Representation and Factorization of Discrete-Time Rational All-Pass Functions

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***Abstract*—In this paper, we obtain a general characterization of discrete-time all-pass rational matrix functions from state-space representations. We establish a general characterization of the solutions of LMI's and Riccati equations. Finally, we derive a complete factorization theory of all-pass functions. Our results are obtained in the most general setting, without introducing any *ad hoc* assumption and can be applied to a variety of problems such as the discrete-time counterpart of the H^∞ model reduction problems solved by Glover in continuous-time.**

I. Introduction

Consider real rational, proper, square matrix-valued functions

$$K(z) := C(zI - A)^{-1}B + D \tag{1}$$

which are *all-pass*, namely such that $K(z)[K(1/z)]^\top = I$. When A is an asymptotically stable matrix, i.e. it has all its eigenvalues inside the unit disk, these functions are called *rational inner*. A rich literature dealing with the characterization and factorization of these functions flourished in the last four decades. Starting from the pioneering work on state space realization of scalar orthogonal filters [5], [16], [18], [19]–[21], there is now a quite complete theory available for matrix inner functions; besides [9] and the work referred to in the paper [17], one may look in Chap. 4 of the recent book [10].

In this paper we leave the well understood framework of stable matrices. This gives rise to a theory that is much more general but much harder to derive. In fact, several previous attempts made in this direction have always resorted to *ad hoc* facilitating assumptions such as D and/or A being non-singular as in many early references like [4] and [2, Chapter 7] (where J-all-pass functions are considered). Another typical assumption is *unmixing*, i.e. absence of reciprocal pairs of eigenvalues of A, which is a facilitating assumption used for example in the book [14]. Clearly unmixing is automatically guaranteed for inner functions where all the poles are inside the unit disk. We consider here the most general case and recover the results for A asymptotically stable, for A invertible and for A unmixed as particular cases.

We provide a completely general characterization and parameterization of discrete-time all-pass matrix functions and use this result to describe in full generality the geometry of the solution set of certain LMI's and of the associated Riccati equations. We also develop a factorization theory and related state-space procedures for the factorization of all-pass functions. Some of the results presented in the main theorem of the next section, parallel the continuous-time fundamental result of Glover's [12, Theorem 5.1] in the most general setting, without introducing any *ad hoc* assumption. A complete discussion of this problem seems to be presented here for the first time. The derivation of the discrete-time version of Glover's model reduction procedure is not a simple transposition of the arguments used in continuous-time as there are several differences which make the endeavor technically much harder. Some of these difficulties are well-known and can be recognized in the literature, for example in the paper [13]. The use of the often advocated Cayley transform (that maps a continuous-time system into a discrete-time one by transforming a continuous state matrix A_c into the discrete counterpart $A_d := (I - A_c)^{-1}(I + A_c)$) requires, for example, invertibility assumption on the matrices $(I - A_c)$ and $(A_d + I)$ which are not met in some applications (see for example the comment in [26, p. 1996]). In this respect, it seems to be an accepted point in the systems and control community that, as stated in [26, p. 559], "*. . . it is generally more appealing to give derivations in the coordinates of the original [discrete-time] data; also algorithms may be more reliable if generated for the specific model class*". *A case in point seems to be discrete-time H^∞ model reduction*. Apparently a discrete version of the continuous-time all-pass dilation of Glover under general hypotheses corresponding to those made in the present paper has been lacking. So far, to our best knowledge, the book literature of the last two or three decades, e.g. [1] or, [26], [11] seems to be just re-proposing continuous-time H^∞ model reduction *without directly addressing* a discrete-time version of Glover's theory.

The results of this paper have many possible applications. Applications to Hankel-norm approximation of rational discrete-time transfer functions may now be pursued by just following the route shown in the paper [12]. In Chapter 16 of the book [14] a slightly less general characterization of discrete all-pass functions is used to do Hankel-norm stochastic model approximation. Stochastic modeling without stability constraints is another direction which has been touched upon in [8], further exposed in

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[14] and can be addressed in wider generality by using the techniques described in this paper. This is a relatively unappreciated area of stochastic modeling which has several applications to smoothing and to non causal estimation [15]. We believe that this setting is worth understanding especially because of a very illuminating isomorphism with LQ control with an indefinite cost function.

The lay-out of this paper is as follows:

Section II contains the statement and proof of the main result. The proof is essentially self-contained save for a technical Lemma from [6] which considerably generalizes a result on controllability due to Wimmer [23], [24].

In Section III we introduce two dual linear matrix inequalities with a rank constraint which define families of square all-pass functions having a fixed pole structure. We prove a geometric characterization of all solutions in terms of A- or $A^\top$- invariant subspaces. When A is non singular these matrix inequalities turn into two dual homogeneous algebraic Riccati equations. A very exhaustive classification and description of the solutions of those Riccati equations is provided. It is well-known, see e.g. [25] that the analysis of algebraic Riccati equations can be reduced to that of homogeneous Riccati equations.

The study of families of solutions of the constrained LMI's of Section III unveils the basic principles and a direct method to characterize and classify the left- and right all pass factors of an arbitrary square all pass rational function. Rational factorization theory was first systematically discussed in the early book [3] quite heavily relying on the assumption of an invertible D matrix. Here we extend the factorization results of Fuhrmann and Hoffmann [9] derived for inner functions, under general hypotheses. When A is non-singular the classification can be given directly in terms of solutions of two dual homogeneous algebraic Riccati equations.

In the concluding section we indicate some possible generalizations to non square matrix functions.

Notation and background results: The image and the kernel of matrix M are denoted by $\mathrm{im}\,(M)$ and $\ker(M)$, respectively, while the transpose and the Moore-Penrose pseudo-inverse of M are denoted by $M^\top$ and M^+, respectively. A technical condition which is often referred to is that of *unmixing*. One says that $A \in \mathbb{R}^{n\times n}$ has *unmixed spectrum* or, briefly, is *unmixed* if it does not have reciprocal pairs $(\lambda, 1/\lambda)$ of eigenvalues. In particular an unmixed matrix cannot have eigenvalues of modulus one. Given a real rational matrix-valued function $K(z)$, we define the *conjugate* rational function

$$K^*(z) := [K(1/z)]^\top,$$

so that $K(z)$ is all-pass when $K(z)K^*(z) = I$. Hence all-pass functions are spectral factors of the identity. In particular, $K(z)$ restricted to the unit circle (i.e. for z such that $|z| = 1$) is a unitary matrix. In the setting of [15, Chap. 16], $K(z)$ can be interpreted as the transfer function of a (possibly a-causal) filter which maps a normalized white noise input into an output which is also a normalized white noise.

In our analysis a crucial role is played by a very powerful and (surprisingly) recent result on Stein (discrete-time Lyapunov) equations [6, Lemma 3.1]. This result will be used repeatedly in the paper so for the benefit of the reader, we shall recall its statement. Using our notation, the lemma reads as follows:

Lemma 1.1: Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$ be given and let Q be a solution of equation $A^\top QA - Q = C^\top C$. Then $\ker(Q)$ is A-invariant and $\ker(Q) \subset \ker(C)$.

This result implies that $\ker(Q)$ is contained in the unobservability subspace

$$\mathscr{N} := \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

of the pair (A, C). In fact, $\mathscr{N}$ can be characterised as the largest A-invariant subspace contained in $\ker(C)$. If A is unmixed, equation $A^\top QA - Q = C^\top C$ has a unique solution (see e.g. [24, Lemma 5.1]) Q, and it has long been known that its kernel is exactly $\mathscr{N}$ [24, Lemma 5.1]. In the general case however, it may have infinitely many solutions but each such solution has a kernel which is A-invariant and is contained in $\ker(C)$. In the extreme case when $A = I$ and $C = 0$, any matrix Q is a solution so that there exist solutions whose kernel is $\{0\}$ even if $\mathscr{N} = \mathbb{R}^n$. Of course, dual considerations can be made for the dual equation $APA^\top - P = BB^\top$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

Another concept worth recalling is that of *inertia* of a symmetric matrix $E = E^\top$. It is defined as the ordered triplet (n_+, n_-, n_0) of integers consisting of the numbers of positive, negative and zero eigenvalues of E (counting multiplicity). Recall that, if E is partitioned as $E = \left[\begin{smallmatrix} E_1 & E_{12} \\ E_{12}^\top & E_2 \end{smallmatrix}\right]$ and E_1 is invertible, one can define $T := \left[\begin{smallmatrix} I & -E_1^{-1}E_{12} \\ 0 & I \end{smallmatrix}\right]$ in such a way that $T^\top ET = \mathrm{diag}(E_1, E_2 - E_{12}^\top E_1^{-1}E_{12})$ so that the inertia of E is the (elementwise) sum of the inertia of E_1 and of the inertia of the *Schur complement* $E_2 - E_{12}^\top E_1^{-1}E_{12}$. Similarly, if E_2 is invertible, the inertia of E is the (elementwise) sum of the inertia of E_2 and of the inertia of the Schur complement $E_1 - E_{12}E_2^{-1}E_{12}^\top$.

II. The Main Result

Theorem 2.1

1) Let (1) be a minimal realization of an $m \times m$ rational discrete-time all-pass function. Then A is non-singular if and only if D is non-singular.
2) Let (1) be a minimal realization of a rational discrete-time all-pass function. Then, there exist
   a) a unique symmetric matrix $P = P^\top$ such that
$$\begin{cases} APA^\top - P = BB^\top \\ BD^\top - APC^\top = 0 \\ DD^\top - CPC^\top = I \end{cases} \tag{2}$$
   b) a unique symmetric matrix $Q = Q^\top$ such that
$$\begin{cases} A^\top QA - Q = C^\top C \\ C^\top D - A^\top QB = 0 \\ D^\top D - B^\top QB = I \end{cases} \tag{3}$$

The matrices P and Q are invertible and satisfy $PQ = I$.

3) Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$ be given (no minimality is now assumed). If there exists $P = P^\top$ satisfying (2) then $K(z)$ given by (1) is all-pass. Similarly, if there exists $Q = Q^\top$ satisfying (3) then $K(z)$ given by (1) is all-pass.
Finally, P is a non-singular solution of (2) if and only if P^{-1} is a (non-singular) solution of (3).

4) Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ be a given reachable pair. Then, $P = P^\top$ is such that

$$APA^\top - P = BB^\top \tag{4}$$

if and only if there exist matrices $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ such that (1) is a minimal realization of an all-pass function $K(z)$ and P is the solution of (2) for the quadruple (A, B, C, D). In this case, P is necessarily non-singular and such that

$$I + B^\top P^{-1} B \geq 0. \tag{5}$$

5) Let $A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{m\times n}$ be a given observable pair. Then, $Q = Q^\top$ is such that

$$A^\top QA - Q = C^\top C \tag{6}$$

if and only if there exist matrices $B \in \mathbb{R}^{n\times m}$ and $D \in \mathbb{R}^{m\times m}$ such that $K(z)$ given by (1) is a minimal realization of an all-pass function and Q is the solution of (3) for the quadruple (A, B, C, D). In this case, Q is necessarily non-singular and such that

$$I + CQ^{-1}C^\top \geq 0. \tag{7}$$

6) Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$ be given. If there exists $P = P^\top$ and $Q = Q^\top$ such that

$$\begin{cases} APA^\top - P = BB^\top \\ A^\top QA - Q = C^\top C \\ PQ = I \end{cases} \tag{8}$$

then there exists a matrix $D \in \mathbb{R}^{m\times m}$ such that $K(z)$ given by (1) is all-pass.

