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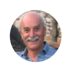

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REALIZATION THEORY FOR MULTIVARIATE STATIONARY GAUSSIAN PROCESSES*

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Abstract. This paper collects in one place a comprehensive theory of stochastic realization for continuous-time stationary Gaussian vector processes which in various pieces has appeared in a number of our earlier papers. It begins with an abstract state space theory, based on the concept of *splitting subspace*. These results are then carried over to the spectral domain and described in terms of Hardy functions. Finally, differential-equations type stochastic realizations are constructed. The theory is coordinate-free, and it accommodates infinite-dimensional representations, minimality and other systems-theoretical concepts being defined by subspace inclusion rather than by dimension. We have strived for conceptual completeness rather than generality, and the same framework can be used for other types of stochastic realization problems.

CONTENTS

1. Introduction.

The following inverse problem is of central importance in stochastic systems theory. Given a stationary Gaussian vector process $\{y(t);\, t\in\mathbb{R}\}$, find a vector-valued stationary Gaussian Markov process $\{x(t);\, t\in\mathbb{R}\}$ of smallest possible dimension so that

$$y(t)=Cx(t) \tag{1.1}$$

for some matrix C, and determine a stochastic differential equation for x. This is the *stochastic realization problem* and the representation is called a *minimal stochastic realization.*

This problem, first formulated by Kalman [21] in 1965, has generated a rather extensive literature. Most notable among the early contributions are the papers by Anderson [2] and Faurre [11], the main focus of which is the realization of spectral factors and the Yakubovich–Kalman–Popov lemma. The more recent work by Ruckebusch [39], Lindquist and Picci [25], and Pavon [36] is geared toward the characterization of Markovian representations in terms of the information carried by the given process. During the last decade, the bulk of the papers on stochastic realization theory have been concerned with geometric state space construction in Hilbert space. Here the forerunners are Akaike [1] and Picci [37], whereas the most comprehensive contributions are due to Lindquist and Picci [26]–[32] and Ruckebusch [40]–[44]. A more extensive bibliography can be found in our survey paper [24].

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There are both conceptual and practical reasons why this problem is important. On the conceptual side, a theory of stochastic realization should give a firm foundation of the idea of *state* and *state space models*. Clearly this is of central importance in setting stochastic systems theory on a sound mathematical basis. The purpose of this paper is to present such a theory in which the idea of state is defined through a fundamental property of conditional independence (splitting), a natural generalization of the property of state in the deterministic theory. This point of view provides a general framework for stochastic modeling in which problems of stochastic systems theory can be set.

Important areas for potential application of this theory include identification, stochastic model reduction, and stochastic control, and there is preliminary evidence that the basic ideas presented here will prove to be fruitful. Moreover, there are already problems in estimation theory which have been successfully tackled by such an approach. Some cases in point are smoothing [49], interpolation [51], and, in general, problems with a noncausal information flow. Possible extensions of the theory presented here to the nonlinear (non-Gaussian) case will provide solution to even wider areas of important applications. For example, realization theory of finite-state processes would provide powerful technics to solve important problems in communication theory.

Stochastic realization theory is *not* a generalization of deterministic input-output realization theory. Characteristic of the stochastic problem is the fact that there are many different (minimal) causality structures which describe the same external behavior, the basic problem being to classify all of them. Note that a similar problem is encountered in J. C. Willems' deterministic realization theory [52] for "signals", a theory which has many points of contact with ours.

This invited paper collects in one place a reasonably self-contained treatment of the geometric theory of stochastic realization which in various pieces has appeared in a number of our previous papers [26]–[32], some of which are published in volumes of limited availability. We have strived for conceptual completeness rather than generality. Consequently, many of the results presented here have generalizations in various directions, some straightforward and others more nontrivial. The basic conceptual framework, however, is the same.

The need for a geometric theory of stochastic realization is illustrated by the problem formulation above. As it stands, the problem may not be meaningful unless the given process has a rational spectral density and hence a finite-dimensional representation is possible. In the general case, a representation of type (1.1) exists only under certain technical conditions (which we do not want to introduce at the beginning). Moreover, the concept of *minimality* needs a natural dimension-free formulation which also covers the infinite-dimensional situation. Finally, a geometric theory is *coordinate-free* and hence allows us to factor out, in the first analysis, the properties of the realizations which depend only on the choice of coordinates and may unduly complicate the picture.

To this end, let us reformulate the above problem in terms of Hilbert space geometry. Let $\{y(t);\, t\in\mathbb{R}\}$ be a stationary Gaussian stochastic vector process which is mean-square continuous and centered. Consider the space $\hat{H}$ of all finite linear combinations of the random variables $\{y_k(t);\, t\in\mathbb{R},\, k=1,2,\cdots,m\}$. Endowed with the inner product $\langle \xi, \eta\rangle := E\{\xi\eta\}$, where $E\{\cdot\}$ denotes mathematical expectation, $\hat{H}$ is a pre-Hilbert space. Let H be the Hilbert space obtained by taking the closure of $\hat{H}$; this is known as the Gaussian space of y [35]. A standard argument [38, p. 15] shows that there is a group $\{U_t;\, t\in\mathbb{R}\}$ of unitary operators on H such that $U_t y_k(s)=y_k(s+t)$ for all $s, t\in\mathbb{R}$ and $k=1,2,\cdots,m$. Since y is mean-square continuous, the group

$\{U_t; t \in \mathbb{R}\}$ is strongly continuous. We shall use the notation $E^X\lambda$ to denote the orthogonal projection of $\lambda \in H$ onto a subspace[1] X of H. This notation is motivated by the fact that $E^X\lambda$ coincides with the conditional expectation $E\{\lambda \mid \mathscr{X}\}$ where $\mathscr{X}$ is the σ-field generated by the random variables in X [9].

Consider the class of subspaces X of H with the properties

(i) $y_k(0) \in X$ for $k = 1, 2, \cdots, m$;

(ii) X is *Markovian* in the sense that

$$\langle \lambda - E^X\lambda, \mu - E^X\mu \rangle = 0 \quad \text{for } \lambda \in X^-, \mu \in X^+$$

where X^- and X^+ are the closed linear hulls of $\{U_tX; t \leqq 0\}$ and $\{U_tX; t \geqq 0\}$ respectively;

(iii) X is *minimal* in the sense that if X_1 is a subspace of X and X_1 satisfies (i) and (ii), then $X_1 = X$.

The term *Markovian* is motivated by the fact, as we shall see below (Proposition 2.1), that (ii) is equivalent to each of the two conditions

$$E^{X^-}\lambda = E^X\lambda \quad \text{for } \lambda \in X^+, \tag{1.2a}$$

$$E^{X^+}\lambda = E^X\lambda \quad \text{for } \lambda \in X^-. \tag{1.2b}$$

For reasons to be reported in § 3 (Proposition 3.1), a subspace X satisfying (i) and (ii) will be called a *Markovian splitting subspace.*

What is then the connection between such subspaces and the stochastic realization problem stated above? Let us *for the moment* assume that X has finite dimension n, and let $\{x_1, x_2, \cdots, x_n\}$ be a basis in X. Then, in view of property (i), there is an $m \times n$ matrix $C = \{c_{ij}\}$ such that $y_i(0) = \sum_{j=1}^n c_{ij}x_j$ for $i = 1, 2, \cdots, m$. Consequently,

$$y(t) = Cx(t) \tag{1.3}$$

where $\{x(t); t \in \mathbb{R}\}$ is the n-dimensional stationary stochastic process defined by setting $x_k(t) := U_t x_k$ for $k = 1, 2, \cdots, n$. Under suitable geometric conditions on X (to be introduced in § 3) this process is purely nondeterministic [38]; *for the sake of this example,* we shall assume that this is the case. Since

$$U_tX = \text{span}\{x_1(t), x_2(t), \cdots, x_n(t)\}, \tag{1.4}$$

condition (1.2a), shifted by U_t, is equivalent to

$$E\{x_k(s) \mid \mathscr{X}_t^-\} = E\{x_k(s) \mid \mathscr{X}_t\} \quad \text{for } s \geqq t, \quad k = 1, 2, \cdots, n \tag{1.5}$$

where $\mathscr{X}_t^-$ and $\mathscr{X}_t$ are the σ-fields generated by $\{x_k(\tau); \tau \leqq t, k = 1, 2, \cdots, m\}$ and $\{x_k(t); k = 1, 2, \cdots, n\}$ respectively. Consequently x is a vector *Markov process.* Finally, as we shall see below, condition (iii) insures that the dimension n is as small as possible. The condition $X \subset H$ is not implied by the original problem formulation, but it is not unnatural since the process y is the only thing given. Such realizations are called *internal* [25]. However, several of the applications mentioned above require that we consider the noninternal situation when H is imbedded in a larger Hilbert space. Although many of our results remain valid in the noninternal setting and others can be generalized [43], [44], we shall restrict ourselves here to a simple prototype problem.

It is well known that a vector Markov process *of the type described above* has a representation

$$x(t) = \int_{-\infty}^{t} e^{A(t-\sigma)} B \, du(\sigma) \tag{1.6}$$

[1] In this paper a *subspace* is assumed to be closed.

where A and B are matrices, u is a vector-valued orthogonal increment process with components in H, and the integral is defined in quadratic mean [11]. Together with (1.3) this yields a *forward* stochastic realization

$$dx = Ax\,dt + B\,du,\qquad y = Cx. \tag{1.7}$$

The forward property is characterized by $X \subset H^-(du)$, where $H^-(du)$ is the subspace generated by the components of the past increments $\{u(t)-u(s);\ t, s \leqq 0\}$. By symmetry and (1.2b), there is also a representation

$$x(t) = -\int_t^{\infty} e^{\bar{A}(\sigma-t)}\bar{B}\,d\bar{u}(\sigma) \tag{1.8}$$

which corresponds to a *backward* stochastic realization

$$dx = -\bar{A}x\,dt + \bar{B}\,d\bar{u},\qquad y = Cx. \tag{1.9}$$

This realization is backward because $X \subset H^+(d\bar{u})$, the subspace generated by the components of the future increments $\{\bar{u}(t)-\bar{u}(s);\ t, s \geqq 0\}$. Characterizing Markovian representations in terms of pairs of realizations, one evolving forward and one backward, is one of the key ideas in [25] and in the present work. It is well known and easy to show that the transfer functions

$$W(s) = C(sI-A)^{-1}B, \tag{1.10a}$$

$$\bar{W}(s) = C(sI+\bar{A})^{-1}\bar{B} \tag{1.10b}$$

are rational spectral factors of y, W having all its poles in the left and $\bar{W}$ all its poles in the right half plane.

It follows from finite-dimensional stochastic realization theory [2], [11], [12] that (1.7) is a minimal stochastic realization if and only if (a) it is *reachable*, i.e. $[B, AB, A^2B, \cdots]$ is full rank, (b) it is *observable*, i.e. $[C', (CA)', (CA^2)', \cdots]$ is full rank (where prime denotes transpose), and (c) W has minimal degree (among spectral factors). Likewise (1.9) is minimal if and only if (a)$'$ it is *controllable*, i.e. $[\bar{B}, \bar{A}\bar{B}, \bar{A}^2\bar{B}, \cdots]$ is full rank, (b)$'$ it is *constructible*, i.e. $[C', (C\bar{A})', (C\bar{A}^2)', \cdots]$ is full rank[2], and (c)$'$ $\bar{W}$ has minimal degree. As can be easily checked, $x(0)$ being a basis in X automatically takes care of conditions (a) and (a)$'$, and hence they will not occur in the geometric theory. Conditions (b), (c), (b)$'$ and (c)$'$ will be given natural geometric and function theoretic characterizations below which hold also in the infinite-dimensional case. We shall see, for example, that minimality is equivalent not only to (b)+(c) or to (b)$'$+(c)$'$ but also to (b)+(b)$'$.

This paper divides naturally into three parts. The first part, consisting of §§ 3–5, is devoted to a characterization of the class of Markovian splitting subspaces and an analysis of their systems-theoretical properties. Section 2 is a preliminary in which we define the concept of *perpendicular intersection*, introduced in [29].

In the second part, consisting of §§ 6 and 7, the geometry is described in terms of Hardy spaces, and the Markovian splitting subspaces are characterized by pairs $(W, \bar{W})$ of spectral factors. This part of the theory has some connections with Lax–Phillips scattering theory [23].

[2] Using the terms *controllable* and *constructible* instead of *reachable* and *observable* when referring to a system evolving backwards is in agreement with accepted terminology in systems theory [22].

Finally, in §§ 8–10, we assign to each Markovian splitting subspace X two stochastic realizations, a forward one with transfer function W and a backward one with transfer function $\bar{W}$, having their systems-theoretical properties prescribed by X. Moreover, we study the relationships between realizations corresponding to different splitting subspaces.

2. Perpendicular intersection. Let A, B and X be subspaces of a real Hilbert space H. We shall say that A and B are *conditionally orthogonal given* X if

$$\langle \alpha - E^X\alpha, \beta - E^X\beta\rangle = 0 \quad \text{for } \alpha \in A, \quad \beta \in B. \tag{2.1}$$

This will be denoted $A \perp B | X$. When X is the trivial subspace, i.e. $X = 0$, this reduces to the usual orthogonality $A \perp B$. Conditional orthogonality is orthogonality after subtracting the components in X. We write $A \vee B$ to denote the vector sum, i.e. the closure of $\{\alpha + \beta \mid \alpha \in A, \beta \in B\}$ and $A \oplus B$ to denote orthogonal direct sum; $A \ominus B$ is the subspace $C \subset A$ such that $B \oplus C = A$; $B^\perp$ is the orthogonal complement of B in H, i.e. $B^\perp = H \ominus B$. Finally, $E^A B = \{E^A \beta \mid \beta \in B\}$. This space may not be closed, and we shall write $\bar{E}^A B$ to denote the closure.

PROPOSITION 2.1. *The following statements are equivalent.*

(i) $A \perp B | X$.

(ii) $B \perp A | X$.

(iii) $(A \vee X) \perp B | X$.

(iv) $E^{A \vee X}\beta = E^X \beta$ *for* $\beta \in B$.

(v) $(A \vee X) \ominus X \perp B$.

(vi) $E^A \beta = E^A E^X \beta$ *for* $\beta \in B$.

Proof. The equivalence between (i), (ii) and (iii) follows directly from the definition. Since $(\beta - E^X\beta) \perp X$, relation (2.1) may be written $\langle \alpha, \beta - E^X\beta\rangle = 0$. Therefore, (iii) is equivalent to $(\beta - E^X\beta) \perp A \vee X$, i.e. $E^{A\vee X}(\beta - E^X\beta) = 0$, which is precisely (iv). Moreover, (i) is equivalent to $(\beta - E^X\beta) \perp A$, i.e. $E^A(\beta - E^X\beta) = 0$, which is the same as (vi). Finally, set $Z := (A \vee X) \ominus X$; then $A \vee X = X \oplus Z$, i.e. $E^{A\vee X}\beta = E^X\beta + E^Z\beta$. Hence (iv) is equivalent to $E^Z\beta = 0$ for $\beta \in B$, i.e. $Z \perp B$. This is (v). □

PROPOSITION 2.2. *Let* $A \perp B | X$. *Then*

$$A \cap B \subset X. \tag{2.2}$$

Proof. Let $\lambda \in A \cap B$. Then $\lambda \perp \lambda | X$, i.e. $\lambda \in X$. □

PROPOSITION 2.3. *Let A and B be subspaces of H. Then*

$$A \perp B | \bar{E}^A B. \tag{2.3}$$

Moreover, any $X \subset A$ such that $A \perp B | X$ contains $\bar{E}^A B$.

To prove this we need the following decomposition.

LEMMA 2.1. *Let A and B be subspaces of H. Then*

$$A = \bar{E}^A B \oplus (A \cap B^\perp). \tag{2.4}$$

Proof. Let $\alpha \in A$ and $\beta \in B$. Then $\langle \alpha, E^A\beta\rangle = \langle \alpha, \beta\rangle$. Consequently, if $\alpha \perp \bar{E}^A B$, then $\alpha \in B^\perp$. □

Proof of Proposition 2.3. If $X \subset A$, $A \perp B | X$ is equivalent to $A \ominus X \perp B$ (Proposition 2.1). In particular, this is satisfied by $X = \bar{E}^A B$ (Lemma 2.1). In general, $A \ominus X \subset A \cap B^\perp$, i.e., $X \supset \bar{E}^A B$ (Lemma 2.1). □

Suppose $A \perp B | X$. Then it follows trivially from the definition that, if $A_1 \subset A$ and $B_1 \subset B$, then $A_1 \perp B_1 | X$. A more interesting question is how far A and B can be *expanded* while remaining conditionally orthogonal given X.

THEOREM 2.1. *Let A_0 and B_0 be subspaces such that $A_0 \vee B_0 = H$, and suppose that $A_0 \perp B_0 | X$. Let $A \supset A_0$ and $B \supset B_0$. Then $A \perp B | X$ if and only if*

$$A \subset A_0 \vee X, \qquad B \subset B_0 \vee X. \tag{2.5}$$

If the upper bounds are attained, i.e. $A = A_0 \vee X$ and $B = B_0 \vee X$, then $X = A \cap B$.

Proof. By Proposition 2.1, $(A_0 \vee X) \perp (B_0 \vee X) | X$. Therefore, $A \perp B | X$ whenever (2.5) holds. Conversely, assume that $A \perp B | X$. Then $(A \vee X) \perp B_0 | X$, and therefore $Z \perp B_0 | X$ when $Z := (A \vee X) \ominus (A_0 \vee X)$, i.e. $Z \perp (B_0 \vee X) \ominus X$ (Proposition 2.1). But, by definition, $Z \perp (A_0 \vee X)$ and therefore $Z \perp A_0 \vee B_0 = H$, i.e. $Z = 0$. Consequently $A \vee X = A_0 \vee X$, i.e. the first of relations (2.5) must hold. The second follows by symmetry. If $A = A_0 \vee X$ and $B = B_0 \vee X$, then $X \subset A \cap B$. But by Proposition 2.2, $A \cap B \subset X$. Hence $X = A \cap B$.

The following proposition describes the geometry of the maximal spaces in Theorem 2.1.

PROPOSITION 2.4. *The following conditions are equivalent.*

(i) $A \perp B | A \cap B$.
(ii) $E^A B = A \cap B$.
(iii) $E^B A = A \cap B$.
(iv) $E^A B = E^B A$.

Proof. First, suppose that (i) holds. Then, by Proposition 2.1 (iv), $E^A B = E^{A \cap B} B = A \cap B$, which is (ii). Condition (iii) follows by symmetry, exchanging A and B. Hence (iv) follows. Conversely, (ii) or (iii) and Proposition 2.3 imply (i). (Note that (ii) implies that $E^A B$ is closed and therefore $\bar{E}^A B = E^A B$.) Finally, if (iv) holds, $E^A B$, and hence $\bar{E}^A B$, is contained in $A \cap B$. But, by Propositions 2.2 and 2.3, $A \cap B \subset \bar{E}^A B$. Hence $A \cap B = \bar{E}^A B$. Consequently (i) holds (Proposition 2.3).

We shall say that two subspaces A and B satisfying the conditions of Proposition 2.4 *intersect perpendicularly*. As we have seen, perpendicular intersection corresponds to maximal A and B in Theorem 2.1. The upper bound is also attained in the inclusion $A \cap B \subset X$ of Proposition 2.2. Note that, for any pair (A, B) of perpendicularly intersecting subspaces, $E^A B$ is closed.

THEOREM 2.2. *Let A and B be subspaces such that $A \vee B = H$. Then the following conditions are equivalent.*

(i) *A and B intersect perpendicularly.*
(ii) $B^\perp \subset A$.
(iii) $H = A^\perp \oplus (A \cap B) \oplus B^\perp$.
(iv) *E^A and E^B commute.*

Proof. Set $X := A \cap B$. If (i) holds, $X = \bar{E}^A B$, and hence $A \ominus X \perp B$ (Lemma 2.1). But, since $X \subset B$ and $A \vee B = H$, $(A \ominus X) \oplus B = H$, and therefore $A \ominus X = B^\perp$, i.e. $A = X \oplus B$. Hence both (ii) and (iii) follow. Each of the conditions (ii) and (iii) implies the existence of a subspace X with the property $H = A^\perp \oplus X \oplus B^\perp$, so that if $\lambda \in H$,

$$E^A E^B \lambda = E^X E^B \lambda + E^{B^\perp} E^B \lambda = E^X \lambda \tag{2.6}$$

and

$$E^B E^A \lambda = E^X E^A \lambda + E^{A^\perp} E^A \lambda = E^X \lambda \tag{2.7}$$

and therefore (iv) follows. It just remains to prove that (iv) implies (i). But, $E^A E^B H = E^B E^A H$ yields $E^A B = E^B A$, i.e. A and B intersect perpendicularly (Proposition 2.4). □

3. The geometry of splitting subspaces. Let H be a real separable Hilbert space, let $\{U_t; t \in \mathbb{R}\}$ be a strongly continuous group of unitary operators on H, and let H^- and H^+ be subspaces enjoying the invariance properties

$$U_t H^- \subset H^- \quad \text{for } t \leqq 0, \tag{3.1a}$$

$$U_t H^+ \subset H^+ \quad \text{for } t \geqq 0 \tag{3.1b}$$

and together spanning H, i.e. $H^- \vee H^+ = H$.

Although these are the only assumptions needed for the geometric theory of §§ 3–5, the situation we have in mind is the one delineated in the Introduction: H is the Gaussian space of an m-dimensional stationary Gaussian vector process, which is mean-square continuous and centered, and $\{U_t; t \in \mathbb{R}\}$ is the group of shifts: $U_t y_k(s) = y_k(s+t)$. Moreover,

$$\begin{aligned} H^- &:= \overline{\text{span}}\,\{y_k(t); t \leqq 0, k = 1, 2, \cdots, m\}, \\ H^+ &:= \overline{\text{span}}\,\{y_k(t); t \geqq 0, k = 1, 2, \cdots, m\} \end{aligned} \tag{3.2}$$

where $\overline{\text{span}}\,\{\cdot\}$ denotes closed linear hull. Hence we shall refer to H^- and H^+ as the *past space* and the *future space* respectively.

We shall say that X is a *splitting subspace* if H^- and H^+ are conditionally orthogonal given X, i.e. $H^- \perp H^+ | X$. According to Proposition 2.1, this is equivalent to each of the two conditions

$$\begin{aligned} E^{H^- \vee X} \lambda &= E^X \lambda \quad \text{for } \lambda \in H^-, \\ E^{H^+ \vee X} \lambda &= E^X \lambda \quad \text{for } \lambda \in H^+. \end{aligned} \tag{3.3}$$

Consequently, a splitting subspace X can be thought of as a "memory" or a "sufficient statistic" containing all information about the past needed in predicting the future, or, equivalently, all the information about the future required to estimate the past. *Splitting subspace* is a concept originally introduced by McKean [34] in a somewhat more restricted sense. A splitting subspace is said to be *minimal* if it contains no proper subspace which is also a splitting subspace. The spaces H, H^- and H^+ are splitting subspaces, but in general they are not minimal.

A subspace X is said to be *Markovian* if the subspaces X^- and X^+ generated by $\{U_t X; t \leqq 0\}$ and $\{U_t X; t \geqq 0\}$ are conditionally orthogonal given X, i.e. $X^- \perp X^+ | X$. This is condition (ii) in § 1, and, as mentioned there, it is equivalent to each of the conditions (1.2) (Proposition 2.1).

We shall now reformulate the geometric problem of § 1, justifying the name *Markovian splitting subspace* introduced there.

