## Acausal Models and Balanced Realizations of Stationary Processes

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#### Abstract

We study acausal realizations of stationary (or stationary-increment) processes. In particular, we characterize the family of models whose corresponding spectral factors have a fixed zero structure. Acausal models with a fixed zero structure are related to each other by a certain group of state-feedback transformations which is naturally parametrized by the solution set of a homogeneous algebraic Riccati equation. Each feedback transformation reflects some of the eigenvalues of the generator matrix $A$ of the representation to a mirror image with respect to the imaginary axis. Dually, acausal models with a fixed "pole structure" are parametrized by a dual Riccati equation and by a corresponding family of output injection transformations. From a general standpoint the results of this study clarify the role played by dual pairs of Riccati equations in spectral factorization and may be relevant to other problem areas than stochastic modeling. One natural application of the concepts discussed in the paper is to stochastic balancing. Balancing of models with an essentially arbitrary eigenvalue location can be accommodated very naturally in this framework. A balancing algorithm involving the solution of a dual pair of Riccati equations is discussed.


## 1. INTRODUCTION

It is well known that every $n$-dimensional second order (wide-sense) stationary Markov process can be represented as the solution of a stochastic differential equation of the form

$$
\begin{equation*}
d x(t)=A x(t) d t+B d w(t), \quad t \in \mathbb{R} \tag{1.1}
\end{equation*}
$$

where $d w$ is a vector (wide-sense) Wiener process, say of dimension $p$, and the matrices $A, B$ have appropriate dimensions $[8,9]$.

From standard spectral representation theory (see, e.g., [29, Chapter 1, Example 8.4]), it follows that Equation (1.1) admits stationary solutions if and only if the rows of the $n \times p$ matrix $(i \omega I-A)^{-1} B$ are square-integrable functions of $i \omega$. This is equivalent to the absence of poles of the function $s \rightarrow(s I-A)^{-1} B$ on the imaginary axis. Equivalently, the purely imaginary eigenvalues of $A$, if any, must be "uncontrollable" for the pair ( $A, B$ ).

When eigenvalues on the imaginary axis are present, it is well known that they must be simple roots of the minimal polynomial of $A$. These eigenvalues, necessarily even in number (say $2 k$ ), then give rise to a sum of $k$ uncorrelated sinusoidal oscillations with random amplitude, the so-called purely deterministic component of the process. This component of the stationary Markov process $x$ obeys a fixed undriven (i.e., deterministic) linear differential equation of the type $\dot{x}=A_{0} x$, which is the restriction of (1.1) to the purely imaginary eigenspace of $A$. The initial conditions are random variables independent of the driving noise $d w$.

For the purely deterministic dynamics of $x$ the statements of this paper either become empty or apply only in a very trivial sense. In order to avoid overburdening the exposition with uninteresting particularizations and repetitions we shall henceforth assume that the purely deterministic part of $x$ has been subtracted. This is really no loss of generality.

We shall then restrict ourselves to purely nondeterministic processes and to representations (1.1) for which
(1) $(A, B)$ is a controllable pair,
(2) $B$ is full column rank.

Such representations of $x$ will be called minimal. From the discussion above it is immediately seen that there are stationary solutions $x$ of the differential equation (1.1) (which are automatically purely nondeterministic Markov processes) if and only if $A$ does not have eigenvalues on the imaginary axis, i.e.,

$$
\begin{equation*}
\operatorname{Re} \lambda(A) \neq 0 \tag{1.2}
\end{equation*}
$$

When $\operatorname{Re} \lambda(A)<0$ [respectively, $\operatorname{Re} \lambda(A)>0$ ], the representation (1.1) is called forward or causal [backward or anticausal]. It is well known [18] how to compute a backward representation starting from a forward one or vice versa. The two representations (in particular the relative A-matrices) are related by a linear state-feedback transformation.

Traditionally the causality of a representation, i.e., the condition $\operatorname{Re} \lambda(A)<0$, has been regarded as being equivalent to stationarity. Instead, as we shall see, stationarity permits a whole family of representations (1.1) of the (same) process $x$, where $A$ can have quite an arbitrary spectrum.

In the next section we shall study and classify the family of all such minimal representations of a stationary p.n.d. Markov process $x$. It will be shown that the relation between two arbitrary representations is still a state-feedback transformation and that the feedback can be computed by solving a homogeneous Riccati equation.

Before entering into this, we need to say how the stochastic differential equation (1.1) has to be interpreted in the general situation where Re $\lambda(A) \neq 0$, but the eigenvalues may otherwise be arbitrarily located in the complex plane.

By decomposing $\mathbb{R}^{n}$ into a direct sum of the stable and antistable manifolds for $A$ and choosing a new basis accordingly, the model (1.1) can be decoupled in the form

$$
\begin{align*}
& d x_{-}(t)=\hat{A}_{-} x_{-}(t) d t+\hat{B}_{-} d w(t)  \tag{1.3a}\\
& d x_{+}(t)=\hat{A}_{+} x_{+}(t) d t+\hat{B}_{+} d w(t) \tag{1.3b}
\end{align*}
$$

with $\operatorname{Re} \lambda\left(\hat{A}_{-}\right)<0$ and $\operatorname{Re} \lambda\left(\hat{A}_{+}\right)>0$.
Now (1.3a) and (1.3b) are interpreted as forward and backward stochastic differential equations with stationary solutions

$$
\begin{align*}
& x_{-}(t)=\int_{-\infty}^{t} e^{\hat{A}_{-}(t-\tau)} \hat{B}_{-} d w(\tau)  \tag{1.4a}\\
& x_{+}(t)=-\int_{t}^{+\infty} e^{\hat{A}_{+}(t-\tau)} \hat{B}_{+} d w(\tau) \tag{1.4b}
\end{align*}
$$

More generally, we shall consider dynamical models of a purely nondeterministic stationary-increment process $\{y(t)\}, y(t) \in \mathbb{R}^{m}$, with a rational
incremental spectral density matrix $\Phi(s)$. It is well known (e.g., [21]) that any such process ${ }^{1}$ can be represented in the form

$$
\begin{align*}
& d x=A x d t+B d w  \tag{1.5a}\\
& d y=C x d t+D d w \tag{1.5b}
\end{align*}
$$

where $A, B, C, D$ arc real matrices of appropriate size and $d w$ is a $p$-dimensional wide-sense Wiener process defined on the real line. Models of the type (1.5) are called "stochastic realizations" of $d y$. Again, in the literature only forward or causal [backward or anticausal] realizations are considered where $\operatorname{Re} \lambda(A)<0[\operatorname{Re} \lambda(A)>0]$. Here we shall instead consider the problem of classifying all realizations of $d y$ with an arbitrary causality structure. In accordance with what has been explained above, we shall just assume that $\operatorname{Re} \lambda(A) \neq 0$. Clearly, if (1.5) is a stochastic realization of $d y$, the transfer function

$$
\begin{equation*}
W(s)=D+C(s I-A)^{-1} B \tag{1.6}
\end{equation*}
$$

is a "factor" of the (incremental) spectral density matrix $\Phi(s)$ of $d y$, i.e.,

$$
\begin{equation*}
W(s) W(-s)^{\prime}=\Phi(s) \tag{1.7}
\end{equation*}
$$

Any solution of (1.7) of minimal McMillan degree $\delta[W]$, i.e., such that $\delta[W]=\frac{1}{2} \delta[\Phi]$, is called a minimal spectral factor $[1]$.