Proof:

1) By assumption we have

$$K(z)K^*(z) = I. \tag{9}$$

Notice that $K(\infty) = D$ so that, by taking the limit $z \to \infty$ in (9), we see that D is non-singular if and only if $K^*(z)$ is bounded at infinity or, equivalently, if and only if $K(z)$ is bounded in a neighborhood of the origin. Since (1) is a minimal realization, this is equivalent to A being non-singular.

2) Let us first assume that D is non-singular. By recalling point 1), we have that A is non-singular as well.

We have the following minimal realizations:

$$K(z)^{-1} = D^{-1} - D^{-1}C(zI - \Gamma)^{-1}BD^{-1},\ \Gamma := A - BD^{-1}C. \tag{10}$$

and

$$\begin{aligned} K^*(z) =&\ B^\top(z^{-1}I - A^\top)^{-1}C^\top + D^\top \\ =&\ D_0^\top - B^\top A^{-\top}(zI - A^{-\top})^{-1}A^{-\top}C^\top, \end{aligned} \tag{11}$$

with $D_0^\top := D^\top - B^\top A^{-\top}C^\top$, so that, by imposing $K(z)^{-1} = K^*(z)$, we conclude that there exists a unique invertible matrix T such that

$$T^{-1}A^{-\top}T = A - BD^{-1}C (= \Gamma) \tag{12a}$$

$$T^{-1}A^{-\top}C^\top = BD^{-1} \tag{12b}$$

$$B^\top A^{-\top}T = D^{-1}C \tag{12c}$$

$$D^{-1} = D^\top - B^\top A^{-\top}C^\top \tag{12d}$$

By inserting (12c) in (12a) and multiplying on the right side by $(A^{-\top}T)^{-1}$, we get

$$T^{-1} - AT^{-1}A^\top = -BB^\top \tag{13}$$

so that the first of (2) admits a solution $P = T^{-1}$. Moreover, by inserting the expression of D^{-1} provided by (12d) in (12b) we get $BD^\top = (T^{-1} + BB^\top)A^{-\top}C^\top$, which, in view of (13), may be written as $BD^\top = AT^{-1}C^\top$, so that $P = T^{-1}$ solves also the second of (2). Finally, by multiplying (12d) on the left side by D and taking into account of (12c), we easily see that $P = T^{-1}$ solves also the third of (2). Similarly we see that from (12) it follows that T solves the three equations (3). The proof that T is symmetric is a bit lengthy and is deferred to Appendix B.

So far we have established our result in the case when D is non-singular. We now show how this case may be viewed as a first step for proving the result in the general setting in which D may be singular. Consider an arbitrary rational proper all pass function $K(z)$ and the corresponding factorization (78) established in Lemma A.1 of the Appendix. Let $K_0(z) := C_0(zI - A_0)^{-1}B_0 + D_0$ be a minimal realization of $K_0(z)$ so that $D_0 = K_0(\infty)$ is non-singular. Then equations (2) with $A = A_0, B = B_0, C = C_0$ and $D = D_0$ have a symmetric solution P_0 which is non-singular. In view of Lemma A.2 we know that $K_1(z) := K_0(z)\bar{K}_1(z)$ has the reachable realization $K_1(z) = C_1\left(zI - A_1\right)^{-1} B_1 + D_1$ where

$$C_1 := [D_{0,2} \mid C_0], A_1 := \begin{bmatrix} 0 & 0 \\ B_{0,2} & A_0 \end{bmatrix}, B_1 := \begin{bmatrix} 0 & I \\ B_{0,1} & 0 \end{bmatrix} U_1,$$

and $D_1 := [D_{0,1} \mid 0]U_1$. Now it is immediate to check by inspection that

$$P_1 := \begin{bmatrix} -I & 0 \\ 0 & P_0 \end{bmatrix} \tag{14}$$

solves equations (2) with $A = A_1,\ B = B_1,\ C = C_1$ and $D = D_1$. We can iteratively repeat this argument for $K_i(z)$, $i = 2, 3, \ldots, k$ and eventually find that $K(z)$ has a reachable realization $K(z) = \bar{C}\left(zI - \bar{A}\right)^{-1}\bar{B} + D$ and that equations (2) with $A = \bar{A}$, $B = \bar{B}$, $C = \bar{C}$ and $D = D$, have a solution $\bar{P}$. Without loss of generality we may assume that $\bar{A}$, $\bar{B}$, $\bar{C}$ are in the Kalman reachability form

$$\bar{C} = [\tilde{C} \mid 0], \qquad \bar{A} = \begin{bmatrix} \tilde{A} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \qquad \bar{B} := \begin{bmatrix} \tilde{B} \\ B_2 \end{bmatrix} \tag{15}$$

and $\bar{P}$ is partitioned conformably as

$$\bar{P} = \begin{bmatrix} \tilde{P} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}. \tag{16}$$

By writing block-wise equations (2) with $A = \bar{A}$, $B = \bar{B}$, $C = \bar{C}$, $D = D$, and $P = \bar{P}$ we see that the $(1,1)$ block $\tilde{P}$ is a symmetric solution of equations (2) with $A = \tilde{A}$, $B = \tilde{B}$, $C = \tilde{C}$, $D = D$ corresponding to the minimal realization

$$K(z) = \tilde{C}\left(zI - \tilde{A}\right)^{-1}\tilde{B} + D. \tag{17}$$

The original minimal realization of $K(z)$ is related to (17) by a change of basis so that there exists a non-singular matrix T such that $A = T^{-1}\tilde{A}T$, $B = T^{-1}\tilde{B}$, $C = \tilde{C}T$. Then it is immediate to check that $P := T^{-1}\tilde{P}T^{-\top}$ is a solution of equations (2) for the original realization (1).

By resorting to a dual argument we establish the existence of a symmetric matrix $Q = Q^\top$ solving (3).

We now prove uniqueness. Assume that P_1 and P_2 are solutions of (2) and let $\Delta := P_1 - P_2$. We need to show that $\Delta = 0$. It is immediate to check that Δ satisfies the equations

$$A\Delta A^\top - \Delta = 0, \quad A\Delta C^\top = 0, \quad C\Delta C^\top = 0. \tag{18}$$

From the second and the third of these equations we see that $\operatorname{im}(\Delta C^\top)$ is contained in the non-observability subspace $\mathscr{N}$ of (A, C) and, since (A, C) is assumed to be observable, this means that $\Delta C^\top = 0$. This implies that $C\Delta = 0$ and, in turn, $CA^k\Delta(A^\top)^k = 0$ for all $k = 0, 1, \ldots, n-1$, so that $CA^k\Delta(A^\top)^n = 0$ for all $k = 0, 1, \ldots, n-1$. This means that $\operatorname{im}(\Delta(A^\top)^n)$ is contained in the unobservable subspace of (A, C), so that, as before, $\Delta(A^\top)^n = 0$. Now, by multiplying the first of (18) on the right side by $(A^\top)^{n-1}$, we get $\Delta(A^\top)^{n-1} = 0$ and, iteratively, $\Delta(A^\top)^{n-2} = 0$, and so on, up to $\Delta = 0$. The proof for equations (3) is dual and is therefore skipped.

We now address the non-singularity of P and Q. Indeed, from observability of the pair (A, C) and Lemma 1.1, it follows that Q is invertible. The invertibility of P follows by a dual argument. It remains to show that $PQ = I$. To this aim, write (2) in the form

$$FXF^\top = X \tag{19}$$

where

$$F := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad X := \begin{bmatrix} P & 0 \\ 0 & -I \end{bmatrix}. \tag{20}$$

Clearly, X is non-singular and

$$X^{-1} = \begin{bmatrix} P^{-1} & 0 \\ 0 & -I \end{bmatrix}. \tag{21}$$

Thus $FXF^\top X^{-1} = I$. Therefore, F is non-singular as well and we have $F^\top X^{-1} = X^{-1}F^{-1}$ or

$$F^\top X^{-1} F = X^{-1}. \tag{22}$$

The expression (21) of X^{-1} implies that P^{-1} is a solution of equations (3). These equations, however, admit a unique solution so that $P^{-1} = Q$ or, equivalently, $PQ = I$.

3) Assume that equations (2) admit a solution $P = P^\top$. Let us compute the product

$$\begin{aligned} \Phi :=\ & K(z)K^*(z) \\ =\ & [C(zI - A)^{-1}B + D][B^\top(z^{-1}I - A^\top)^{-1}C^\top + D^\top]. \end{aligned}$$

The first of equations (2) can be rewritten as

$$\begin{aligned} BB^\top =\ & (zI - A)P(z^{-1}I - A^\top) - zP(z^{-1}I - A^\top) \\ & -z^{-1}(zI - A)P, \end{aligned}$$

so that

$$\begin{aligned} & C(zI - A)^{-1}BB^\top(z^{-1}I - A^\top)^{-1}C^\top \\ & = CPC^\top - zC(zI - A)^{-1}PC^\top \\ & \quad - z^{-1}CP(z^{-1}I - A^\top)^{-1}C^\top. \end{aligned}$$

Moreover, from

$$z(zI - A)^{-1} = I + A(zI - A)^{-1} = I + (zI - A)^{-1}A$$

it follows that

$$\begin{aligned} & C(zI - A)^{-1}BB^\top(z^{-1}I - A^\top)^{-1}C^\top \\ & = -CPC^\top - C(zI - A)^{-1}APC^\top \\ & \quad - CPA^\top(z^{-1}I - A^\top)^{-1}C^\top. \end{aligned}$$

In conclusion, we have

$$\begin{aligned} K(z)K^*(z) =\ & DD^\top - CPC^\top \\ & + C(zI - A)^{-1}(BD^\top - APC^\top) \\ & + (DB^\top - CPA^\top)(z^{-1}I - A^\top)^{-1}C^\top. \end{aligned}$$

By taking into account the second and the third of equations (2), we now get $K(z)K^*(z) = I$.

Assume now that equations (3) admit a solution $Q = Q^\top$. By computing the product $K^*(z)K(z)$ and using the dual of the previous argument, we get $K^*(z)K(z) = I$.

The fact that P is a non-singular solution of (2) if and only if P^{-1} is a non-singular solution of (3) can be shown by defining F and X as in (20) and using the same argument that led to (22).