PROPOSITION 3.1. *The subspace X satisfies the conditions*

(i) $y_k(0) \in X$, $k = 1, 2, \cdots, m$,

(ii) *X is Markovian*

if and only if X is a Markovian splitting subspace.

Proof. (if): Since X is a splitting subspace, it follows from Proposition 2.2 that $H^- \cap H^+ \subset X$. But $y_k(0) \in H^- \cap H^+$ for $k = 1, 2, \cdots, m$, and therefore (i) follows. Condition (ii) is part of the assumption. (only if): Condition (i) implies that $H^- \subset X^-$ and $H^+ \subset X^+$. Hence the splitting property of X follows from the Markovian property. □

The following characterization of the class of splitting subspaces will be of central importance in what follows.

THEOREM 3.1. [28], [29]. *A subspace X is a splitting subspace if and only if*

$$X = S \cap \bar{S} \tag{3.4}$$

for some pair $(S, \bar{S})$ of perpendicularly intersecting subspaces such that $S \supset H^-$ and $\bar{S} \supset H^+$. The correspondence $X \leftrightarrow (S, \bar{S})$ is one-one, S and $\bar{S}$ being given by

$$\begin{aligned} S &= H^- \vee X, \\ \bar{S} &= H^+ \vee X. \end{aligned} \tag{3.5}$$

Proof. (if): Suppose that S and $\bar{S}$ intersect perpendicularly. Then $S \perp \bar{S} | X$ where $X = S \cap \bar{S}$ (Proposition 2.4). But, since $S \supset H^-$ and $\bar{S} \supset H^+$, this implies that $H^- \perp H^+ | X$, i.e. X is a splitting subspace. (only if): Suppose $H^- \perp H^+ | X$. Let S and $\bar{S}$ be defined by (3.5). Then, by Theorem 2.1, $S \perp \bar{S} | X$ and $X = S \cap \bar{S}$. This implies that S and $\bar{S}$ intersect perpendicularly (Proposition 2.4).

(one-one): Suppose that S and $\bar{S}$ are perpendicularly intersecting subspaces such that $S \supset H^-$ and $\bar{S} \supset H^+$. Then $X = S \cap \bar{S}$ is a splitting subspace, i.e. $H^- \perp H^+ | X$. We need to show that $S = H^- \vee X$ and $\bar{S} = H^+ \vee X$. But S contains H^- and X, and $\bar{S}$ contains H^+ and X; hence $S \supset H^- \vee X$ and $\bar{S} \supset H^+ \vee X$. On the other hand, $S \perp \bar{S} | X$ (Proposition 2.4), and therefore $S \subset H^- \vee X$ and $\bar{S} \subset H^+ \vee X$ (Theorem 2.1), establishing the required equalities. □

COROLLARY 3.1. [28]. *In Theorem* 3.1, (3.4) *can be exchanged for $X = E^S \bar{S}$ or $X = E^{\bar{S}} S$.*

Proof. Follows immediately from Proposition 2.4. □

We shall write $X \sim (S, \bar{S})$ to exhibit the unique pair $(S, \bar{S})$ corresponding to X. The geometry of Theorem 3.1 can be illustrated as in Fig. 1. It also illustrates

COROLLARY 3.2. [28]. *A subspace X is a splitting subspace if and only if there are subspaces $S \supset H^-$ and $\bar{S} \supset H^+$ such that*

$$H = S^\perp \oplus X \oplus \bar{S}^\perp. \tag{3.6}$$

The pair $(S, \bar{S})$ is the same as in Theorem 3.1, *i.e. $X \sim (S, \bar{S})$.*

Proof. (if): Relation (3.6) implies that $\bar{S}^\perp \subset S$, and therefore S and $\bar{S}$ intersect perpendicularly (Theorem 2.2). Also, by Theorem 2.2 (iii), $X = S \cap \bar{S}$. Then the rest follows from Theorem 3.1.

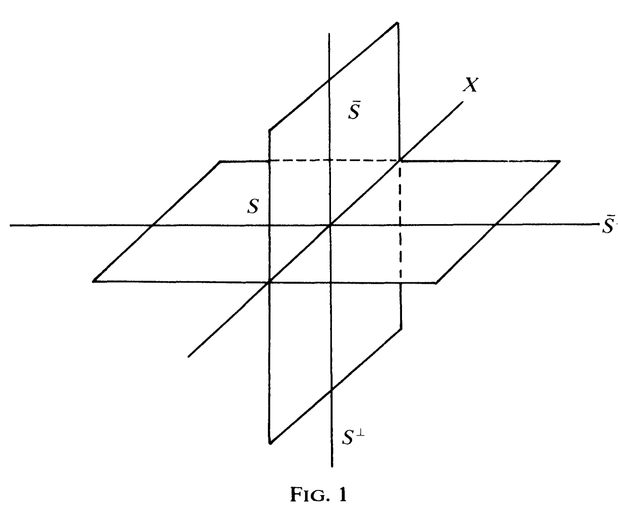


FIG. 1

(only if): In view of Theorem 3.1, it only remains to show that, if S and $\bar{S}$ intersect perpendicularly, (3.6) holds with $X = S \cap \bar{S}$. But this follows from Theorem 2.2. □

Equation (3.6) is analogous to the decomposition in terms of incoming and outgoing subspaces in Lax-Phillips scattering theory [23]. Adding the invariance conditions of Theorem 3.2 below, $\bar{S}^{\perp}$ corresponds to the incoming and $S^{\perp}$ to the outgoing subspace. The parallels will be more apparent in § 7, as we turn to Hardy space representation.

The splitting subspace $X \sim (S, \bar{S})$ is said to be *proper* if both $S^{\perp}$ and $\bar{S}^{\perp}$ are full range.[3] Since $S^{\perp}$ and $\bar{S}^{\perp}$ are the pieces of H in (3.6) which we discard, properness is to a certain extent an indication that the splitting subspace X offers nontrivial data reduction; H, H^- and H^+ are *not* proper.

THEOREM 3.2. [28]. *Let $X \sim (S, \bar{S})$ be a splitting subspace. Then X is Markovian if and only if*

$$U_t S \subset S \quad \textit{for } t \leqq 0, \tag{3.7a}$$

$$U_t \bar{S} \subset \bar{S} \quad \textit{for } t \geqq 0. \tag{3.7b}$$

Proof. (if): Since $X \subset S$, (3.7a) implies that $U_t X \subset S$ for $t \leqq 0$, i.e. $X^- \subset S$. In the same way, (3.7b) implies that $X^+ \subset \bar{S}$. Therefore, since $S \perp \bar{S} | X$, we have $X^- \perp X^+ | X$. (only if): Suppose that $X \sim (S, \bar{S})$ is a Markovian splitting subspace. Then $y_k(0) \in X$ for $k = 1, 2, \cdots, m$ (Proposition 3.1), and therefore $X^- \supset H^-$ and $X^+ \supset H^+$. Moreover, $X^- \perp X^+ | X$, and consequently $X = X^- \cap X^+$ (Theorem 2.1). Hence X^- and X^+ intersect perpendicularly (Proposition 2.4). Then, by Theorem 3.1, $X \sim (X^-, X^+)$ is a splitting subspace. But then, in view of the one-one correspondence $X \leftrightarrow (S, \bar{S})$, we must have $S = X^-$ and $\bar{S} = X^+$, which clearly have the required invariance properties. □

From Theorems 3.1 and 3.2 we see that $H \sim (H, H)$, $H^- \sim (H^-, H)$ and $H^+ \sim (H, H^+)$ are Markovian splitting subspaces, but they are not in general minimal.

Given an arbitrary splitting subspace $X \sim (S, \bar{S})$, how do we find a minimal one contained in it?

LEMMA 3.1. *Let $X \sim (S, \bar{S})$ and $X_0 \sim (S_0, \bar{S}_0)$ be splitting subspaces. Then $X_0 \subset X$ if and only if $S_0 \subset S$ and $\bar{S}_0 \subset \bar{S}$.*

Proof. The if-part follows from (3.4) and the only-if part from (3.5). □

To obtain a minimal splitting subspace, then, we would need to reduce S and $\bar{S}$ as far as possible, while preserving the splitting geometry of Theorem 3.1. By Theorem 2.2, S and $\bar{S}$ intersect perpendicularly if and only if $\bar{S}^{\perp} \subset S$ or, equivalently, $S^{\perp} \subset \bar{S}$. Therefore, in order that $S \supset H^-$, $\bar{S} \supset H^+$, and S and $\bar{S}$ intersect perpendicularly, we must have

$$S \supset H^- \vee \bar{S}^{\perp}, \tag{3.8a}$$

$$\bar{S} \supset H^+ \vee S^{\perp}. \tag{3.8b}$$

We must therefore reduce S and $\bar{S}$ without violating these conditions. The following theorem describes one procedure to do this.

THEOREM 3.3. [29]. *Let $X \sim (S, \bar{S})$ be a splitting subspace. Set $\bar{S}_0 := H^+ \vee S^{\perp}$ and $S_0 := H^- \vee \bar{S}_0^{\perp}$. Then $X_0 \sim (S_0, \bar{S}_0)$ is a minimal splitting subspace such that $X_0 \subset X$. If X is Markovian, then so is X_0.*

Proof. By definition, $S_0 \supset H^-$, $\bar{S}_0 \supset H^+$, and $\bar{S}_0^{\perp} \subset S_0$, i.e. S_0 and $\bar{S}_0$ intersect perpendicularly (Theorem 2.2). Hence $X_0 \sim (S_0, \bar{S}_0)$ is a splitting subspace (Theorem

[3] A subspace M of H is *full range* if the closed linear hull of the shifted spaces $\{U_t M ; t \in \mathbb{R}\}$ is all of H.

3.1). Also, $S^{\perp} \subset \bar{S}_0$, i.e. $\bar{S}_0^{\perp} \subset S$, and $H^- \subset S$, and therefore $S_0 \subset S$. Moreover, (3.8b) may be written $\bar{S}_0 \subset \bar{S}$. Consequently $X_0 \subset X$ (Lemma 3.1).

Next, we show that X_0 is minimal. To this end, suppose that $X_1 \sim (S_1, \bar{S}_1)$ is a splitting subspace such that $X_1 \subset X_0$. Then, by Lemma 3.1, $S_1 \subset S_0$ and $\bar{S}_1 \subset \bar{S}_0$. However, from the splitting geometry (3.8) we have

$$S_1 \supset H^- \vee \bar{S}_1^{\perp}, \tag{3.9a}$$

$$\bar{S}_1 \supset H^+ \vee S_1^{\perp}. \tag{3.9b}$$

Since $\bar{S}_1 \subset \bar{S}_0$, $\bar{S}_1^{\perp} \supset \bar{S}_0^{\perp}$, and therefore (3.9a) yields $S_1 \supset S_0$. Hence $S_1 = S_0$. Furthermore, $X_1 \subset X_0 \subset X$ implies that $S_1 \subset S$ (Lemma 3.1), i.e. $S_1^{\perp} \supset S^{\perp}$, which together with (3.9b) yields $\bar{S}_1 \supset \bar{S}_0$. Thus $\bar{S}_1 = \bar{S}_0$. Consequently $X_1 = X_0$, establishing the minimality of X_0.

It remains to shown that if X is Markovian then so is X_0. In view of Theorem 3.2, this amounts to showing that

$$U_t S_0 \subset S_0 \quad \text{for } t \leqq 0, \tag{3.10a}$$

$$U_t \bar{S}_0 \subset \bar{S}_0 \quad \text{for } t \geqq 0 \tag{3.10b}$$

follows from (3.7). It is well known and easy to show that if a subspace M is invariant under an operator T, i.e. $TM \subset M$, then the orthogonal complement $M^{\perp}$ is invariant under the adjoint T^*, i.e. $T^* M^{\perp} \subset M^{\perp}$. Then, noting that $U_t^* = U_{-t}$, we see that (3.7a) can be written $U_t S^{\perp} \subset S^{\perp}$ for $t \geqq 0$, which together with (3.1b) yields (3.10b). In the same way, (3.10b) and (3.1a) yields (3.10a). □

From this we see, as we could expect, that for minimality we must have equality in (3.8a) and in (3.8b).

COROLLARY 3.3. [28]. *A splitting subspace $X \sim (S, \bar{S})$ is minimal if and only if*

$$\bar{S} = H^+ \vee S^{\perp}, \tag{3.11a}$$

$$S = H^- \vee \bar{S}^{\perp}. \tag{3.11b}$$

Given S, (3.11a) is the smallest subspace $\bar{S}$ containing H^+ and intersecting S perpendicularly. Likewise, given $\bar{S}$, (3.11b) is the smallest subspace containing H^- and intersecting $\bar{S}^{\perp}$ perpendicularly. It follows from Theorem 3.3 that these minimality conditions remain the same if we restrict our analysis to Markovian splitting subspaces. Therefore the properties "minimal" and "Markovian" can be studied separately.

COROLLARY 3.4. [29]. *A Markovian splitting subspace which contains no Markovian splitting subspace as a proper subspace is a minimal splitting subspace.*

The existence of minimal splitting subspaces, finally, is insured by Theorem 3.3.

COROLLARY 3.5. *Each (Markovian) splitting subspace contains a minimal (Markovian) splitting subspace.*

Applying Theorem 3.3 to the Markovian splitting subspaces $H^- \sim (H^-, H)$ and $H^+ \sim (H, H^+)$, we obtain[4] the minimal Markovian splitting subspaces $H^{+/-} \sim (H^-, H^+ \vee (H^-)^{\perp})$ and $H^{-/+} \sim (H^- \vee (H^+)^{\perp}, H^+)$. Introducing

$$N^- := H^- \cap (H^+)^{\perp}, \tag{3.12a}$$

$$N^+ := H^+ \cap (H^-)^{\perp}, \tag{3.12b}$$

we may write[4] $H^{-/+} \sim (H^-, (N^-)^{\perp})$ and $H^{+/-} \sim ((N^+)^{\perp}, H^+)$. Moreover, by Corollary

[4] Recall that, for any subspaces A and B, $(A \vee B)^{\perp} = A^{\perp} \cap B^{\perp}$.

3.2, $H^- = H^{+/-} \oplus N^-$ and $H^+ = H^{-/+} \oplus N^+$, i.e., in view of Lemma 2.1, we have

$$H^{+/-} = \bar{E}^{H^-} H^+, \tag{3.13a}$$

$$H^{-/+} = \bar{E}^{H^+} H^-. \tag{3.13b}$$

Consequently $H^{+/-}$ and $H^{-/+}$ are the forward and backward *predictor spaces.* From Proposition 2.3 we see that $H^{+/-}$ is the only minimal splitting subspace contained in H^-, and $H^{-/+}$ is the only one contained in H^+.

Hence we have identified two minimal splitting subspaces, but where do we look for the others? To answer this question, first note that, since $H^- = H^{+/-} \oplus N^-$ and $H^+ = H^{-/+} \oplus N^+$,

$$H = N^- \oplus H^\square \oplus N^+ \tag{3.14}$$

where $H^\square$ is the *frame space*

$$H^\square = H^{+/-} \vee H^{-/+}. \tag{3.15}$$

Since $(N^+)^\perp \supset H^-$ and $(N^-)^\perp \supset H^+$, $H^\square$ is a splitting subspace (Corollary 3.2), which is Markovian (Theorem 3.2), but in general nonminimal.

THEOREM 3.4. [26]. *The frame space $H^\square$ is the closed linear hull of all minimal splitting subspaces. If X is a minimal splitting subspace, then*

$$H^- \cap H^+ \subset X \subset H^\square. \tag{3.16}$$

Proof. Let $X \sim (S, \bar{S})$ be a minimal splitting subspace. Then it satisfies (3.11). But, $S \supset H^-$ and $\bar{S} \supset H^+$, or, equivalently, $S^\perp \subset (H^-)^\perp$ and $\bar{S}^\perp \subset (H^+)^\perp$, which together with (3.11) yields $\bar{S} \subset (N^-)^\perp$ and $S \subset (N^+)^\perp$. Hence $X \subset H^\square$. In view of (3.15), the minimal splitting subspaces span $H^\square$. The relation $H^- \cap H^+ \subset X$ follows from Proposition 2.2. □

Consequently, as far as minimal splitting subspace construction is concerned, only the frame space $H^\square$ is of interest; the spaces N^- and N^+ in the decomposition (3.14) may be discarded. This observation is of importance in many applications, such as, for example, smoothing [4]. The point here is that, whenever y has a rational spectral density, $H^\square$ is finite-dimensional while of course H is not.

In the event that the past space H^- and future space H^+ intersect perpendicularly, $H^\square = H^- \cap H^+$, and hence, by Theorem 3.4, there is a unique minimal splitting subspace. In the finite-dimensional case, this happens if and only if y has a rational spectral density the numerator polynomial of which is constant.

The special role played by the minimal splitting subspaces $H^{+/-}$ and $H^{-/+}$ is further underlined by the following result. In §§ 4 and 7 we shall identify $H^{+/-}$ and $H^{-/+}$ as the minimum and maximum elements in a certain lattice of splitting subspaces.

THEOREM 3.5. [30]. *Let $X \sim (S, \bar{S})$ be a splitting subspace. Then $\bar{E}^{H^-} X = H^{+/-}$ if and only if $X \perp N^-$ and $\bar{E}^{H^+} X = H^{-/+}$ if and only if $X \perp N^+$.*

Proof. Applying the projection E^{H^-} to $X = E^S \bar{S}$ (Corollary 3.1) and noting that $H^- \subset S$, we obtain $\bar{E}^{H^-} X = \bar{E}^{H^-} \bar{S}$. But $\bar{S} \supset H^+$, and hence $\bar{E}^{H^-} X \supset H^{+/-}$. Conversely, suppose that $\xi \in X$. Then, since $H^- = H^{+/-} \oplus N^-$, $E^{H^-} \xi = E^{H^{+/-}} \xi + E^{N^-} \xi$. But, since $X \perp N^-$, the last term is zero, and consequently $E^{H^-} \xi \in H^{+/-}$. Hence $\bar{E}^{H^-} X \subset H^{+/-}$. This establishes the first part. The second follows by symmetry. □

4. Observability, constructibility, and minimality. Let X be a splitting subspace, and consider the orthogonal decomposition

$$X = \bar{E}^X H^+ \oplus [X \cap (H^+)^\perp] \tag{4.1}$$

given by Lemma 2.1. An element in the subspace $X \cap (H^+)^\perp$ cannot be distinguished from zero by observing the future $\{y(t); t \geqq 0\}$ and is therefore called *unobservable*, in analogy with deterministic systems theory [22, p. 52]. The splitting subspace X is said to be *observable* if the unobservable subspace is trivial, i.e. $X \cap (H^+)^\perp = 0$. Likewise,

$$X = \bar{E}^X H^- \oplus [X \cap (H^-)^\perp] \tag{4.2}$$

and we call X *constructible* if the unconstructible subspace $X \cap (H^-)^\perp = 0$.

The above definitions of observability and constructibility, introduced by Ruckebusch in [42], are in complete agreement with the corresponding concepts in deterministic systems theory. To illustrate this point, let us consider the finite-dimensional stochastic system (1.7), which can be solved to yield

$$y(t) = C e^{At} x(0) + \int_0^t C e^{A(t-\sigma)} B\, du(\sigma). \tag{4.3}$$

Now, X is observable if and only if $\bar{E}^X H^+ = X$, i.e.

$$x_k(0) \in \overline{\operatorname{span}}\, \{\hat{y}_i(t); t \geqq 0, i = 1, 2, \cdots, m\} \tag{4.4}$$

for $k = 1, 2, \cdots, n$, where $\hat{y}_k(t) := E^X y_k(t)$. For $t \geqq 0$, $\hat{y}(t) = C e^{At} x(0)$, since then the components of the second term in (4.3) are orthogonal to X. Therefore $\{\hat{y}(t); t \geqq 0\}$ is the output of the linear dynamic system

$$\begin{aligned} \dot{z} &= Az, \qquad z(0) = x(0), \\ \hat{y} &= Cz, \qquad t \geqq 0. \end{aligned} \tag{4.5}$$

The question of observability of X is thus reduced to determining if $x(0)$ can be solved in terms of $\{\hat{y}(t); t \geqq 0\}$ which happens if and only if (4.5) is observable in the usual sense of deterministic systems theory [22]. Similarly, X is constructible if and only if $x(0)$ can be solved in terms of $\{\hat{y}(t); t \leqq 0\}$. But, from the backward system (1.9), we see that $\{\hat{y}(t); t \leqq 0\}$ is the output of

$$\begin{aligned} \dot{\bar{z}} &= -\bar{A}\bar{z}, \qquad \bar{z}(0) = x(0), \\ \hat{y} &= C\bar{z}, \qquad t \leqq 0 \end{aligned} \tag{4.6}$$

and therefore X is constructible if and only if (4.6) is.

In the general setting, observability and constructibility can be characterized as follows.

THEOREM 4.1. [28]. *Let $X \sim (S, \bar{S})$ be a splitting subspace. Then X is observable if and only if*

$$\bar{S} = H^+ \vee S^\perp \tag{4.7}$$

and constructible if and only if

$$S = H^- \vee \bar{S}^\perp. \tag{4.8}$$

Proof. The observability condition $X \cap (H^+)^\perp = 0$ is equivalent to $X^\perp \vee H^+ = H$, which, in view of Corollary 3.2, can be written $(S^\perp \oplus \bar{S}^\perp) \vee H^+ = H$. Since $H^+ \subset \bar{S} \perp \bar{S}^\perp$, this is equivalent to $(H^+ \vee S^\perp) \oplus \bar{S}^\perp = H$, which is the same as (4.7). The proof of the constructibility part is analogous. □

The following result, first presented in [42] in a somewhat different formulation, is an immediate consequence of Corollary 3.3 and Theorem 4.1.

COROLLARY 4.1. (Ruckebusch). *A splitting subspace is minimal if and only if it is both observable and constructible.*

This statement may at first sight seem analogous to the central result in classical deterministic realization theory that a realization is minimal if and only if it is both observable and reachable. However, as we shall see below, it is in fact of quite a different nature, involving both the forward and the backward realization. This was indicated in § 1. In terms of the discussion there, Corollary 4.1 states that X is minimal if and only if conditions (b) and (b)′ both hold.

Defining the *observability map* $\mathcal{O}: X \to H^+$ to be $\mathcal{O}\xi = E^{H^+}\xi$, the decomposition (4.1) of X into an observable and an unobservable subspace is seen to be identical to the well-known relation

$$X = \overline{\text{range } \mathcal{O}^*} \oplus \ker \mathcal{O} \tag{4.9}$$

[48, p. 205], where $\mathcal{O}^*: H^+ \to X$ is the adjoint operator $\mathcal{O}^*\lambda = E^X\lambda$, and ker denotes null space. Consequently X is observable if and only if $\mathcal{O}$ is one-one or, equivalently, $\mathcal{O}^*$ maps onto a dense subset of X. The splitting subspace X is said to be *exactly observable* if $\mathcal{O}^*$ is onto.

Similarly (4.2) can be written

$$X = \overline{\text{range } \mathcal{C}^*} \oplus \ker \mathcal{C} \tag{4.10}$$

where $\mathcal{C}: X \to H^-$ is the *constructibility map* $\mathcal{C}\xi = E^{H^-}\xi$. The splitting subspace X is constructible if and only if $\mathcal{C}$ is one-one or, equivalently, $\mathcal{C}^*: H^- \to X$ maps densely onto; it is *exactly constructible* if $\mathcal{C}^*$ is onto.