From the spectral factorization equation (1.7) it follows that a minimal spectral factor is naturally defined only up to right multiplication by a constant orthogonal matrix. For this reason two input Wiener processes $d w_{1}$, $d w_{2}$ differing by left multiplication by a constant (say $p \times p$ ) orthogonal matrix will be identified.

Throughout this paper we shall always consider realizations for which
(1) $(A, B, C)$ is a minimal triplet,
(2) $\left[\begin{array}{l}B \\ D\end{array}\right]$ has independent columns.

This is no loss of generality [21].

[^0]With this convention, models (1.5) whose transfer function $W$ is a minimal spectral factor are minimal stochastic realizations of $d y$ in the sense that the state process $x$ has dimension equal to $\frac{1}{2} \delta[\Phi]$, i.e., the smallest possible dimension in the family of all realizations of the form (1.5) [1, 21].

### 1.1. Background on Geometric Theory

For the purposes of this paper, it will be natural to consider also coordinate-free representations of $d y$, by assigning to each model (1.5) the $n$-dimensional space

$$
\begin{equation*}
X=\left\{a^{\prime} x(0) \mid a \in \mathbb{R}^{n}\right\} \tag{1.8}
\end{equation*}
$$

of random variables. ${ }^{2}$ This space is a subspace of the ambient space $\mathbf{H}(d w)$ of the model (1.5), defined as the closure of the linear hull of the random variables $\left\{w_{i}(t)-w_{i}(\tau) ; i=1,2, \ldots, p ; t, \tau \in \mathbb{R}\right\}$ in the topology of the inner product $\langle\xi, \eta\rangle=E\{\xi \eta\}$, where $E\{\cdot\}$ stands for mathematical expectation. The ambient space $\mathbf{H}(d w)$ is naturally equipped with the shift $\left\{U_{t}\right.$; $t \in \mathbb{R}\}$ induced by $d w$, a unitary group of operators on $\mathbf{H}(d w)$ such that $U_{t}\left[w_{i}(\tau)-w_{i}(\sigma)\right]=w_{i}(\tau+t)-w_{i}(\sigma+t)$ for all $i=1,2, \ldots, p$ and $t, \tau, \sigma \in \mathbb{R}$. All random variables of the stochastic system (1.5) belong to $\mathbf{H}(d w)$, and moreover the processes $x$ and $d y$ are stationary with respect to $\left\{U_{t}\right\}$, i.e., $U_{t} x_{i}(\tau)=x_{i}(\tau+t)$ for all $i=1,2, \ldots, n$ and $t, \tau \in \mathbb{R}$, and $U_{t}\left[y_{i}(\tau)-y_{i}(\sigma)\right]=y_{i}(\tau+t)-y_{i}(\sigma+t)$ for all $i=1,2, \ldots, m$ and $t, \tau, \sigma$ $\in \mathbb{R}$.

Defining the past and future output spaces as

$$
\mathbf{H}^{-}=\operatorname{closure}\left\{a^{\prime}[y(t)-y(s)] \mid a \in \mathbb{R}^{m}, t, s \leqslant 0\right\}
$$

and

$$
\mathbf{H}^{+}=\operatorname{closure}\left\{a^{\prime}[y(t)-y(s)] \mid a \in \mathbb{R}^{m}, t, s \geqslant 0\right\}
$$

respectively, it is easy to show and well established in the literature [20, 21, 4] that each $X$, defined as in (1.8), is a minimal Markovian splitting subspace for $\mathbf{H}^{-}$and $\mathbf{H}^{+}$, i.e., in particular renders $\mathbf{H}^{-}$and $\mathbf{H}^{+}$conditionally orthogonal given $X$. Moreover this property captures the concept of the

[^1]stochastic state space model of $d y$ in a coordinate-free way. A fundamental difference from deterministic realization theory is that in the stochastic casc there are in gneral many nonequivalent state-space models. In fact, even models of least complexity (minimal models) may be infinitely many.

Stochastic realizations such that ${ }^{3}$

$$
\begin{equation*}
X \subset \mathbf{H}:=\mathbf{H}^{-} \vee \mathbf{H}^{+} \tag{1.9}
\end{equation*}
$$

are called internal. For these realizations the state space is constructed by using only random variables containcd in the subspacc

$$
\mathbf{H}:=\mathbf{H}(d y)=\operatorname{closure}\left\{a^{\prime}[y(t)-y(s)] \mid a \in \mathbb{R}^{n}, t, s \in \mathbb{R}\right\}
$$

spanned by the process $d y$. It follows that for internal realizations the ambient space can be chosen equal to H. For $d y$ is p.n.d., and $\mathbf{H}(d y)$ can be generated by some vector Wiener process. From this it is easily seen that internal realizations correspond to left-invertible (full-rank) spectral factors [18]. In general, however, $\mathbf{H}$ is only a proper subspace of the ambient space.

Minimality of a Markovian splitting subspace $X$ is understood in the sense of subspace inclusion. For finite-dimensional subspaces this coincides with the notion of minimality in the sense of smallest possible dirrension. Any choice of basis $x(0):=\left[x_{1}(0), \ldots, x_{n}(0)\right]^{\prime}$ in a minimal Markovian splitting subspace $X$, imbedded in its ambient space $\mathbf{H}(d w)$ equipped with the shift induced by $d w$, defines a stationary Markov process $x(t):=U_{t} x(0), t \in \mathbb{R}$, serving as a state process of minimal dimension for $d y$. Different choices of basis in the same $X$ produce state processes which differ merely by a constant (deterministic) similarity transformation. Different choices of the Wiener process $d w$ generating the ambient space $\mathbf{H}(d w)$ will lead to different dynamical models (i.e., realizations) of $d y$. As we shall see, the choice of the Wiener-process generator has to do with the causality structure of the realization.

