this BSR it is trivial to verify that all of the state-affine theory presented here extends to the multidimensional input setting. The details are left to the reader.

In this paper we have employed the techniques of [4] to develop a theory of state-affine systems. Both infinite degree and infinite dimensional systems are studied. An alternate approach to state-affine realization is given in [7]. In [7] the input–output data is arranged in a Hankel matrix. Using this Hankel matrix [7] obtains a realization algorithm for state-affine systems.

In this paper Hankel matrices were not used. Our input–output data are characterized by the transfer function θ for a Volterra series. It is our transform representation that leads to an efficient realization algorithm. The simplicity of the algorithm followed because the operators S, T, and E are naturally suited to act on rational functions. For n-homogeneous systems, our transfer function for a Volterra series degenerates into the regular transfer function used in [1], [2]. In [2], [7] other transform techniques for a Volterra series are also discussed. For further references on transform representation for nonlinear systems see [2], [4], [7].

Finally, we have developed a theory of minimality, span reachability, and observability directly from the BSR and the operator H_Σ. This method degenerates into the approach of [4] when the underlying system is bilinear, i.e., of the form (1.3). As noted earlier, alternate proofs to Corollaries (6.2) and (6.3) are given in [7]. For further comments concerning the applications of shifts to nonlinear systems, see [4].

Acknowledgment

Conversations with Prof. W. J. Rugh of the Johns Hopkins University have been most helpful.

References

[1] S. J. Clancy, G. E. Mitzel, and W. J. Rugh, "On transfer function representations for homogeneous nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 242–249, 1979.
[2] S. J. Clancy and W. J. Rugh, "On the realization problem for stationary homogeneous discrete time systems," *Automatica*, vol. 14, pp. 357–366, 1978.
[3] A. E. Frazho, "Shift operators and bilinear system theory," in *Proc. 1978 IEEE Conf. Decision, Contr.*, pp. 551–556.
[4] A. E. Frazho, "A shift operator approach to bilinear system theory," *SIAM J. Contr.*, pp. 640–658, 1980.
[5] E. G. Gilbert, "Functional expansions for the response of nonlinear differential systems," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 909–921, 1977.
[6] S. I. Marcus, "Optimum nonlinear estimation for a class of discrete-time stochastic systems," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 297–302, 1979.
[7] E. D. Sontag, "Realization theory of discrete-time nonlinear systems: Part I—the bounded case," *IEEE Trans. Circuits Syst.*, vol. CAS-26, pp. 342–356, 1979.

A Characterization of Minimal Square Spectral Factors

LORENZO FINESSO AND GIORGIO PICCI

***Abstract*—Two different well-known approaches to the spectral factorization problem $\Phi(s)=W(s)W'(-s)$ are connected together by relating the geometric properties of the solution set of the underlying algebraic Riccati equation to the structure of the "all-pass" factor of each minimal solution $W(s)$.**

I. Introduction

The aim of this paper is to study the family of all minimal square solutions of the (matrix) spectral factorization problem $\Phi(s)=W(s)W'(-s)$.

Manuscript received March 31, 1980; revised April 21, 1981. Paper recommended by S. I. Marcus, Past Chairman of the Stochastic Control Committee.

L. Finesso is with the Istituto di Elettrotecnica ed Elettronica, Università di Padova, Padova, Italy.

G. Picci is with the Istituto per Ricerche di Dinamica dei Sistemi e di Bioingegneria del C. N. R., Padova, Italy.

It is well known that in the scalar case all minimal spectral factors are obtainable from the minimum phase one by "flipping" zeros to the right-half plane (cf., e.g., [1]). We show that an intuitive generalization of this "zero flipping" process characterizes the matrix factors as well. To obtain this result we have however to dwell on a rather detailed analysis of the algebraic Riccati equation which underlies the spectral factorization problem. Especially, useful in this context are the results derived by J. C. Willems in his important paper [18] connecting the solution set of the ARE and the family of invariant subspaces for a certain pair of "extremal" feedback matrices. These matrices in our context describe the zeros of the minimum and maximum phase spectral factors. In this respect our results make also contact with the geometric approach of C. Martin [11], [12].

The analysis presented here is motivated by the interpretation that minimal square spectral factors have in stochastic realization theory [8], [15], [13]. In fact, looking at spectral factors from this point of view provides a very intuitive interpretation of the above characterization in terms of invariant subspaces. These subspaces support the time evolution of certain "error processes" associated to the (stochastic) state of any minimal spectral factor. Also, the classical description of spectral factors in term of inner–outer factorization [19] can be made very precise in this context. A complete characterization of the inner part of a minimal spectral factor is given. It is shown that all these inner parts are left (inner) divisors of a certain fixed rational matrix inner function and therefore (cf. [4], [7]) in one to one correspondence with invariant subspaces. These subspaces are precisely those describing the geometry of the solutions of the ARE

II. Minimal Square Spectral Factors and ARE

Given an $m\times m$ real rational matrix function $\Phi(s)$ for which[1]

i) $\Phi(s)=\Phi'(-s)$,
ii) $\Phi(s)$ is analytic for $s=i\omega$ and $\Phi(i\omega)\geqslant 0$,
iii) $\Phi(\infty)<\infty$,

the *spectral factorization problem* for $\Phi(s)$, consists in finding real rational matrices $W(s)$, whose poles lie in the left-half complex plane $\{\mathrm{Re}\, s<0\}$ and such that they satisfy the relation

$$\Phi(s)=W(s)W'(-s) \tag{2.1}$$

for all s in the complex plane. Such functions $W(s)$ will be called (stable) *spectral factors*.

In this paper we shall make the following additional assumption on Φ.

iv) $\Phi(i\omega)>0$.

This implies that $\Phi(\infty):=R$, is a symmetric positive definite matrix. The assumption is not an essential one but it simplifies the formalism considerably and in any case it seems to be important enough for the applications to deserve a separate treatment. A *minimal spectral factor* is one having the least McMillan degree [14], i.e., the smallest possible dimension of any minimal realization. Since minimal spectral factors are the solutions of (2.1) of least complexity, the search for spectral factors is usually restricted to the minimal ones.