According to Proposition 2.1 (vi), the splitting property $H^- \perp H^+ | X$ is equivalent to $G = \mathcal{O}\mathcal{C}^*$, where $G: H^- \to H^+$ is the map $G\lambda = E^{H^+}\lambda$. This can be described by the commutative diagram

$$\begin{tikzcd} H^- \arrow[rr, "G"] \arrow[dr, "\mathcal{C}^*"'] & & H^+ \\ & X \arrow[ur, "\mathcal{O}"'] & \end{tikzcd} \tag{4.11}$$

Such a factorization is said to be *canonical* if the first map (here $\mathcal{C}^*$) has a range which is dense in X and the second map (here $\mathcal{O}$) is one-one. In view of Corollary 4.1, we can summarize this in

PROPOSITION 4.1. *Let $G: H^- \to H^+$ be the map $G\lambda = E^{H^+}\lambda$. Then a subspace X is a splitting subspace if and only if the diagram* (4.11) *commutes. This splitting subspace is minimal if and only if the factorization is canonical.*

A splitting subspace X is *exactly canonical* if it is both exactly observable and exactly constructible. These conditions are technical and do not occur in the minimality criteria. However, certain results are much easier to prove in the finite-dimensional case (Theorem 4.3 is a case in point), and the reason for this is that the attribute "exact" is redundant in this case. Thus the technical difficulties are due to the lack of exactness rather than to infinite dimensions. The following lemma, found in [43, p. 28], relates exact canonicity to G having a closed range.

LEMMA 4.1. (Ruckebusch). *If G has a closed range, then all minimal splitting subspaces are exactly canonical. If one splitting subspace is exactly canonical, the G has a closed range.*

Proof. Recall that if a map has a closed range, then so does its adjoint [48, p. 205]; this will be used several times in the proof. Let X be a minimal splitting subspace. Then $G = \mathcal{O}\mathcal{C}^*$, and $\mathcal{C}^*H^-$ is dense in X (Proposition 4.1). Clearly $GH^- = \mathcal{O}\mathcal{C}^*H^- \subset \mathcal{O}X$. We want to show that, if GH^- is closed, then $GH^- = \mathcal{O}X$ so that $\mathcal{O}$, and hence $\mathcal{O}^*$, has a closed range, i.e. X is exactly observable. To this end, let $\xi \in X$ be arbitrary. Then there is a sequence $\{\xi_n\}$ in $\mathcal{C}^* H^-$ such that $\xi_n \to \xi$. But $\mathcal{O}\xi_n \in GH^-$, and, since $\mathcal{O}$ is continuous, $\mathcal{O}\xi_n \to \mathcal{O}\xi \in GH^-$. Hence $\mathcal{O}X = GH^-$ as required. In the same way, we use the adjoint factorization $G^* = \mathcal{C}\mathcal{O}^*$, which is also canonical, to prove that X is exactly constructible. Conversely, assume that X is exactly canonical. Then $\mathcal{C}^* H^- = X$, and therefore, since $\mathcal{O}X$ is closed, $GH^- = \mathcal{O}\mathcal{C}^* H^-$ is closed. □

The following theorem ties together the geometric concept of minimality with that based on dimension.

THEOREM 4.2. *All minimal splitting subspaces have the same dimension.*

Proof. Let us first assume that G has a closed range. Let X_1 and X_2 be any two minimal splitting subspaces. Then there are two canonical factorizations $G = \mathcal{O}_1 \mathcal{C}_1^* = \mathcal{O}_2 \mathcal{C}_2^*$ (Proposition 4.1) which are in fact exactly canonical (Lemma 4.1). Consider the commutative diagram

$$\begin{tikzcd}
& X_1 \arrow[dr, "\mathcal{O}_1"] \arrow[dd, dashed] & \\
H^- \arrow[ur, "\mathcal{C}_1^*"] \arrow[dr, "\mathcal{C}_2^*"'] & & H^+ \\
& X_2 \arrow[ur, "\mathcal{O}_2"'] &
\end{tikzcd}$$

in which $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ are onto and $\mathcal{O}_1$ and $\mathcal{O}_2$ are one-one. Then, using the argument of Kalman [22, pp. 256–258], we see that there is a bijective map from X_1 to X_2 (dotted arrow). Consequently X_1 and X_2 are isomorphic vector spaces, and therefore they have the same dimension. It remains to consider the case in which G does not have a closed range. But then, by Lemma 4.1, no minimal splitting subspace is exactly canonical, and consequently all are infinite-dimensional. Therefore, since H is a separable Hilbert space, all X have dimension $\aleph_0$. □

Next we shall give an alternative characterization of the class of minimal Markovian splitting subspaces which involves only the space S [or the space $\bar{S}$], and consequently, as we shall see below, only the forward [or the backward] realization. As a preliminary, first note that Theorem 4.1 has the following corollary.

COROLLARY 4.2. *The subspace X is an observable splitting subspace if and only if there is a subspace $S \supset H^-$ such that*

$$X = \bar{E}^S H^+. \tag{4.12}$$

It is a constructible splitting subspace if and only if there is a subspace $\bar{S} \supset H^+$ such that

$$X = \bar{E}^{\bar{S}} H^-. \tag{4.13}$$

The subspaces S and $\bar{S}$ are those of Theorem 3.1, *i.e.* $X \sim (S, \bar{S})$.

Proof. Suppose that $X \sim (S, \bar{S})$ is an observable splitting subspace. Then $X = E^S \bar{S}$ (Corollary 3.1), which together with the observability condition (4.7) yields (4.12). Conversely, suppose there is an $S \supset H^-$ such that (4.12) holds. Define $\bar{S} := H^+ \vee S^\perp$. Then S and $\bar{S}$ intersect perpendicularly (Theorem 2.2) and $X = E^S \bar{S}$. Hence $X \sim (S, \bar{S})$ is a splitting subspace (Corollary 3.1) which is observable (Theorem 4.2). The rest follows from the symmetric argument. □

There are now two representations for the class of minimal Markovian splitting subspaces, one based on (4.12), the other on (4.13). We shall only state the first, the second being the symmetric one. Phrased in terms of the finite-dimensional analysis of § 1, Theorem 4.3 states that minimality is equivalent to conditions (b) + (c); this we shall see in § 7.

THEOREM 4.3. [31]. *Assume that N^- and N^+ are full range. Then X is a minimal Markovian splitting subspace if and only if*

$$X = \bar{E}^S H^+ \tag{4.14}$$

for some S satisfying the invariance condition (3.7a) *and*

$$H^- \subset S \subset (N^+)^\perp. \tag{4.15}$$

The correspondence $X \leftrightarrow S$ is one-one, S being given by $S = H^- \vee X$.

Consequently, the class of minimal Markovian splitting subspaces forms a lattice, induced by the subspaces S: the greatest lower bound of X_1 and X_2 is the X corresponding to $S_1 \cap S_2$; the least upper bound corresponds to $S_1 \vee S_2$. Hence $X_1 < X_2$ if and only if $S_1 \subset S_2$. This lattice has the minimum element $H^{+/-}$, corresponding to $S = H^-$, and the maximum element $H^{-/+}$, corresponding to $S = (N^+)^\perp$.

To establish Theorem 4.3 it just remains to prove that *an observable splitting subspace $X \sim (S, \bar{S})$ is minimal if and only if $S \subset (N^+)^\perp$*. Then the rest follows from Corollary 4.2 and Theorem 3.2. The only-if part of this statement is immediate. In fact, since $S = H^- \vee X$ (Theorem 3.1), it follows from Theorem 3.4. The proof of the if-part, however, is more difficult. It can be found in [31]; also see Theorem 7.3 below. (Note that the proof in [28] is incorrect.)

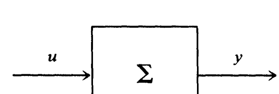
However, in the special case that the map G has a closed range, the proof is easy. Then G maps onto $H^{-/+}$. Moreover, since $X \subset S \perp N^+$, $\mathcal{O}X \subset H^{-/+}$ (Theorem 3.5). Consequently, we may without restriction replace (4.11) by

$$\begin{tikzcd} H^- \arrow[rr, "G"] \arrow[dr, "\mathcal{C}^*"'] & & H^{-/+} \\ & X \arrow[ur, "\mathcal{O}"'] & \end{tikzcd}$$

In this diagram, G is onto. Furthermore, since from the diagram $\mathcal{O}X \supset GH^-$, $\mathcal{O}$ is onto. By observability, $\mathcal{O}$ is one-one and therefore the inverse $\mathcal{O}^{-1}: H^{-/+} \to X$ is well defined and onto. Consequently, $\mathcal{C}^* = \mathcal{O}^{-1}G$ is onto, i.e. X is constructible; hence X is minimal (Corollary 4.1).

5. Reconciliation with systems theory. We wish to pinpoint the similarities and the differences between the state space constructions in deterministic and stochastic realization theory from an abstract systems-theoretical point of view. To this end, let us first briefly review some basic concepts of the standard state space construction in deterministic systems theory [22], [15].

Consider an external description of a continuous-time, constant, linear dynamical system Σ, which we illustrate as a "black box"



with input u and output y. Let U be a space of input functions u which are identically zero for $t > 0$, and let Y be a space of output functions y which are identically zero for $t < 0$. Let $\Gamma: U \to Y$ be the (linear) restricted input-output map defined by Σ. (Consequently, we apply the inputs up to time zero and observe the outputs from time zero on.) The input space U is invariant under the operation $(\sigma_t u)(\tau) = u(\tau + t)$ of shifting the function a distance $t \geqq 0$ to the left, i.e. $\sigma_t U \subset U$ for $t \geqq 0$.

Two inputs $u_1, u_2 \in U$ are (*Nerode*) *equivalent* if the corresponding outputs Γu_1 and Γu_2 coincide, i.e. $u_1 - u_2 \in \ker \Gamma$. Then a minimal state space is obtained by forming the quotient space $X = U/\ker \Gamma$. If R is the projection onto the quotient space, there is an injective map O so that the diagram

$$\begin{tikzcd} U \arrow[rr, "\Gamma"] \arrow[dr, "R"'] & & Y \\ & X \arrow[ur, "O"'] & \end{tikzcd}$$

commutes [46, p. 23]. Hence we have a canonical factorization of Γ through the minimal state space X. (A noncanonical factorization will yield a nonminimal realization [22].) The semigroup $\{e^{At}; t \geqq 0\}$, determining the dynamics of the realization, is then isomorphic to the family of maps making the diagrams

$$\begin{tikzcd} U \arrow[r, "R"] \arrow[d, "\sigma_t"'] & X \arrow[d, dashed, "\sigma_t(X)"] \\ U \arrow[r, "R"] & X \end{tikzcd}$$

commute.

In the *stochastic* realization problem only the output process is given, and therefore the choice of input space is somewhat arbitrary. While the minimal state space in the deterministic theory is essentially unique, there are many solutions to the stochastic problem, each minimal Markovian splitting subspace $X \sim (S, \bar{S})$ giving rise to a minimal state space. As it turns out, each such state space is best described by two realizations, one evolving forward in time having S as input space and H^+ as output space, and another evolving backward with $\bar{S}$ as input space and H^- as output space. In § 6 we shall see that (under suitable conditions) there are two orthogonal increment processes u and $\bar{u}$ such that $S = H^-(du)$ and $\bar{S} = H^+(d\bar{u})$. These processes, called the *generating processes* of X, will be the input processes of respectively the forward and the backward realization of X.

THEOREM 5.1. *Let X be a subspace of H, and set $S := H^- \vee X$ and $\bar{S} := H^+ \vee X$. Then $X \sim (S, \bar{S})$ is a splitting subspace if and only if the diagrams*

$$\begin{tikzcd} S \arrow[rr, "\Gamma_+"] \arrow[dr, "\mathcal{R}"'] & & H^+ \\ & X \arrow[ur, "\mathcal{O}"'] & \end{tikzcd} \qquad \begin{tikzcd} \bar{S} \arrow[rr, "\Gamma_-"] \arrow[dr, "\mathcal{H}"'] & & H^- \\ & X \arrow[ur, "\mathcal{C}"'] & \end{tikzcd}$$

commute, the maps being defined as $\Gamma_+\lambda = E^{H^+}\lambda$, $\Gamma_-\lambda = E^{H^-}\lambda$, $\mathcal{R}\lambda = E^X\lambda$, $\mathcal{H}\lambda = E^X\lambda$, $\mathcal{O}\lambda = E^{H^+}\lambda$, and $\mathcal{C}\lambda = E^{H^-}\lambda$ with domains as indicated. If one diagram commutes, then so does the other. The left factorization is canonical if and only if X is observable, the right one if and only if X is constructible, and both if and only if X is minimal. The maps $\mathcal{R}$ and $\mathcal{H}$ are always onto. Moreover,

$$X \supset S \ominus \ker \Gamma_+, \tag{5.1a}$$

$$X \supset \bar{S} \ominus \ker \Gamma_- \tag{5.1b}$$

with equality in (5.1a) *if and only if the left factorization is canonical and equality in* (5.1b) *if and only if the right factorization is canonical.*

Note that $\mathcal{O}$ and $\mathcal{C}$ are the observability and constructibility maps defined in § 4. Following standard terminology in systems theory, $\mathcal{R}$ is the *reachability map* and $\mathcal{K}$ the *controllability map.*

Proof. By Proposition 2.1 (vi), the left factorization $\Gamma_+ = \mathcal{O}\mathcal{R}$ is equivalent to $H^+ \perp S|X$ and the right one $\Gamma_- = \mathcal{C}\mathcal{K}$ to $H^- \perp \bar{S}|X$. But these conditional orthogonality conditions are both equivalent to $H^- \perp H^+|X$ (Proposition 2.1), the splitting property. Since $X \subset S$ and $X \subset \bar{S}$, $\mathcal{R}$ and $\mathcal{K}$ are obviously onto. Therefore the left factorization is canonical if and only if $\mathcal{O}$ is one-one, i.e. X is observable, and the right one is canonical if and only if $\mathcal{C}$ is one-one, i.e. X is constructible. Then, the minimality statement follows from Corollary 4.1. Since $\ker \Gamma_+ = S \cap (H^+)^\perp$, it follows from Lemma 2.1 that $S \ominus \ker \Gamma_+ = \bar{E}^S H^+$. But, in view of the splitting property (3.3), $\bar{E}^S H^+ = \bar{E}^X H^+$, which is the observable subspace of X; see (4.1). Hence (5.1a) holds, and there is equality if and only if $\bar{E}^X H^+ = X$, i.e. X is observable. The proof of the symmetric statement involving (5.1b) is analogous. □

Observing that $S \ominus \ker \Gamma_+$ and $\bar{S} \ominus \ker \Gamma_-$ are representations of the quotient spaces $S/\ker \Gamma_+$ and $\bar{S}/\ker \Gamma_-$ respectively, the analogy with the deterministic construction is apparent. Note, however, that in order for X to be a minimal splitting subspace, and hence correspond to a minimal state space, *both* diagrams need to be canonical. This is because the input space in the stochastic problem is not fixed but may change with X.

Some of the geometric results of §§ 3 and 4 can be inferred directly from the diagrams. Clearly we always have

$$\ker \mathcal{R} \subset \ker \Gamma_+ \tag{5.2}$$

with equality if and only if $\mathcal{O}$ is one-one. In fact, except for the elements in $\ker \mathcal{R}$ which are sent to the zero point in X and onto the zero point in H^+, there may be a subset of S whose image in X is nontrivial but then mapped onto the zero point of H^+. This happens if and only if $\mathcal{O}$ fails to be one-one. However, $\ker \mathcal{R} = S \ominus X = \bar{S}^\perp$ (Corollary 3.2), and therefore (5.2) can be written $\bar{S}^\perp \subset S \cap (H^+)^\perp$ or, equivalently, $\bar{S} \supset H^+ \vee S^\perp$, i.e. (3.8a). Equality yields the observability condition (4.7). Likewise, the corresponding relation between $\ker \mathcal{K}$ and $\ker \Gamma_-$ yields the constructibility condition (4.8).

Construction of semigroups for the stochastic problem requires that X is Markovian, in which case the input space S is invariant under the shift $\{U_t^*; t \geqq 0\}$ and $\bar{S}$ is invariant under $\{U_t; t \geqq 0\}$ (Theorem 3.2). These shifts play the role of $\{\sigma_t; t \geqq 0\}$ in the deterministic theory.

THEOREM 5.2. [30]. *Let $X \sim (S, \bar{S})$ be a Markovian splitting subspace. For each $t \geqq 0$, let $U_t(X): X \to X$ be the compressed shift $U_t(X)\xi = E^X U_t \xi$ and $U_t(X)^*: X \to X$ its adjoint $U_t(X)^*\xi = E^X U_{-t}\xi$. Then, for $t \geqq 0$, the diagrams*

$$\begin{array}{ccc} S & \xrightarrow{\mathcal{R}} & X \\ {\scriptstyle U_t^*}\downarrow & & \downarrow{\scriptstyle U_t(X)^*} \\ S & \xrightarrow{\mathcal{R}} & X \end{array}, \qquad \begin{array}{ccc} \bar{S} & \xrightarrow{\mathcal{K}} & X \\ {\scriptstyle U_t}\downarrow & & \downarrow{\scriptstyle U_t(X)} \\ \bar{S} & \xrightarrow{\mathcal{K}} & X \end{array}$$

commute. Moreover, $\{U_t(X); t \geqq 0\}$ and $\{U_t(X)^*; t \geqq 0\}$ are strongly continuous contraction semigroups, and for each $\xi \in X$ and $t \geqq 0$,

$$E^S U_t \xi = U_t(X)\xi, \tag{5.3a}$$

$$E^{\bar{S}} U_{-t} \xi = U_t(X)^* \xi. \tag{5.3b}$$

If X is proper, both $U_t(X)$ and $U_t(X)^*$ tend strongly to zero as $t \to \infty$.

Proof. Let $t \geqq 0$ and $\lambda \in \bar{S}$. Then, since $\bar{S} = X \oplus S^{\perp}$ (Corollary 3.2),

$$E^X U_t \lambda = E^X U_t E^X \lambda + E^X U_t E^{S^{\perp}} \lambda. \tag{5.4}$$

Here the last term is zero, for $U_t S^{\perp} \subset S^{\perp} \perp X$ (Theorem 3.2 and Corollary 3.2). Therefore,

$$E^X U_t \lambda = E^X U_t E^X \lambda, \tag{5.5}$$

and consequently $\mathcal{H} U_t = U_t(X)\mathcal{H}$ as required. Also, since $S \perp \bar{S} | X$ and $U_t \lambda \in \bar{S}$, the left member of (5.5) can be exchanged for $E^S U_t \lambda$. Therefore, since $X \subset \bar{S}$, (5.3a) follows. The symmetric argument yields $\mathcal{R} U_t^* = U_t(X)^* \mathcal{R}$ and (5.3b). Since $U_t U_s = U_{t+s}$, it follows from (5.5) that $U_t(X) U_s(X) = U_{t+s}(X)$, i.e. $\{U_t(X); t \geqq 0\}$ is a semigroup, which is strongly continuous since $\{U_t\}$ is. Clearly $U_t(X)$ is a contraction, for U_t is unitary. If X is proper, $\bigcap_{t \leqq 0} U_t S = 0$ and hence, in view of (5.3a) and the identity $E^S U_t = U_t E^{U_t^* S}$, we get $\| U_t(X)\xi \| = \| E^{U_{-t} S} \xi \| \to 0$ as $t \to \infty$ proving the last statement of the theorem. The family $\{U_t(X)^*; t \geqq 0\}$ is merely the adjoint semigroup with the same properties. □

Following the pattern of this section, in §§ 8 and 9 we shall assign to each proper Markovian splitting subspace X two realizations with the systems-theoretical properties of Theorem 5.1, a forward one with input space S and semigroup $\{U_t(X)^*\}$ and a backward one with input space $\bar{S}$ and semigroup $\{U_t(X)\}$. Therefore we shall call $\{U_t(X)^*; t \geqq 0\}$ and $\{U_t(X); t \geqq 0\}$ the *forward* and *backward Markovian semigroups* of X respectively.

6. Generating processes. By representing the random variables as Wiener integrals we shall next derive functional models for the geometric results presented above.

To this end, let us first define a p-dimensional *Wiener process on the real line* $\mathbb{R}$ to be a real centered Gaussian vector process $\{u(t); t \in \mathbb{R}\}$ which has (almost surely) continuous sample functions and independent (and hence orthogonal) increments such that

$$E\{du(t)\, du(t)'\} = I\, dt. \tag{6.1}$$

Although we shall only be interested in the increments of u, it is convenient to set $u(0) = 0$. Defining $H(du)$ to be the Hilbert space generated by the components of $\{u(t) - u(s); t, s \in \mathbb{R}\}$, we have the orthogonal decomposition

$$H(du) = H^-(du) \oplus H^+(du) \tag{6.2}$$

where $H^-(du)$ and $H^+(du)$ are the subspaces corresponding respectively to the increments $\{u(t) - u(s); t, s \leqq 0\}$ and $\{u(t) - u(s); t, s \geqq 0\}$.

It is well known [38] that to any $\eta \in H(du)$ there is a function f in $\mathcal{L}_p^2(\mathbb{R})$, the space of p-dimensional *real* vector functions square-integrable on $\mathbb{R}$, such that

$$\eta = \sum_{i=1}^{p} \int_{-\infty}^{\infty} f_i(-t)\, du_i(t), \tag{6.3}$$

where the integral is defined in quadratic mean. We shall write (6.3) as

$$\eta = \int_{-\infty}^{\infty} f(-t)\, du(t) \tag{6.4}$$

i.e. we shall think of the function f as a row vector and the process u as a column vector; this convention will be maintained through the rest of the paper. Let $I_u : \mathcal{L}_p^2(\mathbb{R}) \to H(du)$ be the map defined by (6.4), i.e. $\eta = I_u f$. Then $\langle I_u f, I_u g \rangle = \int_{-\infty}^{\infty} f(t) g(t)'\, dt$, the inner product of f and g in $\mathcal{L}_p^2(\mathbb{R})$, i.e. I_u is an isometry. Since it is also onto, I_u is unitary.