Basically the computation of (minimal) Markovian realizations of $d y$ can be reduced to solving the (minimal) spectral factorization problem for $\Phi(s)$. Given a minimal spectral factor $W(s)$ of $\Phi(s)$ with a minimal representation as in (1.6), unless $W$ is left-invertible there is however some arbitrariness involved in the choice of the ambient space to produce an actual realization

[^2]of $d y$. In particular it may be possible to have minimal representations of $d y$ with the same transfer function but with different Wiener processes as inputs. For processes with a rational spectrum this arbitrariness can be eliminated by choosing once and for all a suitably restricted "universal" ambient space which contains just one possible choice of ambient space $\mathbf{I I}(d w)$ for any minimal analytic (or coanalytic) spectral factor [21]. Once the universal ambient space is fixed, a one-to-one correspondence between minimal analytic (or coanalytic) spectral factors and minimal Markovian splitting subspaces $X$ is established. The family of all minimal Markovian splitting subspaces for $d y$, restricted in the above sense, is denoted by $\mathscr{X}$.

Traditionally the spectral factorization problem only regards minimal analytic (or coanalytic) spectral factors. This restriction leads to forward (i.e., causal) realizations or to the corresponding backward or anticausal models. Acausal realizations are obtained from a more general type of spectral factorization where the poles of the factors are not restricted to the right or left half plane.

Once $\Phi(s)$ is known in rational form, say

$$
\begin{equation*}
\Phi(s)=Z(s)+Z(-s)^{\prime}, \quad Z(s)=J+C(s I-A)^{-1} G \tag{1.10}
\end{equation*}
$$

with A asymptotically stable, the so-called positive-real lemma equations of Yakubovich, Kalman, and Popov provide an explicit parametrization of all minimal (analytic) spectral factors $W(s)$ in terms of symmetric positive definite matrices, solutions of a certain linear matrix inequality related to the decomposition (1.10) $[1,2,9,10,18]$.

Although the validity of the positive-real lemma as a way of characterizing positivity of a quadratic functional (or of a rational convolution operator) can be generalized considerably and has in particular been shown to be independent of any stability assumptions on $A$ (this is really classical Riccati theory and goes back to [33] and [23]), it must be stressed that the specific role played by the positive-real lemma in spectral factorization, namely as a tool for parametrizing the family $\mathscr{W}$ of minimal spectral factors, is not immediately generalizable to the nonanalytic setup. ${ }^{4}$ To our knowledge, this problem has never been seriously addressed before. Below we shall also discuss the related question of parametrizing families of acausal realizations of the

[^3]process $d y$ by means of symmetric solutions of a suitably generalized positive-real lemma.

### 1.2. Application to Balancing

Stochastic balancing has been introduced, for discrete-time models, in [5, 6]. The original definition of a balanced model requires a rather "ad hoc" simultaneous diagonal structure of a pair of covariance matrices which can be computed as the maximal and minimal solution of a certain Riccati equation. The definition is given in terms of matrices, and no direct statistical interpretation is evident. It has been argued in the work of Desai and Pal that this structure provides a convenient starting point for a heuristic stochastic model reduction technique based on discarding "small" singular values (for a survey of this circle of ideas one may consult [3]). This notion of balancing and its applications to model reduction has been object of much interest in recent years [26, 12, 15-17]. In regard to stochastic model reduction by "truncated balancing" (as discussed in the literature), we would however venture to say that at the present state of knowledge, it is not yet clear whether the method really lies on firm theoretical grounds.

It is trivial to do balancing for stable systems, but (even deterministic) models where $A$ is not necessarily a stable matrix pose conceptual problems. Computational techniques have been suggested, e.g., the so-called $L Q G$ balancing [14, 25], to deal with the case of unstable systems. This seems however also hard to motivate directly, as it is not clear what kind of system-theoretic properties of the transformed model one is actually trying to achieve by the LQG procedure.

In this respect we shall see that acausal modeling has a natural application to balancing of unstable systems.

In Section 5 of this paper we will interpret stochastic balancing of a finite-dimensional stochastic system of the type (1.5) as the problem of selecting a special basis in the state space (the Markovian splitting subspace $X$ associated to the model) in such a way that the coupling between past and future histories of the process $d y$ is reflected in a canonical "Hotelling-like" structure in the statc spacc. This will lead to a representation-independent notion of stochastic balancing, which can be described purely in geometric terms. Then we shall deal with the problem of computing the balancing transformation for a model with arbitrary pole locations, i.e., a possibly unstable model [we shall still place the restriction (1.2) on the poles of the system, however].

By laking the point of view of stochastic realization theory we may view any "physical" model like (1.5) just as a particular stochastic realization of the output signal $d y$. In this sense technical problems related to the stability of $A$
become irrelevant, as getting a specific location of the eigenvalues of $A$ is merely a matter of choosing an appropriate representative in the equivalence class of all possible acausal stochastic realizations of the output process.

That balancing of (not necessarily stable) state-space models of random signals should be related to understanding the equivalence class of all possible (minimal) acausal state-space representations of the signal is an idea that is also contained in the work of S. Weiland [32]. Weiland's work is exposed in the style and terminology of the (deterministic) "behaviourist" school. Our setup is different, and the final results don't quite seem to agree. In particular, it seems a bit artificial that a Riccati equation eventually pops up in the balancing theory of [32]. Indeed, it is well known that the family of all minimal acausal state-space models of a given deterministic $L^{2}$ signal (behavior) is described essentially by the action of the full feedback group on a particular state-space representation, and by this action the eigenvalues of A can be placed anywhere except on the imaginary axis. So there cannot be eigenvalue flipping with respect to the imaginary axis taking place in the deterministic (behavioral) context. The phenomenon of eigenvalue flipping is related in a very essential way to spectral factorization and the associated solution set of a Riccati equation. These tools play instead a very natural role in the stochastic setup. Stabilization by a certain pole flipping feedback, computed via the solution of a Riccati equation, is incidentally exactly what is done by "LQG balancing."

## 2. ACAUSAL MODELS OF STATIONARY MARKOV PROCESSES

The following is the central result of this section.
Theorem 2.1. Let $x$ be a stationary Markov process admitting a minimal representation (1.1). Then

$$
\begin{equation*}
d x(t)=F x(t) d t+G d v(t) \tag{2.1}
\end{equation*}
$$

is also a minimal representation of $x$ if and only if $F, G$, and dv are generated by the transformation

$$
\begin{align*}
F & =A+B B^{\prime} Q  \tag{2.2}\\
G & =B V  \tag{2.3}\\
d v(t) & =d w(t)-B^{\prime} Q x(t) d t \tag{2.4}
\end{align*}
$$

where $V$ is a $p \times p$ orthogonal matrix and $Q$ is a symmetric solution of the algebraic Riccati equation

$$
\begin{equation*}
A^{\prime} Q+Q A+Q B B^{\prime} Q=0 \tag{2.5}
\end{equation*}
$$

Proof. Sufficiency: Let $Q=Q^{\prime}$ be a solution of the Riccati equation (2.5), and define $F, G$, and do by (2.2) etc. Substituting (2.2) and (2.4) into Equation (1.1), it is seen that the process $x(t)$ can be represented as in (2.1). What needs to be checked is that (2.1) is a minimal representation of $x$ in the sense defined above.