The *stochastic realization problem* is tightly connected to spectral factorization ([1], [6], [8]). It can be formulated as follows. Let $\{y(t)\}$ be an m-dimensional Gaussian stochastic process on the real line, with stationary increments and incremental spectral density matrix $\Phi(s)$ (i.e., $E[d\hat{y}\, d\hat{y}^{*}]=\Phi(i\omega)\,d\omega$, where $d\hat{y}$ is the spectral measure of $\{y(t)\}$, [7]). We look for representations of $\{y\{t\}\}$ of the form,

$$\begin{aligned} dx(t)&=Ax(t)\,dt+B\,du(t) \\ dy(t)&=Cx(t)\,dt+D\,du(t) \end{aligned} \tag{2.2}$$

where $\{u(t)\}$ is some Gaussian, say p-dimensional, process with orthogonal stationary increments and normalized incremental variance (a "Wiener process" on R)

$$E[du(t)\,du(t)']=I\,dt. \tag{2.3}$$

The transfer function of the model (2.2) is the $m\times p$ matrix

$$W(s):=D+C(sI-A)^{-1}B, \tag{2.4}$$

[1] Throughout this paper prime denotes transpose, "*" denotes complex conjugate transpose; $A\geqslant 0$ ($A>0$) means that the matrix A is positive semidefinite (positive definite).

and it is clear that a necessary condition for $\{y(t)\}$ to be representable in the form (2.2) is that $W(s)$ satisfies (2.1), i.e., $W(s)$ must be a spectral factor of $\Phi(s)$. Note that, if $\Phi(s)$ satisfies assumption iv) above, it follows from (2.4) and (2.1) that $DD'=R>0$, and hence D must be of rank m at least. This implies that the number of components, p, of the driving noise process $\{u(t)\}$ in (2.2) is greater or equal than m. The preceding observations refer to the spectral density of $\{y(t)\}$, i.e., to its second-order description only. For the output of the system (2.2) to be *the same process as* $\{y(t)\}$ (or more precisely to be a.s. equal to $y(t)$ for each t) however, the input process has to be chosen in a special way.

A linear system $\{A,B,C,D\}$ *plus* an orthogonal increments process $\{u(t)\}$ for which representation (2.2) holds (a.s. for each t) will be called a (strong) *realization* of the process $\{y(t)\}$. From now on we shall agree on taking $\{A,B,C,D\}$ as a *minimal* realization. This is no loss of generality.

A special class of realizations are the so-called *internal* or *output induced* realizations, for which $\{u(t)\}$ is prescribed to be "y-measurable," i.e., obtainable for each t, as a (nonnecessarily causal) function of the random variables $\{y(s);\ s\in R\}$.

It is relatively easy to see that, under assumption iv) internal realizations must have a *square* transfer function matrix $W(s)$ [8]. In this situation the number of components, p, of the noise process is the minimum possible (i.e., m); and $\{u(t)\}$ can be generated by passing $\{y(t)\}$ through the "whitening filter" whose transfer function is $W^{-1}(s)$.

The first important characterization of square minimal spectral factors is that they are parametrized in a one to one way by the solutions of a certain matrix algebraic Riccati equation. To obtain this characterization decompose the spectral density matrix as

$$\Phi(s)=Z(s)+Z'(-s) \tag{2.5}$$

with $Z(s)$ *positive real* [2]. It is known that there is only one such decomposition for which the McMillan degree of $Z(s)$, $\delta(Z)$, is one-half of the McMillan degree of $\Phi(s)$, [1], [2]. Let

$$Z(s)=1/2R+C(sI-A)^{-1}G \tag{2.6}$$

be a minimal realization of $Z(s)$. By positive realness and condition ii), the $n\times n$ matrix A has all its eigenvalues in $\{\mathrm{Re}\ s<0\}$. There are well-known computational procedures for determining (A,G,C,R) from Φ [5], and in the sequel we shall regard this description as part of the data of the problem. Define the function $\Lambda: R^{n\times n}\to R^{n\times n}$ as

$$\Lambda(P):=AP+PA'+(G-PC')R^{-1}(G-PC')' \tag{2.7}$$

and let $\mathcal{P}_o:=\{P|P=P';\ \Lambda(P)=0\}$ be the set of all symmetric $n\times n$ solutions to the algebraic Riccati equation $\Lambda(P)=0$. The following theorem is taken from [1], [6], [18].

Theorem 2.1: All minimal square solutions to the spectral factorization problem are given, modulo multiplication on the right by a constant unitary matrix, by[2]

$$W(s)=R^{1/2}+C(sI-A)^{-1}B \tag{2.8}$$

where

$$B=(G-PC')R^{-1/2} \tag{2.9}$$

and P is a solution of the ARE $\Lambda(P)=0$.

Formulas (2.8) and (2.9) display the one to one correspondence between the family of minimal square spectral factors and the set $\mathcal{P}_o$. In the following we shall refer to the *numerator matrix*, Γ, of $W(s)$, defined as

$$\Gamma:=A-BR^{-1/2}C. \tag{2.10}$$

Clearly, the eigenvalues of Γ are *the zeros* of the spectral factor $W(s)$, since by a well-known formula [14] the inverse $W^{-1}(s)$ of $W(s)$ is given by

$$W^{-1}(s)=R^{-1/2}-R^{-1/2}C(sI-\Gamma)^{-1}BR^{-1/2}. \tag{2.11}$$

Theorem 2.2 [1], [6], [18]: All solutions in $\mathcal{P}_o$ are positive definite and there are a minimal and a maximal element, P_* and P^*, respectively, such that

$$P_*<P^* \tag{2.12a}$$

and

$$P_*\leq P\leq P^* \qquad \text{for all } P \text{ in } \mathcal{P}_o. \tag{2.12b}$$

Moreover, let B_* and B^* be defined by (2.9) *with P set equal to P_* and P^*, respectively, and consider the numerator matrices,*

$$\Gamma_*:=A-B_*R^{-1/2}C, \tag{2.13}$$

$$\Gamma^*:=A-B^*R^{-1/2}C; \tag{2.14}$$

then all eigenvalues of $\Gamma_*(\Gamma^*)$ have strictly negative (respectively, positive) real part.

Each $P\in\mathcal{P}_o$ can be interpreted as the covariance matrix of the state process $\{x(t)\}$ of the corresponding stochastic realization. Note that, since P is positive definite and solves the Lyapunov equation,

$$AP+PA'+BB'=0 \tag{2.15}$$

all realizations (2.8) of the minimal spectral factors $W(s)$ are minimal. From (2.12) it follows that there is a minimal and a maximal variance realization whose state processes we shall denote by $\{x_*(t)\}$ and $\{x^*(t)\}$, respectively. The corresponding input noises $\{u_*(t)\}$ and $\{u^*(t)\}$ can then be obtained as outputs of the whitening filters,

$$dx_*(t)=\Gamma_*x_*(t)\,dt+B_*R^{-1/2}\,dy(t) \tag{2.16a}$$

$$du_*(t)=-R^{-1/2}Cx_*(t)\,dt+R^{-1/2}\,dy(t) \tag{2.16b}$$

and

$$dx^*(t)=\Gamma^*x^*(t)\,dt+B^*R^{-1/2}\,dy(t) \tag{2.17a}$$

$$du^*(t)=-R^{-1/2}Cx^*(t)\,dt+R^{-1/2}\,dy(t) \tag{2.17b}$$

which are realizations of the inverses of the "minimal" and "maximal" spectral factors, $W_*(s)$ and $W^*(s)$, corresponding to P_* and P^*. The process $\{u_*(t)\}$ is the (forward) *innovation* of $\{y(t)\}$ and $\{u^*(t)\}$ can be related to the so-called *backward innovation* of $\{y(t)\}$ [8].