4) One direction is an immediate consequence of point 2). For the converse, as we have seen in the proof of point 2) invertibility of P only depends on reachability of the pair (A, B) and on the fact that P solves the *first* of equations (2) which is indeed (4). Therefore, P is invertible. Let $(n_+, n - n_+, 0)$ be the inertia of P which is equal to the inertia of $Q := P^{-1}$. Let

$$E := \begin{bmatrix} -Q & 0 & A^\top \\ 0 & I_m & B^\top \\ A & B & -Q^{-1} \end{bmatrix}.$$

The inertia of E is given by the inertia of $\left[\begin{smallmatrix} -Q & 0 \\ 0 & I_m \end{smallmatrix}\right]$, i.e. $(m + n - n_+, n_+, 0)$, plus the inertia of the corresponding

Schur complement S which is given by

$$\begin{aligned} S :=& -Q^{-1} - [A \;\; B] \begin{bmatrix} -Q & 0 \\ 0 & I_m \end{bmatrix}^{-1} \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \\ =& -P + APA^\top - BB^\top = 0_{n\times n}. \end{aligned}$$

In conclusion, the inertia of E is $(m+n-n_+, n_+, n)$. On the other hand, the inertia of E is also given by the inertia of $-Q^{-1} = -P$, i.e. $(n-n_+, n_+, 0)$, plus the inertia of the corresponding Schur complement W which is given by

$$\begin{aligned} W :=& \begin{bmatrix} -Q & 0 \\ 0 & I_m \end{bmatrix} - \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} (-Q^{-1})^{-1} [A \;\; B] \\ =& \begin{bmatrix} A^\top QA - Q & A^\top QB \\ B^\top QA & B^\top QB + I \end{bmatrix}. \end{aligned} \tag{23}$$

Hence the inertia of W is given by the inertia of E, i.e. $(m+n-n_+, n_+, n)$ minus the inertia of $-Q^{-1} = -P$, i.e. $(n-n_+, n_+, 0)$, which amounts to $(m, 0, n)$. Thus, W is positive semidefinite and has rank equal to m. Therefore, there exists a full row-rank matrix $[C \mid D] \in \mathbb{R}^{m\times(n+m)}$ such that $W = [C \mid D]^\top [C \mid D]$. This means that for the given A and B and for the C and D obtained above, there exists a $Q = P^{-1}$ solving (3). Then, in view of point 3), the corresponding $K(z)$ given by (1) is all-pass.

We now prove (5). Indeed, we have already proved that P^{-1} solves (3) and from the third of these equations (5) follows immediately.

It remains to show that (A, C) is an observable pair. To address this issue we exploit (3) whose validity we have already proven. Assume now by contradiction that the pair (A, C) is not observable and let V be a full column-rank matrix whose columns (at least one by the contradiction assumption) form a basis for the unobservable subspace $\mathscr{N}$ of the pair (A, C). Since $\mathscr{N}$ is A-invariant, there exists a matrix K such that

$$AV = VK. \tag{24}$$

By multiplying the first of (3) on the right side by V we get $A^\top QAV = QV$. We now multiply the first of (3) on the right side by AV and on the left side by $A^\top$: We get $(A^\top)^2 QA^2 V = A^\top QAV = QV$. We can iterate this argument and multiply the first of (3) on the right side by $A^k V$ and on the left side by $(A^\top)^k$, $k = 2, 3, \ldots$, getting

$$(A^\top)^l QA^l V = QV, \qquad l = 1, 2, \ldots. \tag{25}$$

We now show that

$$U := QA^n V \neq 0, \tag{26}$$

where n is the dimension of A. In fact, from (25) we get $(A^\top)^n U = (A^\top)^n QA^n V = QV$ and since Q is non-singular and V has full column-rank this yields (26). We now consider the second of equations (3). From this equation, we get $D^\top C = B^\top QA$, and by right-multiplication by V, we get

$$B^\top QAV = 0 \tag{27}$$

so that

$$B^\top QA^l V = B^\top QAVK^{l-1} = 0, \qquad l = 1, 2, \ldots, n-1. \tag{28}$$

Thus, for any $l = 0, 1, \ldots, n-1$, we have

$$\begin{aligned} B^\top (A^\top)^l U =& B^\top (A^\top)^l QA^n V = B^\top (A^\top)^l QA^l VK^{n-l} \\ =& B^\top QVK^{n-l} = B^\top QAVK^{n-l-1} = 0. \end{aligned} \tag{29}$$

In conclusion, $\operatorname{im}(U) \neq \{0\}$ is contained in the unobservable subspace of the pair $(A^\top, B^\top)$ and this is a contradiction because (A, B) is, by assumption, reachable, so that $(A^\top, B^\top)$ is observable.

5) This point is the dual of the previous one.

6) Since P is clearly invertible we can use the same argument employed in the proof of point 4) to show that

$$\begin{aligned} W :=& \begin{bmatrix} -Q & 0 \\ 0 & I_m \end{bmatrix} - \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} (-Q^{-1})^{-1} [A \;\; B] \\ =& \begin{bmatrix} A^\top QA - Q & A^\top QB \\ B^\top QA & B^\top QB + I \end{bmatrix} \end{aligned}$$

is positive semidefinite and has rank equal to m. Therefore, there exists a full row-rank matrix $[C_0 \mid D_0] \in \mathbb{R}^{m\times(n+m)}$ such that $W = [C_0 \mid D_0]^\top [C_0 \mid D_0]$. In particular,

$$A^\top QA - Q = C^\top C = C_0^\top C_0$$

so that there exists an orthogonal matrix U such that $C = UC_0$. Let $D := UD_0$. Therefore,

$$\begin{aligned} W :=& [C_0 \mid D_0]^\top [C_0 \mid D_0] \\ =& [C_0 \mid D_0]^\top U^\top U [C_0 \mid D_0] = [C \mid D]^\top [C \mid D]. \end{aligned}$$

In conclusion, we have

$$D^\top D = I + B^\top QB \tag{30}$$

and

$$D^\top C = B^\top QA \tag{31}$$

These two equations together with the second of (8) give (3) and hence, in view of point 3), $K(z) = C(zI - A)^{-1}B + D$ is all-pass. ∎

Remark 2.1: We shall now pinpoint similarities and differences with the inner case. When A is asymptotically stable, an all-pass filter is actually inner. In this case the results of Theorem 2.1 become much easier to derive. In fact, in the inner setting, the key idea (which is actually just based on Parseval relation), described in [16] for the scalar case (but easily extendable to the matrix case), that $K(z)K^*(z) = 1$ is equivalent to the impulse response sequence of the filter being an orthonormal sequence, requires asymptotic stability and makes only sense in a ℓ^2-type context. In the language of dissipative systems this is equivalent to the existence of a *lossless* state space realization as was also noticed by [20]. This is definitely not the case when A is not asymptotically stable.

In the inner case, again by stability, the matrix $-P$ is positive definite[1] and is indeed the reachability Gramian of the system. Likewise $-Q$ is positive definite and is the observability Gramian. In the general all-pass case the impulse response need not be in ℓ^2, orthogonality of the impulse response does not make sense and P and Q are in general indefinite and do not have a Gramian interpretation. Moreover in the inner case, where $-P$ is positive definite and can therefore be factored as $-P = TT^\top$, after a change of basis in the state space of $K(z)$ induced by the matrix T^{-1}, equation (19) reads as $FF^\top = I$, showing that F is an orthogonal matrix. As a consequence, in this basis, it becomes apparent that, in the words of [20], "the instantaneous output energy plus the instantaneous increase in state energy is precisely equal to the instantaneous input energy." This "energy balance" is valid only for the inner case in which the relation $FF^\top = I$ implies that all the eigenvalues of F have modulus equal to 1. In our general case we only have (19) which clearly implies that F is similar to $F^{-\top}$ so that we can only conclude that the eigenvalues of F come in reciprocal pairs $(\lambda, 1/\lambda)$ which is a much weaker form of balancing.

Remark 2.2: In point 4) of Theorem 2.1 the assumption of reachability of (A, B) can probably be eliminated for the first part of the result. More precisely, we suggest the following conjecture whose proof, however, seems to be non-trivial.

Conjecture 2.1: Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ be given. Then, there exists $P = P^\top$ such that $APA^\top - P = BB^\top$ if and only if there exist matrices $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ such that (1) is an observable realization of an all-pass function. Of course, a dual conjecture could be made for point 5).

Remark 2.3: Consider an all-pass function $K(z)$ represented by (1). Clearly, for any orthogonal matrix U, $K(z)U$ is still all-pass. The two functions $K(z)$ and $K(z)U = C(zI - A)^{-1}BU + DU$ correspond to the same dynamics and it is natural to regard them as equivalent. By considering the polar decomposition of D we see that for any given D there is a unique $D_0 = DU$ such that $D_0 = D_0^\top \geq 0$. Therefore, from now on, whenever convenient, we can safely assume, without loss of generality, that the "D" matrix of the all-pass function $K(z)$ is symmetric and positive semidefinite.

Remark 2.4: Consider point 4) (or 5)) of Theorem 2.1. If A is unmixed, once A and B are given, the solution P of (4) is uniquely determined and hence also the matrices C and D for which $K(z) = C(zI - A)^{-1}B + D$ is all-pass are uniquely determined up to multiplication on the left side by a common orthogonal matrix. This is not the case when A is not unmixed. In this case, for any particular solution P of (4) there exists a particular pair of matrices C and D (essentially different, i.e. not differing by multiplication on the left side by a common orthogonal matrix) for which $K(z) = C(zI - A)^{-1}B + D$ is all-pass. Notice, however, that, once A, B *and* P are fixed, the matrices C and D are always uniquely determined up to multiplication on the left side by a common orthogonal matrix.

Similar considerations can be made for 5). For example, let $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right]$ and $C = I$. In this case the set of all solutions Q of (6) can be parametrized as $Q = \left[\begin{smallmatrix} 1/3 & q \\ q & -4/3 \end{smallmatrix}\right]$ with q being a real parameter. For example, for $q = 0$, we get $B_0 = \left[\begin{smallmatrix} 3 & 0 \\ 0 & -3/4 \end{smallmatrix}\right]$ and $D_0 = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right]$, where the degree of freedom corresponding to the choice of an arbitrary orthogonal matrix multiplying both B_0 and D_0 on the right side, has been fixed in such a way that $D_0 = D_0^\top \geq 0$. Since D_0 is non-singular this procedure does not leave any further degree of freedom. For $q = 1/6$, we get $B_{1/6} = \left[\begin{smallmatrix} 2.85 & 0.57 \\ 0.14 & -0.71 \end{smallmatrix}\right]$ and $D_{1/6} = \left[\begin{smallmatrix} 1.95 & 0.14 \\ 0.14 & 0.52 \end{smallmatrix}\right]$, where, again, the degree of freedom corresponding to the arbitrary orthogonal matrix has been fixed in such a way that $D_{1/6} = D_{1/6}^\top \geq 0$. In conclusion, the two solutions corresponding to $q = 0$ and $q = 1/6$ lead to all-pass functions with different dynamical properties.

Corollary 2.1: A square matrix A can be the state matrix of a minimal realization of an all-pass function if and only if none of the eigenvalues of A has modulus equal to 1.