It is not hard to see that the vector process

$$\hat{u}(i\omega) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}-1}{it}\, du(t) \tag{6.5}$$

defined on the imaginary axis $\mathbb{I}$, has much the same properties as u with[5]

$$E\{d\hat{u}(i\omega)\, d\hat{u}(i\omega)^*\} = \frac{1}{2\pi} I\, d\omega, \tag{6.6}$$

and therefore we can think of it as a vector *Wiener process on the imaginary axis.* Also, it is well known [38, p. 147] that, for each $f \in \mathscr{L}_p^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} f(-t)\, du(t) = \int_{-\infty}^{\infty} \hat{f}(i\omega)\, d\hat{u}(i\omega) \tag{6.7}$$

where $\omega \to \hat{f}(i\omega)$ is the Fourier transform

$$\hat{f}(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt \tag{6.8}$$

extended to all of $\mathscr{L}_p^2(\mathbb{R})$ in the usual manner [10], [38]. The space $\mathscr{L}_p^2(\mathbb{I})$ of all such $\hat{f}$ is a Hilbert space with inner product

$$(\hat{f}, \hat{g}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(i\omega)\hat{g}(-i\omega)'\, d\omega,$$

and the map $\mathscr{F}: \mathscr{L}_p^2(\mathbb{R}) \to \mathscr{L}_p^2(\mathbb{I})$ defined by $\hat{f} = \mathscr{F}f$ is unitary. (We define $\hat{f}$ in the style of the Laplace transform in order to conform with usual nomenclature in systems theory.) In view of (6.7), the map $I_{\hat{u}}: \mathscr{L}_p^2(\mathbb{I}) \to H(du)$ defined by

$$I_{\hat{u}}\hat{f} = \int_{-\infty}^{\infty} \hat{f}(i\omega)\, d\hat{u}(i\omega) \tag{6.9}$$

is also unitary, and the diagram

$$\begin{tikzcd} \mathscr{L}_p^2(\mathbb{R}) \arrow[rr, "\mathscr{F}"] \arrow[dr, "I_u"'] & & \mathscr{L}_p^2(\mathbb{I}) \arrow[dl, "I_{\hat{u}}"] \\ & H(du) & \end{tikzcd}$$

commutes. Taking $f(-t)$ to be the indicator function of the interval $[0, t]$, (6.7) yields the spectral representation

$$u(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t}-1}{i\omega}\, d\hat{u}(i\omega). \tag{6.10}$$

If we let f vary over all functions in $\mathscr{L}_p^2(\mathbb{R})$ which vanish on the negative [positive] real axis, (6.4) generates $H^-(du)$ $[H^+(du)]$. This motivates the introduction of the *Hardy spaces* $\mathscr{H}_p^2$ and $\bar{\mathscr{H}}_p^2$. Let $\mathscr{H}_p^2$ $[\bar{\mathscr{H}}_p^2]$ be the subspace of those $\hat{f} \in \mathscr{L}_p^2(\mathbb{I})$ for which $f := \mathscr{F}^{-1}\hat{f}$ vanishes on the negative [positive] real line. Then, $H^-(du) = I_{\hat{u}}\mathscr{H}_p^2$ and $H^+(du) = I_{\hat{u}}\bar{\mathscr{H}}_p^2$, i.e.

$$\mathscr{L}_p^2(\mathbb{I}) = \mathscr{H}_p^2 \oplus \bar{\mathscr{H}}_p^2 \tag{6.11}$$

[5] Here * denotes conjugate transpose.

is the isomorphic image of (6.2) under $I_{\bar{u}}^{-1}$. Clearly $\bar{\mathcal{H}}_p^2=\{i\omega\to g(-i\omega)|g\in\mathcal{H}_p^2\}$, and therefore $\bar{\mathcal{H}}_p^2$ is sometimes called the *conjugate* Hardy space. The Hardy spaces $\mathcal{H}_p^2$ and $\bar{\mathcal{H}}_p^2$ can also be defined as *bona fide* function spaces of functions analytic in the open right and left complex half planes respectively, the limits of which as the imaginary axis is approached perpendicularly are the elements of $\mathcal{H}_p^2$ and $\bar{\mathcal{H}}_p^2$ as defined above [14], [18], [45]. Therefore the functions of $\mathcal{H}_p^2$ will sometimes be called *analytic* and those of $\bar{\mathcal{H}}_p^2$ *coanalytic*. In the same way we define $\mathcal{H}_{p\times p}^\infty$ to be the space of all $p\times p$-matrix-valued functions bounded and analytic in the open right half plane, or, alternatively, as the corresponding subspace of $\mathcal{L}_{p\times p}^\infty(\mathbb{I})$. Here we shall think of these functions as defined on the imaginary axis, but it is useful to keep the other interpretation in mind.

Our program now is to assign to each proper Markovian splitting subspace $X\sim(S,\bar{S})$ a pair $(u,\bar{u})$ of Wiener processes on the real line such that $H^-(du)=S$, $H^+(d\bar{u})=\bar{S}$, and $H(du)=H(d\bar{u})=H$. Through the isomorphisms $I_{\hat{u}}$ and $I_{\hat{\bar{u}}}$ we shall then transform the geometry of §§ 2–5 to the Hardy space $\mathcal{H}_p^2$ in which the appropriate mappings take a particularly simple form.

Recall that the given m-dimensional process $\{y(t);\, t\in\mathbb{R}\}$ is stationary, Gaussian, mean-square continuous, and centered. From now on, we shall also assume that y is *purely nondeterministic* in the (strong) sense that both $(H^-)^\perp$ and $(H^+)^\perp$ are full range. Then, y has a spectral representation

$$y(t)=\int_{-\infty}^{\infty} e^{i\omega t}\, d\hat{y}(i\omega) \tag{6.12}$$

where $\{\hat{y}(s);\, s\in\mathbb{I}\}$ is an independent-increment process such that

$$E\{d\hat{y}(i\omega)\, d\hat{y}(i\omega)^*\}=\frac{1}{2\pi}\Phi(i\omega)\, d\omega. \tag{6.13}$$

Here the $m\times m$-matrix function Φ is the *spectral density* of y, and $\hat{y}$ is given by

$$\hat{y}(i\omega)=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-i\omega t}-1}{it}\, y(t)\, dt, \tag{6.14}$$

where the limit is in quadratic mean [9].

Since y is purely nondeterministic, $\Phi(i\omega)$ has a constant rank $p\leqq m$ (for almost all ω), and it admits a factorization

$$W(s)\,W(-s)'=\Phi(s) \tag{6.15}$$

where W is an $m\times p$-matrix function whose rows belong to $\mathcal{L}_p^2(\mathbb{I})$ [38, p. 114]. There are many such W, and we call them *full-rank spectral factors*. More specifically, the condition that $(H^-)^\perp[(H^+)^\perp]$ is full range implies that there are W with rows in $\mathcal{H}_p^2(\bar{\mathcal{H}}_p^2)$.

To each full-rank spectral factor W we associate a unique p-dimensional Wiener process (on the real line), namely (6.10) with

$$d\hat{u}=W^{-L}\, d\hat{y} \tag{6.16}$$

where W^{-L} is any left inverse of W, i.e. a $p\times m$-matrix function such that $W^{-L}W=I$. Although, in general, W has more than one left inverse, u is uniquely determined by (6.16). In fact, let W^{-L} and $W^{-L}+\Delta$ be two left inverses. Then $\Delta W=0$, and consequently, because of (6.15), $\Delta\Phi\Delta^*\, d\omega=0$, and therefore the uniqueness is established. For example, we may take $W^{-L}=(W'W)^{-1}W'$. Despite the fact that $WW^{-L}\neq I$ in general, $WW^{-L}\, d\hat{y}=d\hat{y}$, i.e.

$$d\hat{y}=W\, d\hat{u}. \tag{6.17}$$

To see this, form $E\{(I-WW^{-L})\, d\hat{y}\, d\hat{y}^*(I-WW^{-L})^*\}$, which, in view of (6.13) and (6.15), equals zero.

The class $\mathcal{U}$ of Wiener processes u defined in this way is characterized as follows.

PROPOSITION 6.1. [28]. *Let u be a vector Wiener process defined on the real line. Then $u \in \mathcal{U}$ if and only if $H(du) = H$. In this case, for each $t \in \mathbb{R}$,*

$$U_t I_{\hat{u}} = I_{\hat{u}} \chi_t \tag{6.18}$$

where $\chi_t : \mathcal{L}_p^2(\mathbb{I}) \to \mathcal{L}_p^2(\mathbb{I})$ is multiplication by $e^{i\omega t}$.

Proof. First suppose that $u \in \mathcal{U}$. Then, by (6.17),

$$y(t) = \int e^{i\omega t} W \, d\hat{u} \tag{6.19}$$

and consequently, in view of (6.7), $y_k(t) \in H(du)$ for $k = 1, 2, \cdots, m$. Hence $H \subset H(du)$. On the other hand, it follows from (6.10), (6.16) and (6.14) that $H(du) \subset H$, and therefore $H(du) = H$. Conversely, assume that $H(du) = H$. Then, by (6.7), there exists a matrix function W with rows in $\mathcal{L}_p^2(\mathbb{I})$ such that $y(0) = \int W\, d\hat{u}$. From this it is seen that W is a spectral factor and that $d\hat{y} = W\, d\hat{u}$, but it remains to show that W is full rank. However, since $H(du) \subset H$, for each $t \in \mathbb{R}$, there is a matrix function G_t such that $u(t) = \int G_t \, d\hat{y}$; i.e. $u(t) = \int G_t W \, d\hat{u}$. Hence, by (6.10), $G_t W = (e^{i\omega t} - 1)/i\omega I$ for all $t \in \mathbb{R}$. Consequently, W must have full rank. □

COROLLARY 6.1. *Let $u \in \mathcal{U}$. Then, for $k = 1, 2, \cdots, p$,*

$$U_t[u_k(\tau) - u_k(\sigma)] = u_k(\tau + t) - u_k(\sigma + t) \tag{6.20}$$

and consequently $U_t H^-(du)$ is the subspace generated by the components of $\{u(\tau) - u(\sigma) | \tau, \sigma \leqq t\}$.

Proof. In view of (6.10),

$$u(\tau) - u(\sigma) = \int_{-\infty}^{\infty} \frac{e^{i\omega\tau} - e^{i\omega\sigma}}{i\omega} \, d\hat{u} \tag{6.21}$$

and therefore (6.20) follows from (6.18). □

How are the processes in $\mathcal{U}$ related to each other? It is immediately clear that, if u_1 and u_2 correspond to the spectral factors W_1 and W_2 respectively, then

$$d\hat{u}_2 = W_2^{-L} W_1 \, d\hat{u}_1. \tag{6.22}$$

The $p \times p$-matrix function $W_2^{-L} W_1$ is uniquely defined (independent of the choice of left inverse), because, just as above, $d\hat{u}_2 = P_1 \, d\hat{u}_1$ and $d\hat{u}_2 = P_2 \, d\hat{u}_1$ imply that $(P_1 - P_2)(P_1 - P_2)^* = 0$, i.e. $P_1 = P_2$. Also, it follows from (6.15) that the values of $W_2^{-L} W_1$ on $\mathbb{I}$ are unitary matrices. Therefore we can think of u_2 being obtained by passing u_1 through a filter with the transfer function $W_2^{-L} W_1$:

$$\xrightarrow{\;u_1\;} \boxed{W_2^{-L}\, W_1} \xrightarrow{\;u_2\;}. \tag{6.23a}$$

In engineering language, such an object is called an *all-pass filter.* Moreover, for any $f \in \mathcal{L}_p^2(\mathbb{I})$, $I_{\hat{u}_2} f = I_{\hat{u}_1} f W_2^{-L} W_1$, i.e.

$$I_{\hat{u}_1}^{-1} I_{\hat{u}_2} = M_{W_2^{-L} W_1} \tag{6.23b}$$

where, here as in the sequel, $M_Q : \mathcal{L}_2(\mathbb{I}) \to \mathcal{L}_2(\mathbb{I})$ denotes multiplication from the right by Q, i.e. $M_Q f = fQ$.

Clearly, if W is a full-rank spectral factor, then so is WT for any constant unitary $p\times p$ matrix T. However, the corresponding processes in $\mathcal{U}$ are related to each other by a trivial coordinate transformation, and therefore we shall regard them as equivalent. The transformation (6.23) is interesting only if $W_2^{-L}W_1$ is nonconstant.

A matrix function $Q\in\mathcal{H}^\infty_{p\times p}$ with the property that $Q(i\omega)$ are unitary matrices for almost all ω is said to be *inner* [14], [18], [45]. In particular, $W_2^{-L}W_1$ is inner if it belongs to $\mathcal{H}^\infty_{p\times p}$. The following lemma, which is a corollary of a famous theorem by Beurling, generalized to vector functions by Lax [14], [18], [45], states that the all-pass filter (6.23a) is *causal* if and only if $W_2^{-L}W_1$ is inner.

LEMMA 6.1. [28]. *Let u_1 and u_2 be two processes in $\mathcal{U}$, and let W_1 and W_2 be the corresponding spectral factors. Then $W_2^{-L}W_1$ is inner if and only if*

$$H^-(du_2)\subset H^-(du_1). \tag{6.24}$$

Proof. Set $\mathcal{Z}:=I_{\hat{u}_1}^{-1}H^-(du_2)$. Since $U_tH^-(du_2)\subset H^-(du_2)$ for $t\leqq 0$ (Corollary 6.1), $\chi_t\mathcal{Z}\subset\mathcal{Z}$ for $t\leqq 0$ (Proposition 6.1). A subspace $\mathcal{Z}$ with this property is called *invariant.* Moreover, since $H^-(du_2)$ is full range, then so is $\mathcal{Z}$, in the sense that the closed linear hull of $\{\chi_t\mathcal{Z};\ t\in\mathbb{R}\}$ is all of $\mathcal{L}^2_p(\mathbb{I})$. Since $I_{\hat{u}_1}^{-1}H^-(du_1)=\mathcal{H}^2_p$, (6.24) is equivalent to $\mathcal{Z}\subset\mathcal{H}^2_p$. Now, by the Beurling-Lax theorem, the invariant full range subspaces of $\mathcal{H}^2_p$ are precisely the subspaces of the form $\mathcal{H}^2_pQ$ where Q is inner. But, in view of (6.23b) $\mathcal{Z}=\mathcal{H}^2_pW_2^{-L}W_1$. Therefore, if $W_2^{-L}W_1$ is inner, (6.24) holds. Conversely, if (6.24) holds, $W_2^{-L}W_1$ must be inner. In fact, if $\mathcal{H}^2_pQ_1=\mathcal{H}^2_pQ_2$ where both Q_1 and Q_2 take unitary values in $\mathbb{I}$, then $Q_1=TQ_2$ where T is a constant unitary matrix [18]. □

Referring to the alternative definitions of $\mathcal{H}^2_p$ and $\bar{\mathcal{H}}^2_p$, a full-rank spectral factor with rows in $\mathcal{H}^2_p$ will be called *analytic*, and one with rows in $\bar{\mathcal{H}}^2_p$ *coanalytic.* Let $\mathcal{U}^-$ and $\mathcal{U}^+$ respectively be the corresponding subclasses of $\mathcal{U}$.

LEMMA 6.2. [28]. *Let $u\in\mathcal{U}$. Then, $u\in\mathcal{U}^-$ if and only if $H^-(du)\supset H^-$, and $u\in\mathcal{U}^+$ if and only if $H^+(du)\supset H^+$.*

Proof. By definition, $u\in\mathcal{U}^-$ is equivalent to $W_k\in\mathcal{H}^2_p$ for $k=1,2,\cdots,m$, where W_k is the kth row of the spectral factor W corresponding to u. Under the isomorphism $I_{\hat{u}}$ this is equivalent to

$$y_k(0)\in H^-(du)\quad\text{for } k=1,2,\cdots,m. \tag{6.25}$$

For this to hold it is clearly sufficient that $H^-\subset H^-(du)$. Conversely, suppose that (6.25) holds. Then, in view of Corollary 6.1, $y_k(t)\subset H^-(du)$ for $t\leqq 0$ and $k=1,2,\cdots,m$. This implies that $H^-\subset H^-(du)$. The proof of the symmetric statement is analogous. □

We are now in a position to tie together the results of this section with the geometric theory presented in the beginning of the paper. The link is provided by the following theorem.

THEOREM 6.1. [28]. (i) *Let S be a subspace such that $S\supset H^-$ and $S^\perp$ is full range. Then*

$$U_tS\subset S\quad\textit{for } t\leqq 0 \tag{6.26}$$

if and only if there are an analytic full-rank spectral factor W and a corresponding $u\in\mathcal{U}^-$ such that

$$S=H^-(du). \tag{6.27}$$

The spectral factor W and the process u are unique modulo multiplication from the right (respectively the left) by the same constant unitary matrix.

(ii) *Let $\bar{S}$ be a subspace such that $\bar{S} \supset H^+$ and $\bar{S}^\perp$ is full range. Then*

$$U_t \bar{S} \subset \bar{S} \quad \text{for } t \geqq 0 \tag{6.28}$$

if and only if there are a coanalytic full-rank spectral factor $\bar{W}$ and a corresponding $\bar{u} \in \mathcal{U}^+$ such that

$$\bar{S} = H^+(du). \tag{6.29}$$

Here $\bar{W}$ and $\bar{u}$ enjoy the same uniqueness properties as in (i).

Proof. (i) Let $v \in \mathcal{U}$ be arbitrary, and let V be the corresponding spectral factor. (Since there are full-rank spectral factors [38, p. 114], $\mathcal{U}$ is nonempty.) Set $\mathcal{X} := I_{\hat{v}}^{-1} S$. In view of Proposition 6.1, (6.26) is equivalent to

$$\chi_t \mathcal{X} \subset \mathcal{X} \quad \text{for } t \leqq 0. \tag{6.30}$$

Moreover, since both S and $S^\perp$ are full range, then so are $\mathcal{X}$ and $\mathcal{X}^\perp$ (in the sense of the proof of Lemma 6.1). Therefore, there is a $p \times p$ matrix function $i\omega \to Q(i\omega)$ taking values which are unitary matrices such that $\mathcal{X} = \mathcal{H}_p^2 Q$ [18]. Define $W := VQ^{-1}$. Clearly W is a full-rank spectral factor; let $u \in \mathcal{U}$ be the corresponding element in $\mathcal{U}$. The function Q is unique up to multiplication by a constant regular matrix [18] and hence the same is true for W and u. Then, by (6.23b), $I_{\hat{v}}^{-1} I_{\hat{u}} = M_Q$, i.e. $I_{\hat{v}} M_Q = I_{\hat{u}}$. Therefore, since $S = I_{\hat{v}} \mathcal{X}$, we have $S = I_{\hat{v}} M_Q \mathcal{H}_p^2 = I_{\hat{u}} \mathcal{H}_p^2 = H^-(du)$. Since $S \supset H^-$, Lemma 6.2 implies that $u \in \mathcal{U}^-$. This concludes the if-part of (i); the only-if part follows immediately from Corollary 6.1. The proof of (ii) is analogous. $\square$

Consequently, each proper Markovian splitting subspace $X \sim (S, \bar{S})$ is completely determined by a pair $(u, \bar{u})$ of Wiener processes, one in $\mathcal{U}^-$ and the other in $\mathcal{U}^+$; in fact

$$X = H^-(du) \cap H^+(d\bar{u}). \tag{6.31}$$

The processes are called respectively the *forward and backward generating processes* of X.

7. Hardy space representation of Markovian splitting subspaces. The goal of any description of dynamic phenomena is to obtain differential (or difference) equation representations of the relevant state variables, such as (1.7) and (1.9). To achieve this goal, in this section we go through an intermediate step in which the basic objects representing the dynamics are pairs of transfer functions $(W, \bar{W})$, one causal and the other anticausal. We shall arrive at a concrete coordinate-free state-space description in terms of analytic functions which can be computed from W and $\bar{W}$. The appropriate mathematical setting for representing causal and anticausal transfer functions is the theory of Hardy spaces. Notice that, while in the finite-dimensional setting differential equations can be obtained through straightforward algebraic calculations involving the appropriate analytic functions (§ 8), the general situation requires considerably more care (§ 9). The advantage of working with transfer function descriptions, i.e. the Hardy space setting, is that very detailed structural information about the state-space representations is obtained without having to introduce unnecessary finite-dimensionality conditions from the beginning.

Our next task is therefore to transfer the splitting subspace geometry to the Hardy space setting. To this end we need the following lemma.

LEMMA 7.1. [29]. *Let $u_1, u_2 \in \mathcal{U}$ be such that $H^-(du_1) \vee H^-(du_2) = H$, and let W_1 and W_2 be the corresponding spectral factors. Then $H^-(du_1)$ and $H^+(du_2)$ intersect perpendicularly if and only if $W_2^{-L} W_1$ is an inner function.*

Proof. By Theorem 2.2 (ii), $H^-(du_1)$ and $H^+(du_2)$ intersect perpendicularly if and only if $H^-(du_2)\subset H^-(du_1)$. But this is equivalent to $W_2^{-L}W_1$ being *inner* (Lemma 6.1). □

As we have seen, each proper Markovian splitting subspace $X\sim(S,\bar{S})$ is characterized by a pair of generating processes $(u,\bar{u})$; we have $S=H^-(du)$ and $\bar{S}=H^+(d\bar{u})$. Let $(W,\bar{W})$ be the corresponding pair of spectral factors. The condition $S\supset H^-$ is equivalent to W being analytic, $\bar{S}\supset H^+$ to $\bar{W}$ being coanalytic (Lemma 6.2), and the perpendicular intersection of S and $\bar{S}$ to

$$K := \bar{W}^{-L}W \tag{7.1}$$

being inner (Lemma 7.1). The function K is called the *structural function* of X and will play a major role in what follows. It corresponds to the *scattering matrix* in Lax-Phillips scattering theory [23].

Now, by Corollary 3.2, we have $X=S\ominus\bar{S}^{\perp}$, i.e. $X=H^-(du)\ominus H^-(d\bar{u})$, and therefore, in view of (6.23b), $I_{\hat{u}}X=\mathcal{H}_p^2\ominus(\mathcal{H}_p^2K)$. (Remember that $I_{\hat{u}}$ is unitary, and therefore orthogonality is preserved.) Define $\mathcal{H}(J):=\mathcal{H}_p^2\ominus(\mathcal{H}_p^2J)$ for any inner function J. Then

$$X=\int_{-\infty}^{\infty}\mathcal{H}(K)\,d\hat{u}. \tag{7.2}$$

Together with $d\hat{u}=W^{-L}\,d\hat{y}$ this yields a representation in terms of y. Consequently, we have established the following Hardy space version of Theorem 3.1.

THEOREM 7.1. [28]. *A subspace X is a proper Markovian splitting subspace if and only if*

$$X=\int_{-\infty}^{\infty}\mathcal{H}(\bar{W}^{-L}W)\,W^{-L}\,d\hat{y} \tag{7.3}$$

for some pair $(W,\bar{W})$ of full-rank spectral factors such that W is analytic, $\bar{W}$ is coanalytic, and $K:=\bar{W}^{-L}W$ is inner. The correspondence $X\leftrightarrow(W,\bar{W})$ is one-one (modulo multiplication from the right by constant unitary matrices).

Instead applying $I_{\hat{u}}$ to $X=\bar{S}\ominus S^{\perp}$ (Corollary 3.2), we have the symmetric representation

$$X=\int_{-\infty}^{\infty}\bar{\mathcal{H}}(K^*)\,d\hat{\bar{u}} \tag{7.4}$$

where $\bar{\mathcal{H}}(J):=\bar{\mathcal{H}}_p^2\ominus\bar{\mathcal{H}}_p^2J$ for each conjugate inner function J. (A function J is *conjugate inner* if its inverse J^* is inner.) Consequently, since $d\hat{\bar{u}}=\bar{W}^{-L}d\hat{y}$, we can replace (7.3) by

$$X=\int_{-\infty}^{\infty}\bar{\mathcal{H}}(W^{-L}\bar{W})\,\bar{W}^{-L}\,d\hat{y} \tag{7.5}$$

in Theorem 7.1.

Which pairs of spectral factors $(W,\bar{W})$ correspond to *minimal* splitting subspaces? To answer this question we need to take a closer look at the classes of analytic and coanalytic full-rank spectral factors.

By assumption, H^- satisfies the conditions of Theorem 6.1 (i), and hence there is a $u_-\in\mathcal{U}^-$ such that

$$H^-(du_-)=H^-. \tag{7.6}$$

This is the (*forward*) *innovation process* of y. Let W_- denote the corresponding analytic spectral factor. Since $d\hat{y} = W_- \, d\hat{u}_-$, (6.12) yields $I_{\hat{u}_-}^{-1} ay(t) = \chi_t a W_-$ for any row vector a in $\mathbb{R}^m$, and therefore applying $I_{\hat{u}_-}^{-1}$ to (7.6) we obtain

$$\overline{\text{span}}\, \{\chi_t a W_-;\ t \leqq 0,\ a \in \mathbb{R}^m\} = \mathcal{H}_p^2. \tag{7.7}$$

Such a function W_- is called *outer* [14], [18]. There is only one spectral factor with this property. Consequently, we shall call W_- the *outer* (or *minimum phase*) *spectral factor.*

All other analytic full-rank spectral factors have the property $H^-(du) \supset H^-$, where u is the corresponding process in $\mathcal{U}^-$ (Lemma 6.2), i.e. $H^-(du) \supset H^-(du_-)$. Therefore, $Q := W_-^{-L} W$ is inner (Lemma 6.1) so that we have the inner-outer factorization

$$W = W_- Q. \tag{7.8}$$

To see this, use the following lemma.