Let $x(t)$ be the state process of any minimal realization, it is shown in [8] that

$$E[x(t)|y(s), s\leq t]=x_*(t) \tag{2.18}$$

and so the minimum variance realization is actually a steady-state Kalman–Bucy filter, the filter being moreover the same for all minimal realizations of $\{y(t)\}$. On the other hand, if P is the covariance of $\{x(t)\}$,

$$E[P^{-1}x(t)|y(s), s\geq t]=(P^*)^{-1}x^*(t) \tag{2.19}$$

for all minimal realizations. This means that, after the change of basis $\bar{x}(t):=P^{-1}x(t)$, all minimal realizations have the same "backward" (or completely anticausal) Kalman–Bucy filter, the latter being obtained from the maximum variance realization by the change of basis $\bar{x}_*(t):=(P^*)^{-1}x^*(t)$ (see [8] for further details).

III. Minimal All-Pass Factors

It descends from standard theory (see, for instance, [19]) that all square spectral factors (not even necessarily rational) can uniquely be decomposed as a product

$$W(s)=W_*(s)U(s) \tag{3.1}$$

of an *outer* (or "minimum phase") factor $W_*(s)$ and an *inner* part $U(s)$. $W_*(s)$ is analytic together with its inverse on the whole right-half plane $\{\mathrm{Re}\,s>0\}$, while $U(s)$ is a square $m\times m$ matrix function which is analytic and bounded on $\{\mathrm{Re}\,s>0\}$ and unitary on the imaginary axis. In other words,

$$U(s)U'(-s)=I \tag{3.2}$$

for all complex s. Rational inner functions will be called *all pass*. Note

[2] $R^{1/2}$ is the symmetrix positive definite square root of R.

that inner functions are only determined up to multiplication on the right by an arbitrary constant $m \times m$ unitary matrix and therefore it is no loss of generality to normalize $U(s)$ in such a way that $U(\infty)=I$.

The abuse of notation in (3.1) is justified, as the minimal variance spectral factor $W_*(s)$ introduced in Section II is the (unique) outer spectral factor of $\Phi(s)$. The all-pass part of the other minimal square spectral factors has a particularly interesting structure that we now prepare to investigate. Let

$$dx(t)=Ax(t)\,dt+B\,du(t) \tag{3.3a}$$

$$dy(t)=Cx(t)\,dt+R^{1/2}\,du(t) \tag{3.3b}$$

be any minimal internal realization and let us define the *error process* $\tilde{x}(t):=x(t)-x_*(t)$ [notice that $\tilde{x}(t)$ is uncorrelated to $x_*(t)$]. It is easily checked that $\tilde{x}(t)$ satisfies the stochastic differential equation

$$d\tilde{x}(t)=A\tilde{x}(t)\,dt+\left(B-B_*\right)du(t)+B_*\left(du(t)-du_*(t)\right). \tag{3.4}$$

Now, after eliminating $du(t)$ or $du_*(t)$ from (3.4) by using (2.16b) and (3.3b), we find that $\{\tilde{x}(t)\}$ satisfies either one of the two differential equations

$$d\tilde{x}(t)=\Gamma\tilde{x}(t)\,dt+\left(B-B_*\right)du_*(t) \tag{3.5}$$

$$d\tilde{x}(t)=\Gamma_*\tilde{x}(t)\,dt+\left(B-B_*\right)du(t) \tag{3.6}$$

which are indeed related by the transformation,

$$du_*(t)=R^{-1/2}C\tilde{x}(t)\,dt+du(t). \tag{3.7}$$

Putting now (3.6) and (3.7) together, we single out a dynamical system driven by the noise process $\{u(t)\}$, which produces as output $\{u_*(t)\}$. Its transfer function is

$$U(s):=I+R^{-1/2}C\left(sI-\Gamma_*\right)^{-1}\left(B-B_*\right), \tag{3.8}$$

and since $\{u_*(t)\}$ has incremental spectral density $I\,d\omega$ it follows that $U(s)$ must satisfy (3.2), i.e., *$U(s)$ is an all-pass function*. By the above computation, the realization (3.3) can actually be decomposed into the *cascade* of two subsystems, the first (having the transfer function (3.8) and state vector $\tilde{x}(t)$) which transforms the original input noise $\{u(t)\}$ into $\{u_*(t)\}$, and the other, producing $\{y(t)\}$ as the output corresponding to the input $\{u_*(t)\}$. The latter can be taken as the minimum variance realization. Note that this decomposition corresponds to a factorization of the transfer function $W(s)$ which is exactly of the type (3.1). By uniqueness, we than have the following result.

Lemma 3.1: A spectral factor $W(s)$ is minimal square if and only if its inner part is of the form (3.8) *with B given by* (2.9) *for some $P \in \mathcal{P}_o$, and Γ_* and B_* as defined in Theorem 2.2.*

Let us denote by $U^*(s)$ the inner part of the maximum variance spectral factor $W^*(s)$. Its state equations are

$$d\tilde{x}^*(t)=\Gamma_*\tilde{x}^*(t)\,dt+\left(B^*-B_*\right)du^*(t) \tag{3.9a}$$

$$du_*(t)=R^{-1/2}C\tilde{x}^*(t)\,dt+du^*(t) \tag{3.9b}$$

where $\tilde{x}^*(t)=x^*(t)-x_*(t)$ is the "maximal" error process. The covariance matrix of $\tilde{x}^*(t)$ is easily computed to be

$$\Sigma^*:=P^*-P_* \tag{3.10}$$

and by asymptotic stability of Γ_*, it satisfies the Lyapunov equation

$$\Gamma_*\Sigma^*+\Sigma^*\Gamma_*'+\left(B^*-B_*\right)\left(B^*-B_*\right)'=0. \tag{3.11}$$

As $\Sigma^*>0$ (Theorem 2.2) it follows that the pair (Γ_*, B^*-B_*) is controllable [5], and hence the realization (3.9) of $U^*(s)$ is minimal. Incidentally, (3.11) provides an easy way to compute P^*, [and hence $W^*(s)$] once P_* is known ([1], [18]) and also the following useful identity.