Trivially, $(F, G)$ is controllable and $G$ has independent columns. Hence to prove sufficiency we just need to show that $v$ is a (wide-sense) Wiener process. Note that the process $x$ is thereafter represented as the solution of a bona fide linear stochastic differential equation (2.1) with ( $F, G$ ) a controllable pair. Stationarity of $x$ then implies $\operatorname{Re} \lambda(F) \neq 0$.

To show that $v$ is a (wide-sense) Wiener process, define $K(s):=I-$ $B^{\prime} Q(s I-A)^{-1} B$, so that the spectral measures (Fourier transforms) $d \hat{v}$ and $d \hat{w}$ of the stationary increment processes $d v$ and $d w$ in (2.4) are related by $d \hat{v}(i \omega)=K(i \omega) d \hat{\omega}(i \omega)$. The process $d v$ is wide-sense Wiener iff $K(i \omega)$ is a unitary matrix. Set $K:=B^{\prime} Q$; the following lemma characterizes the matrices $K$ which make $K(s)$ an all-pass function, i.e., a solution of the spectral factorization equation

$$
\begin{equation*}
K(s) K(-s)^{\prime}=I \tag{2.6}
\end{equation*}
$$

Lemma 2.1. Let the pair ( $A, B$ ) be controllable, and consider the square rational matrix $K(s):=I-K(s I-A)^{-1} B$. Then (2.6) holds if and only if

$$
\begin{equation*}
K=B^{\prime} Q \tag{2.7}
\end{equation*}
$$

with $Q$ a symmetric solution of the homogeneous Riccati equation

$$
A^{\prime} Q+Q A+Q B B^{\prime} Q=0
$$

Proof. We write the rational matrix $K(s) K(-s)^{\prime}$ as

$$
K(s) K(-s)^{\prime}=I+\left[\begin{array}{ll}
-K & B^{\prime}
\end{array}\right]\left[\begin{array}{cc}
s I-A & -B B^{\prime} \\
0 & s I+A^{\prime}
\end{array}\right]^{-1}\left[\begin{array}{c}
B \\
K^{\prime}
\end{array}\right]
$$

which shows that $K(s)$ solves (2.6) if and only if the (Hamiltonian) triple

$$
\mathscr{C}:=\left[\begin{array}{ll}
-K & B^{\prime}
\end{array}\right], \mathscr{A}:=\left[\begin{array}{cc}
A & B B^{\prime}  \tag{2.8}\\
0 & -A^{\prime}
\end{array}\right], \mathscr{B}:=\left[\begin{array}{c}
B \\
K^{\prime}
\end{array}\right]
$$

provides a (nonminimal) realization with state space of dimension $2 n$ of the $p \times p$ zero transfer function. Now, the corresponding controllability and observability matrices are seen to have the form

$$
\mathscr{R}=\left[\begin{array}{cccc}
B & A B+B+\cdots & A^{2} B+A B+\cdots+B+\cdots & \cdots \\
K^{\prime} & -A^{\prime} K^{\prime} & \left(A^{\prime}\right)^{2} K^{\prime} & \cdots
\end{array}\right]=:\left[\begin{array}{l}
\mathscr{R}_{1} \\
\mathscr{R}_{2}
\end{array}\right]
$$

and

$$
\mathscr{O}=\left[\begin{array}{cc}
-K & B^{\prime} \\
-K A & -B^{\prime} A^{\prime}+\cdots+B^{\prime} \\
-K A^{2} & B^{\prime}\left(A^{\prime}\right)^{2}+\cdots+B^{\prime} A^{\prime}+\cdots+B^{\prime} \\
\vdots & \vdots
\end{array}\right]=:\left[\begin{array}{ll}
\mathscr{O}_{1} & \mathscr{O}_{2}
\end{array}\right]
$$

respectively. Moreover, by controllability of the pair ( $A, B$ ), it follows that the dimension of the controllable subspace $\mathscr{X}_{c}=\operatorname{im} \mathscr{R}$ is

$$
\begin{equation*}
\operatorname{dim} \mathscr{X}_{c}=\operatorname{rank} \mathscr{R} \geqslant \operatorname{rank} \mathscr{R}_{1}=n, \tag{2.9}
\end{equation*}
$$

while the dimension of the unobservable subspace $\mathscr{X}_{n 0}=\operatorname{ker} \mathscr{O}$ is

$$
\begin{equation*}
\operatorname{dim} \mathscr{X}_{n 0}=2 u-\operatorname{rank} \mathscr{O} \leqslant 2 n-\operatorname{rank} \mathscr{O}_{2}=n . \tag{2.10}
\end{equation*}
$$

Hence, $\operatorname{dim} \mathscr{\mathscr { O }}_{n 0} \leqslant n \leqslant \operatorname{dim} \mathscr{X}_{c}$. Therefore, the transfer function associated to the triple (2.8) is zero (equivalently, $\mathscr{X}_{c} \subset \mathscr{P}_{n 0}$ ) if and only if $\mathscr{X}_{c}=\mathscr{X}_{n 0}$ and hence $\operatorname{dim} \mathscr{X}_{c}=\operatorname{dim} \mathscr{X}_{n 0}=n$. In this case, being the observability submatrix $\mathscr{O}_{2}$ of full column rank $n=\operatorname{rank} \mathscr{O}$, there must exist a unique $n \times n$ matrix $Q$ such that $\mathscr{O}_{1}+\mathscr{O}_{2} Q=0$, i.e., we can write the unobservability subspace as

$$
\mathscr{X}_{n 0}=\operatorname{im}\left[\begin{array}{c}
I  \tag{2.11}\\
Q
\end{array}\right] .
$$

Also, the equality $\mathscr{X}_{c}=\mathscr{X}_{n 0}$ implies that the controllability subspace can be written as

$$
\mathscr{X}_{c}=\operatorname{ker}\left[\begin{array}{ll}
-Q & I \tag{2.12}
\end{array}\right],
$$

or, equivalently, $\mathscr{R}_{2}-Q \mathscr{R}_{1}=0$. Conversely, it is clear that if (2.11) and (2.12) hold for some square matrix $Q$, then $\mathscr{X}_{c}=\mathscr{X}_{n 0}$. Now, recall that the unobservability subspace $\mathscr{Z}_{n 0}$ associated to the pair ( $\mathscr{C}, \mathscr{A}$ ) can be characterized as the maximal $\mathscr{A}$-invariant subspace contained in ker $\mathscr{O}$, while the controllability subspace $\mathscr{R}_{c}$ for the pair $(\mathscr{A}, \mathscr{B})$ is the minimal $\mathscr{A}$-invariant subspace containing im $\mathscr{B}$. Also, note that, in view of (2.9) and (2.10), maximality of $\mathscr{P}_{n 0}$ and minimality of $\mathscr{X}_{c}$ are automatically guaranteed by the form of (2.11) and (2.12), respectively. Then the four conditions