LEMMA 7.2. *Let W_1 and W_2 be full-rank spectral factors. Then*

$$W_1 W_1^{-L} W_2 = W_2. \tag{7.9}$$

Proof. Form $(W_1 W_1^{-L} W_2 - W_2)(W_1 W_1^{-L} W_2 - W_2)^*$. Since $W_2 W_2^* = W_1 W_1^*$, we see that this is zero. □

Likewise, Theorem 6.1 (ii) implies that there is a $\bar{u}_+ \in \mathcal{U}^+$ such that

$$H^+(d\bar{u}_+) = H^+. \tag{7.10}$$

This is the *backward innovation process* of y. The corresponding spectral factor $\bar{W}_+$ satisfies

$$\overline{\text{span}}\, \{\chi_t a \bar{W}_+;\ t \geqq 0,\ a \in \mathbb{R}^m\} = \bar{\mathcal{H}}_p^2 \tag{7.11}$$

and is therefore called the *conjugate outer spectral factor.* In the same way as above, we show that any coanalytic full-rank spectral factor $\bar{W}$ can be written

$$\bar{W} = \bar{W}_+ \bar{Q} \tag{7.12}$$

where $\bar{Q} := \bar{W}_+^{-L} \bar{W}$ is conjugate inner. The factorizations (7.8) and (7.12) are unique (modulo trivial coordinate transformations).

Consequently each proper Markovian splitting subspace is characterized by a triplet $(K, Q, \bar{Q}^*)$ consisting of the structural function K and the *forward and backward spectral inner factors* Q and $\bar{Q}^*$. These define three causal all-pass filters with the following inputs and outputs.

$$\xrightarrow{\;u\;} \boxed{\;K\;} \xrightarrow{\;\bar{u}\;} \tag{7.13a}$$

$$\xrightarrow{\;u\;} \boxed{\;Q\;} \xrightarrow{\;u_-\;} \tag{7.13b}$$

$$\xrightarrow{\;\bar{u}_+\;} \boxed{\;\bar{Q}^*\;} \xrightarrow{\;\bar{u}\;} \tag{7.13c}$$

We shall call $(K, Q, \bar{Q}^*)$ the *inner triplet* of X.

Our question on minimality can now be answered in terms of certain coprimeness conditions on these inner functions. Before turning to this, let us define a few concepts. If P_1 and P_2 are inner functions, then so is $P_3 := P_1 P_2$; P_1 is a *left inner divisor* of $P_3(P_1|_L P_3$ for short) and P_2 is a *right inner divisor* of $P_3(P_2|_R P_3)$: (These notations will also be used for conjugate inner functions.) Two inner functions are *left* (*right*) *coprime* if they have no nontrivial (i.e. nonconstant) common left (right) inner divisor. If P_1 and P_2 are right (left) coprime, then there is no cancellation in the factorization $P = P_1 P_2^*$ ($P = P_1^* P_2$); we say that it is *coprime.* If there is such a factorization of P, it is unique (modulo multiplications from the right (respectively the left) by a constant unitary matrix) [14].

THEOREM 7.2. [29]. *The proper Markovian splitting subspace X with inner triplet $(K, Q, \bar{Q}^*)$ is observable if and only if K and $\bar{Q}^*$ are left coprime and constructible if and only if K and Q are right coprime.*

Proof. Let $(u, \bar{u})$ be the generating processes of X. Then the constructibility condition $S = H^- \vee \bar{S}^\perp$ can be written $H^-(du) = H^-(du_-) \vee H^-(d\bar{u})$. Applying $I_{\bar{u}}^{-1}$ to this, and using (6.23b), we obtain $\mathscr{H}_p^2 = (\mathscr{H}_p^2 Q) \vee (\mathscr{H}_p^2 K)$ which holds if and only if Q and K are right coprime [18]. In the same way we see that the observability condition $\bar{S} = H^+ \vee S^\perp$ is equivalent to the conjugate inner functions $\bar{Q}$ and K^* being right coprime, which is the same as K and $\bar{Q}^*$ being left coprime. □

The interplay between the past and the future of y can be described by the all-pass filter

(7.14)
$$\xrightarrow{u_-} \boxed{T_0} \xrightarrow{\bar{u}_+}$$

transforming the forward innovation process u_- into the backward innovation process $\bar{u}_+$; it has the transfer function $T_0 := \bar{W}_+^{-L} W_-$. This is *not* a causal all-pass filter, unless H^- and H^+ intersect perpendicularly. For each proper Markovian splitting subspace X with inner triplet $(K, Q, \bar{Q}^*)$, the function T_0 has the factorization

(7.15)
$$T_0 = \bar{Q} K Q^*.$$

In view of Lemma 7.2, this follows by simple calculation, but it can also be seen by putting the boxes in (7.13) in series, after having reversed (7.13b) and (7.13c). By Theorem 7.2 and Corollary 4.1, X is minimal if and only if there is no cancellation in (7.15), i.e. the factorizations $T := \bar{Q}K$ and $\bar{T} := KQ^*$ are coprime.

What has been established so far in this section holds under the assumption that X is proper. Therefore, we may ask under what conditions at least all minimal splitting subspaces are proper.

THEOREM 7.3. [28]. *Set $T_0 := \bar{W}_+^{-L} W_-$, and let N^- and N^+ be given by* (3.12). *Then the following statements are equivalent.*

(i) *All minimal splitting subspaces are proper.*

(ii) *Both N^- and N^+ are full range.*

(iii) *There are inner functions J_1, J_2, J_3, and J_4 such that*

(7.16)
$$T_0 = J_1 J_2^* = J_3^* J_4.$$

Proof. (i)⇒(iii): The predictor space $H^{+/-}$ is a minimal splitting subspace such that $Q = I$. Hence the second of the factorizations (7.16) follows from (7.15). In the same way the first of relations (7.16) follows from the fact that $H^{-/+}$ is a minimal splitting subspace with $\bar{Q} = I$. (iii)⇒(ii): The first of relations (7.16) yields $W_- J_2 =$

$\bar{W}_+ J_1$. Since J_1 is inner, the spectral factor $W := W_- J_2$ is analytic. Let $u \in \mathcal{U}^-$ be the corresponding generating process. Since $\bar{W}_+^{-L} W = J_1$ is inner, $H^-(d\bar{u}_+) \subset H^-(du)$, i.e. $H^+ = H^+(d\bar{u}_+) \supset H^+(du)$. Moreover, since $u \in \mathcal{U}^-$, $H^-(du) \supset H^-$ (Lemma 6.2), or, equivalently, $(H^-)^\perp \supset H^+(du)$. Hence, $N^+ := H^+ \cap (H^-)^\perp \supset H^+(du)$. Therefore, since $H^+(du)$ is full range, then so is N^+. The second relation (7.16) yields $W_- J_4^* = \bar{W}_+ J_3^*$. Here $\bar{W} := \bar{W}_+ J_3^*$ is a coanalytic spectral factor, and therefore the corresponding $\bar{u} \in \mathcal{U}^+$ satisfies $H^+(d\bar{u}) \supset H^+$, or, equivalently $(H^+)^\perp \supset H^-(d\bar{u})$. Also, $\bar{W}^{-L} W_- = J_4$ is inner, and hence, by Lemma 6.1, $H^-(d\bar{u}) \subset H^-$. Consequently, $N^- := H^- \cap (H^+)^\perp \supset H^-(d\bar{u})$, showing that N^- is full range. (ii)$\Rightarrow$(i): Let $X \sim (S, \bar{S})$ be a minimal splitting subspace. Then, from the proof of Theorem 3.4, we see that $S \subset (N^+)^\perp$ and $\bar{S} \subset (N^-)^\perp$, i.e. $S^\perp \supset N^+$ and $\bar{S}^\perp \supset N^-$. Therefore, if N^+ and N^- are full range, then the same must be true for $S^\perp$ and $\bar{S}^\perp$. Hence X is proper. □

A unitary function T_0 has the property (7.16) if and only if it is *strictly noncyclic*, i.e. the orthogonal complement in $\mathcal{H}_p^2$ of the range of the Hankel operator $H_{T_0}: \bar{\mathcal{H}}_p^2 \to \mathcal{H}_p^2$ defined by $H_{T_0} f = P^{\mathcal{H}_p^2} T_0 f$ (where $P^{\mathcal{X}}$ denotes the orthogonal projection onto $\mathcal{X}$) is full range [14, p. 254]. Therefore, with a slight misuse of notation, we shall say that the process y is *strictly noncyclic* if the conditions of Theorem 7.3 hold. For example, a scalar process y with spectral density $\Phi(i\omega) = (1+\omega^2)^{-3/2}$ will not satisfy these conditions; in this case $H^{+/-} = H^-$ and $H^{-/+} = H^+$ [10, p. 99]. However, it can be shown that all processes with rational spectral density are strictly noncyclic.

COROLLARY 7.1. *Suppose that y is strictly noncyclic. Then the predictor spaces $H^{+/-}$ and $H^{-/+}$, defined by* (3.13), *are proper. Let $(K_-, Q_-, \bar{Q}_-^*)$ and $(K_+, Q_+, \bar{Q}_+^*)$ respectively be their inner triplets. Then $Q_- = I$ and $\bar{Q}_+ = I$; the other inner functions are the unique solutions of the coprime factorizations*

$$T_0 = \bar{Q}_- K_- = K_+ Q_+^*. \tag{7.17}$$

Proof. The factorization (7.17) was derived in the first part of the proof of Theorem 7.3. Since $H^{+/-}$ and $H^{-/+}$ are minimal, the coprimeness follows from Theorem 7.2 and Corollary 4.1. □

Now, in § 3, we saw that $H^{+/-} \sim (H^-, (N^-)^\perp)$, and hence its generating processes are $(u_-, \bar{u}_-)$, where u_- is the innovation process of y and $\bar{u}_- \in \mathcal{U}^+$ is determined, through Theorem 6.1, by

$$H^+(d\bar{u}_-) = (N^-)^\perp. \tag{7.18}$$

The analytic spectral factor is the outer spectral factor W_-, and the coanalytic one is $\bar{W}_- := \bar{W}_+ \bar{Q}_-$. In the same way, $H^{-/+}$ has generating processes $(u_+, \bar{u}_+)$, where $u_+ \in \mathcal{U}^-$ is defined by

$$H^-(du_+) = (N^+)^\perp \tag{7.19}$$

and $\bar{u}_+$ is the backward innovation of y, and its spectral factors are $W_+ := W_- Q_+$ and $\bar{W}_+$, the conjugate outer spectral factor.

Next, we shall take a closer look at the minimal Markovian splitting subspaces of a strictly noncyclic process y.

THEOREM 7.4. [31]. *Suppose that y is strictly noncyclic. Let $X \sim (S, \bar{S})$ be a Markovian splitting subspace. Then the following conditions are equivalent.*

(i) *X is minimal.*
(ii) *X is observable and $S \subset (N^+)^\perp$.*
(iii) *X is constructible and $\bar{S} \subset (N^-)^\perp$.*

The proof of this theorem[6], which can be found in [31] and will not be repeated here, is based on the observation that the structural functions of any two X satisfying (ii) or (iii) have the same invariant factors. The *invariant factors* of a $p\times p$ inner function K are scalar inner functions $k_1, k_2, \cdots, k_p$ defined in the following way. Set $\gamma_0=1$, and, for $i=1,2,\cdots,p$, define γ_i to be the greatest common inner divisor of all $i\times i$ minors of K. Then set $k_i := \gamma_i/\gamma_{i-1}$ for $i=1,2,\cdots,p$. Clearly these functions are inner, for γ_{i-1} divides γ_i. (Two inner functions with the same invariant factors are called *quasi-equivalent* [14].) Consequently we have the following important corollary, the significance of which will become evident in § 10.

THEOREM 7.5. [31]. *Suppose that y is strictly noncyclic. Let K_1 and K_2 be the structural functions of two minimal Markovian splitting subspaces. Then K_1 and K_2 have the same invariant factors.*

To illustrate this result, let us consider the following example [31]. Let y be a two-dimensional process with the rational spectral density

$$\Phi(s)=\frac{1}{(s^2-1)(s^2-4)}\begin{bmatrix} 17-2s^2 & -(s+1)(s-2) \\ -(s-1)(s+2) & 4-s^2 \end{bmatrix}.$$

Then it can be seen that the structural function of $H^{+/-}$ is

$$K_-(s)=\frac{s-1}{(s+1)(s+2)}\begin{bmatrix} s-1.2 & 1.6 \\ 1.6 & s+1.2 \end{bmatrix}$$

and that the one of $H^{-/+}$ is

$$K_+(s)=\frac{s-1}{(s+1)(s+2)}\begin{bmatrix} s-70/37 & 24/37 \\ 24/37 & s+70/37 \end{bmatrix}.$$

These functions look quite different, but they have the same invariant factors, namely

$$k_1(s)=\frac{s-1}{s+1} \quad\text{and}\quad k_2(s)=\frac{(s-1)(s-2)}{(s+1)(s+2)},$$

and are therefore quasi-equivalent.

In the scalar case ($m=1$), quasi-equivalence reduces to equality.

COROLLARY 7.2. *Suppose that y is scalar and strictly noncyclic. Then all minimal Markovian splitting subspaces have the same structural function.*

Conditions (ii) and (iii) of Theorem 7.4 suggest the following definitions for minimality of spectral factors [41]. An analytic full-rank spectral factor W is *minimal* if the corresponding $u\in\mathcal{U}^-$ satisfies the condition $H^-(du)\subset(N^+)^\perp$; a coanalytic full-rank spectral factor $\bar{W}$ is *minimal* if its $\bar{u}\in\mathcal{U}^+$ satisfies $H^+(d\bar{u})\subset(N^-)^\perp$. These definitions are justified by the following result.

COROLLARY 7.3. *Let y be strictly noncyclic. Then there is a one-one correspondence between the class of minimal Markovian splitting subspaces X and the class of minimal analytic (coanalytic) spectral factors $W[\bar{W}]$ (modulo multiplication from the right by constant unitary matrices). The correspondence $X\leftrightarrow W$ $[X\leftrightarrow\bar{W}]$ is that of Theorem* 7.1.

Proof. In view of Theorem 7.4 and the observability condition $\bar{S}=H^+\vee S^\perp$ (Theorem 4.1), there is a one-one correspondence between minimal $X\sim(S,\bar{S})$ and $u\in\mathcal{U}^-$ such that $S=H^-(du)\subset(N^+)^\perp$, i.e. to minimal analytic spectral factors. Here the correspondence $u\leftrightarrow S$ is by Theorem 6.1 and is hence modulo the transformations described there. The proof of the symmetric statement is analogous. □

[6] Theorem 7.4 was first stated in [28], but there is a nontrivial gap in the proof. The same incomplete argument was used in [41], [43].

Note, however, that a Markovian splitting subspace X need not be minimal even if both its analytic and coanalytic spectral factors are minimal; the only thing we can say in this case is that $X \subset H^{\square}$, the frame space.

In § 8, we show that, if $W[\bar{W}]$ is rational, it is minimal if and only if its degree is minimal. This is the concept of minimality mentioned in § 1.

The scalar version of the following result is due to Ruckebusch [41].

PROPOSITION 7.2. [41], [28]. *Suppose that y is strictly noncyclic. Then* (i) $W := W_- Q$ *is a minimal analytic spectral factor if and only if* $Q|_L Q_+$; *and* (ii) $\bar{W} := \bar{W}_+ \bar{Q}$ *is a minimal coanalytic spectral factor if and only if* $\bar{Q}|_L \bar{Q}_-$.

Proof. Let $u \in \mathcal{U}^-$ be the Wiener process of W. Then, by definition, W is minimal if and only if $H^-(du) \subset H^-(du_+)$, which is equivalent to $P := W^{-L} W_+$ being inner (Lemma 6.1). Now, in view of Lemma 7.2, $Q_+ := W_-^{-L} W_+ = W_-^{-L} W W^{-L} W_+ = QP$. Therefore, W is minimal if and only if $Q|_L Q_+$. This establishes (i); (ii) is proved in the same way. □

Now, by Corollary 7.3 and Proposition 7.2, there is a one-one correspondence (modulo trivial transformations) between minimal Markovian splitting subspaces X and left inner divisors Q of Q_+. This provides a parametrization $\{X_Q;\, Q|_L Q_+\}$ of the class of minimal Markovian splitting subspaces which introduces a natural *partial ordering* of this class, under which $X_{Q_1} < X_{Q_2}$ if and only if $Q_1|_L Q_2$. Here there are a minimal element $X_I = H^{+/-}$ and a maximal element $X_{Q_+} = H^{-/+}$. Obviously this is the lattice structure described in the end of § 4. (A similar parameterization can of course be obtained in terms of the conjugate inner functions $\bar{Q}$ such that $\bar{Q}|_L \bar{Q}_-$.)

Given a left inner divisor Q of Q_+, how do we determine X_Q? The inner triplet $(K, Q, \bar{Q}^*)$ can be determined from the factorization (7.15) as described in the following lemma.

LEMMA 7.3. *Suppose y is strictly noncyclic. Let Q be a left inner divisor of* Q_+, *and define* $T := T_0 Q$. *Then, T has a unique (modulo constant unitary factors) coprime factorization*

$$T = \bar{Q} K \tag{7.20}$$

where K is inner, $\bar{Q}$ *is conjugate inner and K and* $\bar{Q}^*$ *are left coprime. Moreover,* $(K, Q, \bar{Q}^*)$ *is the inner triplet of* X_Q.

Proof. Let $(K, Q, \bar{Q})$ be the inner triplet of X_Q. Then (7.20) follows from (7.15). Since X_Q is observable, K and $\bar{Q}^*$ are left coprime (Theorem 7.2). As pointed out above, the coprime factorization is unique, in the sense described in the lemma [14]. Since we do not distinguish between equivalent inner triplets (differing only by constant unitary factors), the lemma follows. □

(For the relationship between the factorization (7.20) and the corresponding Hankel operators, the reader is referred to [30].) Consequently, in view of Theorem 7.1, we have the following representation theorem for the class of minimal Markovian splitting subspaces.

THEOREM 7.6. *Suppose that y is strictly noncyclic. Then a subspace X of H is a minimal Markovian splitting subspace if and only if*

$$X = \int_{-\infty}^{\infty} \mathcal{H}(K) Q^* \, d\hat{u}_- \tag{7.21}$$

for some $Q|_L Q_+$, *where K is the inner factor in the coprime factorization* (7.20) *and* u_- *is the innovation process of y.*

An alternative formulation of this theorem goes as follows. (Here $P^{\mathcal{X}}$ denotes orthogonal projection onto the subspace $\mathcal{X}$ and $\bar{P}^{\mathcal{X}}\mathcal{Y}$ the closure of $P^{\mathcal{X}}\mathcal{Y}$.)

THEOREM 7.7. [28]. *Suppose that y is strictly noncyclic. Then a subspace X of H is a minimal Markovian splitting subspace if and only if*

$$X = \int_{-\infty}^{\infty} [\bar{P}^{\mathscr{H}_p^2 Q^*}(\bar{\mathscr{H}}_p^2 T_0)] \, d\hat{u}_- \tag{7.22}$$

for some left inner divisor Q of Q_+.

Proof. We need to show that X_Q is given by (7.22). Let $u \in \mathscr{U}^-$ be the forward generating process of X_Q, and let W be the corresponding analytic spectral factor. Then $W^{-L}W_- = Q^*$. Now, in view of Corollary 4.2 and (7.10),

$$X_Q = \bar{E}^{H^-(du)} H^+(d\bar{u}_+). \tag{7.23}$$

By (6.23b), $H^-(du)$ and $H^+(d\bar{u}_+)$ correspond, under the isomorphism $I_{\hat{u}}$, to $\mathscr{H}_p^2 Q^*$ and $\bar{\mathscr{H}}_p^2 T_0$ respectively, and therefore (7.22) follows from (7.23). □

Of course, there are also backward versions of Theorems 7.6 and 7.7 in which $\bar{Q}$ plays the role of Q.

8. Stochastic realizations: the finite-dimensional case.

PROPOSITION 8.1. *Let X be a proper Markovian splitting subspace. Then X is finite-dimensional if and only if its structural function K is rational.*

Proof. Let $(u, \bar{u})$ be the generating processes of X. Then, by Corollary 3.1, $X = E^{H^-(du)} H^+(d\bar{u})$, the isomorphic image of which under $I_{\bar{u}}^{-1}$ is $P^{\mathscr{H}_p^2}(\bar{\mathscr{H}}_p^2 K)$. Consequently, X is isomorphic to the range of the Hankel operator $H_K: \bar{\mathscr{H}}_p^2 \to \mathscr{H}_p^2$ defined by $H_K f = P^{\mathscr{H}_p^2} fK$, which, by [14, Thm. 3.8, p. 256], is finite-dimensional if and only if K is rational. □

Now, let X be a finite-dimensional, but not necessarily minimal, Markovian splitting subspace with structural function K and generating processes $(u, \bar{u})$. Then K is rational, and there is a coprime factorization

$$K(s) = \bar{D}(s) D(s)^{-1} \tag{8.1}$$

where D and $\bar{D}$ are real invertible $p \times p$ polynomial matrices which are right coprime, i.e. any common right divisor is unimodular[7] [14], [47]. The matrix polynomial D and $\bar{D}$ are unique (modulo a common unimodular factor). To maintain the symmetry between the past and the future in our presentation we also note that

$$K^*(s) = D(s) \bar{D}(s)^{-1}. \tag{8.2}$$

The following result shows that $\mathscr{H}(K)$, the isomorphic image of X under $I_{\hat{u}}$, consists of rational functions which are *strictly proper*, i.e., in each component, the numerator polynomial is of lower degree than the denominator polynomial.

THEOREM 8.1. [29]. *Let the inner function K have the polynomial-matrix-fraction representation* (8.1). *Then*

$$\mathscr{H}(K) = \{gD^{-1} | g \in \mathbb{R}^p[s];\ gD^{-1} \text{ strictly proper}\} \tag{8.3}$$

where $\mathbb{R}^p[s]$ is the vector space of p-dimensional row vectors of polynomials.

For the proof we need the following lemma.