Lemma 3.2 [18]: The matrices Γ^* and $-\Gamma_*'$ are similar, in fact,

$$\Gamma^*=-\Sigma^*\Gamma_*'\Sigma^{*-1}. \tag{3.12}$$

We now define the "dual" error process $\check{x}(t):=x^*(t)-x(t)$, where $x(t)$ is the state of any internal realization. By following essentially the same route as done for $x(t)$, we find that $\check{x}(t)$ satisfies

$$d\check{x}(t)=\Gamma^*\check{x}(t)\,dt+\left(B^*-B\right)du(t) \tag{3.13}$$

or

$$d\check{x}(t)=\Gamma\check{x}(t)\,dt+\left(B^*-B\right)du^*(t) \tag{3.14}$$

with $u(t)$ and $u^*(t)$ related by

$$du(t)=R^{-1/2}C\check{x}(t)\,dt+du^*(t). \tag{3.15}$$

Equations (3.14) and (3.15), can be thought as defining a linear dynamical systems which accepts as input the process $\{u^*(t)\}$ and returns $\{u(t)\}$ as the corresponding output. Its transfer function is

$$V(s):=I+R^{-1/2}C(sI-\Gamma)^{-1}(B^*-B). \tag{3.16}$$

Again, the relationship $V(s)V'(-s)=I$ follows from the fact that $\{u(t)\}$ is a normalized orthogonal increments process. It is not hard to check that $V(s)$ is actually inner. This follows in fact from (3.14) and (3.15) which, together, provide a realization for the inverse, $V^{-1}(s)$ of $V(s)$. We can write

$$V^{-1}(s)=I-R^{-1/2}C(sI-\Gamma^*)^{-1}(B^*-B)$$

and by Theorem 2.2 all poles of $V^{-1}(s)$ have strictly positive real part. To conclude, just recall that $V^{-1}(s)=V'(-s)$.

Consider now the linear system obtained by cascading $V(s)$ and $W(s)$ (in that order). This system has a realization which is obtained combining (3.3), (3.14), and (3.15). Once the intermediate variable $\{u(t)\}$ is eliminated, it defines a linear dynamic relationship between the input $\{u^*(t)\}$ and the output $\{y(t)\}$. The corresponding transfer function is precisely $W^*(s)$. So we have the following.

Lemma 3.3: For any minimal square spectral factor $W(s)$ there is a corresponding all-pass function, $V(s)$, for which

$$W^*(s)=W(s)V(s). \tag{3.17}$$

For each $W(s)$, the related $V(s)$ is given by (3.16) *where B, Γ and B^* are defined in* (2.9), (2.10), *and Theorem 2.2, respectively.*

Given any rational $m \times m$ matrix function, $R(s)$, a *left divisor* of $R(s)$ is any square matrix, $R_1(s)$, such that,

$$R(s)=R_1(s)R_2(s) \tag{3.18}$$

for some matrix $R_2(s)$. If in addition $R_1(s)$ (and $R_2(s)$) are also rational, we say that the factorization is *coprime* (or *minimal* [17] if the McMillan degrees $\delta(R_1)$, $\delta(R_2)$ of R_1 and R_2, add up to $\delta(R)$, the degree of R.

Proposition 3.1: The all-pass function $U(s)$ defined in (3.8) *is a left inner divisor of $U^*(s)$. In fact, $U^*(s)$ admits the coprime factorization,*

$$U^*(s)=U(s)V(s) \tag{3.19}$$

where $V(s)$ is given by (3.16).

Proof: Follows from (3.17), Lemma 3.1 and the uniqueness of the factorization (3.1) for $W(s)$. The coprimeness is a consequence [14, Theorem 7.2 (v)] since no pole-zero cancellations can occur between all-pass functions.

From coprimeness, i.e., $\delta(U)+\delta(V)=\delta(U^*)=n$ it follows that the realizations (3.8) and (3.16) are not minimal. Minimal realizations for $U(s)$ and $V(s)$ will be investigated in Section V.

IV. ARE and Invariant Subspaces

To each solution $P \in \mathcal{P}_o$ of the ARE we can associate a pair of matrices

$$\Pi^+:=\left(P-P_*\right)\left(P^*-P_*\right)^{-1} \tag{4.1}$$

$$\Pi^-:=\left(P^*-P\right)\left(P^*-P_*\right)^{-1}, \tag{4.2}$$

which satisfy the identity $\Pi^- = I - \Pi^+$ and permit us to express P as

$$P = \Pi^- P_* + \Pi^+ P^*. \tag{4.3}$$

Let $x(t)$ be the state process of the internal realization corresponding to P. Then a direct computation [15], [3] shows that the relative error process satisfies

$$\tilde{x}(t) = \Pi^- \tilde{x}^*(t) \tag{4.4}$$

and by taking covariances and postmultiplying by $\Sigma^{*-1} = (P^* - P_*)^{-1}$ it follows that $(\Pi^+)^2 = \Pi^+$, i.e., Π^+ is a *projection*. Let $\mathcal{X}$ be the range space of Π^+ and $\mathcal{Y}$ those of Π^-. It easily follows from (4.1) that[3]

$$\mathcal{Y} = \mathcal{R}[(\Pi^+)']^\perp = \mathcal{R}(\Sigma^{*-1}\Pi^+\Sigma^*)^\perp = [\Sigma^{*-1}\mathcal{X}]^\perp = \Sigma^*\mathcal{X}^\perp. \tag{4.5}$$

This pair of subspaces and the associated projection operators have a special importance in the theory of ARE as pointed out by J. C. Willems [18].

Theorem 4.1: An $n \times n$ matrix P is a solution to the ARE $\Lambda(P) = 0$ *if and only if P can be expressed in the form* (4.3) *for a pair of projection operators Π^+ and $\Pi^- = I - \Pi^+$ such that*

i) $\mathcal{X} := \mathcal{R}(\Pi^+)$ is Γ_*-invariant,
ii) $\mathcal{Y} := \mathcal{R}(\Pi^-)$ is Γ^*-invariant,
iii) $\mathcal{X} \oplus \mathcal{Y} = R^n$.

Moreover, P is symmetric if and only if $\mathcal{Y} = \Sigma^*\mathcal{X}^\perp$, where Σ^* is the matrix defined by (3.10).