$$
\begin{align*}
& \mathscr{E} \mathscr{X}_{n 0}=0,  \tag{2.13}\\
& \mathscr{A} \mathscr{X}_{n 0} \subset \mathscr{X}_{n 0} \tag{2.14}
\end{align*}
$$

and

$$
\begin{gather*}
\mathscr{X}_{c}^{\perp} \mathscr{B}=0,  \tag{2.15}\\
\mathscr{A} \mathscr{X}_{c} \subset \mathscr{X}_{c} \tag{2.16}
\end{gather*}
$$

are necessary and sufficient, alone, to characterize the subspaces

$$
\mathscr{X}_{n 0}=\operatorname{im}\left[\begin{array}{c}
I \\
Q
\end{array}\right] \quad \text { and } \quad \mathscr{X}_{c}=\operatorname{ker}\left[\begin{array}{ll}
-Q & I
\end{array}\right]
$$

respectively. Equations (2.13)-(2.16), together with (2.11) and (2.12), yield the equivalent set of equations

$$
\begin{align*}
-K+B^{\prime} Q & =0  \tag{2.17}\\
-A^{\prime} Q & =Q\left(A+B B^{\prime} Q\right) \tag{2.18}
\end{align*}
$$

which are exactly (2.7) and (2.5) in the statement of the theorem, and

$$
\begin{align*}
K^{\prime}-Q B & =0  \tag{2.19}\\
-Q A & =\left(Q B B^{\prime}+A^{\prime}\right) Q \tag{2.20}
\end{align*}
$$

However, we observe that if $Q$ solves these four equations, then $Q^{\prime}$ is also a solution. Thus, by uniqueness, we conclude that necessarily $Q-Q^{\prime}$ and we can disregard (2.19) and (2.20) as redundant.

Lemma 2.1 clearly proves sufficiency. Necessity follows by subtracting (1.1) from (2.1) and rearranging the difference as

$$
\begin{equation*}
G d v=B d w \quad(F \quad A) x(t) d t \tag{2.21}
\end{equation*}
$$

Since both representations are minimal, $G$ and $B$ are full column rank. On the other hand, by standard arguments in semimartingale representation theory (see e.g., [19]), the quadratic variations $G G^{\prime}$ and $B B^{\prime}$ in the identiry (2.21) coincide. Therefore $G=B V$ with $V$ an orthogonal $p \times p$ matrix. Since $V d v$ is indistinguishable from $d v$, we may rewrite (2.21) as

$$
\begin{equation*}
B(d v-d w)=(F-A) x(t) d t \tag{2.22}
\end{equation*}
$$

Now multiply from the left by a left inverse, $B^{-L}$, of $B$, and set $K:=$ $B^{-L}(F-A)$. Using (1.1) to eliminate $x$ from (2.22), one obtains a relation between the spectral measures (Fourier transforms) $d \hat{v}$ and $d \hat{w}$ of $d v$ and $d \omega$, of the type $d \hat{v}(i \omega)=K(i \omega) d \hat{w}(i \omega)$, where $K(s):=I-K(s I-A)^{-1} B$. The process $d v$ is wide-sense Wiener, and hence $K(s)$ must be all-pass. Invoking Lemma 2.1, we see that $K$ must be related to a solution of the homogeneous Riccati equation (2.5) by the formula (2.7).

On the other hand, if we multiply (2.22) from the right by $x(t)^{\prime}$ and take expectations, recalling that $x(t)$ has a positive definite variance matrix, we see that

$$
\begin{equation*}
\operatorname{Im}(F-A) \subset \operatorname{Im} B \tag{2.23}
\end{equation*}
$$

From this inclusion $B B^{-L}(F-A)=F-A=B B^{\prime} Q$ readily follows. This concludes the proof of the theorem.

It may be worth noticing that the homogeneous Ricccati equation (2.5) relates to spectral factorization of the spectral density matrix of the process $x$, i.e.,

$$
\begin{align*}
\Phi_{x}(s) & =(s I-A)^{-1} B B^{\prime}\left(-s I-A^{\prime}\right)^{-1} \\
& =(s I-F)^{-1} G G^{\prime}\left(-s I-F^{\prime}\right)^{-1} \tag{2.24}
\end{align*}
$$

In this respect Theorem 2.1 is not the standard way of parametrizing the solutions of the spectral factorization equation in stochastic modeling, where one keeps the pole structure of the factors fixed, and describes instead the different zero structure of the factors. The result describes instead the geometry of the poles of the different factors. For, assuming $\operatorname{Re} \lambda(A)<0$ and recalling that $B$ has independent columns, it is easy to check via the positive-real lemma equations that the spectral density matrix of a stationary Markov process admits just one stable minimal degree spectral factor (up to right multiplication by an orthogonal matrix). For a problem of this kind there is no "zero flipping" taking place at all.

Let $\mathscr{A}_{x}$ denote the set of all $A$-matrices representing a given Markov process $x$. To study this set we introduce the following notations:

$$
\begin{align*}
\Omega_{A}(Q) & :=A^{\prime} Q+Q A+Q B B^{\prime} Q  \tag{2.25}\\
\mathscr{Q}_{A} & :=\left\{Q=Q^{\prime} \mid \Omega_{A}(Q)=0\right\} \tag{2.26}
\end{align*}
$$

By Theorem 2.1 we can write $\mathscr{A}_{x}=A+B B^{\prime} \mathscr{Q}_{A}$. Further, let $Q_{1}, Q_{2} \in \mathscr{Q}_{A}$ and let $A_{1}:=A+B B^{\prime} Q_{1}$; then the following identity is well known (e.g., [30]):

$$
\begin{equation*}
\Omega_{A}\left(Q_{2}\right)-\Omega_{A}\left(Q_{1}\right)=\Omega_{A_{1}}\left(Q_{2}-Q_{1}\right) \tag{2.27}
\end{equation*}
$$

The identity implies that the solution sets $\mathscr{Q}_{A}$ corresponding to different A-matrices merely differ by translation. This observation is stated below and will be used repeatedly in the following.

Lemma 2.2. Let $F=A+B B^{\prime} Q_{0}$ with $Q_{0} \in \mathscr{Q}_{A}$; then

$$
\begin{equation*}
\mathscr{Q}_{F}=\mathscr{Q}_{A}-Q_{0} \tag{2.28}
\end{equation*}
$$

in the sense that $P \in \mathscr{Q}_{F}$ if and only if there is $a Q \in \mathscr{Q}_{A}$ such that $P=Q-Q_{0}$.