Proof: Let us first assume that none of the eigenvalues of A has modulus 1. Without loss of generality, we can choose a basis in which $A = \mathrm{diag}(A_s, A_u)$, where all eigenvalues of A_s have modulus smaller than 1 and all eigenvalues of A_u have modulus greater than 1. Choose B to be the identity and partition it conformably with A so as $B = \mathrm{diag}(I, I)$; it is easy to see that the pair (A, B) is reachable and that Equation (4) admits a symmetric, block-diagonal solution $P = \mathrm{diag}(P_s, P_u)$. Therefore, in view of point 4 of Theorem 2.1, the pair (A, B) can be completed to a quadruple (A, B, C, D) of matrices providing a minimal realization of an all-pass function. Conversely, assume that A has an eigenvalue of modulus 1 and let v be the corresponding eigenvector and v^* be its transpose conjugate. By multiplying the first of Equations (3) on the right side by v and on the left side by v^*, we get $v^*C^\top Cv = 0$ so that $Cv = 0$. Thus v is in the null space of C and hence the pair (A, C) is not observable. In conclusion, A cannot be the state matrix of a minimal realization of an all-pass function. ■

Remark 2.5: Consider a minimal realization (1) of an all-pass function $K(z)$. Clearly, there is a one-to-one correspondence between the poles of $K(z)$ and the eigenvalues of the A matrix. Therefore, as a consequence of Corollary 2.1, an all-pass function may well feature pairs of reciprocal poles and, in general, a given constellation of points in the complex plane (naturally having complex-conjugate symmetry) can be chosen as the set poles of an all-pass function if and only if none of these points has modulus 1. Futhermore, since the zeros of an all-pass function are the reciprocals of its poles, it is also clear none of the zeros of an all-pass function can have modulus 1.

However, the existence of all-pass functions with pairs of reciprocal poles (or zeros) is a possibility that can only happen in the multivariate case. For scalar all-pass functions, cancellation clearly prevents the presence of such pairs of poles (or zeros).

[1]The reason for our choice of sign, namely for not replacing P with $-P$ and Q with $-Q$ is that on one hand, in the general case P and Q are, anyway, indefinite and, on the other hand, our choice leads to the algebraic Riccati equations (52) and (53) that have the standard form of those arising in optimal control and Kalman filtering.

Indeed, a rational scalar function cannot feature a pole and a zero in the same point of the complex plane while this is possible in the multivariate case.

III. LMI's and Homogeneous Algebraic Riccati Equations

All-pass functions can be seen as spectral factors of a spectral density function identically equal to the identity matrix; i.e. $\Phi(z) \equiv I$. This point of view turns out to be useful for classification of all-pass functions having a pre-assigned pole dynamics. It is a classical result in system and control theory [22] that rational spectral factorization can be cast in terms of linear matrix inequalities (LMI). This point of view will be used here. In this section we shall consider square spectral factors which are all-pass.

To fix the pole dynamics we may either assign a reachable pair $(A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m})$ or an observable pair $(C \in \mathbb{R}^{m\times n},\ A \in \mathbb{R}^{n\times n})$. These are two "dual" structural data which will be fixed hereafter. Accordingly, define

$$M(P) := \begin{bmatrix} APA^\top - P & APC^\top \\ CPA^\top & CPC^\top + I \end{bmatrix},$$

and

$$N(Q) := \begin{bmatrix} A^\top QA - Q & A^\top QB \\ B^\top QA & B^\top QB + I \end{bmatrix}$$

and consider the two dual, constrained linear matrix inequalities (CLMI),

$$\begin{cases} M(P) \geq 0 \\ \operatorname{rank}[M(P)] = m \end{cases} \tag{32}$$

$$\begin{cases} N(Q) \geq 0 \\ \operatorname{rank}[N(Q)] = m \end{cases} \tag{33}$$

The following is an immediate corollary of Theorem 2.1.

Corollary 3.1: Let $P = P^\top$ be a solution of (32) and let

$$M(P) = \begin{bmatrix} G \\ L \end{bmatrix} \begin{bmatrix} G^\top & L^\top \end{bmatrix} \tag{34}$$

be a factorization of full rank m. Then

$$K_L(z) := C(zI - A)^{-1}G + L \tag{35}$$

is a (in general non minimal) realization of a square all-pass function. Dually, let $Q = Q^\top$ be a solution of (33) and let

$$N(Q) = \begin{bmatrix} H & J \end{bmatrix}^\top \begin{bmatrix} H & J \end{bmatrix} \tag{36}$$

be a factorization of full rank m. Then

$$K_R(z) := H(zI - A)^{-1}B + J \tag{37}$$

is a realization of a square all-pass function.

Clearly $P = 0$ and $Q = 0$ are always solutions of the inequalities (32) and (33) and it may well happen that these inequalities admit no other solutions save for these trivial ones. We need to exclude these uninteresting circumstances. We shall henceforth assume that there is a D such that the matrix function with (minimal) realization

$$K(z) := C(zI - A)^{-1}B + D \tag{38}$$

is all-pass. By Theorem 2.1 this happens if and only if equations (34) with $G = B$ and $L = D$ hold for a nonsingular $P \equiv P_0$ or, equivalently, if and only if (36) with $H = C$ and $J = D$ hold for a nonsingular $Q \equiv Q_0$. In fact, P_0 and Q_0 turn out to be such that $P_0 Q_0 = I$. In the next section it will be shown that each $K_R(z)$ is a right factor of $K(z)$ and each $K_L(z)$ is a left factor of $K(z)$.

Notational convention: From now on, the problem data will be a minimal realization (A, B, C, D) of a square all-pass function as in (38). The unique solutions of (2) and (3) will be denoted by P_0 and Q_0, respectively and we shall reserve the symbols P and Q for generic solutions of (32) and (33).

Theorem 3.1: Let (38) be a minimal realization of a square all-pass function. Then

1) (i) For each solution $P = P^\top$ of (32), the subspace

$$\mathcal{Y} = \ker(P) \tag{39}$$

is $A^\top$-invariant.

(ii) The set of non-singular solutions of (32) can be parametrized as:

$$\mathbb{P} = \{P_\Delta : \Delta \in \mathcal{D}_p\}, \tag{40}$$

where $P_\Delta := (P_0^{-1} + \Delta)^{-1}$, P_0 is the unique solution of (2), and $\mathcal{D}_p$ is the vector space of solutions of $A^\top \Delta A - \Delta = 0$. If A is unmixed, then $\mathcal{D}_p = \{0\}$ and (32) admits a unique non-singular solution $P_\Delta = P_0$, which is the unique solution of (2). If A is not unmixed, then $\mathbb{P}$ contains infinitely many solutions.

(iii) Let P_Δ be a non-singular solution of (32); then to any $A^\top$-invariant subspace $\mathcal{Y}$ there corresponds a solution P of (32) given by

$$P := \left[(I - \Pi)P_\Delta^{-1}(I - \Pi)\right]^+ \tag{41}$$

where Π is the orthogonal projector onto $\mathcal{Y}$. The kernel of P is $\mathcal{Y}$. If A is unmixed, equation (41), with $P_\Delta = P_0$ being the unique solution of (2), parametrizes the set of all solutions of (32) in terms of $A^\top$-invariant subspaces.

2) (i) For each solution $Q = Q^\top$ of (33), the subspace

$$\mathcal{X} = \ker(Q). \tag{42}$$

is A-invariant.

(ii) The set $\mathbb{Q}$ of non-singular solutions of (33) can be parametrized as:

$$\mathbb{Q} = \{Q_\Delta : \Delta \in \mathcal{D}_q\} \tag{43}$$

where $Q_\Delta := (Q_0^{-1} + \Delta)^{-1}$, Q_0 is the unique solution of (3), and $\mathcal{D}_q$ is the vector space of solutions of $A\Delta A^\top - \Delta = 0$. If A is unmixed, then $\mathcal{D}_q = \{0\}$ and (33) admits a unique non-singular solution $Q_\Delta = Q_0$, which is the unique solution of (3). If A is not unmixed, than $\mathbb{Q}$ contains infinitely many solutions.

(iii) Let Q_Δ be a non-singular solution of (33), then to any A-invariant subspace $\mathcal{X}$, there corresponds a solution

Q of (33) given by

$$Q := \left[(I-\Pi)Q_\Delta^{-1}(I-\Pi)\right]^+ \tag{44}$$

where Π is the orthogonal projector onto $\mathcal{X}$. The kernel of Q is $\mathcal{X}$. If A is unmixed, equation (44), with $Q_\Delta = Q_0$ being the unique solution of (3), parametrizes the set of all solutions of (33) in terms of A-invariant subspaces.

Proof: We prove only point 2), as the proof of point 1) is dual.

(i) It is clear that (33) is equivalent to existence of two matrices $H \in \mathbb{R}^{m\times n}$ and $J \in \mathbb{R}^{m\times m}$ such that $[H \mid J]$ has full row-rank and $N(Q) = [H \mid J]^\top [H \mid J]$. Therefore, if Q is a solution of (33) then $A^\top QA - Q = H^\top H$, so that, in view of [6, Lemma 3.1], $\mathcal{X} := \ker(Q)$ is A-invariant.

(ii) Clearly the solution Q_0 of (3) is a non-singular solution of (33) and the corresponding matrices H and J, introduced before, coincide with C and D of (38). Then in view of Theorem 2.1, point 3), we have

$$AQ_0^{-1}A^\top - Q_0^{-1} = BB^\top. \tag{45}$$

Let now $\tilde{Q}_0$ be another non-singular solution of (33) and C_0 and D_0 be such that $N(\tilde{Q}_0) = [C_0 \mid D_0]^\top [C_0 \mid D_0]$. Equivalently, $\tilde{Q}_0$ is a non-singular solution of (3) corresponding to the quadruple (A, B, C_0, D_0). Using again Theorem 2.1, point 3), we have that $\tilde{Q}_0^{-1}$ is a solution of (2) corresponding to the same quadruple, so that, in particular, $A\tilde{Q}_0^{-1}A^\top - \tilde{Q}_0^{-1} = BB^\top$. Comparing the latter with (45), we see that $\tilde{Q}_0^{-1} = Q_0^{-1} + \Delta$ where Δ is a solution of the homogeneous Lyapunov equation $A\Delta A^\top - \Delta = 0$. If A is unmixed, this equation has a unique solution $\Delta = 0$ so that $\tilde{Q}_0 = Q_0$.

Assume now that A is not unmixed. Then equation $A\Delta A^\top - \Delta = 0$ has a non-trivial vector space $\mathcal{D}_q$ of solutions and the previous argument shows that any non-singular solution of (33) has the form $(Q_0^{-1} + \Delta)^{-1}$. It remains to show that for each $\Delta \in \mathcal{D}_q$, $(Q_0^{-1} + \Delta)$ is nonsingular and $(Q_0^{-1} + \Delta)^{-1}$ is a solution of (33). Observe that $A[Q_0^{-1} + \Delta]A^\top - [Q_0^{-1} + \Delta] = BB^\top$ for any $\Delta \in \mathcal{D}_q$. Since (A, B) is reachable, any $\tilde{P}_\Delta := Q_0^{-1} + \Delta$ is invertible by the dual of [6, Lemma 3.1] and, in view of Theorem 2.1, point 4), there exist two matrices C_Δ and D_Δ such that $C_\Delta(zI - A)^{-1}B + D_\Delta$ is a minimal realization of a rational all-pass function and $\tilde{P}_\Delta$ is the solution of (2) corresponding to the quadruple $(A, B, C_\Delta, D_\Delta)$. This is equivalent to $Q_\Delta := \tilde{P}_\Delta^{-1} = (Q_0^{-1} + \Delta)^{-1}$ solving (3) for the same quadruple so that Q_Δ is a solution of (33) which therefore has infinitely many solutions.