Theorem 4.1 gives an alternative way of parametrizing spectral factors. Indeed since there is a one correspondence between $\mathcal{P}_o$ and the family of invariant subspaces $\mathcal{X}$ for Γ_*, the spectral factors can be parametrized directly in terms of $\mathcal{X}$ (or, better, in terms of the projection Π^+). The formulas can be derived from (2.9) and (2.10) by using (4.3). For example, from $P - P_* = \Pi^+(P^* - P_*)$ we can see that

$$B - B_* = \Pi^+(B^* - B_*), \qquad \Gamma - \Gamma_* = \Pi^+(\Gamma^* - \Gamma_*). \tag{4.6}$$

Let Π be a projection operator on R^n and $\mathcal{L}$ its range space. *The compression of $A: R^n \to R^n$ to the subspace $\mathcal{L}$*, is the linear operator $[A]_\mathcal{L} := \Pi A \Pi$. The concept is used in the following lemma which established an important relation existing between the numerator matrix Γ of any minimal square spectral factor and the "extreme" numerators Γ_* and Γ^*.

Lemma 4.1: Let Γ be the numerator matrix of the spectral factor corresponding to $P \in \mathcal{P}_o$ and $(\mathcal{X}, \mathcal{Y})$ the associated pair of complementary subspaces. Then both $\mathcal{X}$ and $\mathcal{Y}$ are invariant for Γ and on these subspaces Γ coincides with the compression of Γ^* and Γ_* to $\mathcal{X}$ and $\mathcal{Y}$, respectively,

$$\Gamma\Pi^+ = \Pi^+\Gamma^*\Pi^+, \qquad \Gamma\Pi^- = \Pi^-\Gamma_*\Pi^-. \tag{4.7}$$

In fact, the following relations hold

$$\Gamma\Pi^+ = \Pi^+\Gamma^*, \qquad \Gamma\Pi^- = \Pi^-\Gamma_*. \tag{4.8}$$

Proof: By taking differentials in (4.4) and substituting (3.5) and its analog for $\tilde{x}^*(t)$, we obtain

$$\Gamma\tilde{x}(t)\,dt + (B - B_*)\,du_*(t) = \Pi^+\Gamma^*\tilde{x}^*(t)\,dt + \Pi^+(B^* + B_*)\,du_*(t)$$

which by virtue of (4.6) and (4.4) reduces to

$$\Gamma\Pi^+\tilde{x}^*(t) = \Pi^+\Gamma^*\tilde{x}^*(t).$$

Right multiplying by $\tilde{x}^*(t)'$ and taking expectations, gives the first formula in (4.8). Then the first part of (4.7) and Γ-invariance of $\mathcal{X}$ follow immediately. Notice now that the second relation in (4.6) can be rewritten as

$$\Gamma = \Pi^-\Gamma_* + \Pi^+\Gamma^*.$$

Premultiply this identity by Π^+ (recalling that $\Pi^+\Pi^- = 0$) and insert $\Gamma\Pi^+$ in place of $\Pi^+\Gamma^*$, getting

$$\Pi^+\Gamma = \Gamma\Pi^+.$$

This formula proves that the direct sum decomposition $R^n = \mathcal{X} \oplus \mathcal{Y}$ is actually reducing for Γ and hence,

$$\Pi^-\Gamma = \Gamma\Pi^-$$

holds as well. Premultiplying the expression for Γ given above by Π^- yields then the second halves of (4.8) and (4.7).

Let now $T := [X|Y]$ be a basis matrix in R^n obtained by picking basis matrices X and Y in $\mathcal{X}$ and $\mathcal{Y}$, respectively, and denote by

$$T^{-1} := \left[\frac{W}{Z}\right] \tag{4.9}$$

the inverse of T partitioned conformably. Since $WY = 0$ and $ZX = 0$, Z' and W' are bases for $\mathcal{X}^\perp$ and $\mathcal{Y}^\perp$, respectively. Let us introduce a change of basis in R^n, described by the matrix T. It is clear that the numerator matrices Γ_*, Γ^* assume thereby the structure

$$\hat{\Gamma}_* = \begin{bmatrix} \Gamma_{*11} & \Gamma_{*12} \\ 0 & \Gamma_{*22} \end{bmatrix} \qquad \hat{\Gamma}^* = \begin{bmatrix} \Gamma_{11}^* & 0 \\ \Gamma_{21}^* & \Gamma_{22}^* \end{bmatrix} \tag{4.10}$$

where

$$\Gamma_{*11} = W\Gamma_*X, \qquad \Gamma_{*12} = W\Gamma_*Y, \qquad \Gamma_{*22} = Z\Gamma_*Y \tag{4.11}$$

$$\Gamma_{11}^* = W\Gamma^*X, \qquad \Gamma_{21}^* = Z\Gamma^*X, \qquad \Gamma_{22}^* = Z\Gamma^*Y. \tag{4.12}$$

Also, since $\mathcal{X} \oplus \mathcal{Y}$ is a direct sum of reducing subspaces for Γ we obtain

$$\hat{\Gamma} = \begin{bmatrix} \Gamma_{11}^* & 0 \\ 0 & \Gamma_{*22} \end{bmatrix}. \tag{4.13}$$

The diagonal blocks $\hat{\Gamma}_{11}$ and $\hat{\Gamma}_{22}$ coincide with Γ_{11}^* and Γ_{*22} because of (4.7). This fact can also be checked by a direct computation. In fact, $\hat{\Gamma}_{11}$ is given by

$$W\Gamma X = W\Gamma\Pi^+X = W\Pi^+\Gamma^*X$$

and $W\Pi^+ = W$, since W' is a basis for $\mathcal{Y}^\perp = \mathcal{N}[(I - \Pi^+)']$. It follows from (4.12) (first equality) that $\Gamma_{11} = \Gamma_{11}^*$. That $\hat{\Gamma}_{22} = \Gamma_{*22}$ follows by dual arguments.

Formula (4.13) provides a parametrization of the *zeros* of any minimal spectral factor $W(s)$ in terms of the uniquely associated invariant subspace $\mathcal{X}$. Let us denote by $\sigma_+(\Gamma)$ and $\sigma_-(\Gamma)$ the unstable (i.e., lying on $\operatorname{Re} s > 0$) and stable part of the spectrum of Γ and by $\lambda_+(\Gamma)$ and $\lambda_-(\Gamma)$ the corresponding generalized eigenspaces. The structure of all minimal square spectral factors is then described in the following theorem.