Proof. If $P$ is a difference of two elements of $\mathscr{Q}_{\mathrm{A}}$ as in the statement of the lemma, then use (2.27) to check that $P \in Q_{F}$. Conversely, note that $A=F+B B^{\prime}\left(-Q_{0}\right)$ with $-Q_{0} \in Q_{F}$. Rewriting (2.27) with $A$ set equal to $F, Q_{2}=P, Q_{1}=-Q_{0}$ (both in $\mathscr{Q}_{P}$ ), and $A_{1}=A$, it follows that $\Omega_{A}(P+$ $\left.Q_{0}\right)=0$, i.e., $P+Q_{0} \in \mathbb{Q}_{A}$.

Clearly, $\mathscr{A}_{x}$ should be independent of the particular starting matrix $A$. This in fact follows immediately from (2.28) of Lemma 2.2 above, as

$$
\begin{equation*}
A+B B^{\prime} \mathscr{Q}_{A}=F+B B^{\prime} \mathscr{Q}_{F} \tag{2.29}
\end{equation*}
$$

The following result describes how the feedback transformation (2.2) changes the eigenstructure of the starting matrix $A$. It appeared in [27] at about the same time of the publication of Scherer's paper [30], where a slightly more general result of the same type is given (Theorem 2).

We shall say that A has unmixed spectrum if none of the eigenvalues of $A$ do occur in opposite pairs, i.e., $-\sigma(A) \cap \sigma(A)=\varnothing$. Under this condition it is well known that the Lyapunov equation $A P+P A^{\prime}+B B^{\prime}=0$ has a unique symmetric solution $P=P^{\prime}$ (this condition is not necessary, however, for the existence of solutions; compare [34, 28]). If in addition ( $A, B$ ) is a controllable pair, the solution $P$ is nonsingular (see e.g., [11]).

Theorem 2.2. Let $(A, B)$ be controllable, and assume that A has unmixed spectrum. Then there exists a one-to-one correspondence betwen the solution set $\mathscr{Q}_{A}$ of the algebraic Riccati equation (2.5) and the family of A-invariant subspaces of $A$. The correspondence is the map assigning to each $Q \in \mathscr{Q}_{\mathrm{A}}$ its kernel

$$
\begin{equation*}
Q \leftrightarrow \operatorname{Ker} Q \tag{2.30}
\end{equation*}
$$

Moreover, for every solution $Q=Q^{\prime}$ of (2.5) the feedback law (2.2) leaves unaltered the restriction of $A$ to the $A$-invariant subspace Ker $Q$, while it renders the map, say $A_{2}$, induced by $A$ on the quotient space $\mathbb{R}^{n} / \operatorname{Ker} Q$ similar to $-A_{2}^{\prime}$.

In particular, the eigenvalues of the map induced by $A$ on the quotient space $\mathbf{R}^{n} /$ Ker $Q$ are reflected into a symmetric set with respect to the imaginary axis.

The feedback law (2.2) hence "flips" some of the eigenvalues of $A$ with respect to the imaginary axis. In fact, it is easy to see by purely algebraic arguments that an arbitrary $A$ (not necessarily with unmixed spectrum) can be transformed by (2.2) into a matrix with spectrum in the left half plane. For in the basis of (1.3a), (1.3b) the Riccati equation (2.5) has a block-diagonal solution $\hat{Q}:=\operatorname{diag}\left\{0, P_{2}^{-1}\right\}$, where $P_{2}$ is the negative definite solution of the Lyapunov equation

$$
\begin{equation*}
\hat{A}_{+} P_{2}+P_{2} \hat{A}_{+}^{\prime}+\hat{B}_{+} \hat{B}_{+}^{\prime}=0 \tag{2.31}
\end{equation*}
$$

Hence

$$
\hat{A}+\hat{B} \hat{B}^{\prime} \hat{Q}=\left[\begin{array}{cc}
\hat{A}_{-} & *  \tag{2.32}\\
0 & -P_{2} \hat{A}_{+}^{\prime} P_{2}^{-1}
\end{array}\right]
$$

has all the eigenvalues in the left half plane.
It follows from standard theory (compare e.g., [33, 23]) that under controllability of $(A, B)$, the algebraic Riccati equation (2.5) possesses a maximal and a minimal solution (with respect to the usual semidefinite ordering of matrices), denoted $Q_{+}, Q_{-}$, respectively. The gap $Q_{+}-Q_{-}$is positive definite, and its inverse

$$
\begin{equation*}
P:=\left(Q_{+}-Q_{-}\right)^{-1} \tag{2.33}
\end{equation*}
$$

solves either of the Lyapunov equations

$$
\begin{align*}
& A_{-} P+P A_{-}^{\prime}+B B^{\prime}=0  \tag{2.34}\\
& A_{+} P+P A_{+}^{\prime}-B B^{\prime}=0 \tag{2.35}
\end{align*}
$$

where

$$
\begin{equation*}
A_{-}=A+B B^{\prime} Q_{-}, \quad A_{+}=A+B B^{\prime} Q_{+} \tag{2.36}
\end{equation*}
$$

It follows by Lyapunov theory that $\operatorname{Re} \lambda\left(A_{-}\right)<0$ and $\operatorname{Re} \lambda\left(A_{+}\right)>0$, respectively. These classical results are actually extremely easy to prove in the present context, for the homogeneous Riccati Equation (2.5). We shall actually implicitly rederive them in the course of justifying the following theorem, which tells how to compute the variance of the Markov process $x$ starting from a general acausal representation (1.1).

Theorem 2.3. The linear feedback laws (2.2) corresponding to the maximal (minimal) solution $Q_{+}\left(Q_{-}\right)$of the Riccati equation (2.5) transform the Markovian representation (1.1) into the backward (forward) differential-equation model of $x$ with antistable (stable) infinitesimal generator matrices $A_{+}\left(A_{-}\right)$given by Equation (2.36). Moreover, the inverse $P$ of
the gap matrix defined in (2.33) is equal to the variance matrix of the process $x$, namely, $P=E x(t) x(t)^{\prime}$.