(iii) Let $\mathcal{X}$ be an A-invariant subspace. Consider an orthogonal change of basis induced by the matrix $T = [V_\perp \mid V]$, where the columns of V form a basis for $\mathcal{X}$ and the columns of $V_\perp$ form a basis for $\mathcal{X}^\perp$. In this basis we have

$$T^\top \mathcal{X} = \operatorname{im} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{46}$$

and

$$\bar{A} := T^{-1}AT = T^\top AT = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix}. \tag{47}$$

Partition $\bar{B} := T^{-1}B = T^\top B$ conformably as $\bar{B} = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$. Let Q_Δ be a non-singular solution of (33) and let C_Δ and D_Δ be such that $N(Q_\Delta) = [C_\Delta \mid D_\Delta]^\top [C_\Delta \mid D_\Delta]$. Equivalently, Q_Δ is the non-singular solution of (3) corresponding to an all-pass function described by the quadruple $(A, B, C_\Delta, D_\Delta)$. Hence, in the new basis $\bar{Q}_\Delta := T^\top Q_\Delta T$ is a non-singular solution of (3) corresponding to the quadruple $(\bar{A}, \bar{B}, \bar{C}_\Delta, D_\Delta)$, with $\bar{C}_\Delta := C_\Delta T$. In view of Theorem 2.1, point 3), $\bar{Q}_\Delta^{-1}$ is a non-singular solution of (2) corresponding to the same quadruple. Partition such a $\bar{Q}_\Delta^{-1}$ conformably with $\bar{A}$ as

$$\bar{Q}_\Delta^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} \tag{48}$$

and note that it must in particular satisfy the first equation of (2) so that the block of index (1,1) must satisfy the reduced Lyapunov equation

$$A_1 P_{11} A_1^\top = P_{11} + B_1 B_1^\top. \tag{49}$$

Since the pair (A, B) is reachable, the pair (A_1, B_1) is reachable as well, so that from Theorem 2.1, point 4), it follows that P_{11} is invertible and there exist C_1 and D_1 such that P_{11} is the unique solution of (2) corresponding to a reduced quadruple (A_1, B_1, C_1, D_1) and hence, P_{11}^{-1} is the unique solution of (3) corresponding to the same quadruple. It is now a matter of direct computation to check that

$$\bar{Q} := \begin{bmatrix} P_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{50}$$

is a solution of (3) corresponding to the quadruple $(\bar{A}, \bar{B}, [C_1 \mid 0], D_1)$. Therefore, $Q := T\bar{Q}T^\top$ is a solution of (3) corresponding to the quadruple $(A, B, [C_1 \mid 0]T^\top, D_1)$ and hence, it is also a solution of (33). The fact that $\ker[Q] = \mathcal{X}$ is a direct consequence of (50). We need to show that (44) is a coordinate-free representation of Q. By observing that $T^\top \Pi T = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ and

$$\begin{aligned} (I-\Pi)Q_\Delta^{-1}(I-\Pi) &= (I-\Pi)TT^\top Q_\Delta^{-1} TT^\top (I-\Pi) \\ &= (I-\Pi)T \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} T^\top (I-\Pi), \end{aligned}$$

it is a straightforward computation to show that

$$\left[(I-\Pi)Q_\Delta^{-1}(I-\Pi)\right]^+ = T \begin{bmatrix} P_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^\top = Q. \tag{51}$$

The last thing that remains to be proven is the fact that when A is unmixed, *all* solutions of (33) are parametrized in terms of A-invariant subspaces by (44), with $Q_\Delta = Q_0$. We have already shown that in this case (33) has a unique non-singular solution which coincides with the unique solution Q_0 of (3). The representation of the other (singular) solutions can be obtained by a procedure similar to the one introduced above. Indeed, assume that Q is a singular solution of (33) and let H, and J be such that $N(Q) = [H \mid J]^\top [H \mid J]$. As already proved, $\ker[Q]$ is A-invariant so that we can perform a change of coordinates such that in the new basis Q has the structure of the right-hand side of (50), with P_{11} being a non-singular matrix, A has the structure of the right-hand side of (47), and $B = [B_1^\top \mid B_2^\top]^\top$ and $H = [H_1 \mid H_2]$ are partitioned conformably. It is now a matter of direct computation to check that P_{11}^{-1} is a solution of (3) corresponding to the quadruple (A_1, B_1, H_1, J) so that P_{11} is a solution of (2) corresponding to the same quadruple. Hence, P_{11} satisfies the Lyapunov equation $A_1 P_{11} A_1^\top - P_{11} = B_1 B_1^\top$. But since A is unmixed, A_1 is also unmixed so that P_{11} is uniquely determined by the Lyapunov equation. As a consequence, there is a unique Q with the given kernel which necessarily coincides with the one given by the right-hand side of (44) with $Q_\Delta = Q_0$ and $\mathcal{X} = \ker[Q]$. ■

Remark 3.1: Let $\mathcal{P}_\Delta$ and $\mathcal{Q}_\Delta$ denote the set of solutions of (32) and (33) described by (41) and (44) for a specific Δ. While when A is unmixed (and hence $\mathcal{D}_p = \mathcal{D}_q = \{0\}$ so that we necessarily have $\Delta = 0$) the families $\mathcal{P}_0$ and $\mathcal{Q}_0$ constitute the entire set of solutions of the LMI's (32) and (33), it is not clear if this also holds for the case of a mixed A even if one takes the union of the sets $\mathcal{P}_\Delta$ with respect to $\Delta \in \mathcal{D}_p$ or the union of the sets $\mathcal{Q}_\Delta$ with respect to $\Delta \in \mathcal{D}_q$. The theorem provides a bijective correspondence between the family $\mathcal{Q}_0$ of solutions of (33) and the family of A-invariant subspaces. When A is not unmixed, (33), besides $\mathcal{Q}_0$, has infinitely many other families of solutions each of which being likewise parametrized by A-invariant subspaces. Each of these families corresponds to a non-singular solution $Q_\Delta \in \mathbb{Q}$ of (33) where $\mathbb{Q}$ is the set of non-singular solutions parametrized by (43). The family $\mathcal{Q}_0$ corresponding to $\Delta = 0$ will play an important role in the following.

Similar considerations can be made for the dual family $\mathcal{P}_0$ of solutions of (32) which, in case of unmixed A constitutes the set of all solutions of (32) and in case of a mixed A is just one of infinitely many families of solutions of (32).

Remark 3.2: There is an obvious bijective correspondence between the set of A-invariant subspaces and that of $A^\top$-invariant subspaces. Indeed, $\mathcal{X}$ is A-invariant if and only if $\mathcal{Y} := \mathcal{X}^\perp$ is $A^\top$-invariant. This correspondence induces a bijective correspondence between the sets $\mathcal{P}_0$ and $\mathcal{Q}_0$. In fact, to any solution $Q = \left[(I-\Pi)Q_0^{-1}(I-\Pi)\right]^+ \in \mathcal{Q}_0$ there corresponds a solution $P = \left[\Pi P_0^{-1}\Pi\right]^+ \in \mathcal{P}_0$. To see this, just note that the orthogonal projector $\Pi_\mathcal{Y}$ onto $\mathcal{Y} := \mathcal{X}^\perp$ is equal to $(I - \Pi)$, with Π being the orthogonal projector onto $\mathcal{X}$. In this case we shall call P and Q *complementary solutions* of the LMI's (32) and (33). Indeed for complementary solutions we have

$$\operatorname{rank} P + \operatorname{rank} Q = n\,.$$

Of course, when A is not unmixed, a similar correspondence holds for any pair of families $\mathcal{P}_\Delta$ and $\mathcal{Q}_{\Delta'}$ of solutions of (32) and (33) respectively, where $\mathcal{P}_\Delta$ is the family corresponding to a certain $P_\Delta \in \mathbb{P}$ and $\mathcal{Q}_{\Delta'}$ is the family corresponding to $Q_{\Delta'} := P_\Delta^{-1} \in \mathbb{Q}$ (with $\Delta' := P_\Delta - P_0$).

A. The Case of Non-Singular A: Riccati Equations

In case of a non-singular A matrix, equations (32) and (33) reduce, respectively, to the following homogeneous algebraic Riccati equations (ARE)

$$P = APA^\top - APC^\top(I + CPC^\top)^{-1}CPA^\top \tag{52}$$

and

$$Q = A^\top QA - A^\top QB(I + B^\top QB)^{-1}B^\top QA. \tag{53}$$

The equivalence of the two representations is stated in the following proposition.

Proposition 3.1: Let (38) be a minimal realization of a rational discrete-time all-pass function and assume that A is non-singular. Then $P = P^\top$ is a solution of (32) if and only if it is a solution of (52) and $Q = Q^\top$ is a solution of (33) if and only if it is a solution of (53).

Proof: We prove only the equivalence of (33) and (53) as the other equivalence is dual. Let Q be a solution of (33). Then there exist $H \in \mathbb{R}^{m\times n}$ and $J \in \mathbb{R}^{m\times m}$ such that $N(Q) = [H \mid J]^\top [H \mid J]$. In view of Theorem 2.1, point 3), $H(zI - A)^{-1}B + J$ is all-pass. After eliminating the non-observable part of this realization we obtain a minimal realization say $\bar{C}(zI - \bar{A})^{-1}\bar{B} + J$ of the same all-pass function where the $\bar{A}$ matrix clearly remains non-singular. This, in particular implies that J is also non-singular so that $I + B^\top QB = J^\top J$ is strictly positive definite and hence invertible. Then, $\operatorname{rank}[N(Q)] = m$ implies that the Schur complement of $I + B^\top QB$ in $N(Q)$ vanishes which is equivalent to Q being a solution of (53). Conversely, let $Q = Q^\top$ be an arbitrary solution of (53). To show that Q satisfies the LMI (33) it is enough to show that $I + B^\top QB$ is positive semi-definite and, hence, positive defnite. In fact, in this case we can use, in the opposite direction, the previous argument based on the Schur complement. The Riccati equation (53) can be written as

$$QA^{-1} = A^\top Q - A^\top QB(I + B^\top QB)^{-1}B^\top Q \tag{54}$$

from which it is easy to see that $\ker(Q)$ is A^{-1}-invariant and hence A-invariant. Select a basis where A has the form shown in the right-hand side of (47), $B = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$ is partitioned conformably and Q has the same structure of the right-hand side of (50) where P_{11} is non singular so that $Q_{11} := P_{11}^{-1}$ is also non-singular. Then substituting $Q = \operatorname{diag}(Q_{11},\, 0)$ into (53) it is immediate to see that P_{11}^{-1} satisfies

$$P_{11}^{-1} = A_1^\top P_{11}^{-1} A_1 - A_1^\top P_{11}^{-1} B_1 (I + B_1^\top P_{11}^{-1} B_1)^{-1} B_1^\top P_{11}^{-1} A_1 \tag{55}$$

so that, by using the Sherman-Morrison-Woodbury formula, we get $A_1 P_{11} A_1^\top = P_{11} + B_1 B_1^\top$. Since (A, B) is reachable, (A_1, B_1) is also reachable and Theorem 2.1, point 4), implies that $I + B_1^\top Q_{11} B_1$ is positive semidefinite. Observing that $I + B^\top QB = I + B_1^\top Q_{11} B_1$ concludes the proof. ■

Notice that the ARE's (52) and (53) do not impose explicitly any positivity condition: the previous result shows that these conditions are automatically met when A is non-singular. On the contrary, when A is singular, it seems that one needs to impose explicitly the positivity condition in (32) and (33): this may be merely due to a technical difficulty and we conjecture that the LMI (32) has the same solution set of the equation $\operatorname{rank}[M(P)] = m$ and dually, the LMI (33) has the same solution set of equation $\operatorname{rank}[N(Q)] = m$.