Theorem 4.2: Let $W(s) = R^{1/2} + C(sI - A)^{-1}B$ be the minimal square spectral factor which corresponds to the Γ_*-invariant subspace $\mathcal{X}$. Its numerator Γ is then the direct sum

$$\Gamma = [\Gamma^*]_\mathcal{X} \oplus [\Gamma_*]_\mathcal{Y} \tag{4.14}$$

of the compressions $[\Gamma^*]_\mathcal{X}$, $[\Gamma_*]_\mathcal{Y}$, of Γ^* to $\mathcal{X}$ and, respectively, of Γ_* to the complementary subspace $\mathcal{Y} = \Sigma^*\mathcal{X}^\perp$. The spectrum of Γ is accordingly decomposed as,

$$\sigma_+(\Gamma) = \sigma([\Gamma^*]_\mathcal{X}), \qquad \sigma_-(\Gamma) = \sigma([\Gamma_*]_\mathcal{Y}) \tag{4.15a}$$

$$\lambda_+(\Gamma) = \mathcal{X}, \qquad \lambda_-(\Gamma) = \mathcal{Y}. \tag{4.15b}$$

Consequently, the whitening filter $W(s)^{-1}$ can be decomposed as

$$R^{1/2}W^{-1}(s)R^{1/2} = I - CX(sI - \Gamma_{11}^*)^{-1}WB^* - CY(sI - \Gamma_{*22})^{-1}ZB_* \tag{4.16}$$

where Γ_{11}^* and Γ_{*22} are matrix representations of $[\Gamma^*]_\mathcal{X}$ and $[\Gamma_*]_\mathcal{Y}$ relative to any basis $T = [X|Y]$ in $\mathcal{X} \oplus \mathcal{Y}$ [whose inverse is partitioned as in (4.9)].

Proof: All that needs to be proven is the decomposition (4.16), since the rest follows from (4.13). In order to do that, notice that if we change basis in the realization (2.11) of $W^{-1}(s)$, by using the above defined

[3] Symbols $\mathcal{R}$ and $\mathcal{N}$ stay for "range" and "nullspace;" $\perp$ means orthogonal complement.

matrix T, $\hat{\Gamma}:=T^{-1}\Gamma T$ assumes the structure (4.13) and $T^{-1}B$ can be partitioned as

$$T^{-1}B=\begin{bmatrix} WB \\ ZB \end{bmatrix}.$$

Let us now use the representation (4.6) for B and notice that $W\Pi^{-}=0$, $W\Pi^{+}=W$, and similarly $Z\Pi^{-}=Z$, $Z\Pi^{-}=0$. A simple computation leads then to (4.16). □

Remarks: Since Γ^* is similar to $-\Gamma'_*$ [cf. (3.12)], it follows that Γ^*_{11} is similar to Γ'_{*11}, and hence the first equation in (4.15a) can also be rewritten in the form

$$\sigma_+(\Gamma)=-\sigma(\Gamma_{*|\mathfrak{X}}) \tag{4.17}$$

which is the vector analog of the "zero flipping" rule.

Formula (4.16) shows that the input noise $\{u(t)\}$ in any internal realization can be obtained by combining a causal (asymptotically stable) and a completely anticausal (totally unstable) whitening filter. This additive decomposition can be related to smoothing formulas of the "Mayne–Fraser" type [3].

V. Inner Divisors and Invariant Subspace

In this section we investigate the relations which exist between invariant subspaces and an alternative parametrization of the minimal square spectral factors, namely, their all-pass part, $U(s)$, which was defined in Section III. There are recent results which connect divisor theory for rational matrices and invariant subspaces ([4], [17]). These results, once specialized to all-pass functions will provide a neat isomorphism between the lattice of Γ_*-invariant subspaces $\mathfrak{X}$, and the lattice of all left inner divisors, $U(s)$, of the maximal inner function $U^*(s)$ defined in (3.9).

We start with a lemma on minimal realizations of all-pass functions.

Lemma 5.1: Every rational inner function, $U(s)$, can be minimally represented as

$$U(s)=I-H(sI-F)^{-1}QH' \tag{5.1a}$$

or as

$$U(s)=I-G'Q^{-1}(sI-F)^{-1}G \tag{5.1b}$$

where F is an asymptotically stable matrix, (F,G) is a controllable pair, Q is the unique positive definite solution of the matrix Lyapunov equation

$$FQ+QF'+GG'=0 \tag{5.2}$$

and G and H are related by the transformation,

$$H=-G'Q^{-1}. \tag{5.3}$$

Proof: Let $U(s)$ have a minimal realization $\{F,G,H,J\}$, i.e.,

$$U(s)=J+H(sI-F)^{-1}G.$$

Since $U(\infty)=J$, J is a unitary matrix and since $U(s)$ is defined up to multiplication by an arbitrary unitary matrix, we can as well assume $J=I$. From $U^{-1}(s)=U'(-s)$ we obtain

$$I-H(sI-\bar{F})^{-1}G=I-G'(sI+F')^{-1}H'$$

where $\bar{F}:=F-GH$. The matrix triples $(\bar{F},G,H)$ and $(-F',H',G')$ must then be similar, i.e., there has to exist some nonsingular T for which

$$F-GH=-TF'T^{-1} \tag{5.4a}$$

$$H=G'T^{-1} \tag{5.4b}$$

$$G=TH'. \tag{5.4c}$$

From (5.4a, b) it follows that

$$F+TF'T^{-1}=GG'T^{-1}$$

and hence $Q:=-T$ must satisfy the Lyapunov equation

$$FQ+QF'+GG'=0$$

which, by controllability, has a unique positive definite solution. Notice that the numerator matrix $\bar{F}$, of $U(s)$, is then given by formula (5.4a), i.e.,

$$\bar{F}=-QF'Q^{-1}. \tag{5.5}$$

Let us now consider the problem of *describing all (coprime) all-pass factorizations of* a given rational inner function $U(s)$.

$$U(s)=U_1(s)U_2(s). \tag{5.6}$$

The factor $U_1(s)(U_2(s))$ in (5.6) is called *a left (right) all-pass divisor of* $U(s)$. Either $U_1(s)$ or $U_2(s)$ determine the factorization uniquely.

We shall assume that $U(s)$ is described by a minimal realization of the form (5.1a) of dimension n. The corresponding minimal state space realizations of $U_i(s)$ will be denoted by

$$U_i(s)=I-H_i(sI-F_i)^{-1}Q_iH_i', \qquad i=1,2. \tag{5.7}$$

Lemma 5.2: There is a one to one correspondence between the set of inner factorizations of $U(s)$ and the family of subspaces $\mathfrak{X}$ of R^n which are F-invariant,

i) $F\mathfrak{X}\subset\mathfrak{X}$.

To each $\mathfrak{X}$ in R^n, satisfying i), associate the complementary subspace $\mathfrak{Y}$ defined as,

ii) $\mathfrak{Y}=Q\mathfrak{X}^{\perp}$.