LEMMA 8.1. *If K is rational, the space $\mathscr{H}(K)$ consists of strictly proper rational functions.*

Proof. Set $k := \det K$. Then $\mathscr{H}_p^2 k \subset \mathscr{H}_p^2 K$ [14, p. 187], and consequently $\mathscr{H}(K) \subset \mathscr{H}(kI)$. Therefore, it is no restriction to study the scalar case $p = 1$. In fact, if K is

[7] A matrix polynomial is *unimodular* if it has a polynomial inverse.

rational, then so is k. So, if we can prove that the scalar $\mathcal{H}(k)$ consists of rational functions, the same holds true for $\mathcal{H}(kI)$ and thus for $\mathcal{H}(K)$. A scalar rational inner function k is a finite Blaschke product [14], [18], i.e. a *finite* product of coprime functions $k_i(s) := (s-s_i)^{\nu_i}(s+\bar{s}_i)^{-\nu_i}$, where, for each i, s_i is a complex number, $\bar{s}_i$ its complex conjugate, and ν_i an integer. Then $\mathcal{H}^2 k = \bigcap_i \mathcal{H}^2 k_i$, and hence $\mathcal{H}(k) = \bigvee_i \mathcal{H}(k_i)$, so it is enough to show that any $\mathcal{H}(k_i)$ consists of rational functions. To this end, we quote from [10, p. 34] that

$$e_j(s) = \frac{1}{s+\bar{s}_i}\left[\frac{s-s_i}{s+\bar{s}_i}\right]^j, \qquad j=0,1,2,\cdots \tag{8.4}$$

is an orthogonal basis in $\mathcal{H}^2$. However, $e_j k_i = e_{j+\nu_i}$, and hence $\mathcal{H}^2 k_i$ is spanned by $\{e_{\nu_i}, e_{\nu_{i+1}}, \cdots\}$. Therefore, $\mathcal{H}(k_i)$ is the span of $\{e_0, e_1, \cdots, e_{\nu_{i-1}}\}$, which is a space of strictly proper rational functions. Consequently, the same is true for $\mathcal{H}(K)$, as required. □

Proof of Theorem 8.1. It is not hard to show that

$$\mathcal{H}(K) = \{f \in \mathcal{H}_p^2 \mid fK^* \in \bar{\mathcal{H}}_p^2\} \tag{8.5}$$

[18, p. 75]. In view of (8.2), this may be written

$$\mathcal{H}(K) = \{gD^{-1} \in \mathcal{H}_p^2 \mid g\bar{D}^{-1} \in \bar{\mathcal{H}}_p^2\}. \tag{8.6}$$

Since $gD^{-1} \in \mathcal{H}(K)$ is rational (Lemma 8.1), then so is g. Any rational g such that $gD^{-1} \in \mathcal{H}_p^2$ and $g\bar{D}^{-1} \in \bar{\mathcal{H}}_p^2$ must be analytic in the whole complex plane, and hence $g \in \mathbb{R}^p[s]$. By Lemma 8.1, gD^{-1} is strictly proper. □

COROLLARY 8.1. [29]. *Let X be a finite-dimensional Markovian splitting subspace with structural function* (8.1). *Then the corresponding spectral factors W and $\bar{W}$ are strictly proper rational. In fact,*

$$W(s) = N(s)D(s)^{-1}, \tag{8.7a}$$

$$\bar{W}(s) = N(s)\bar{D}(s)^{-1} \tag{8.7b}$$

for some $m \times p$-matrix polynomial N.

Proof. By the definition (7.1), $W = \bar{W}K$ (Lemma 7.2), i.e. $\bar{W} = WK^*$, and therefore, in view of (8.5) and the fact that W is analytic and $\bar{W}$ is coanalytic, the rows of W belong to $\mathcal{H}(K)$. Hence, by Theorem 8.1, W is strictly proper rational and has a representation (8.7a). However, (7.1) and (8.1) yield $\bar{W}\bar{D} = WD$, which is precisely N. Hence (8.7b) follows. Since $\bar{W}$ is square-integrable, it must be strictly proper. □

It is important to note that the factorization (8.7) need *not* be coprime. The significance of this will be made clear below.

We proceed to construct a basis in X. To this end, we shall choose the arbitrary unimodular factor in (8.1) so that (i) if n_i is the degree of the ith column of $[\begin{smallmatrix} D \\ \bar{D} \end{smallmatrix}]$, then $n_1 + n_2 + \cdots + n_p = n$, where n is the common degree of det D and det $\bar{D}$; and (ii) D and $\bar{D}$ are *column proper*, i.e. the highest-degree coefficient matrices D_h and $\bar{D}_h$ are full rank; here D_h ($\bar{D}_h$) is the constant matrix whose ith column consists of the coefficients of s^{n_i} in the ith column of D ($\bar{D}$). It is always possible to choose D and $\bar{D}$ in this way, and there are procedures to achieve it [13], [20], [47]. With this representation, we have

$$D(s) = D_h\{\text{diag}\{s^{n_1}, s^{n_2}, \cdots, s^{n_p}\} + D_0\Pi(s)\}, \tag{8.8a}$$

$$\bar{D}(s) = \bar{D}_h\{\text{diag}\{s^{n_1}, s^{n_2}, \cdots, s^{n_p}\} + \bar{D}_0\Pi(s)\} \tag{8.8b}$$

where diag $\{s^{n_1}, s^{n_2}, \cdots, s^{n_p}\}$ is the $p \times p$ matrix with $s^{n_1}, s^{n_2}, \cdots, s^{n_p}$ on the diagonal and zeros elsewhere, the $n \times p$-matrix polynomial $\Pi(s)$ is the transpose of

$$\Pi(s)' := \begin{bmatrix} s^{n_1-1}, \cdots, s, 1 & & \\ & s^{n_2-1}, \cdots, s, 1 & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ & & s^{n_p-1}, \cdots, s, 1 \end{bmatrix} \tag{8.9}$$

(where empty spaces are zeros), and D_0 and $\bar{D}_0$ are constant $p \times n$ matrices.

LEMMA 8.2. [29]. *The n rows of the $n \times p$ matrix $\Pi(s)D(s)^{-1}$ of rational functions form a basis in $\mathscr{H}(K)$.*

Proof. The rows of ΠD^{-1} are clearly linearly independent. It remains to show that they span $\mathscr{H}(K)$. In view of Theorem 8.1, this amounts to showing that gD^{-1} is strictly proper for precisely those $g \in \mathbb{R}^p[s]$ which can be written $a\Pi(s)$ for some row vector $a \in \mathbb{R}^n$, i.e. for those $g \in \mathbb{R}^p[s]$ with $\deg g_i < n_i$ for $i = 1, 2, \cdots, p$. By Cramer's rule, $[D(s)^{-1}]_{ij} = (-1)^{i+j}\Delta_{ji}(s)/\Delta(s)$, where $\Delta := \det D$ and Δ_{ji} is the determinant of the matrix obtained by deleting row j and column i in D. Hence, Δ_{ji} is a sum of products of one element from each of all columns of D except the ith, and consequently $\deg \Delta_{ji} \leqq n - n_i$. Since D_h is full rank, for each i, there is a j such that equality holds. In fact, in each column i of D there is a row j such that deleting row j and column i in D_h leaves a nonsingular matrix. Hence in forming Δ_{ji} there is at least one product that contains only factors of highest degree. Since $\deg \Delta = n$, gD^{-1} is therefore strictly proper if and only if $\deg g_i < n_i$ for $i = 1, 2, \cdots, p$ as required. □

THEOREM 8.2. [29]. *Let X be a finite-dimensional Markovian splitting subspace with structural function* (8.1) *and spectral factors* (8.7). *Then, for each $t \in \mathbb{R}$, the components of the random vector*

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t}\Pi(i\omega)N(i\omega)^{-L}\, d\hat{y} \tag{8.10}$$

form a basis in U_tX. Moreover,

$$y(t) = Cx(t), \tag{8.11}$$

where the matrix C is uniquely determined by identifying coefficients of like power of s in

$$N(s) = C\Pi(s). \tag{8.12}$$

The process x also has the representations

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t}\Pi(i\omega)D(i\omega)^{-1}\, d\hat{u}, \tag{8.13a}$$

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t}\Pi(i\omega)\bar{D}(i\omega)^{-1}\, d\hat{\bar{u}} \tag{8.13b}$$

where $(u, \bar{u})$ are the generating processes of X.

Proof. In view of Lemma 8.2, it is immediately clear from (7.2) that $\{x_1(0), x_2(0), \cdots, x_n(0)\}$, as defined by (8.13a), is a basis in X. Then, it follows from (6.18) that $\{x_1(t), x_2(t), \cdots, x_n(t)\}$ is a basis in U_tX for each $t \in \mathbb{R}$. From (6.16) and (8.7a) we have that $D^{-1}\, d\hat{u} = N^{-L}\, d\hat{y}$, and hence (8.10) and (8.13a) are equivalent. The equivalence of (8.13a) and (8.13b) follows from $d\hat{\bar{u}} = K\, d\hat{u}$ and (8.1). In the proof of Corollary 8.1, we saw that the rows of W belong to $\mathscr{H}(K)$. Therefore, by Lemma 8.2, there is an $m \times n$ matrix C such that $W(s) = C\Pi(s)D(s)^{-1}$; hence, in view of (8.7a), (8.12) holds. It is easy to see that C is uniquely determined by this relation. Inserting this expression for W in (6.12)+(6.17) and observing (8.13a) we have (8.11). □

In particular, it follows from Theorem 8.2 that

$$U_tX = \{ax(t) | a \in \mathbb{R}^n\} \tag{8.14}$$

which is precisely (1.4). Hence x is a *state process* of X.

It remains to find two Markovian representations for the state process x, a forward one generated by u and a backward one driven by $\bar{u}$. To this end, we define the $n \times n$ matrices A and $\bar{A}$ and the $n \times p$ matrices B and $\bar{B}$ as

$$\begin{aligned} A &:= J - \Pi(0)D_0, & \bar{A} &:= -J + \Pi(0)\bar{D}_0, \\ B &:= \Pi(0)D_h^{-1}, & \bar{B} &:= \Pi(0)\bar{D}_h^{-1}. \end{aligned} \tag{8.15}$$

Here J is the block diagonal matrix

$$J = \operatorname{diag}\{J_{n_1}, J_{n_2}, \cdots, J_{n_p}\} \tag{8.16}$$

where J_k is the $k \times k$ shift matrix with ones on the superdiagonal and zeros elsewhere. The pair $[J, \Pi(0)]$ is known as the *Brunovsky canonical form*, and $\{n_1, n_2, \cdots, n_p\}$ are known as its *indices*.

THEOREM 8.3. *Let $\{x(t); t \in \mathbb{R}\}$ be the state process* (8.10) *of the Markovian splitting subspace*

$$X = \{ax(0) | a \in \mathbb{R}^n\} \tag{8.17}$$

and let A, B, $\bar{A}$, and $\bar{B}$ be defined by (8.15). *Then A and $\bar{A}$ have the same eigenvalues, all located in the open left half plane, and $[B, AB, A^2B, \cdots]$ and $[\bar{B}, \bar{A}\bar{B}, \bar{A}^2\bar{B}, \cdots]$ have full rank. Moreover, x has the two representations*

$$x(t) = \int_{-\infty}^{\infty} e^{A(t-\sigma)} B \, du(\sigma), \tag{8.18a}$$

$$x(t) = -\int_{-\infty}^{\infty} e^{\bar{A}(\sigma-t)} \bar{B} \, d\bar{u}(\sigma) \tag{8.18b}$$

where $(u, \bar{u})$ are the generating processes of X and the integrals are defined in quadratic mean.

Proof. A simple calculation yields $(sI - J)\Pi(s) = \Pi(0) \operatorname{diag}\{s^{n_1}, s^{n_2}, \cdots, s^{n_p}\}$, and consequently $(sI - A)\Pi(s) = BD(s)$, i.e.

$$\Pi(s)D(s)^{-1} = (sI - A)^{-1}B. \tag{8.19}$$

It is well known and easy to show that $[B, AB, A^2B, \cdots]$ has full rank, and therefore (8.19) has degree n [20], [47]. Consequently $\det D(s)$ and $\det(sI - A)$ have the same zeros, i.e. the eigenvalues of A and the zeros of $\det D(s)$ coincide. In the same way, we see that

$$\Pi(s)\bar{D}(s)^{-1} = (sI + \bar{A})^{-1}\bar{B}, \tag{8.20}$$

and therefore, since $[\bar{B}, \bar{A}\bar{B}, \bar{A}^2\bar{B}, \cdots]$ is full rank [20], [47], the eigenvalues of $\bar{A}$ are the zeros of $\det \bar{D}(-s)$. In view of (8.1), $\det K = \det \bar{D} / \det D$, which is a finite Blaschke product [14], [18]. Such a function has all its poles in the open left half plane, and the zeros of its numerator polynomial are related to those of its denominator polynomial by a simple change of sign. Consequently, the zeros of $\det D(s)$ and $\det \bar{D}(-s)$ coincide, i.e. A and $\bar{A}$ have the same eigenvalues, and they are located in the open left half plane. The rows of (8.19) belong to $\mathcal{H}_p^2$ (Lemma 8.2), and the inverse Fourier transform of $(i\omega I - A)^{-1}B$ is $e^{At}B$ for $t \geqq 0$ and zero otherwise. Consequently, in view of (6.7) and (6.18), (8.18a) follows from (8.13a). In the same way, (8.18b) follows from (8.13b). In fact, $(i\omega I+\bar{A})^{-1}$ has inverse Fourier transform $e^{-\bar{A}t}$ for $t\leqq 0$ and zero otherwise, its rows belonging to $\bar{\mathscr{H}}^2_p$. □

Therefore, given a finite-dimensional Markovian splitting subspace X with generating processes $(u, \bar{u})$, there are a forward stochastic realization

$$\Sigma\begin{cases} dx = Ax\,dt + B\,du, \\ y = Cx \end{cases} \tag{8.21a}$$

and a backward one

$$\bar{\Sigma}\begin{cases} dx = \bar{A}x\,dt + \bar{B}\,d\bar{u}, \\ y = Cx \end{cases} \tag{8.21b}$$

such that (8.17) holds; this follows from (8.11) and (8.18). We shall call them the *standard (forward and backward) realizations* of X. The fact that Σ is forward and $\bar{\Sigma}$ is backward is seen from (8.18), but it can also be illustrated by (3.6) rewritten as

$$H = H^-(d\bar{u})\oplus X\oplus H^+(du), \tag{8.22}$$

i.e. the components of the state $x(0)$ are orthogonal to the future increments of u and to the past increments of $\bar{u}$.

From Theorem 8.3 it also follows that Σ is always reachable and $\bar{\Sigma}$ is always controllable, with these terms defined as in § 1. The circumstances under which Σ is observable and $\bar{\Sigma}$ is constructible is described by the following theorem.

THEOREM 8.4. *Let X be a finite-dimensional Markovian splitting subspace with spectral factors $(W, \bar{W})$ and standard realizations* (8.21). *Then, W is the transfer function of Σ, and the following conditions are equivalent.*

(i) *X is observable.*
(ii) *Σ is observable.*
(iii) *The factorization $W = ND^{-1}$ of Corollary* 8.1 *is coprime, i.e. N and D are right coprime.*

Symmetrically, $\bar{W}$ is the transfer function of $\bar{\Sigma}$ and the following conditions are equivalent.

(iv) *X is constructible.*
(v) *$\bar{\Sigma}$ is constructible.*
(vi) *The factorization $\bar{W} = N\bar{D}^{-1}$ is coprime.*

Proof. We shall only consider the first part. The second follows by symmetry. In view of (8.12), (8.19) and (8.7a), we have

$$W(s) = C(sI-A)^{-1}B \tag{8.23}$$

and consequently W is the transfer function of Σ. Then, the equivalence of (ii) and (iii) follows from [14, p. 41] or [20, p. 439], so it only remains to show that (i) and (ii) are equivalent. With $S = H^-(du)$, it follows from (8.17a) and (8.11) that

$$E^S ay(t) = aC\,e^{At}x(0) \tag{8.24}$$

for any row vector $a\in\mathbb{R}^m$, and consequently

$$\bar{E}^S H^+ = \overline{\text{span}}\,\{aCe^{At};\, t\geqq 0,\, a\in\mathbb{R}^m\}x(0). \tag{8.25}$$

By Corollary 4.2, the left member of (8.25) equals X if and only if X is observable. On the other hand, Σ is observable if and only if the range of $\{e^{A't}C';\, t\geqq 0\}$ is dense in $\mathbb{R}^n$ [22]. Therefore, it follows from (8.17) that (i) and (ii) are equivalent. □

We shall say that a finite-dimensional stochastic realization is *minimal* if there is no other realization with a state process x of smaller dimension. Together with Theorem 8.2, the following result implies that Σ and $\bar{\Sigma}$ are minimal if and only if X is minimal.

THEOREM 8.5. *A finite-dimensional splitting subspace is minimal if and only if its dimension is minimal.*

Proof. Let X be a splitting subspace. First, assume that there is a splitting subspace X_1 of smaller dimension than X. By Corollary 3.5, X_1 contains a minimal splitting subspace X_2. Since $\dim X_2 \leqq \dim X_1 < \dim X$, Theorem 4.2 implies that X is nonminimal. Second, suppose that X is not minimal. Then it contains a minimal splitting subspace as a proper subspace (Corollary 3.5), and thus X cannot have minimal dimension. □

By Theorem 8.4 and Corollary 4.1, it is not enough for the stochastic realization Σ to be both reachable and observable to be minimal as is the case in deterministic realization theory; for this to happen the backward realization $\bar{\Sigma}$ must be constructible also, or, alternatively, the analytic spectral factor W must be minimal in the sense described in § 7. In the finite-dimensional case under discussion, minimality of spectral factors can be related to their degrees, as the following result shows.

COROLLARY 8.2. *Let X be a finite-dimensional Markovian splitting subspace with spectral factors $(W, \bar{W})$. Then*

$$\dim X \geqq \deg W \tag{8.26}$$

with equality if and only if X is observable, and

$$\dim X \geqq \deg \bar{W} \tag{8.27}$$

with equality if and only if X is constructible. Moreover, $W[\bar{W}]$ is minimal if and only if its degree is as small as possible.

Proof. By Theorem 8.2, $\dim X$ equals n, the degree of $\det D$. But, since $W = ND^{-1}$, $\deg W \leqq n$, with equality if and only if N and D are right coprime, which, in view of Theorem 8.4, holds if and only if X is observable. Now, suppose that X is observable. Then, $\deg W = \dim X$. Since W is minimal if and only if X is minimal (Corollary 7.4) and X is minimal if and only if $\dim X$ is minimal (Theorem 8.5), W is minimal if and only if $\deg W$ is minimal. The proofs of the statements concerning $\bar{W}$ are analogous. □

In view of Theorem 7.4, we have also established the following result.

COROLLARY 8.3. *Let X be a finite-dimensional Markovian splitting subspace with spectral factors $(W, \bar{W})$ and standard realizations* (8.21). *Then the following conditions are equivalent.*

(i) *X is minimal.*
(ii) *Σ is minimal.*
(iii) *Σ is observable and W is minimal.*
(iv) *$\bar{\Sigma}$ is minimal.*
(v) *$\bar{\Sigma}$ is constructible and $\bar{W}$ is minimal.*
(vi) *Σ is observable and $\bar{\Sigma}$ is constructible.*

As an application of Theorem 8.4, let us give an alternative characterization of the class of minimal Markovian splitting subspaces in the case that y is a *scalar* process. Then, N, D, and $\bar{D}$ are scalar and $\bar{D}(s) = D(-s)$, for K is a finite scalar Blaschke product. Minimality of X requires that both Condition (iii) and Condition (vi) in Theorem 8.4 are satisfied, i.e. $N(s)$ and $\psi(s) := D(s)D(-s)$ are coprime. This is clearly equivalent to coprimeness of $\varphi(s) := N(s)N(-s)$ and $\psi(s)$. Therefore, we can characterize the class of minimal Markovian splitting subspaces in the following way.

Write the rational density Φ of y as $\Phi=\varphi/\psi$ where φ and ψ are coprime polynomials. For each polynomial solution N of

$$N(s)N(-s)=\varphi(s) \tag{8.28}$$

form

$$X=\int_{-\infty}^{\infty}\left\{\frac{g(s)}{N(s)}\,\middle|\,\deg g<\frac{1}{2}\deg\psi\right\}d\hat{y} \tag{8.29}$$

where $\deg g<n$ means that g is a polynomial of degree less than n. It follows from (8.10), (8.17), and what has been said above that this procedure produces precisely the minimal splitting subspaces of y.

9. Stochastic realizations: the general case. In § 8, given a Markovian splitting subspace X of finite dimension n, we constructed a state process $\{x(t);\, t\in\mathbb{R}\}$ taking values in $\mathbb{R}^n$ and forward and backward differential equation representations for it. The main point of this construction is a convenient choice of basis in X. In this basis the matrix representations e^{At} and $e^{\bar{A}t}$ of the Markov semigroups $\{U_t(X)^*\}$ and $\{U_t(X)\}$ can be found almost by inspection from the matrix fraction representation (8.1) of the structural function K. This immediately leads to the forward and backward realizations (8.21) of the process y in the familiar state space form. So, in the finite-dimensional case, the passage from any solution of the abstract realization problem, i.e. a Markovian splitting subspace $X\sim(S,\bar{S})$ and a corresponding Markov semigroup $\{U_t(X);\, t\in\mathbb{R}\}$, is merely a question of coordinatization.

On the other hand, the theory developed up to § 8 is absolutely independent of any restrictions of the dimension of X. The natural question to ask at this point is thus the following. Given a Markovian splitting subspace of possibly infinite dimensions, when is it possible to obtain differential equations representations for $\{y(t);\, t\in\mathbb{R}\}$ of the type (1.7) and (1.9)?

This is basically a *representation problem* in which one seeks a global description in terms of local or infinitesimal data. As such it has no meaningful solution in general. Obtaining differential equation representations for a process with nonrational spectrum necessarily involves restrictions of a technical nature (essentially smoothness conditions) on the underlying spectral factors. The elucidation of these conditions is one of the goals of this section. Note that there are several possible mathematical frameworks for infinite-dimensional Markov processes as solutions of stochastic differential equations (e.g. [17] and [49]), all of which coincide when specialized to the finite-dimensional case. Here we shall work in a setting which looks most natural to us, but other approaches are possible.

The problem dealt with in this section might seem relevant only from a purely theoretical point of view. However, we remark that many engineering problems involve random processes with nonrational spectra, e.g. turbulence, wave spectra, gyroscopic noise, etc. In practical problems, these spectra must be approximated, and finite-dimensional approximate realizations must be constructed. Understanding the exact structure of the infinite-dimensional state space models for these processes is probably the best way to gain insight into the approximation process and to design efficient finite-dimensional filters.

An important feature of the construction in § 8 is that $x(0)$ is a basis in X so that the *state space* $\mathbb{R}^n$, i.e. the space in which the state process $\{x(t);\, t\in\mathbb{R}\}$ takes values, and the splitting subspace X have the same dimension (and are therefore isomorphic). Choosing the state space in this way insures that the forward realization Σ is reachable and the backward realization $\bar{\Sigma}$ is controllable. Of course we could have achieved the same thing by taking as the state space any vector space $\mathscr{X}$ isomorphic to X such as, for example, an n-dimensional vector space of polynomials in the style of Fuhrmann [14], thereby obtaining a coordinate-free representation.

In this section we shall assume that X is a possibly infinite-dimensional (not necessarily minimal) proper Markovian splitting subspace with spectral factors $(W, \bar{W})$ and generating processes $(u, \bar{u})$. As before, it is reasonable to take as the state space a Hilbert space isomorphic to X. In this paper, we shall choose $\mathscr{X} := I_u^* X$ as the state space of the forward realization and $\bar{\mathscr{X}} := I_{\bar{u}}^* X$ as the state space in the backward one. then, by (6.7), (7.2) and (7.4), $\mathscr{X} = \mathscr{F}^* \mathscr{H}(K)$ and $\bar{\mathscr{X}} = \mathscr{F}^* \mathscr{H}(K^*)$ where K is the structural function of X.

As explained in § 5, the forward realization should, in an abstract sense, be a stochastic dynamical system with input u and semigroup $\{U_t(X)^*; t \geqq 0\}$. With our present choice of state space we should therefore take

$$e^{At} := I_u^* U_t(X)^* I_u. \tag{9.1}$$

Of course, as should be, $\{e^{At}; t \geqq 0\}$ is a strongly continuous contraction semigroup on $\mathscr{X}$ (Theorem 5.2), and the infinitesimal generator A is in general an unbounded operator with domain $\mathscr{D}(A)$ dense in $\mathscr{X}$.

In the same way, the backward realization should have input $\bar{u}$ and a semigroup isomorphic to $\{U_t(X); t \geqq 0\}$. We take

$$e^{\bar{A}t} := I_{\bar{u}}^* U_t(X) I_{\bar{u}} \tag{9.2}$$

defining a strongly continuous contraction semigroup on the state space $\bar{\mathscr{X}}$ of the backward realization.