As a direct consequence of Theorem 3.1 and Proposition 3.1, we have the following corollary.

Corollary 3.2: Let (38) be a minimal realization of a rational bi-proper discrete-time all-pass function. Then

1) The unique solution $P_0 = P_0^\top$ of (2) is also a non-singular solution of the homogeneous Riccati equation (52). This solution generates the family $\mathcal{P}_0$ of symmetric solutions of (52) as described by equation (41), where $P_\Delta = P_0$ and where Π is the orthogonal projector onto an $A^\top$-invariant subspace $\mathcal{Y}$. The elements $P = P^\top$ of this family are in a one-to-one correspondence with the set of $A^\top$-invariant subspaces.

   If A is unmixed then P_0 is the only non-singular solution of (52) and $\mathcal{P}_0$ is the set of all solutions of (52).

2) The unique solution $Q_0 = Q_0^\top$ of (3) is also a non-singular solution of the homogeneous Riccati equation (53). This solution generates the family $\mathcal{Q}_0$ of symmetric solutions of (53) as described by equation (44), where $Q_\Delta = Q_0$ and where Π is the orthogonal projector onto an A-invariant subspace $\mathcal{X}$. The elements of this family are in a one-to-one correspondence with the set of A-invariant subspaces $\mathcal{X}$.

   If A is unmixed then Q_0 is the only non-singular solution of (53) and $\mathcal{Q}_0$ is the set of all solutions of (53).

IV. Factorization of All-Pass Functions

In this section we discuss a remarkable relation between solutions of the constrained LMI's (or ARE) and all pass divisors. The background facts are established in the following lemma.

Lemma 4.1: Let (38) be a minimal realization of a rational discrete-time all-pass function and let Q_0 be the unique solution of (3). Let $P \in \mathcal{P}_0$ and let $Q \in \mathcal{Q}_0$ be the complementary solution of (33) associated to P in the sense described in Remark 3.2. Let $\mathcal{X} := \ker Q$ be the A-invariant subspace corresponding to Q and $\mathcal{Y} := \ker P = \mathcal{X}^\perp$ be the $A^\top$-invariant subspace corresponding to P. Then, one can select a basis such that, $\mathcal{X}, \mathcal{Y}, A, B, C, Q, P$ and Q_0 have the following structure

$$\mathcal{X} = \operatorname{im}\begin{bmatrix} 0 \\ I \end{bmatrix}, \ \mathcal{Y} = \operatorname{im}\begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{56}$$

$$A = \begin{bmatrix} A_r & 0 \\ A_{21} & A_l \end{bmatrix}, \ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ C = [C_1 \mid C_2], \tag{57}$$

$$Q = \begin{bmatrix} P_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \ P = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22}^{-1} \end{bmatrix}, \ Q_0 = \begin{bmatrix} P_{11}^{-1} & 0 \\ 0 & Q_{22} \end{bmatrix}. \tag{58}$$

Proof: Perform a preliminary change of basis as in equation (46) of the proof of Theorem 3.1 (now we use a slightly different notation) so that $\mathcal{X}, \mathcal{Y}$ are given by (56), and A has the block-triangular structure $A = \left[\begin{smallmatrix} \bar{A}_r & 0 \\ \bar{A}_{21} & A_l \end{smallmatrix}\right]$. In this basis, partition Q_0 and $P_0 = Q_0^{-1}$ as $Q_0 = \left[\begin{smallmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{smallmatrix}\right]$ and $P_0 = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{smallmatrix}\right]$. We have already proved that in this basis $Q = \left[\begin{smallmatrix} P_{11}^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. We now show that Q_{22} is invertible. Partition B and C conformably with Q_0 as $B = \left[\begin{smallmatrix} B_1 \\ \bar{B}_2 \end{smallmatrix}\right]$, and $C = [\bar{C}_1 \mid C_2]$. Matrix Q_0 is the solution of (3) the first of which, in the chosen basis, yields,

$$A_l^\top Q_{22} A_l - Q_{22} = C_2^\top C_2. \tag{59}$$

Since (A, C) is observable, (A_l, C_2) is observable as well so that from (59) it follows that Q_{22} is non-singular, [6, Lemma 3.1]. Considering now (41), where we set $P_\Delta = Q_0^{-1}$, and Π is the orthogonal projector onto $\mathcal{Y}$, we see that in the same basis $P = \left[\begin{smallmatrix} 0 & 0 \\ 0 & Q_{22}^{-1} \end{smallmatrix}\right]$. Since Q_0 and Q_{22} are non-singular, the Schur complement $Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^\top$ is also non-singular and $P_{11} = (Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^\top)^{-1}$. Perform now a further change of basis induced by $T = \left[\begin{smallmatrix} I & 0 \\ -Q_{22}^{-1}Q_{12}^\top & I \end{smallmatrix}\right]$. Although the two subspaces (56) are no longer orthogonal they are still in direct sum, matrix A in (57) is modified by just changing $\bar{A}_{21}$ into $A_{21} := \bar{A}_{21} + Q_{22}^{-1}Q_{12}^\top A_1 - A_2 Q_{22}^{-1} Q_{12}^\top$, and in (57) we have $B_2 := \bar{B}_2 + Q_{22}^{-1}Q_{12}^\top B_1$, and $C_1 := \bar{C}_1 - C_2 Q_{22}^{-1} Q_{12}^\top$. ■

Theorem 4.1: Let (38) be a minimal realization of a rational discrete-time all-pass function. Let $\mathcal{P}_0$ be the family of solutions of (32) associated to the (unique) solution P_0 of (2) and $\mathcal{Q}_0$ be the family of solutions of (33) associated to the (unique) solution Q_0 or (3), as described in Remark 3.1.[2]

1) For each $P \in \mathcal{P}_0$, let G and L be such that $[G^\top \mid L^\top]$ has full row-rank and

$$M(P) = [G^\top \mid L^\top]^\top [G^\top \mid L^\top]. \tag{60}$$

   Then

$$K_L(z) := C(zI - A)^{-1}G + L \tag{61}$$

   is a (non-minimal) realization of a left all-pass divisor of $K(z)$. The McMillan degree n_l of $K_L(z)$ is equal to the rank of P.

   Conversely, any left all-pass divisor of $K(z)$ has the structure (61), where $[G^\top \mid L^\top]$ has full row-rank and satisfies (60) for a suitable $P \in \mathcal{P}_0$.

2) For each $Q \in \mathcal{Q}_0$, let H and J be such that $[H \mid J]$ has full row-rank and

$$N(Q) = [H \mid J]^\top [H \mid J]. \tag{62}$$

   Then

$$K_R(z) := H(zI - A)^{-1}B + J \tag{63}$$

   is a (non-minimal) realization of a right all-pass divisor of $K(z)$. The McMillan degree n_r of $K_R(z)$ is equal to the rank of Q.

   Conversely, any right all-pass divisor of $K(z)$ has the structure (63), where $[H \mid J]$ has full row-rank and satisfies (62) for a suitable $Q \in \mathcal{Q}_0$.

Proof: We prove only point 1) as point 2) is dual. We first observe that $K_L(z)$ is all-pass; in fact, P is a solution of (2) associated to the quadruple (A, G, C, L) so that in view of point 3) of Theorem 2.1, $K_L(z)$ is all-pass.

Let $P \in \mathcal{P}_0$ and $Q \in \mathcal{Q}_0$ be complementary solutions of (32) and (33), respectively, as described in Remark 3.2. Select a basis as in Lemma 4.1 so that $\mathcal{X}, \mathcal{Y}, A, B, C, Q, P$ and Q_0 have the

[2] As already observed, under the additional assumption that A is unmixed, $\mathcal{P}_0$ is the family of *all* symmetric solutions of (32) and $\mathcal{Q}_0$ is the family of *all* symmetric solutions of (33).

structure described in (56), (57) and (58). In the chosen basis, compute $M(P)$ to obtain

$$M(P) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_l Q_{22}^{-1} A_l^\top - Q_{22}^{-1} & A_l Q_{22}^{-1} C_2^\top \\ 0 & C_2 Q_{22}^{-1} A_l^\top & C_2 Q_{22}^{-1} C_2^\top + I \end{bmatrix} \quad (64)$$

so that G must have the block structure, $G = \left[\begin{smallmatrix} 0 \\ G_l \end{smallmatrix}\right]$ and $K_L(z)$ defined in (61) has the following realization

$$K_L(z) = C_2(zI - A_l)^{-1} G_l + L \quad (65)$$

(it could be shown that this realization is minimal but this result will come as a byproduct at the end of the proof). Observe now that (A, C) is observable so that (A_l, C_2) is observable as well. Now since Q_{22}^{-1} is a solution of (2) associated with the quadruple (A_l, G_l, C_2, L), then Q_{22} must be a solution of (3) associated with the same quadruple. In particular, from the second of equations (3) we get $A_l^\top Q_{22} G_l = C_2^\top L$ which may be rewritten as $[A_l^\top Q_{22} \mid -C_2^\top]\left[\begin{smallmatrix} G_l \\ L \end{smallmatrix}\right] = 0$. In this factorization, the matrix $[A_l^\top Q_{22} \mid -C_2^\top]$ has full row-rank; in fact, (A_l, C_2) is observable so that $[A_l^\top \mid C_2^\top]$ has full row-rank; hence $[A_l^\top \mid -C_2^\top]$ has full row-rank as well; furthermore since Q_{22} is non-singular also $[A_l^\top Q_{22} \mid -C_2^\top]$ has full row-rank. The right matrix $\left[\begin{smallmatrix} G_l \\ L \end{smallmatrix}\right]$ has full column-rank; in fact, we have already observed that its transpose has full row-rank. In conclusion, $[A_l^\top Q_{22} \mid -C_2^\top] \in \mathbb{R}^{n \times (n+m)}$ has rank n so that its kernel has dimension m and hence the m linearly independent columns of the matrix $\left[\begin{smallmatrix} G_l \\ L \end{smallmatrix}\right]$ are a basis for $\ker[A_l^\top Q_{22} \mid -C_2^\top]$.