This subspace is $\bar{F}$-invariant, i.e.,

iii) $\bar{F}\mathfrak{Y}\subset\mathfrak{Y}$

and for any change of basis in R^n defined by a matrix $T=[X|Y]$, with $\mathfrak{X}=\text{span } X$, $\mathfrak{Y}=\text{span } Y$, *one has*

iv) $$T^{-1}FT=\begin{bmatrix} F_1 & -Q_1H_1'H_2 \\ 0 & F_2 \end{bmatrix}, \qquad HT=[H_1\ H_2]$$

v) $$T^{-1}Q(T^{-1})'=\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \qquad T^{-1}QH'=\begin{bmatrix} Q_1H_1' \\ Q_2H_2' \end{bmatrix}$$

where F_i, H_i, Q_i, $i=1,2$ define, through (5.7), an all-pass factorization of $U(s)$.

Proof: Let us first show that any $\mathfrak{X}$ satisfying the invariance condition i), yields an all-pass factorization of $U(s)$. From (5.5) it follows that $\mathfrak{Y}$ defined by ii) is indeed $\bar{F}$-invariant. Also, the matrix T clearly transforms F to an upper block diagonal form of the type shown in iv).

If we partition T^{-1} as $[W'Z']'$ and recall that by definition of T,

W' is a basis for $\mathfrak{Y}^{\perp}$

Z' is a basis for $\mathfrak{X}^{\perp}$

it follows from ii) that QZ' is a basis for $\mathfrak{Y}$ and hence in the partitioned matrix

$$T^{-1}Q(T^{-1})'=\begin{bmatrix} WQW' & WQZ' \\ ZQW' & ZQZ' \end{bmatrix} \tag{5.8}$$

the off-diagonal blocks are zero. Letting $Q_1:=WQW'$ and $Q_2:=ZQZ'$, v) follows.

Notice that any pair of complementary subspaces satisfying i) and iii) corresponds ([4]) to a coprime factorization of $U(s)$ of the form (5.6) where $U_i(s)$ are rational matrices. That $U_i(s)$ are inner follows however from the special choice ii) of the complementary subspace $\mathfrak{Y}$ which leads, via v), to the special structure (5.7) of the factors. In order to check this fact we shall compute the upper right block F_{12}, of the transformed F matrix. Since $W\bar{F}Y=0$, by iii), we can write

$$F_{12}=WFY=W(F-\bar{F})Y=WQ(Q^{-1}F+F'Q^{-1})Y=-WQH'HY \tag{5.9}$$

[in the last passage we have used (5.2) and (5.3)]. Using the second relation in v) one obtains

$$F_{12}=-Q_1H_1'H_2$$

which proves iv). Notice now that iv) and v) provide an explicit state-space realization of the cascade $U_1(s)U_2(s)$ where $U_i(s)$ are the all-pass functions (5.7).

To prove the converse one needs to show that to any inner factorization of $U(s)$, (5.6), with $U_i(s)$ given by (5.7) there correspond complementary subspaces $\mathcal{X}$ and $\mathcal{Y}$ with the stated properties.

If we write down explicitely the state-space realization of the cascade $U_1(s)U_2(s)$ and compare to the given realization (5.1a) of $U(s)$ we see that there has to exist an invertible $n\times n$ matrix T such that iv) and v) hold.

In fact, the first equation in v) comes from the diagonal structure of the solution of the Lyapunov equation for the cascade realization. Now, from the first relation in iv) it follows that there must exist an invariant subspace $\mathcal{X}$ for F such that $F_1=F|_{\mathcal{X}}$. In fact, T has the structure $T=[X|Y]$ where $\mathcal{X}=sp\ X$, and Y is a basis for some complementary subspace $\mathcal{Y}$.

Let $T^{-1}=[W'Z']'$. Notice that W' and Z' can be given the same meaning as in the first part of the proof. From the first equation in v) if then follows that

$$WQZ'=0$$

which implies that QZ' is the orthogonal complement in R^n of $\mathcal{Y}^{\perp}$, i.e., that QZ' is a basis for $\mathcal{Y}$. Since Z' is a basis for $\mathcal{X}^{\perp}$, ii) follows. □

The all-pass function of special interest to us is the maximal all-pass factor $U^*(s)$ defined by (3.9),

$$U^*(s)=I-R^{-1/2}C(sI-\Gamma_*)^{-1}\Sigma^*C'R^{-1/2}. \qquad (5.10)$$

The numerator matrix of $U^*(s)$ is $\Gamma_*+\Sigma^*C'R^{-1}C=\Gamma^*$ and so the inverse $U^{*-1}(s)$ has the realization

$$U^*(s)^{-1}=I+R^{-1/2}C(sI-\Gamma^*)^{-1}\Sigma^*C'R^{-1/2}. \qquad (5.11)$$

An easy identification of symbols, namely

$$F=\Gamma_*,\ H=R^{-1/2}C,\ Q=\Sigma^*$$

leads to the following corollary of Lemma 5.2.

Corollary 5.1: The (partially ordered) set of left inner divisors of $U^*(s)$ is in one to one correspondence with the family of Γ_*-invariant subspaces $\mathcal{X}$ of R^n. Each left divisor, $U_1(s)$, has a minimal realization

$$U_1(s)=I-R^{-1/2}CX(sI-\Gamma_{*11})^{-1}W\Sigma^*C'R^{-1/2}, \qquad (5.12)$$

and the relative right divisor $U_2(s)$ (such that $U^*(s)=U_1(s)U_2(s)$) has a corresponding minimal realization,

$$U_2(s)=I-R^{-1/2}CY(sI-\Gamma_{*22})^{-1}Z\Sigma^*C'R^{-1/2} \qquad (5.13)$$

where X and Y are bases in $\mathcal{X}$ and $\Sigma^*\mathcal{X}^{\perp}$, respectively, and W, Z, Γ_{*11}, Γ_{*22} have the same meaning as in Section IV.

It is not hard to see that $U_1(s)$ and $U_2(s)$ coincide with the all-pass functions $U(s)$ and $V(s)$ of Proposition 3.1. In fact, the matrix $(B-B_*)$ of (3.8) can, by using (4.6), (3.12), be put in the form

$$B-B_*=-\Pi^{+}\Sigma^*C'R^{-1/2}$$

and similarly, (B^*-B) appearing in (3.16) can be rewritten as

$$B^*-B=-\Pi^{-}\Sigma^*C'R^{-1/2}.$$

The usual change of basis leads us now to consider the matrices

$$\widehat{B-B_*}:=-\begin{bmatrix}W\\Z\end{bmatrix}\Pi^{+}\Sigma^*C'R^{-1/2}, \qquad (5.14)$$

$$\widehat{B^*-B}:=-\begin{bmatrix}W\\Z\end{bmatrix}\Pi^{-}\Sigma^*C'R^{-1/2}. \qquad (5.15)$$

Since $W\Pi^{+}=W$ and $Z\Pi^{+}=0$ (equivalent to $W\Pi^{-}=0$ and $Z\Pi^{-}=Z$), combining (5.14), the expression of $\hat{\Gamma}_*$ in (4.10) and $R^{-1/2}C[X|Y]$ gives (5.12). Similarly (5.15), (4.10) and $R^{-1/2}C[X|Y]$ give (5.13).