It remains to determine maps $B: \mathbb{R}^P \to \mathscr{X}$ and $C: \mathscr{X} \to \mathbb{R}^m$ for the forward realization and $\bar{B}: \mathbb{R}^P \to \bar{\mathscr{X}}$ and $\bar{C}: \bar{\mathscr{X}} \to \mathbb{R}^m$ for the backward realization having the appropriate properties. We would like these maps to be bounded.

We begin with the forward realization. Let $\xi \in X$ be arbitrary, and let $f \in \mathscr{X}$ be the corresponding point in the state space, i.e. $\xi = I_u f$. Then

$$U_t \xi = \int_{-\infty}^{\infty} f(-\sigma)\, du(\sigma + t) = \int_{-\infty}^{\infty} f(t - \sigma)\, du(\sigma). \tag{9.3}$$

But, $\mathscr{F} f \in \mathscr{H}(K) \subset \mathscr{H}_p^2$, and therefore f vanishes on negative real line so that

$$U_t \xi = \int_{-\infty}^{t} f(t - \sigma)\, du(\sigma). \tag{9.4}$$

Consequently, since $S = H^-(du)$, (5.3a) yields

$$U_t(X) \xi = \int_{-\infty}^{0} f(t - \sigma)\, du(\sigma). \tag{9.5}$$

It follows from (9.1) that $U_t(X)\xi = I_u e^{A^* t} f$, and hence

$$(e^{A^* t} f)(\tau) = \begin{cases} f(t + \tau) & \text{for } \tau \geqq 0, \\ 0 & \text{for } \tau < 0. \end{cases} \tag{9.6}$$

Therefore, whenever defined, $A^* f$ is the derivative of f in the $\mathscr{L}^2$ sense [3].

Now, following a standard construction [3], define $\mathscr{Z}$ to be the domain $\mathscr{D}(A^*)$ of the unbounded operator A^* equipped with the graph topology

$$\langle f, g \rangle_{\mathscr{Z}} = \langle f, g \rangle + \langle A^* f, A^* f \rangle \tag{9.7}$$

where now $\langle f, g\rangle := \int_0^\infty f(t)g(t)'\,dt$ is the inner product in $\mathscr{X}$. Since an infinitesimal generator such as A^* is a closed operator with a dense domain [48], $\mathscr{Z}$ is a Hilbert space which is densely embedded in $\mathscr{X}$. The topology of $\mathscr{Z}$ is stronger than that of $\mathscr{X}$, and therefore all continuous linear functionals on $\mathscr{X}$ are continuous on $\mathscr{Z}$ as well. Consequently, we can think of the dual space $\mathscr{X}^*$ as embedded in the dual space $\mathscr{Z}^*$. Then, identifying $\mathscr{X}^*$ with $\mathscr{X}$, we have

$$\mathscr{Z} \subset \mathscr{X} \subset \mathscr{Z}^* \tag{9.8}$$

where $\mathscr{Z}$ is dense in $\mathscr{X}$ which in turn is dense in $\mathscr{Z}^*$. We shall write (f, f^*) to denote the value of the functional $f^* \in \mathscr{Z}^*$ evaluated at $f \in \mathscr{Z}$ (or, by reflexivity, the value at f^* of f regarded as a functional on $\mathscr{Z}^*$). Clearly, the bilinear form (f, f^*) coincides with the inner product $\langle f, f^*\rangle$ whenever $f^* \in \mathscr{X}$. Since A^*f is the derivative of f, $\mathscr{Z}$ is a subspace of the Sobolev space $H^1(\mathbb{R}^+)$, and $\mathscr{Z}^*$ is a space of distributions [3].

Next, define $D: \mathscr{Z} \to \mathscr{X}$ to be the differentiation operator on $\mathscr{Z}$. Then $Df = A^*f$ for all $f \in \mathscr{Z}$, but, since $\|Df\| \leqq \|f\|_{\mathscr{Z}}$, D is a continuous map. Its adjoint $D^*: \mathscr{X} \to \mathscr{Z}^*$ is the extension of A to $\mathscr{X}$, because $(f, D^*g) = \langle A^*f, g\rangle$. We collect some well-known properties of D in the following lemma.

LEMMA 9.1. *The map $(I-D): \mathscr{Z} \to \mathscr{X}$ is bijective, and it has a bounded inverse $(I-D)^{-1}: \mathscr{X} \to \mathscr{Z}$. Moreover,*

$$\|f\|_{\mathscr{Z}}^2 \leqq \|(I-D)f\|^2 \leqq 2\|f\|_{\mathscr{Z}}^2. \tag{9.9}$$

Proof. Since $\{U_t(X); t \geqq 0\}$ is a strongly continuous contraction semigroup (Theorem 5.2), then so is $\{e^{A^*t}; t \geqq 0\}$. Consequently, D is dissipative, i.e. $\langle Df, f\rangle \leqq 0$ for all $f \in \mathscr{Z}$, and $(I-D)$ maps $\mathscr{Z}$ onto $\mathscr{X}$ [48, p. 250]. The dissipative property implies that

$$\|(I-D)f\|^2 \geqq \|f\|^2 + \|Df\|^2 \tag{9.10}$$

and therefore $(I-D)$ is also injective. Hence, $(I-D)^{-1}: \mathscr{X} \to \mathscr{Z}$ is defined on all of $\mathscr{X}$, and, due to (9.10), $\|(I-D)^{-1}g\|_{\mathscr{Z}} \leqq \|g\|$, i.e. $(I-D)^{-1}$ is a bounded map. The first of inequalities (9.9) is precisely (9.10), whereas the second follows from the inequality $(a-b)^2 \leqq 2(a^2+b^2)$. □

We shall construct a shift realization much along the lines of infinite-dimensional deterministic realization theory [5], [6], [14], [15], [19]. Note, however, that, in comparison with this work, our set-up has been transposed. This is necessary in order to obtain the appropriate relation between observability (constructibility) of X and its standard forward (backward) realization, as we shall see below.

Let $f \in \mathscr{Z}$. Since $\mathscr{Z}$ is a *bona fide* function space, we can evaluate f at each point, and consequently, in view of (9.6),

$$f(t) = (e^{A^*t}f)(0). \tag{9.11}$$

Since $\mathscr{Z}$ is a subspace of the Sobolev space $H^1(\mathbb{R}^+)$; the evaluation operator is bounded [3], [16]. However, we want it defined on $\mathscr{X}$, and for this we need the operator $(I-D)$ of Lemma 9.1. Since A^* commutes with e^{A^*t}, then so does $(I-D)$. Therefore, (9.11) yields

$$f(t) = [(I-D)^{-1}\, e^{A^*t}(I-D)f](0). \tag{9.12}$$

Now, recalling that $(I-D)^{-1}$ maps $\mathscr{X}$ onto $\mathscr{Z}$ (Lemma 9.1),

$$B^*g = [(I-D)^{-1}g](0) \tag{9.13}$$

defines a bounded map $B^*: \mathscr{X} \to \mathbb{R}^p$. Let $B: \mathbb{R}^p \to \mathscr{X}$ be its adjoint. Then, (9.12) may be written

$$f(t)=B^*e^{A^*t}(I-D)f, \tag{9.14}$$

and therefore, if e_k is the kth unit axis vector in $\mathbb{R}^p$,

$$f_k(t)=\langle B^*e^{A^*t}(I-D)f, e_k\rangle_{\mathbb{R}^p}=\langle (I-D)f, e^{At}Be_k\rangle, \qquad k=1,2,\cdots,p. \tag{9.15}$$

Together, (9.4) and (9.15) yield, for each $\xi \in I_u\mathcal{Z}$, the representation

$$U_t\xi=\sum_{k=1}^{p}\int_{-\infty}^{t}\langle g, e^{A(t-\sigma)}Be_k\rangle du_k(\sigma) \tag{9.16}$$

where $g=(I-D)I_u^*\xi$. It can be shown that if the structural function K is analytic in some strip $-\alpha<\operatorname{Re}(s)\leqq 0$ of the complex plane, the integral

$$x(t)=\int_{-\infty}^{t} e^{A(t-\sigma)}B\,du(\sigma) \tag{9.17}$$

is well defined [32], and hence it defines an $\mathcal{X}$-valued *state process* $\{x(t); t\in\mathbb{R}\}$, i.e. a Hilbert-space-valued process with nuclear covariance operator [16]. If so, (9.16) can be written

$$U_t\xi=\langle g, x(t)\rangle. \tag{9.18}$$

If the integral (9.17) is not well defined, we can interpret the state process $\{x(t); t\in\mathbb{R}\}$ as a *generalized stochastic process* in the sense of [17], in which case (9.18) is merely shorthand for (9.16), rather than a *bona fide* inner product.

Note that, when g varies over $\mathcal{X}$, f ranges over $\mathcal{Z}$ (Lemma 9.1), and hence ξ ranges over $I_u\mathcal{Z}$ which is dense in X. Consequently,

$$X=\operatorname{cl}\{\langle g, x(0)\rangle \mid g\in\mathcal{X}\} \tag{9.19}$$

where cl stands for closure (in the topology of H). This should be compared with (8.17) in the finite-dimensional case, of which it is a generalization: recall that the state space $\mathbb{R}^n$ corresponds to $\mathcal{X}$ here.

It is important to note that we must take closure in (9.19). This means that processes with components of type $\{U_t\xi; t\in\mathbb{R}\}$ can be represented as outputs of a stochastic dynamical system with state process $\{x(t); t\in\mathbb{R}\}$ if and only if $\xi\in I_u\mathcal{Z}$, which is only a dense subset of X. Therefore, in particular, we must have

$$y_k(0)\in I_u\mathcal{Z} \quad \text{for } k=1,2,\cdots,m \tag{9.20}$$

in order to have y as an output. This condition can be characterized in the following ways.

PROPOSITION 9.1. *Let X be a proper Markovian splitting subspace with analytic spectral factor W. Let w denote the inverse Fourier transform of W and Γ the infinitesimal generator of $\{U_t(X); t\geqq 0\}$. Then, the following conditions are equivalent to* (9.20).

(i) *$y_k(0)\in\mathcal{D}(\Gamma)$ for $k=1,2,\cdots,m$.*

(ii) *The rows $w_1, w_2, \cdots, w_m$ of w belong to $\mathcal{Z}$.*

(iii) *The rows of $i\omega W(i\omega)-N$ belong to $\mathcal{H}_p^2$ for some constant $m\times p$ matrix N.*

Proof. First note that, by construction, $\mathcal{D}(\Gamma)=I_u\mathcal{Z}$, and therefore (9.20) and (i) are the same. Since $I_u=I_{\hat{u}}\mathcal{F}$, it follows from (6.12)+(6.17) that

$$w_k=I_u^*y_k(0) \quad \text{for } k=1,2,\cdots,m. \tag{9.21}$$

Hence the equivalence of (i) and (ii) is immediate; that of (ii) and (iii) follows from [23, Lemma 3.1]. □

If the conditions of Proposition 9.1 are satisfied, the inner products

$$(9.22)\qquad (Cg)_k = \langle (I-D)w_k, g\rangle, \qquad k=1,2,\cdots,m$$

are well defined, and, they define a bounded operator $C:\mathscr{X}\to\mathbb{R}^p$ such that

$$(9.23)\qquad y(t)=\int_{-\infty}^{t} C\,e^{A(t-\sigma)}B\,du(\sigma),$$

as can be seen from (9.16) and (9.21). If the integral (9.17) is well defined, this may be written

$$(9.24)\qquad y(t)=Cx(t);$$

otherwise we may interpret (9.24) in the generalized sense mentioned above, i.e. simply as (9.23). We shall call (9.23) the *standard forward realization* of X.

How natural are the conditions of Proposition 9.1? For any (forward) stochastic realization

$$(9.25)\qquad \begin{aligned} dx &= Ax\,dt + B\,du,\\ y &= Cx \end{aligned}$$

with x a strong solution, we must have

$$(9.26)\qquad E^{H^-(du)}a[y(h)-y(0)]=\int_0^h E^{H^-(du)}aCAx(t)\,dt$$

for any row vector $a\in\mathbb{R}^m$ and $h\geqq 0$. Using (5.3a), it is easy to see that this implies $\|[U_t(X)-I]y(0)\|\leqq kh$ and hence, as in [33], Condition (i) of Proposition 9.1, providing a justification for this condition. However, it should be noted that, even if (9.17) is well defined, it is not automatically true that x is a strong solution of the stochastic differential equation in (9.25) [8].

Next, we shall investigate the systems-theoretical properties of the realization (9.23). Let us begin with reachability. Recall that (9.23) is *reachable* if $\bigcap_{t\geqq 0}\ker B^*\,e^{A^*t}=0$ [14]. But, in view of (9.14), $B^*\,e^{A^*t}g=0$ for all $t\geqq 0$ if and only if $f:=(I-D)^{-1}g$ is identically zero, i.e. if and only if $g=0$. Hence, (9.23) is reachable.

The realization (9.23) is said to be *observable* if $\bigcap_{t\geqq 0}\ker C\,e^{At}=0$ [14]. To determine if this holds, form

$$(9.27)\qquad (C\,e^{At}g)_k=\langle (I-D)w_k, e^{At}g\rangle=\langle (I-D)\,e^{A^*t}w_k, g\rangle$$

where we have used the fact that D and e^{A^*t} commute. Define the vector space

$$(9.28)\qquad \mathcal{M}:=\operatorname{span}\{e^{A^*t}w_k;\ t\geqq 0,\ k=1,2,\cdots,m\}.$$

From (9.27) it follows then that (9.23) is observable if and only if $(I-D)\mathcal{M}$ is dense in $\mathscr{X}$. However, in view of (9.9), this is equal to $\mathcal{M}$ being dense in $\mathscr{Z}$ (in $\mathscr{Z}$ topology).

On the other hand, $X\sim(S,\bar{S})$ is an observable splitting subspace if and only if the vector space

$$(9.29)\qquad M:=\operatorname{span}\{E^S y_k(t);\ t\geqq 0,\ k=1,2,\cdots,m\}$$

is dense in X (Corollary 4.3). In view of Theorem 5.2, $E^S y_k(t)=U_t(X)y_k(0)$, and therefore, by (9.1) and (9.21), $M=I_u\mathcal{M}$.

Now, suppose that (9.23) is observable. Then $\mathcal{M}$ is dense in $\mathscr{Z}$ and hence in $\mathscr{X}$ (weaker topology). Consequently, M is dense in X, i.e. X is observable. Next, suppose that X is observable. Then M is dense in X, and hence $\mathcal{M}$ is dense in $\mathscr{X}$. Therefore, we have the situation

(9.30) $$\mathcal{M} \subset \mathcal{Z} \subset \mathcal{X}$$

where the vector space $\mathcal{M}$ is dense in the Hilbert space $\mathcal{X}$. Since the topology of $\mathcal{Z}$ is stronger than that of $\mathcal{X}$, (9.30) does not automatically imply that $\mathcal{M}$ is dense in $\mathcal{Z}$ as required; $\mathcal{Z}$ is said to be *normal* if this favorable situation occurs [3, p. 101]. However, it can be shown that the dissipative property of $\{e^{A^*t}; t \geqq 0\}$ implies that $\mathcal{Z}$ is normal [32]. Consequently, the realization (9.23) is observable if and only if X is observable.

We collect these observations in the following theorem.

THEOREM 9.1. [32]. *Let X be a proper Markovian splitting subspace with forward generating process u, and let $\mathcal{X} := I_u X$. Then*

(9.31) $$X = \text{cl}\left\{ \sum_{k=1}^{p} \int_{-\infty}^{t} \langle g, e^{A(t-\sigma)} Be_k \rangle \, du_k(\sigma) | g \in \mathcal{X} \right\}$$

where $\{e^{At}; t \geqq 0\}$ is the strongly continuous contraction semigroup on $\mathcal{X}$ defined by (9.1) *and $B: \mathbb{R}^p \to \mathcal{X}$ is the adjoint of* (9.13). *If the structural function of X is analytic in some strip $-\alpha < \text{Re}(s) \leqq 0$ of the complex plane, the integral*

(9.32) $$x(t) = \int_{-\infty}^{t} e^{A(t-\sigma)} B \, du(\sigma)$$

is well defined and defines an $\mathcal{X}$-valued random process $\{x(t); t \in \mathbb{R}\}$ in terms of which (9.31) *can be written*

(9.33) $$X = \text{cl}\{\langle g, x(0)\rangle | g \in \mathcal{X}\}.$$

If the conditions of Proposition 9.1 *are satisfied, there is a map $C: \mathcal{X} \to \mathbb{R}^m$, defined by* (9.22), *such that*

(9.34) $$y(t) = \int_{-\infty}^{t} C e^{A(t-\sigma)} B \, du(\sigma)$$

which, in the case that (9.32) *is well defined, yields*

(9.35) $$y(t) = Cx(t).$$

This is a reachable forward realization which is observable if and only if X is observable.

The construction of the corresponding backward realization is analogous, exchanging $\mathbb{R}^+$ for $\mathbb{R}^-$ everywhere. Let $\bar{\mathcal{Z}}$ be $\mathcal{D}(\bar{A}^*)$ equipped with graph topology, and let $\bar{D}: \bar{\mathcal{Z}} \to \bar{\mathcal{X}}$ be the (bounded) differentiation operator on $\bar{\mathcal{Z}}$. Let $\langle \cdot, \cdot \rangle$ denote the inner product in $\bar{\mathcal{X}}$. Then, we can proceed as above to obtain, for each $\xi \in I_{\bar{u}} \bar{\mathcal{Z}}$, the representation

(9.36) $$U_t \xi = \sum_{k=1}^{p} \int_{t}^{\infty} \langle g, e^{\bar{A}(\sigma-t)} \bar{B} e_k \rangle \, d\bar{u}_k(\sigma)$$

where $g = (I - \bar{D})^{-1} I_{\bar{u}}^* \xi$, and $\bar{B}: \mathbb{R}^p \to \bar{\mathcal{X}}$ is the adjoint of

(9.37) $$\bar{B}^* g = [(I - \bar{D})^{-1} g](0).$$

Now, if one of the three equivalent conditions

(i) $y_k(0) \in \mathcal{D}(\Gamma^*)$, $k = 1, 2, \cdots, m$,

(9.38) (ii) $\bar{w}_k := I_{\bar{u}}^* y_k(0) \in \bar{\mathcal{Z}}$, $k = 1, 2, \cdots, m$,

(iii) the rows of $i\omega \bar{W}(i\omega) - \bar{N}$ belong to $\mathcal{H}_p^2$ for some constant $m \times p$ matrix $\bar{N}$

hold, we may define a bounded linear operator $\bar{C}: \bar{\mathscr{X}} \to \mathbb{R}^m$ by the relations

$$(\bar{C}g)_k = \langle (I - \bar{D})\bar{w}_k, g \rangle, \qquad k = 1, 2, \cdots, m \tag{9.39}$$

and then we have the *standard backward realization*

$$y(t) = \int_t^\infty \bar{C} e^{\bar{A}(\sigma - t)} \bar{B} \, d\bar{u}(\sigma). \tag{9.40}$$

Following the convention set up in §§ 1 and 8, we shall say that (9.40) is *controllable* if $\bigcap_{t \geqq 0} \ker \bar{B}^* e^{\bar{A}^* t} = 0$ and *constructible* if $\bigcap_{t \geqq 0} \ker \bar{C} e^{\bar{A} t} = 0$. It is then easy to check that Theorem 9.1 has the following "backward" version.

THEOREM 9.2. [32]. *Let X be a proper Markovian splitting subspace with backward generating process $\bar{u}$. Set $\bar{\mathscr{X}} := I_{\bar{u}} X$. Then*

$$X = \mathrm{cl} \left\{ \sum_{k=1}^p \int_t^\infty \langle g, e^{\bar{A}(\sigma - t)} \bar{B} e_k \rangle \, du_k(\sigma) | g \in \bar{\mathscr{X}} \right\} \tag{9.41}$$

where $\{e^{\bar{A}t}; t \geqq 0\}$ is the strongly continuous contraction semigroup (9.2) *on $\bar{\mathscr{X}}$, and $\bar{B}: \mathbb{R}^p \to \bar{\mathscr{X}}$ is the adjoint of* (9.37). *If the structural function of X is analytic in some strip $-\alpha < \mathrm{Re}(s) \leqq 0$ of the complex plane, there is an $\bar{\mathscr{X}}$-valued random process $\{\bar{x}(t); t \in \mathbb{R}\}$ defined by*

$$\bar{x}(t) = \int_t^\infty e^{\bar{A}(\sigma - t)} \bar{B} \, d\bar{u}(\sigma) \tag{9.42}$$

so that (9.41) *may be written*

$$X = \mathrm{cl} \{ \langle g, \bar{x}(0) \rangle | g \in \bar{\mathscr{X}} \}. \tag{9.43}$$

Moreover, if the conditions (9.38) *hold, there is a map $\bar{C}: \bar{\mathscr{X}} \to \mathbb{R}^p$, defined by* (9.39), *such that* (9.40) *holds, and hence, if* (9.42) *is well defined,*

$$y(t) = \bar{C}\bar{x}(t). \tag{9.44}$$

This is a controllable backward realization which is constructible if and only if X is constructible.

Consequently, for X to have both a forward and backward realization we must have

$$y_k(0) \in \mathscr{D}(\Gamma) \cap \mathscr{D}(\Gamma^*), \qquad k = 1, 2, \cdots, m. \tag{9.45}$$

Questions of this sort are studied in [33].

10. State space isomorphism. There is an important difference between stochastic and deterministic realization theory which manifests itself already in the finite-dimensional case. In the deterministic theory, there is an essentially unique minimal realization (modulo trivial coordinate transformations). This is not the case in the stochastic theory. Two different minimal Markovian splitting subspaces give rise to realizations with probabilistically different state processes. Therefore, it is important to investigate the relationship between realizations of different minimal X.

In this section we shall study the class of standard forward realizations (9.23) of minimal Markovian splitting subspaces; the corresponding results for backward realizations are analogous and will not be mentioned. Our main goal is to clarify the connections between triplets (A, B, C) of such forward realizations. In the finite-dimensional case, this link is provided by the Yakubovic-Kalman-Popov or Positive Real Lemma, to which we shall return below.

For the rest of the paper we shall *assume that y is strictly noncyclic.* Then the class of minimal Markovian splitting subspaces can be parameterized by the left inner divisors Q of Q_+ (Theorem 7.6), and this parametrization, denoted $\{X_Q; Q|_L Q_+\}$, induces a lattice structure on the class under which $X_{Q_2} < X_{Q_1}$ if and only if $Q_2|_L Q_1$; see § 7.

Let $(K_1, Q_1, \bar{Q}_1^*)$ and $(K_2, Q_2, \bar{Q}_2^*)$ be the inner triplets of two minimal Markovian splitting subspaces, X_{Q_1} and X_{Q_2}. Then, it follows from (7.15) that

$$\bar{Q}_2^* \bar{Q}_1 K_1 = K_2 Q_2^* Q_1. \tag{10.1}$$

LEMMA 10.1. *The following statements are equivalent.*

(i) $X_{Q_2} < X_{Q_1}$.

