly, $P_{Q,+}$, which leads to flipping all the zeros of W_Q, must be the only invertible solution of ARE (10).

The first step is to establish the relation between the unique nonsingular solutions P_+ and $P_{Q,+}$ of (4) and (10), respectively. This is the content of the following lemma.

Lemma 3.1: The nonsingular solutions P_+ and $P_{Q,+}$ are related by the formula

$$P_{Q,+}^{-1} = Q + P_+^{-1}. \tag{13}$$

Proof: It is immediate to check that P_+^{-1} is the unique solution of the Stein equation

$$P_+^{-1} - \Gamma^\top P_+^{-1}\Gamma + C^\top D^{-\top}D^{-1}C = 0. \tag{14}$$

Similarly, $P_{Q,+}^{-1}$ is the unique solution of the Stein equation

$$P_{Q,+}^{-1} - \Gamma^\top P_{Q,+}^{-1}\Gamma + C_Q^\top D_Q^{-\top}D_Q^{-1}C_Q = 0. \tag{15}$$

Therefore, the difference $\Delta := P_{Q,+}^{-1} - P_+^{-1}$ is also the unique solution of the equation

$$\Delta - \Gamma^\top\Delta\Gamma + C_Q^\top D_Q^{-\top}D_Q^{-1}C_Q - C^\top D^{-\top}D^{-1}C = 0. \tag{16}$$

We now compute

$$\begin{aligned} R_Q &:= C_Q^\top D_Q^{-\top} D_Q^{-1} C_Q - C^\top D^{-\top} D^{-1} C \\ &= C^\top D^{-\top} \Delta_Q D^{-1} C + A^\top Q B \Delta_Q^{-1} B^\top Q A \\ &\quad - C^\top D^{-\top} B^\top Q A - A^\top Q B D^{-1} C \\ &\quad - C^\top D^{-\top} D^{-1} C \\ &= C^\top D^{-\top} B^\top Q B D^{-1} C - C^\top D^{-\top} B^\top Q A \\ &\quad - A^\top Q B D^{-1} C + A^\top Q B \Delta_Q^{-1} B^\top Q A \\ &= C^\top D^{-\top} B^\top Q B D^{-1} C - C^\top D^{-\top} B^\top Q A \\ &\quad - A^\top Q B D^{-1} C + A^\top Q A - Q \\ &= \Gamma^\top Q \Gamma - Q \end{aligned} \tag{17}$$

where the penultimate equality follows from the fact that Q is a solution of (7) and the last equality can be checked by plugging the definition $\Gamma := A - BD^{-1}C$ into the final expression and developing the products.

Identity (17), together with (16), gives

$$\Delta - \Gamma^\top \Delta \Gamma = Q - \Gamma^\top Q \Gamma \tag{18}$$

and, by uniqueness, $\Delta := P_{Q,+}^{-1} - P_+^{-1} = Q$, so that (13) follows. □

We are now ready for the second step: To derive an explicit formula for expressing an arbitrary solution P_Q as a function of P and Q. To this end, we shall use the following lemma, which is a particular case of [1, Th. 2.2]. An analogous result is Statement 1(iii) in [11, Th. 3.1] although referring to the specific case of all-pass functions.

Lemma 3.2: The solution P of the Riccati equation (4) corresponding to a $\Gamma^\top$-invariant subspace $\mathcal{S}$ can be expressed by the formula

$$P = \left[(I - \Pi_{\mathcal{S}}) P_+^{-1} (I - \Pi_{\mathcal{S}})\right]^\dagger \tag{19}$$

where † denotes the Moore–Penrose pseudoinverse and $\Pi_{\mathcal{S}}$ is the orthogonal projector onto the subspace $\mathcal{S} = \ker P$.

Formula (19) allows to compute the solution of a homogeneous ARE having a specified kernel. It only depends on two ingredients: The unique invertible solution of the ARE (which may be regarded as the "most informative" among the solutions) and the kernel of the desired solution. Intuitively, it says that the inverse of the invertible part of the desired solution P is equal to the inverse of the only invertible solution P_+ restricted to the orthogonal complement of the specified kernel.

We are now ready to present our main result.

Theorem 3.1: Let P be an arbitrary solution of (4) and Q be an arbitrary solution of (7). Then the unique solution P_Q of (10) such that $\ker(P) = \ker(P_Q)$ can be expressed by the formula

$$P_Q = [PP^\dagger Q P P^\dagger + P^\dagger]^\dagger \tag{20}$$

which generalizes (13).