Proof. We proceed in three steps.
(1) Let $A_{-} \in \mathscr{A}_{x}$ have stable eigenvalues, and let $P$ be the positive definite solution of the Lyapunov equation (2.34). Trivially, $P^{-1} \in \mathscr{Q}_{A}$, and the matrix $A_{+}:-A_{-}+B B^{\prime} P^{-1}$ has only unstable eigenvalues (in facl, $A_{+}$ yields the backward, or anticausal, representation of $x$ ). So it is obvious that all solutions $Q \in \mathscr{Q}_{A_{-}}$of the Riccati equation $\Omega_{A_{-}}(Q)=0$ are positive semidefinite and, dually, all solutions $\bar{Q} \in \mathscr{Q}_{A_{+}}$are negative semidefinite. Therefore,

$$
\begin{equation*}
\min \mathscr{Q}_{A_{-}}=0, \quad \max \mathscr{Q}_{A_{+}}=0 \tag{2.37}
\end{equation*}
$$

(2) Let now $Q_{+}, Q_{-} \in \mathscr{Q}_{A}$ be as in the statement of the theorem, and let $A_{+}, A_{-}$be defined as in (2.36). From Lemma 2.2, $\mathscr{Q}_{A}=\mathscr{Q}_{A_{-}}+Q_{-}$and also $\mathscr{O}_{A}=\mathscr{O}_{A_{+}}+Q_{+}$. Hence maximal and minimal elements exist for the sets on both sides of these equalities and

$$
\begin{equation*}
\min \mathscr{Q}_{A}=Q_{-}, \quad \max \mathscr{Q}_{A}=Q_{+} \tag{2.38}
\end{equation*}
$$

as claimed.
(3) From the definition of $A_{+}$and Lemma 2.2, $\mathscr{Q}_{A_{+}}=\mathscr{Q}_{A_{-}}-P^{-1}$. Therefore

$$
\begin{equation*}
\min \mathscr{Q}_{A_{+}}=-P^{-1}, \quad \max \mathscr{Q}_{A_{-}}=P^{-1} \tag{2.39}
\end{equation*}
$$

In particular, max $\mathscr{Q}_{A_{-}}-\min \mathscr{Q}_{A_{-}}=P^{-1}$. Since the set $\mathscr{Q}_{A}$ is just the translate of $\mathscr{Q}_{A_{-}}$by a fixed symmetric matrix,

$$
\begin{equation*}
\max \mathscr{Q}_{A}-\min \mathscr{Q}_{A}=\max \mathscr{Q}_{A_{-}}-\min \mathscr{Q}_{A_{-}}=P^{-1} \tag{2.40}
\end{equation*}
$$

and the matrix in the second member of this equality is clearly the inverse of the variance of $x$.

This concludes the proof.

## 3. ACAUSAL MARKOVIAN REPRESENTATIONS

Our first concern in this section will be to characterize the family of all minimal realizations

$$
\begin{align*}
& d x=A x d t+B d w  \tag{3.1a}\\
& d y=C x d t+D d w \tag{3.1b}
\end{align*}
$$

of the stationary-increment process $d y$, with the same state process $x$. Since $\left[\begin{array}{l}B \\ D\end{array}\right]$ is of full column rank, it follows that the increments of $d w$ are functions of the joint history of $x$ and $d y$ so that $\mathbf{H}(d w)=\mathbf{H}(d y) \vee \mathbf{H}(x)$. Hence any other realization with the same $x$ process must be driven by a Wiener process $d v$ such that $\mathbf{H}(d v)=\mathbf{H}(d w)$. In particular, any such $d v$ must have the same dimension $p$ as $d w$ (the integer $p$ is in fact just the multiplicity, or rank, of the joint process [ $x, d y]^{\prime}$ [29]).

Theorem 2.1 has in the present setting the following analog.
Theorem 3.1. The stochastic system

$$
\begin{align*}
& d z=F z d t+G d v  \tag{3.2a}\\
& d y=H z d t+J d v \tag{3.2b}
\end{align*}
$$

is a minimal stochastic realization of $d y$ with the same state process $x$ of the realization (3.1), i.e., $z(t)=x(t)$ a.s. for all $t \in \mathbb{R}$, if and only if $(F, G, H, J, d v)$, and $(A, B, C, D, d w)$ are related by the transformation

$$
\begin{align*}
F & =A+B B^{\prime} Q  \tag{3.3a}\\
{\left[\begin{array}{c}
G \\
J
\end{array}\right] } & =\left[\begin{array}{c}
B \\
D
\end{array}\right] V  \tag{3.3b}\\
H & =C+D B^{\prime} Q  \tag{3.3c}\\
d v(t) & =d w(t)-B^{\prime} Q x(t) d t \tag{3.3d}
\end{align*}
$$

where $V$ is a $p \times p$ orthogonal matrix and $Q$ is a symmetric solution of the homogeneous algebraic Riccati equation,

$$
\begin{equation*}
A^{\prime} Q+Q A+Q B B^{\prime} Q=0 . \tag{3.4}
\end{equation*}
$$

Proof. The idea of the proof is that we are really characterizing the family of acausal representations of the same joint process $\left[\begin{array}{l}x \\ y\end{array}\right]$. We shall hence write (3.1) and (3.2) respectively as

$$
\begin{align*}
& {\left[\begin{array}{l}
d x \\
d y
\end{array}\right]=\mathscr{A}\left[\begin{array}{l}
x \\
y
\end{array}\right] d t+\mathscr{B} d w,}  \tag{3.5}\\
& {\left[\begin{array}{l}
d z \\
d y
\end{array}\right]=\mathscr{F}\left[\begin{array}{l}
x \\
y
\end{array}\right] d t+\mathscr{G} d v,} \tag{3.6}
\end{align*}
$$

where

$$
\begin{align*}
& \mathscr{A}:=\left[\begin{array}{ll}
A & O \\
C & O
\end{array}\right], \quad \mathscr{B}:=\left[\begin{array}{c}
B \\
D
\end{array}\right],  \tag{3.7}\\
& \mathscr{F}:=\left[\begin{array}{ll}
F & O \\
H & O
\end{array}\right], \mathscr{G}:=\left[\begin{array}{c}
G \\
J
\end{array}\right] . \tag{3.8}
\end{align*}
$$

Necessity: Let (3.1) and (3.2) be two minimal realizations, and assume that $x=z$. By comparing the two representations for the joint process we obtain

$$
\mathscr{G} d v=\mathscr{B} d w+(\mathscr{A}-\mathscr{F})\left[\begin{array}{l}
x  \tag{3.9}\\
y
\end{array}\right] d t
$$

from which (3.3b) readily follows (as both $\mathscr{G}$ and $\mathscr{B}$ have independent columns), and by arguing as in the proof of (2.23), Theorem 2.1 , one obtains $\operatorname{im}(\mathscr{A}-\mathscr{F}) \subset \operatorname{im} \mathscr{B}$, that is,

$$
\operatorname{im}\left[\begin{array}{rr}
A & -F  \tag{3.10}\\
C & H
\end{array}\right] \subset \operatorname{im}\left[\begin{array}{l}
B \\
D
\end{array}\right]
$$

which is the same as saying that there must be a matrix $K$ such that

$$
\begin{equation*}
A-F=-B K, \quad C-H=-D K \tag{3.11}
\end{equation*}
$$

Now, identifying (without loss of generality) the Wiener processes $d v$ and $V d v$, we can rewrite the relation (3.9) as $d v=d w-K x(t) d t$, which, using
(3.1a) and Lemma 2.1, leads to the conclusion that ( $F, G, H, J, d v$ ) must indeed be given by (3.3). Sufficiency follows by paraphrasing the sufficiency proof of Theorem 2.1.