Now use the fact that Q_0 is a solution of equations (3) associated with the quadruple (A, B, C, D). From the lower block of the second of these equations, we get

$$[A_l^\top Q_{22} \mid -C_2^\top] \begin{bmatrix} B_2 \\ D \end{bmatrix} = 0. \quad (66)$$

Similarly, from the left-lower block of the first of the same equations, we get

$$[A_l^\top Q_{22} \mid -C_2^\top] \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix} = 0. \quad (67)$$

Hence, there exist matrices D_r and C_r such that

$$\begin{bmatrix} B_2 \\ D \end{bmatrix} = \begin{bmatrix} G_l \\ L \end{bmatrix} D_r; \qquad \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix} = \begin{bmatrix} G_l \\ L \end{bmatrix} C_r. \quad (68)$$

It is now a matter of direct computation to see that

$$\begin{aligned} K(z) &= [LC_r \mid C_2] \left(zI - \begin{bmatrix} A_r & 0 \\ G_l C_r & A_l \end{bmatrix} \right)^{-1} \begin{bmatrix} B_1 \\ G_l D_r \end{bmatrix} + LD_r \\ &= [C_2(zI - A_l)^{-1} G_l + L][C_r(zI - A_r)^{-1} B_1 + D_r] \\ &= K_L(z) \hat{K}_R(z) \end{aligned} \quad (69)$$

where we have introduced the rational function $\hat{K}_R(z) := C_r(zI - A_r)^{-1} B_1 + D_r$. Note that, since $K(z)$ and $K_L(z)$ are all-pass, $\hat{K}_R(z)$ is necessarily all pass. To show that $K_L(z)$ is a left divisor of $K(z)$ it remains only to observe that $K(z)$ has a minimal realization of dimension n and that $n = n_l + n_r$ where n_l is the dimension of A_l and n_r is the dimension of A_r. As a byproduct, (65) is a minimal realization of $K_L(z)$ and $C_r(zI - A_r)^{-1} B_1 + D_r$ is a minimal realization of $\hat{K}_R(z)$. Finally, by construction, the McMillan degree n_l of $K_L(z)$ equals the dimension of Q_{22} or, equivalently, the rank of P.

Conversely, let $K(z) = \hat{K}_L(z) \hat{K}_R(z)$ with $\hat{K}_L(z) := C_l(zI - A_l)^{-1} B_l + D_l$, and $\hat{K}_R(z) := C_r(zI - A_r)^{-1} B_r + D_r$, being minimal realizations of all-pass functions and assume that the McMillan degree of $K(z)$ equals the sum of the McMillan degrees of $\hat{K}_L(z)$ and $\hat{K}_R(z)$. Then, up to a change of basis which does not affect the result that we need to establish, we have that

$$A = \begin{bmatrix} A_r & 0 \\ B_l C_r & A_l \end{bmatrix}; \quad B = \begin{bmatrix} B_r \\ B_l D_r \end{bmatrix}. \quad (70)$$

$$C = [D_l C_r \mid C_l]; \quad D = D_l D_r. \quad (71)$$

Hence, without loss of generality, we assume that the matrices A, B, C, D of (38) have the expressions (70) and (71). Since $\hat{K}_L(z)$ and $\hat{K}_R(z)$ are all-pass functions, there exist an invertible matrix P_l solving equations (2) associated with the quadruple (A_l, B_l, C_l, D_l) and an invertible matrix P_r solving equations (2) associated with the quadruple (A_r, B_r, C_r, D_r). By exploiting (70) and (71), it is straightforward to check that, in the selected basis, $\operatorname{diag}(P_r, P_l)$ is the (unique) solution of (2) associated with the quadruple (A, B, C, D). Hence, we have $P_0 = \left[\begin{smallmatrix} P_r & 0 \\ 0 & P_l \end{smallmatrix}\right]$. Let $\mathcal{Y} = \operatorname{im}\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ be an $A^\top$-invariant subspace so that

$$P := \left[(I - \Pi) P_0^{-1} (I - \Pi)\right]^+ = \begin{bmatrix} 0 & 0 \\ 0 & P_l \end{bmatrix} \in \mathcal{P}_0. \quad (72)$$

By direct computation, it is also straightforward to check that

$$M(P) = [G^\top \mid L^\top]^\top [G^\top \mid L^\top] \quad (73)$$

where $G := \left[\begin{smallmatrix} 0 \\ B_l \end{smallmatrix}\right]$ and $L := D_l$. Now define, as in (61), the left factor $K_L(z)$ associated with P, G and L given by (72) and (73). By eliminating the non-reachable part of this $K_L(z)$, we see that $K_L(z) = \hat{K}_L(z)$. ■

Theorem 4.1 provides a one to one correspondence between the family $\mathcal{P}_0$ of solutions of (32) and left all-pass factors of $K(z)$ defined up to multiplication from the right side by a constant orthogonal matrix U. Similarly, Theorem 4.1 also provides a one to one correspondence between the family $\mathcal{Q}_0$ of solutions of (33) and right factors of $K(z)$ defined up to multiplication from the left side by a constant orthogonal matrix U. On the other hand, a left factor $K_L(z)$ of $K(z)$ is associated with a right factor $K_R(z)$ by the factorization relation $K(z) = K_L(z) K_R(z)$. Given a factorization of this type, it is natural to ask what is the relation between the solution $P \in \mathcal{P}_0$ associated with $K_L(z)$ and the solution $Q \in \mathcal{Q}_0$ associated with the corresponding $K_R(z)$. The following result addresses this question and shows that P and Q are related by the same bijective correspondence introduced in Remark 3.2.

Proposition 4.1: Let (38) be a minimal realization of a rational discrete-time all-pass function and let $K(z) = K_L(z) K_R(z)$ be a minimal factorization of $K(z)$. The matrices

$P \in \mathcal{P}_0$ and $Q \in \mathcal{Q}_0$ associated with $K_L(z)$ and $K_R(z)$, respectively, by Theorem 4.1 satisfy the relation $\ker[P] = (\ker[Q])^\perp$ and are therefore a complementary pair.

Proof: As in the proof of Theorem 4.1, let $P \in \mathcal{P}_0$ and let $Q \in \mathcal{Q}_0$ be the corresponding solution of (33) as described in Remark 3.2, i.e. the only element of $\mathcal{Q}_0$ such that $\ker[P] = (\ker[Q])^\perp$. We select a basis as in Lemma 4.1 so that $\mathfrak{X}, \mathfrak{Y}, A, B, C, Q, P$ and Q_0 have the structure described in (56), (57) and (58). Consider a left factor $K_L(z)$ associated with P: as we have seen in the proof of Theorem 4.1, the corresponding right factor $\hat{K}_R(z)$ (that satisfies equation (69)) has a minimal realization of the form $\hat{K}_R(z) = C_r(zI - A_r)^{-1}B_1 + D_r$. Let P_r be the solution of (2) associated with the quadruple (A_r, B_1, C_r, D_r). By taking (57) and (68) into account, we easily see by a direct computation that $\mathrm{diag}(P_r, Q_{22}^{-1})$ is the solution of (2) associated to the quadruple (A, B, C, D). Since the solution $P = \mathrm{diag}(P_{11}, Q_{22}^{-1})$ of this equation is unique, we have $P_r = P_{11}$. On the other hand, we know that the right factor $K_R(z)$ associated with the matrix Q is given by (63) and, by duality, has a minimal realization of the form $K_R(z) = H_r(zI - A_r)^{-1}B_1 + J$. Now we compare the all-pass functions $\hat{K}_R(z)$ and $K_R(z)$ and we see that they have the same state and input matrices and that the solutions of the equation (2) associated to the minimal quadruple (A_r, B_1, C_r, D_r) and of the equation (2) associated to the minimal quadruple (A_r, B_1, H_r, J), coincide. Hence, $K_R(z)$ and $\hat{K}_R(z)$ differ by multiplication on the left side by a constant orthogonal matrix. ■

In the case when $K(z)$ is bi-proper, or equivalently, A and D are non-singular, we know that (32) and (33) reduce to ARE's. Moreover, for any given solution P of (32), (or, equivalently, of (52)) we can provide an explicit expression for the matrices G and L by solving (60). The following corollary connects solutions of ARE's and all-pass factorizations.

Corollary 4.1: Let (38) be a minimal realization of a rational bi-proper discrete-time all-pass function. Let $\mathcal{P}_0$ be the family of solutions of (52) associated with the solution P_0 of (2) and $\mathcal{Q}_0$ be the family of solutions of (53) associated with the solution Q_0 or (3), as described in Corollary 3.2.[3]

1) For each $P \in \mathcal{P}_0$, the function

$$K_L(z) := C(zI - A)^{-1}G + L, \tag{74}$$

with

$$\begin{cases} L := (I + CPC^\top)^{1/2} \\ G := APC^\top L^{-\top} \end{cases} \tag{75}$$

is a (non-minimal) realization of a left all-pass divisor of $K(z)$.

Conversely, any left all-pass divisor of $K(z)$ is given up to multiplication from the right side by a constant orthogonal matrix by (74), (75).

2) For each $Q \in \mathcal{Q}_0$, the function

$$K_R(z) := H(zI - A)^{-1}B + J, \tag{76}$$

with

$$\begin{cases} J := (I + B^\top QB)^{1/2} \\ H := J^{-\top}B^\top QA \end{cases} \tag{77}$$

is a (non-minimal) realization of a right all-pass divisor of $K(z)$.

Conversely, any right all-pass divisor of $K(z)$ is given—up to multiplication on the left side by a constant orthogonal matrix by (76), (77).

[3]Similarly to the general case, under the additional assumption that A is unmixed, $\mathcal{P}_0$ is the family of *all* symmetric solutions of (52) and $\mathcal{Q}_0$ is the family of *all* symmetric solutions of (53).

V. Conclusion

In this paper we have provided a completely general characterization of discrete-time all-pass matrix functions in the same spirit of the continuous-time result of Glover's [13, Theorem 5.1]. Applications to some class of LMI's, to homogeneous Riccati equations and to the factorization of all-pass functions are discussed. The characterization is presented for square all-pass matrix functions but a generalization to non-square functions can be pursued along the same lines.

Our analysis has been carried over for the case of *real* rational all-pass functions because this is the interesting case in a large majority of control applications. We remark, anyway, that the complex case can be dealt with by exactly the same arguments and computations. The results and all the formulas are the same with the only precaution that $M^\top$ should be understood as the *transpose conjugate* of the matrix M and, for a rational function $K(z)$, $K^*(z)$ should be interpreted as $[K(1/\bar{z})]^\top$, where $\bar{z}$ is the complex conjugate of z and, as just specified, $\cdot^\top$ denotes transposition and conjugation.

Appendix A
Factorization of All-Pass Functions Which are Singular at Infinity

Lemma A.1: Let $K(z)$ be an $m \times m$ rational proper discrete-time all-pass function. Then $K(z)$ can be written as

$$K(z) = K_0(z)\bar{K}_1(z)\bar{K}_2(z)\ldots\bar{K}_k(z) \tag{78}$$

where $K_0(z)$ is a rational discrete-time all-pass function such that $K_0(\infty)$ is non-singular and the $\bar{K}_i(z)$'s are rational proper all-pass functions (whose only pole is in the origin) having a realization of the following form

$$\bar{K}_i(z) = \begin{bmatrix} I_{m-p_i} & 0 \\ 0 & 0 \end{bmatrix} U_i + \begin{bmatrix} 0 \\ I_{p_i} \end{bmatrix} (zI_{p_i} - 0)^{-1}[0 \mid I_{p_i}]U_i \tag{79}$$

where U_i is a constant orthogonal matrix.