(ii) $V_1 := Q_2^* Q_1$ *is inner.*

(iii) $V_2 := \bar{Q}_2^* \bar{Q}_1$ *is inner.*

If these conditions are satisfied, then

$$K_1 V_1^* = V_2^* K_2 \tag{10.2}$$

with K_1 *and* V_1 *right coprime and* K_2 *and* V_2 *left coprime.*

Proof. Let $X_{Q_1} \sim (S_1, \bar{S}_1)$ and $X_{Q_2} \sim (S_2, \bar{S}_2)$. Then, by Lemma 6.1, (ii) is equivalent to

$$S_2 \subset S_1 \tag{10.3}$$

and (iii) is equivalent to $\bar{S}_2^\perp \subset \bar{S}_1^\perp$, i.e.

$$\bar{S}_2 \supset \bar{S}_1. \tag{10.4}$$

But, since X_{Q_1} and X_{Q_2} are minimal, (10.3) and (10.4) are equivalent (Corollary 3.3), establishing the equivalence of (ii) and (iii). Now, $Q_1 = Q_2 V_1$. Hence (i) and (ii) are equivalent, and, since K_1 and Q_1 are right coprime (Theorem 7.2 and Corollary 4.1), then so are K_1 and V_1. Likewise, since $\bar{Q}_2^* = V_2 \bar{Q}_1^*$, the left coprimeness of K_2 and V_2 follows from that of K_2 and $\bar{Q}_2^*$ (Theorem 7.2 and Corollary 4.1). Relation (10.2) is the same as (10.1). □

The following theorem describes the intertwining of the triplets (A_1, B_1, C_1) and (A_2, B_2, C_2) corresponding to two minimal Markovian splitting subspaces, X_1 and X_2, which are ordered.

THEOREM 10.1. *Let* X_1 *and* X_2 *be two minimal Markovian splitting subspaces such that* $X_2 < X_1$, *and let* Σ_1 *and* Σ_2 *be the corresponding standard forward realizations with state spaces* $\mathscr{X}_1$ *and* $\mathscr{X}_2$. *Then the map* $R: \mathscr{X}_1 \to \mathscr{X}_2$ *defined by*

$$Rf = P^{\mathscr{X}_2} \mathscr{F} M_{Q_1^* Q_2} \mathscr{F}^* f \tag{10.5}$$

is injective with dense range, and the following diagram commutes,

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "B_1"] & \mathscr{X}_1 \arrow[r, "e^{A_1 t}"] \arrow[d, "R"] & \mathscr{X}_1 \arrow[d, "R"] \arrow[dr] & \\ & & & \mathbb{R}^m \\ \mathbb{R}^p \arrow[r, "B_2"] & \mathscr{X}_2 \arrow[r, "e^{A_2 t}"] & \mathscr{X}_2 \arrow[ur] & \end{tikzcd} \tag{10.6}$$

where indices refer to Σ_1 *and* Σ_2.

Proof. Let K_1 and K_2 be the structural functions of X_1 and X_2, and let $\Sigma_t(K_i): \mathscr{H}(K_i) \to \mathscr{H}(K_i)$ be the restricted shifts

$$\Sigma_t(K_i) f = P^{\mathscr{H}(K_i)} \chi_t f \tag{10.7}$$

for $t \geqq 0$ and $i=1,2$. Since there are inner functions V_1 and V_2 such that $V_2 K_1 = K_2 V_1$ (Lemma 10.1), there is a map $\hat{R}^*: \mathscr{H}(K_2) \to \mathscr{H}(K_1)$ such that

$$\hat{R}^* \Sigma_t(K_2) = \Sigma_t(K_1) \hat{R}^* \tag{10.8}$$

[14, Thm. 14.8, p. 203]. This map is given by

$$\hat{R}^* f = P^{\mathscr{H}(K_1)} M_{Q_2^* Q_1} f \tag{10.9}$$

and, in view of the coprimeness conditions of Lemma 10.1, $\hat{R}^*$ is injective with dense range [14, Thm. 14.11, p. 206]. Therefore the same is true for the adjoint $\hat{R}: \mathscr{H}(K_1) \to \mathscr{H}(K_2)$ and for $R := \mathscr{F}^* \hat{R} \mathscr{F}: \mathscr{X}_1 \to \mathscr{X}_2$, which is the map of the theorem. It remains to show that the diagram commutes. To this end, first note that $e^{A_i^* t} = \mathscr{F}^* \Sigma_t(K_i) \mathscr{F}$ for $i = 1, 2$, and therefore (10.8) is equivalent to

$$R\, e^{A_1 t} = e^{A_2 t} R. \tag{10.10}$$

Then the same intertwining must hold for the resolvents, i.e. in particular

$$R(I - A_1)^{-1} = (I - A_2)^{-1} R \tag{10.11}$$

(Lemma 9.1), and therefore

$$(I - A_1^*) R^* = R^* (I - A_2^*). \tag{10.12}$$

Now, if W_1 and W_2 are the analytic spectral factors of X_1 and X_2, then $W_1 = W_2 Q_2^* Q_1$. But, in view of (8.5), $a W_1 \in \mathscr{H}(K_1)$ for any row vector $a \in \mathbb{R}^m$, and hence $a W_1 = \hat{R}^* a W_2$. Consequently

$$a w_1 = R^* a w_2 \tag{10.13}$$

where $w_1 := \mathscr{F}^* W_1$ and $w_2 := \mathscr{F}^* W_2$. Now, from the definition (9.22) it is easy to see that

$$C_i^* a = (I - A_i^*) a w_i \tag{10.14}$$

for $i = 1, 2$. (Recall that $A^* f = Df$.) Consequently, in view of (10.12) and (10.13), $C_1^* = R^* C_2^*$, i.e.

$$C_1 = C_2 R. \tag{10.15}$$

This together with (10.10) proves that the diagram commutes. □

The parts of diagram (10.6) involving B_1 and B_2 add nothing to the theorem but have been added to remind the reader that the two horizontal chains of arrows realize different functions, namely w_1 and w_2. This situation differs of course from that in the deterministic "state space isomorphism" theorems [22, p. 258].

A map which is injective with dense range such as R in Theorem 10.1 will be called *quasi-invertible.* In the finite-dimensional case, this is the same as invertible, and therefore, in this case, the condition $X_2 < X_1$ of Theorem 10.1 is unnecessary, for we have also a diagram with the arrows reversed. In particular, the semigroups $\{e^{A_1 t}; t \geqq 0\}$ and $\{e^{A_2 t}; t \geqq 0\}$ are then similar.

In the infinite-dimensional situation, a natural generalization of similarity is quasisimilarity. We say that the semigroups $\{e^{A_1 t}; t \geqq 0\}$ and $\{e^{A_2 t}; t \geqq 0\}$ are *quasisimilar* if there are quasi-invertible maps $R_1: \mathscr{X}_1 \to \mathscr{X}_2$ and $R_2: \mathscr{X}_2 \to \mathscr{X}_1$ such that

$$\begin{aligned} R_1 e^{A_1 t} &= e^{A_2 t} R_1, \\ R_2\, e^{A_2 t} &= e^{A_1 t} R_2. \end{aligned} \tag{10.16}$$

Only the first of relations (10.16) is given by Theorem 10.1, and then only if $X_2 < X_1$. If we also had the other, the ordering assumption would be unnecessary also in the infinite-dimensional case, for quasisimilarity is an equivalence relation [14, p. 74]. That this favorable situation actually happens follows from the next theorem, the proof of which can be found in [31].

THEOREM 10.2. [31]. *Let Σ_1 and Σ_2 be the forward standard realizations corresponding to two arbitrary minimal Markovian splitting subspaces. Then the corresponding semigroups $\{e^{A_1 t}; t \geqq 0\}$ and $\{e^{A_2 t}; t \geqq 0\}$ are quasisimilar, i.e. they satisfy* (10.16).

This implies that, as far as the rectangular part of the diagram (10.6) is concerned, the ordering condition $X_2 < X_1$ can be dispensed with. Whether this is true for the diagram as a whole is as yet an open question.

By [14, Thm. 15.18, p. 220], the semigroups are quasisimilar if and only if the corresponding structural functions K_1 and K_2 are quasi-equivalent, i.e. have the same invariant factors, and therefore Theorem 10.2 is equivalent to Theorem 7.5. This allows us to draw the conclusion that the infinitesimal generators A corresponding to minimal Markovian splitting subspaces have the same eigenvalues. To see this, just note that these eigenvalues are the poles of the common determinant of the structural functions [23, Thm. 3.2, p. 70], [14, Thm. 13.8, p. 195].

Theorem 10.2 can also be stated in terms of Jordan models. For a discussion of this concept, see, for example, [14, p. 214].

COROLLARY 10.1. [31]. *All semigroups $\{e^{At}; t \geqq 0\}$ corresponding to minimal Markovian splitting subspaces have the same Jordan model, i.e. they are all quasisimilar to the direct sum*

$$\Sigma_t(k_1) \oplus \Sigma_t(k_2) \oplus \cdots \oplus \Sigma_t(k_p) \tag{10.17}$$

where $k_1, k_2, \cdots, k_p$ are the common invariant factors of the structural functions, and the restricted shifts $\Sigma_t(k_i)$, $i = 1, 2, \cdots, p$ and $t \geqq 0$, are defined as in (10.7) *but for a scalar Hardy space.*

As an application of Theorem 10.1, we shall next derive an infinite-dimensional version of the Positive Real Lemma equations. For this we shall need the following two lemmas.

LEMMA 10.2. *Let A and B be defined by* (9.1) *and* (9.13). *Then*

$$AP + PA^* + BB^* = 0 \tag{10.18}$$

where $P: \mathscr{X} \to \mathscr{X}$ is the positive self-adjoint operator

$$P = (I - A)^{-1}(I - A^*)^{-1}. \tag{10.19}$$

Proof. Let $f_i \in \mathscr{X}$, $i = 1, 2$. Then, recalling that $A^* f = Df$ for $f \in \mathscr{X}$, where D is the differentiation operator, integration by parts yields

$$\langle A^* f_1, f_2 \rangle + \langle f_1, A^* f_2 \rangle = \int_0^\infty (\dot{f}_1 f_2' + f_1 \dot{f}_2') \, dt = -f_1(0) f_2(0)'. \tag{10.20}$$

Also, by the definition (9.13),

$$\begin{aligned} \langle (I - A^*) f_1, BB^*(I - A^*) f_2 \rangle &= \langle B^*(I - A^*) f_1, B^*(I - A^*) f_2 \rangle_{\mathbb{R}^p} \\ &= f_1(0) f_2(0)'. \end{aligned} \tag{10.21}$$

Now, let $g_i \in \mathscr{X}$, $i = 1, 2$, be arbitrary. Then, by Lemma 9.1, $f_i := (I - A^*)^{-1} g_i \in \mathscr{X}$ for $i = 1, 2$. Inserting this into (10.20) and (10.21) and adding the relations, we obtain

$$\langle g_1, APg_2 \rangle + \langle g_1, PA^* g_2 \rangle + \langle g_1, BB^* g_2 \rangle = 0 \tag{10.22}$$

where we have used the fact that A^* and $(I - A^*)^{-1}$ commute. This yields (10.18). □

The operator P is actually the *state covariance operator* in the sense that

$$E\{\langle g_1, x(0)\rangle\langle g_2, x(0)\rangle\} = \langle g_1, Pg_2\rangle. \tag{10.23}$$

To see this, note that, by (9.4) and (9.18),

$$\langle g, x(0)\rangle = \int_{-\infty}^{0} [(I-A^*)^{-1}g](-\sigma)\,du(\sigma) \tag{10.24}$$

where, in general, the left member should be understood in the sense of (9.16). In passing, we recall that the state process $\{x(t);\, t\in\mathbb{R}\}$ is a *bona fide* $\mathscr{X}$-valued random process if and only if the operator P is nuclear [16].

LEMMA 10.3. *Let* $\Lambda:\mathbb{R}^+\to\mathbb{R}^{m\times m}$ *be the covariance*

$$\Lambda(t) = E\{y(t)y(0)'\}, \qquad t \geqq 0 \tag{10.25}$$

and let A and C be defined by (9.1) *and* (9.22). *Then*

$$\Lambda(t) = C\,e^{At}PC^* \tag{10.26}$$

where P is the state covariance operator (10.19).

Proof. Since $C^*a = (I-A^*)aw$ for any row vector $a\in\mathbb{R}^m$, and $(I-A)^{-1}$ commutes with e^{At}, we have

$$C\,e^{At}PC^*a = C(I-A)^{-1}\,e^{At}aw, \tag{10.27}$$

and therefore

$$[C\,e^{At}PC^*]_{kj} = \langle (I-A^*)w_k, (I-A)^{-1}\,e^{At}w_j\rangle_{\mathscr{X}} = \langle w_k, e^{At}w_j\rangle_{\mathscr{X}}. \tag{10.28}$$

But $\mathscr{F}^*\,e^{At}\mathscr{F} = \Sigma_t(K)^*$ and $W_k = \mathscr{F}w_k$. Hence

$$[C\,e^{At}PC^*]_{kj} = \langle W_k, P^{\mathscr{H}(K)}e^{-i\omega t}\,W_j\rangle_{\mathscr{H}(K)} = \langle W_k, e^{-i\omega t}\,W_j\rangle_{\mathscr{L}^2_p(\mathbb{I})} \tag{10.29}$$

because, by (8.5), $W_k\in\mathscr{H}(K)$. Consequently, (10.26) follows from the Bochner representation

$$\Lambda(t) = \int_{-\infty}^{\infty} e^{i\omega t}\Phi(i\omega)\,d\omega. \tag{10.30}$$

□

To compare the standard forward realizations of different minimal Markovian splitting subspaces, we must reduce them to the same state space. In view of the ordering condition of Theorem 10.1, the most suitable choice of common state space is $\mathscr{X}_I$, the state space of the minimal element $X_I := H^{+/-}$ of the lattice. Given the standard forward realization Σ_Q of an arbitrary minimal Markovian splitting subspace X_Q, the reduction will be according to the diagram

$$\begin{tikzcd} & \mathscr{X}_Q \arrow[r, "e^{A_Q t}"] \arrow[d, "R_Q"] & \mathscr{X}_Q \arrow[d, "R_Q"] \arrow[dr, "C_Q"] & \\ \mathbb{R}^p \arrow[ur, "B_Q"] \arrow[dr, "B"'] & & & \mathbb{R}^m \\ & \mathscr{X}_I \arrow[r, "e^{At}"] & \mathscr{X}_I \arrow[ur, "C"'] & \end{tikzcd} \tag{10.31}$$

defining a new (A, B, C) for X_Q which has state space $\mathscr{X}_I$. Here R_Q is the map (10.5) with $Q_1 := Q$ and $Q_2 := I$. Then, when X_Q varies over the lattice of minimal Markovian splitting subspaces, $A := A_I$ and $C := C_I$ are fixed, whereas B varies.

THEOREM 10.2. *Let X_Q be an arbitrary minimal Markovian splitting subspace, and let (A, B, C) be defined by* (10.31). *Then*

$$\begin{aligned} &AP+PA^*+BB^*=0,\\ &PC^*=G \end{aligned} \tag{10.32}$$

where the positive self-adjoint operator $P: \mathscr{X}_I \to \mathscr{X}_I$, defined by

$$P := (I-A)^{-1}R_Q R_Q^*(I-A^*)^{-1} \tag{10.33}$$

is the state covariance operator in the fixed state space representation, and $G: \mathbb{R}^m \to \mathscr{X}$ is given by

$$G := (I-A)^{-1}(I-A^*)^{-1}C^*. \tag{10.34}$$

Proof. By Lemma 10.2, $A_Q P_Q + P_Q A_Q^* + B_Q B_Q^* = 0$, where $P_Q := (I-A_Q)^{-1} \times (I-A_Q^*)^{-1}$. Transforming this via (10.31) and (10.11) yields the first of relations (10.32). To derive the second relation (10.32), reduce the representation (10.26) to the fixed state space $\mathscr{X}_I$. Comparing the expressions for $\Lambda(t)$ thus obtained corresponding to Σ_Q and Σ_I respectively, we have

$$C\,e^{At}[PC^* - P_I C^*] = 0 \quad \text{for all } t \geqq 0. \tag{10.35}$$

Since Σ_I is observable (Theorem 9.1), this implies that $PC^* = P_I C^*$, which is precisely G. □

We have thus shown that all standard forward realizations $\{\Sigma_Q;\, Q|_L Q_+\}$ reduced to the common fixed state space $\mathscr{X}_I$ satisfy equations akin to those of the Positive Real Lemma [2], [11], [12]. Note, however, that in our case the representation is coordinate-free.

REFERENCES

[1] H. AKAIKE, *Markovian representation of stochastic processes by canonical variables*, SIAM J. Control, 13 (1975), pp. 165–173.

[2] B. D. O. ANDERSON, *The inverse problem of stationary covariance generation*, J. Statist. Phys., 1 (1969), pp. 133–147.

[3] J.-P. AUBIN, *Applied Functional Analysis*, Wiley-Interscience, New York, 1979.

[4] F. BADAWI, A. LINDQUIST AND M. PAVON, *A stochastic realization approach to the smoothing problem*, IEEE Trans. Automat. Control, 24 (1979), pp. 878–888.

[5] J. S. BARAS AND R. W. BROCKETT, *H^2-functions and infinite-dimensional realization theory*, SIAM J. Control, 13 (1975), pp. 221–241.

[6] J. S. BARAS AND P. DEWILDE, *Invariant subspaces methods in linear multivariable distributed systems and lumped distributed network synthesis*, Proc. IEEE, 64 (1976), pp. 160–178.

[7] A. BENSOUSSAN, *Filtrage optimal des systèmes linéaires*, Dunod, Paris, 1971.

[8] R. F. CURTAIN AND A. J. PRITCHARD, *Infinite Dimensional Linear Systems*, Springer-Verlag, Berlin, 1978.

[9] J. L. DOOB, *Stochastic Processes*, John Wiley, New York, 1953.

[10] H. DYM AND H. P. MCKEAN, *Gaussian Processes, Function Theory, and the Inverse Spectral Problem*, Academic Press, New York, 1976.

[11] P. FAURRE, *Réalisations markoviennes de processus stationnaires*, Research Report No. 13, March 1973, IRIA Laboria, Le Chesney, France.

[12] P. FAURRE, M. CLERGET AND F. GERMAIN, *Opérateurs rationnels positifs*, Dunod, Paris, 1979.

[13] G. D. FORNEY, JR., *Minimal bases of rational vector spaces, with applications to multivariable linear systems*, SIAM J. Control, 13 (1975), pp. 453–520.

[14] P. A. FUHRMANN, *Linear Operators and Systems in Hilbert Space*, McGraw-Hill, New York, 1981.

[15] ———, *On realization of linear systems and applications to some questions in stability*, Math. Systems Theory, 8 (1974), pp. 132–141.

[16] I. M. GELFAND AND N. YA. VILENKIN, *Generalized Functions*, Vol. 4, Academic Press, New York, 1964.

[17] I. GIKHMAN AND A. V. SKOROKHOD, *Introduction to the Theory of Random Processes*, Saunders, Philadelphia, 1965.

[18] H. HELSON, *Lectures on Invariant Subspaces*, Academic Press, New York, 1976.

[19] J. W. HELTON, *Systems with infinite-dimensional state space: The Hilbert space approach*, Proc. IEEE, 64 (1976), pp. 145-160.

[20] T. KAILATH, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[21] R. E. KALMAN, *Linear stochastic filtering-reappraisal and outlook*, Proc. Symp. System Theory, Polytechnic Institute of Brooklyn, 1965, pp. 197-205.

[22] R. E. KALMAN, P. L. FALB AND M. A. ARBIB, *Topics in Mathematical System Theory*, McGraw-Hill, New York, 1979.

[23] P. D. LAX AND R. S. PHILLIPS, *Scattering Theory*, Academic Press, New York, 1967.

[24] A. LINDQUIST, M. PAVON AND G. PICCI, *Recent trends in stochastic realization theory*, in Prediction Theory and Harmonic Analysis: The Pesi Masani Volume, V. Mandrekar and H. Salehi, eds., North-Holland, Amsterdam, 1983.

[25] A. LINDQUIST AND G. PICCI, *On the stochastic realization problem*, SIAM J. Control Optim., 17 (1979), pp. 365-389.

[26] ———, *A state-space theory for stationary stochastic processes*, Proc. 21st Midwest Symposium on Circuits and Systems, Ames, IO, August, 1978.

[27] ———, *A Hardy space approach to the stochastic realization problem*, Proc. 1978 Conference on Decision and Control, San Diego, pp. 933-939.

[28] ———, *Realization theory for multivariate stationary gaussian processes* I: *State space construction*, Proc. 4th International Symposium on the Mathematical Theory of Networks and Systems, July 1979, Delft, Holland, pp. 140-148.

[29] ———, *Realization theory for multivariate stationary gaussian processes* II: *State space theory revisited and dynamical representations of finite dimensional state spaces*, in Proc. 2nd International Conference on Information Sciences and Systems, Patras, Greece, July 1979, Reidel, Dordrecht, pp. 108-129.

[30] ———, *State space models for gaussian stochastic processes*, in Stochastic Systems: The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J. C. Willems, eds., Reidel, Dordrecht, 1981.

[31] ———, *On a condition for minimality of Markovian splitting subspaces*, Systems Control Lett., 1 (1982), pp. 264-269.

[32] ———, *Infinite dimensional stochastic realizations of continuous-time stationary vector processes*, Topics in Operators and Systems, H. Dym and I. Gohberg, eds., Birkhäuser Verlag, Berlin, 1984.

[33] ———, *Forward and backward semimartingale representations for stationary increment processes*, Stochastics, to appear.

[34] H. P. MCKEAN, *Brownian motion with several dimensional time*, Theory Prob. Appl., VIII (1963), pp. 353-354.

[35] J. NEVEU, *Processus aléatoires gaussiens*, Les Presses de l'Université de Montréal, Montreal, 1968.

[36] M. PAVON, *Stochastic realization and invariant directions of the matrix Riccati equation*, SIAM J. Control Optim., 18 (1980), pp. 155-180.

[37] G. PICCI, *Stochastic realization of Gaussian processes*, Proc. IEEE, 64 (1976), pp. 112-122.

[38] YU. A. ROZANOV, *Stationary Random Processes*, Holden-Day, New York, 1967.

[39] G. RUCKEBUSCH, *Répresentations markoviennes de processus gaussiens stationnaires*, Ph.D. thesis, Univ. Paris VI, 1975.

[40] ———, *Répresentations markoviennes de processus gaussiens stationnaires*, C.R. Acad. Sc. Paris, Ser. A, 282 (1976), pp. 649-651.

[41] ———, *Factorisations minimales de densités spectrales et répresentations markoviennes*, Proc. 1re Colloque AFCET-SMF, Palaiseau, France, 1978.

[42] ———, *A state space approach to the stochastic realization problem*, Proc. 1978 International Symposium on Circuits and Systems, New York.

[43] ———, *Théorie géométrique de la répresentation markovienne*, Thèse de doctorat d'etat, Univ. Paris VI, 1980.

[44] ———, *On the structure of minimal Markovian representations*, Nonlinear Stochastic Problems, R. Bucy and J. M. F. Moura, eds., Reidel, Dordrecht, 1983.

[45] B. SZ.-NAGY AND C. FOIAS, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970.

[46] S. MACLANE AND G. BIRKHOFF, *Algebra*, Macmillan, New York, 1967.

[47] W. A. WOLOVICH, *Linear Multivariate Systems*, Springer-Verlag, Berlin, 1974.

[48] K. YOSIDA, *Functional Analysis*, Springer-Verlag, Berlin, 1965.
[49] F. BADAWI, A. LINDQUIST AND M. PAVON, *A stochastic realization approach to the smoothing problem*, IEEE Trans. Automat. Control, 24 (1979), pp. 878-888.
[50] A. V. BALAKRISHNAN, *Applied Functional Analysis*, Springer-Verlag, Berlin, 1976.
[51] M. PAVON, *New results on the interpolation problem for continuous-time stationary increment processes*, SIAM J Control Optim., 22 (1984), pp. 133-142.
[52] J. C. WILLEMS, *System theoretic models for the analysis of physical systems*, Ricerche di Automatica, 10 (1979), pp. 71-104.