At this point we have more than enough material to prove the following important characterization of minimal square spectral factors.

Theorem 5.1: A square spectral factor $W(s)$ is minimal if and only if its inner part is a left inner divisor of $U^*(s)$.

Proof: If $W(s)$ is minimal, we have already seen that its inner part $U(s)$ is a left divisor of $U^*(s)$ (Proposition 3.1). Vice versa, if $U_1(s)$ divides $U^*(s)$ on the left, then we have just checked that $U_1(s)$ is the inner part, $U(s)$, of some minimal spectral factor.

Theorem 5.1 was proven, for the scalar discrete time case in [16] and for $\Phi(\infty)=0$, in [9] and [10]. In these references the result is obtained by rather sophisticated analytical tools (theory of Hardy spaces). Our approach instead is completely elementary.

The structure of inner divisors of $U^*(s)$ helps to understand the pole-zero cancellation process which takes place when we multiply $W_*(s)$ and $U(s)$ [or alternatively $W^*(s)$ and $V^{-1}(s)$)] together. From (5.12) and (5.13) it follows that the inverses of $U_1(s)(=U(s))$ and $U_2(s)(=V(s))$ are given by

$$U^{-1}(s)=I+R^{-1/2}CX(sI-\Gamma^*_{11})^{-1}W\Sigma^*C'R^{-1/2}$$

and

$$V^{-1}(s)=I+R^{-1/2}CY(sI-\Gamma^*_{22})^{-1}Z\Sigma^*C'R^{-1/2}.$$

In fact, the respective numerator matrices are given by

$$\Gamma_{*11}-W\Sigma^*C'R^{-1}CX=W\Gamma_*X+W(\Gamma^*-\Gamma_*)X=\Gamma^*_{11}$$

and

$$\Gamma_{*22}-Z\Sigma^*C'R^{-1}CY=Z\Gamma_*Y+Z(\Gamma^*-\Gamma_*)Y=\Gamma^*_{22}$$

[compare with (4.11)].

If we now look back at (4.4) we may note that the dynamics of the state vector associated with $U(s)$ [i.e., the error process $\tilde{x}(t)$] takes entirely place in the invariant subspace $\mathcal{X}$ of Γ_*. Since $\Gamma_{*\mathcal{X}}$ is Γ_{*11} (in the appropriate coordinate system) and $\tilde{x}(t)$ evolves according to a pole zero configuration described by Γ_{*11} and Γ^*_{11}, respectively, the effect of cascading $W_*(s)$ with $U(s)$ is the neat substitution of the upper left block Γ_{*11}, of $\hat{\Gamma}_*$, with Γ^*_{11}, thus leading to the structure (4.13) for Γ.

The same type of heuristic explanation works for the cascade of $W^*(s)$ and $V^{-1}(s)$; now we just have to refer to the "dual" error process $\check{x}(t)=\Pi^{-}\tilde{x}^*(t)$.

References

[1] B. D. O. Anderson, "The inverse problem of stationary covariance generation," *J. Stat. Phys.*, vol. 1, pp. 133–147, 1969.

[2] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis*. Englewood Cliffs, NJ: Prentice Hall, 1973.

[3] F. Badawi, A. Lindquist, and M. Pavon, "A stochastic realization approach to the smoothing problem," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 878–888, 1979.

[4] H. Bart, I. Gohberg, and M. A. Kaashoek, *Minimal Factorizations of Matrix and Operator Functions*. New York: Birkhäuser, 1980.

[5] R. W. Brockett, *Finite Dimensional Linear Systems*. New York: Wiley, 1970.

[6] P. Faurre, "Realisations Markoviennes de processus stationnaires." IRIA (Laboria), Le Chesnay, France, Tech. Rep. 13, Mar. 1973.

[7] I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*. Philadelphia, PA: Saunders, 1969.

[8] A. Lindquist and G. Picci, "On the stochastic realization problem," *SIAM J. Contr. Optimiz.*, vol. 17, pp. 365–389 1979.

[9] A. Lindquist and G. Picci, "Realization theory for multivariate stationary Gaussian processes I: state space construction," in *Proc. 4th Int. Symp. Math. Theory of Networks and Syst.*, Delft, Holland, July 1979, pp. 140–148.

[10] A. Lindquist and G. Picci, "Realization theory for multivariate stationary Gaussian processes II: state space theory revisited and dynamical representations of finite fimensional state space," in *Proc. 2nd Int. Conf. Inform. Sci. Syst.*, Patras, Greece, July 1979.

[11] C. Martin, "Grassman manifolds and global properties of the Riccati equation," *Int. Symp. Operator Theory of Networds and Syst.*, Lubbock, TX, Aug. 1977, pp. 82–85.

[12] C. Martin, "Grassman manifolds, Riccati equations and feedback invariants of linear systems," in *Proc. NATO ASI and AMS Summer Seminar in Appl. Math. on Algebraic and Geometric Methods in Linear Syst. Theory*, to be published.

[13] M. Pavon, "Stochastic realization and invariant directions of the matrix Riccati equation," *SIAM J. Contr. Optimiz.*, Mar. 1980.

[14] H. H. Rosenbrock, *State-Space and Multivariable Theory*. London, England: Nelson, 1970.

[15] G. Ruckebusch, "Representations Markoviennes de processus Gaussiens stationnaires," thesis, Univ. of Paris VI, 1975.

[16] G. Ruckebusch, "Factorisations minimales de densités spectrales et représentations markoviennes," in *Proc. IRE Colloque AFCET-SMF*, Palaiseau, France, 1978.

[17] P. Van Dooren, P. DeWilde, "Minimal factorization of rational matrices," in *Proc. 1978 Decision Contr. Conf.*, San Diego, CA, Jan. 1979, pp. 170–172.

[18] J. C. Willems, "Least squares stationary optimal control and the algebraic Riccati equation," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 621–634, 1971.

[19] D. C. Youla, "On the factorization of rational matrices," *IRE Trans. Inform. Theory*, vol. IT-7, pp. 172–189, 1961.