Proof: Since $(I - \Pi_{\mathcal{S}})$ projects onto the range space of P, a basic property of the Moore–Penrose pseudoinverse [16, p. 421] implies that $(I - \Pi_{\mathcal{S}}) = PP^\dagger$ so that (19) can be rewritten as $P = \left[PP^\dagger P_+^{-1} PP^\dagger\right]^\dagger$ and hence

$$P^\dagger = PP^\dagger P_+^{-1} PP^\dagger . \tag{21}$$

Now, since P and P_Q have the same kernel, they also have the same image so that the orthogonal projectors on this image may be written in two ways as

$$I - \Pi_{\mathcal{S}} = PP^\dagger = P_Q P_Q^\dagger . \tag{22}$$

Thus, the analog of formula (19) for P_Q yields

$$P_Q = \left[(I - \Pi_{\mathcal{S}}) P_{Q,+}^{-1} (I - \Pi_{\mathcal{S}})\right]^\dagger = \left[PP^\dagger P_{Q,+}^{-1} PP^\dagger\right]^\dagger \tag{23}$$

where $P_{Q,+}$ is the only nonsingular solution of (10) (such a solution corresponds to the $\Gamma^\top$-invariant subspace $\{0\}$). Hence, after inserting (13), we get $P_Q = [PP^\dagger (Q + P_+^{-1}) PP^\dagger]^\dagger$, and, finally, by using (21), we obtain the explicit expression (20) for P_Q depending only on P and Q. □

A. State-Space Formulas for the Spectral Factor W

Next we show how our result can be employed to derive state-space formulas for a model (1) with an arbitrary state space and causality configuration. Let us consider two arbitrary $\Gamma^\top$- and A-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ which is to say two arbitrary zero- and pole-flipping transformations of the singularities of $W_0(z)$ associated with the desired state space and causality configuration for the model (1). Let P and Q be the solutions of the Riccati equations (4) and (7) corresponding to the invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ and consider the left lower path in the commutative diagram (11) so that the zero flipping is done after a pole flipping defined by Q. The relevant Riccati solution P_Q is given in formula (20) so that the desired realization of $W(z)$ can be written in a closed form as

$$W(z) := C_Q (zI - A_Q)^{-1} B_{P_Q} + D_{P_Q} \tag{24}$$

with

$$\begin{aligned} B_{P_Q} &:= (B_Q D_Q^\top + A_Q P_Q C_Q^\top)(D_Q D_Q^\top + C_Q P_Q C_Q^\top)^{-1/2} \\ D_{P_Q} &:= (D_Q D_Q^\top + C_Q P_Q C_Q^\top)^{1/2} \end{aligned}$$

where P_Q is given by (20), and $A_Q, B_Q, C_Q,$ and D_Q are given by (9).

B. Remarks

1) There is symmetry in flipping poles and zeros and indeed the roles of AREs (4) and (7) are completely interchangeable. Hence, an analogous formula holds for expressing Q_P in terms of Q and P and a dual procedure would work to obtain a realization such as (24) by following the upper right path of (11), i.e., computing P first and then performing the appropriate pole flipping defined by Q_P.

2) The commutative diagram (11) can be viewed as a restricted version of the following diagram corresponding to the "total" flipping of singularities:

$$\begin{array}{ccc} W_0 & \xrightarrow{P_+} & W_{P_+} \\ {\scriptstyle Q_+}\downarrow & & \downarrow{\scriptstyle Q_{P_+,+}} \\ W_{Q_+} & \xrightarrow[P_{Q_+,+}]{} & W_{++} \end{array}$$

where P_+ and Q_+ are the invertible solutions of AREs (4) and (7), respectively. Hence, P_+ flips *all* the zeros and Q_+ flips *all* the poles of W_0, $P_{Q_+,+}$ flips all the zeros of W_{Q_+}, and $Q_{P_+,+}$ flips all the poles of W_{P_+}. In this way, the spectral factor W_{++} has all the singularities reflected to reciprocal positions with respect to those of W_0.

3) We have developed our theory for the discrete-time case. With obvious modifications and a much simpler derivation, similar results hold for the continuous-time case.

4) Concerning the general causality structure considered in this paper, we may just hint at several possible connections, such as generalized time-symmetric stochastic models, as those considered in [15],

stationary processes defined on a one-dimensional spatial domain, or on *line transects* in two-dimensional spatial domains as those considered by Whittle [21]. In another direction, a general causality structure is involved in models of stationary *reciprocal processes* of which stationary processes on the line may be seen as limits when their boundary conditions are moved to $\pm\infty$.

IV. Conclusion

We have discussed the classification of general (not necessarily analytic outside of the unit circle) square spectral factors in terms of the solutions of two AREs. We have also described the construction of general spectral factors, which involve both zero and pole flipping on an arbitrary reference spectral factor.

Among the possible applications of our theory, we mention steady-state smoothing (see [10] and references therein). Indeed, the smoothing algorithm is based on two (causal and anticausal) models that must be computed in order to implement the Mayne–Fraser two-filter formula.

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