Proof: Consider a minimal realization $K(z) = C(zI - A)^{-1}B + D$. If D is non-singular, $K_0(z) = K(z)$ and we are done. If D is singular, we resort to the Silverman algorithm as described in [7]. Assume the matrix D has q_1 linearly independent columns, with $0 \le q_1 < m$. Let V_1 be an orthogonal matrix such that $DV_1 = \begin{bmatrix} D_{11} & 0 \end{bmatrix}$, with $D_{11} \in \mathbb{R}^{m \times q_1}$ being full column rank. Let us partition $BV_1 = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}$ conformably, obtaining the following block structure,

$$\tilde{K}_1(z) := K(z)V_1 = C(zI - A)^{-1}\begin{bmatrix} B_{11} & B_{12} \end{bmatrix} + \begin{bmatrix} D_{11} & 0 \end{bmatrix}, \tag{80}$$

and let

$$\hat{K}_1(z) := \tilde{K}_1(z)\begin{bmatrix} I_{q_1} & 0 \\ 0 & zI_{m-q_1} \end{bmatrix}. \tag{81}$$

Clearly, $\hat{K}_1(z)$ is all-pass as it is the product of all-pass functions. Moreover, $\hat{K}_1(z)$ can be written as

$$\hat{K}_1(z) = [\hat{K}_{11}(z) \mid \hat{K}_{12}(z)]$$

where

$$\hat{K}_{11}(z) := D_{11} + CB_{11}z^{-1} + CAB_{11}z^{-2} + \ldots$$

and

$$\hat{K}_{12}(z) := CB_{12} + CAB_{12}z^{-1} + CA^2B_{12}z^{-2} + \ldots$$

so that $\hat{K}_1(z)$ has the following realization

$$\hat{K}_1(z) = C(zI - A)^{-1}\begin{bmatrix} B_{11} & AB_{12} \end{bmatrix} + \begin{bmatrix} D_{11} & CB_{12} \end{bmatrix}. \tag{82}$$

At this point, either $\begin{bmatrix} D_{11} & CB_{12} \end{bmatrix}$ is right-invertible, or we may iterate the above procedure by introducing another orthogonal matrix V_2, such that

$$\begin{bmatrix} D_{11} & CB_{12} \end{bmatrix} V_2 = \begin{bmatrix} D_{21} & 0 \end{bmatrix},$$

with $D_{21} \in \mathbb{R}^{m\times q_2}$ of full column rank and $q_2 \geq q_1$; we define the new all-pass function

$$\tilde{K}_2(z) := \hat{K}_1(z)V_2 = C(zI - A)^{-1}\begin{bmatrix} B_{21} & B_{22} \end{bmatrix} + \begin{bmatrix} D_{21} & 0 \end{bmatrix}, \tag{83}$$

where $\begin{bmatrix} B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & AB_{12} \end{bmatrix} V_2$.

Notice that $K(z)$ is all-pass so that it has full rank (as a rational matrix function) and at each step it is multiplied by a full rank all-pass matrix. Hence, at each step the pole at infinity of the factor $\mathrm{diag}(I_{q_i}, zI_{m-q_i})$ must cancel a zero at infinity because the product remains proper and its McMillan degree does not increase. Since $K(z)$ is rational, it has finitely many zeros and hence, after a finite number of steps (say k) of the above procedure, we get a rational proper all pass function

$$\tilde{K}_k(z) = K(z)\prod_{i=1}^{k} V_i \begin{bmatrix} I_{q_i} & 0 \\ 0 & zI_{m-q_i} \end{bmatrix}, \tag{84}$$

without zeros at infinity i.e. such that $\tilde{K}_k(\infty)$ is non-singular. Now we set $K_0(z) := \tilde{K}_k(z)$, so that

$$K(z) = K_0(z)\left[\prod_{i=1}^{k} V_i \begin{bmatrix} I_{q_i} & 0 \\ 0 & zI_{m-q_i} \end{bmatrix}\right]^{-1}. \tag{85}$$

Finally, by setting $p_i := q_{k+1-i}$, $i = 1, 2, \ldots, k$, and $U_i := V_{k+1-i}^\top$, $i = 1, 2, \ldots, k$, and observing that

$$\begin{bmatrix} I_{p_i} & 0 \\ 0 & zI_{m-p_i} \end{bmatrix}^{-1} = \begin{bmatrix} I_{m-p_i} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_{p_i} \end{bmatrix}(zI_{p_i} - 0)^{-1}[0 \mid I_{p_i}] \tag{86}$$

we obtain (78) and (79). ∎

Lemma A.2: Let $K(z)$ be an $m \times m$ rational proper discrete-time all-pass function factored as in (78). Consider a reachable realization

$$K_i(z) = C_i(zI - A_i)B_i + D_i \tag{87}$$

of $K_i(z) := K_0(z)\bar{K}_1(z)\bar{K}_2(z)\ldots\bar{K}_i(z)$. Partition B_i and D_i as $B_i = [B_{i,1} \mid B_{i,2}]$ and $D_i = [D_{i,1} \mid D_{i,2}]$, where $B_{i,1}$ and $D_{i,1}$ have $m - p_{i+1}$ columns. Then a reachable realization of $K_{i+1}(z) := K_i(z)\bar{K}_{i+1}(z)$ is given by

$$\begin{aligned} K_{i+1}(z) = [D_{i,2} \mid C_i]\left(zI - \begin{bmatrix} 0 & 0 \\ B_{i,2} & A_i \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 & I \\ B_{i,1} & 0 \end{bmatrix} U_{i+1} \\ +[D_{i,1} \mid 0]U_{i+1}. \end{aligned} \tag{88}$$

Proof: The realization (88) is the result of a direct computation. The fact that this realization is reachable may be easily seen by using the PBH test. In fact, as a consequence of the fact that $[A_i - \lambda I \mid B_{i,1} \mid B_{i,2}]$ has full row-rank for all $\lambda \in \mathbb{C}$, we immediately see that also

$$\begin{bmatrix} -\lambda I & 0 & 0 & I \\ B_{i,2} & A_i - \lambda I & B_{i,1} & 0 \end{bmatrix}$$

has full row-rank for all $\lambda \in \mathbb{C}$. ∎

Appendix B
Proof of Symmetry of T:

Somehow in the same spirit of [12], we shall show that

$$U := T^{-1}T^\top \tag{89}$$

satisfies

$$A = U^{-1}AU, \tag{90a}$$

$$B = U^{-1}B, \tag{90b}$$

$$C = CU. \tag{90c}$$

This means that U is a similarity transform that leaves unchanged the triple (A, B, C) of the system. Since (A, B, C) is, by assumption, a minimal realization, this means that $U = I$, or that $T = T^\top$.

We start with (90c). Solving (12b) and (12c) for B we get

$$B = T^{-1}A^{-\top}C^\top D \tag{91}$$

and

$$B = AT^{-\top}C^\top D^{-\top} \tag{92}$$

By inserting in the latter the expression of $D^{-\top}$ obtained by transposing (12d), we get

$$B = AT^{-\top}C^\top D - AT^{-\top}C^\top CA^{-1}B \tag{93}$$

Now we take the inverse of both sides of (12a) and use the Sherman-Morrison-Woodbury formula thus obtaining

$$\begin{aligned} T^{-1}A^\top T = {} & A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} \\ = {} & A^{-1} + A^{-1}BD^\top CA^{-1}. \end{aligned} \tag{94}$$

From (92) we get $BD^\top = AT^{-\top}C^\top$ which, plugged into the left-hand side of (94), yields

$$T^{-1}A^\top T = A^{-1} + T^{-\top}C^\top CA^{-1}. \tag{95}$$

The latter provides an expression for $T^{-\top}C^\top CA^{-1}$ which, plugged into the left-hand side of (93) gives

$$B = AT^{-\top}C^\top D + B - AT^{-1}A^\top TB \tag{96}$$

so that $T^{-\top}C^\top D = T^{-1}A^\top TB$, or $B = T^{-1}A^{-\top}TT^{-\top}C^\top D$. By comparing the latter with (91), we eventually get

$$C^\top = TT^{-\top}C^\top \tag{97}$$

which, by recalling that $U := T^{-1}T^\top$, readily implies (90c).

We now use a dual argument to obtain (90b). Solving (12b) and (12c) for C we get

$$C = DB^\top A^{-\top}T \tag{98}$$

and

$$C = D^{-\top}B^\top T^\top A \tag{99}$$

By inserting in the latter the expression of $D^{-\top}$ obtained by transposing (12d), we get

$$C = DB^\top T^\top A - CA^{-1}BB^\top T^\top A \tag{100}$$

From (99) we get $D^\top C = B^\top T^\top A$ which, plugged into the left-hand side of (94), yields $T^{-1}A^\top T = A^{-1} + A^{-1}BB^\top T^\top$. The latter provides an expression for $A^{-1}BB^\top T^\top$ which, plugged into the left-hand side of (100) gives

$$C = DB^\top T^\top A + C - CT^{-1}A^\top TA \tag{101}$$

so that $DB^\top T^\top = CT^{-1}A^\top T$, or $C = DB^\top T^\top T^{-1}A^{-\top}T$. By comparing the latter with (98), we eventually get

$$B^\top = B^\top T^\top T^{-1} \tag{102}$$

which, by recalling that $U := T^{-1}T^\top$, readily implies (90b).

We now prove (90a). We multiply equation (12a) on the left side by U^{-1} and on the right side by U. By taking into account (90b) and (90c), we get

$$U^{-1}AU = T^{-\top}A^{-\top}T^\top + BD^{-1}C. \tag{103}$$

On the other hand, by transposing the first and the last member of (94) and multiplying on the left side by $T^{-\top}$ and on the right side by $T^\top$ we get

$$\begin{aligned} A = {} & T^{-\top}A^{-\top}T^\top + T^{-\top}A^{-\top}C^\top DB^\top A^{-\top}T^\top \\ = {} & T^{-\top}A^{-\top}T^\top + T^{-\top}A^{-\top}C^\top DD^{-1}DB^\top A^{-\top}T^\top. \end{aligned} \tag{104}$$

Moreover, by inserting in the right-hand side of the latter the expressions of $A^{-\top}C^\top D$ and $DB^\top A^{-\top}$ obtained from (91) and (98), respectively, we get

$$\begin{aligned} A = {} & T^{-\top}A^{-\top}T^\top + \underbrace{T^{-\top}T}_{U^{-1}}BD^{-1}C\underbrace{T^{-1}T^\top}_{U} \\ = {} & T^{-\top}A^{-\top}T^\top + BD^{-1}C. \end{aligned} \tag{105}$$

Finally, by comparing the latter with (103), we get (90a). ∎

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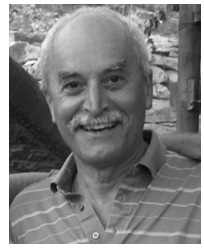

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