As we shall comment below, the stochastic realizations obtained by the "pole-flipping" family of feedback transformations (3.3) all have the same zeros. This may be roughly taken to mean that the zeros of the corresponding spectral factors are left invariant by the transformation and coincide with the zeros of the fixed reference minimal spectral factor

$$
\begin{equation*}
W(s)=C(s I-A)^{-1} B+D \tag{3.12}
\end{equation*}
$$

A more precise statement requires a brief digression on the notion of zero dynamics and will be reported below.

### 3.1. Relation with the Zero Dynamics

In this subsection we shall assume that the spectral density of the process $d y$ is coercive, i.e., there is a $c>0$ such that $\Phi(i \omega) \geqslant c I$ for all $\omega \in \mathbb{R}$; in particular, $R:=\Phi(\infty)>0$. This is the same as $D D^{\prime}>0$ in any minimal realization, and by suitable postmultiplication by an orthogonal matrix we may even assume $D$ of the form $D=\left[R^{1 / 2} O\right]$, where $R^{1 / 2}$ is the symmetric square root of $R$. This permits us to rewrite the realization (3.1) in standard form as

$$
\begin{align*}
& d x=A x d t+B_{1} d w_{1}+B_{2} d w_{2} \\
& d y=C x d t+R^{1 / 2} d w_{1} \tag{3.13}
\end{align*}
$$

with $d w_{1}$ and $d w_{2}$ uncorrelated Wiener processes of dimensions $m$ and $p-m$, respectively. Eliminating the noise $d w_{1}$ produces a state representation

$$
\begin{equation*}
d x=\Gamma x d t+B_{1} R^{-1 / 2} d y+B_{2} d w_{2} \tag{3.14}
\end{equation*}
$$

in feedback form, where $\Gamma$ is the numerator matrix

$$
\begin{equation*}
\Gamma=A-B_{1} R^{-1 / 2} C \tag{3.15}
\end{equation*}
$$

Now it is well known [33] that coercivity of $\Phi$ guarantees absence of eigenvalues of $\Gamma$ on the imaginary axis. The representation (3.14) can
therefore be given a meaning by integrating unstable modes backward in time and stable modes forward in time, in complete analogy with the meaning of an acausal differential equation for the state process $x$ as was stipulated in the beginning of this paper.

We shall call (3.14) the inverse dynamics of the realization (3.13). The inverse dynamics can be defined also for more general systems than those which can be brought to standard form, at the price of some additional complications which we prefer to avoid here. It can be seen as a way of exhibiting the decomposition of the state space into an internal component, which is a subspace made up only of functionals of the output process $d y$, and an external component, which is instead made up of functionals of the external noise $d w_{2}$.

A related concept of zero dynamics ${ }^{5}$ in the stochastic framework has been discussed in [21] and in a recent paper [22]. In the last reference it is shown that the unreachable subspace $\left\langle\Gamma \mid B_{2}\right\rangle^{\perp}$ of $\left(\Gamma, B_{2}\right)$, denoted $\mathscr{S}^{*}$, is just the coordinate space of the internal subspace $X \cap \mathbf{H}$ of $X$, in the sense that $X \cap \mathbf{H}=\left\{a^{\prime} x(0) \mid a \in \mathscr{S}^{*}\right\}$ and that the eigenvalues of $\left.\Gamma^{\prime}\right|_{\mathscr{P}^{*}}$ are the so-called invariant zeros of the system ( $A, B, C,\left[R^{1 / 2}, 0\right]$ ) or, equivalently, of the spectral factor $W(s)$, transfer function of the realization (3.13).

While the "internal" subsystem ( $\Gamma, B_{1} R^{-1 / 2}$ ) restricted to $\mathscr{S}^{*}$ is related to the internal subspace and to the invariant zero structure, the "external" dynamics ( $\Gamma, B_{2}, d w_{2}$ ), restricted to the complementary subspace $\mathscr{R}^{*}:=$ $\left\langle\Gamma \mid B_{2}\right\rangle$, generates the external portion of the state space, $E^{X} \mathbf{H}^{\perp}$ [21, Section 11.3], and the "assignable" zeros of the realization (3.13). In this sense two realizations with state process admitting the same inverse dynamics will certainly have the same zero structure.

Our interest in the inverse-dynamics representation of $x$ comes from the fact that (3.14) is invariant under the feedhack transformation (3.3):

Corollary 3.1. Two minimal acausal realizations of the stationaryincrement process $d y$ with the same state process $x$ have the same inverse dynamics (and hence the same zero structure).

Proof. A simple coordinate-free proof could be given by noting that the same $x$ implies the same state space $X$ and hence the same internal and external components of $X$.

An alternative proof can be obtained by comparing the inverse dynamics of two realizations like (3.1) and (3.2), related by the eigenvalue flipping

[^4]feedback (3.3). Without loss of generality we may assume $V=I$ in (3.3b). Then, with obvious notation, we obtain
\[

$$
\begin{equation*}
\Gamma_{F}:=F-G_{1} R^{-1 / 2} H=A-B_{1} R^{-1 / 2} C+B_{2} B_{2}^{\prime} Q=\Gamma+B_{2} B_{2}^{\prime} Q \tag{3.16}
\end{equation*}
$$

\]

where $Q$ is a solution of (3.4). Using this expression, the inverse dynamics of (3.2) can be written

$$
\begin{equation*}
d x=\Gamma_{F} x d t+G_{2} d v_{2}=\Gamma x d t+B_{2} d v_{2}+B_{2} B_{2}^{\prime} Q x d t \tag{3.17}
\end{equation*}
$$

Now, premultiplying (3.3d) by $\left[0, B_{2}\right]$, we get $B_{2} d v_{2}=B_{2} d w_{2}-$ $B_{2} B_{2}^{\prime} Q x d t$, which, substituted into (3.17), yields the same inverse dynamics of (3.1).

Realizations of the form (3.1) related by a "pole-flipping" feedback can then be said to "have the same zeros." In particular, two minimal internal (i.e., corresponding to a square spectral factor) realizations will have the same numerator matrix $\Gamma$ and the same matrix $B$. Also, note that the inverse of a square spectral factor $W(s)=C(s I-A)^{-1} B+D$ is given by $W(s)^{-1}=$ $-C(s I-\Gamma)^{-1} B D^{-1}+D^{-1}$.

## 4. FIXED POLE DYNAMICS AND THE ACAUSAL POSITIVE REAL LEMMA

Our goal in this section will be to obtain a result "dual" to Theorem 3.1, i.e., a parametrization of all minimal acausal stochastic realizations of the form (3.1) where now the pole dynamics is kept fixed. Much in analogy with the causal or analytic situation, the parametrization can be given in terms of solutions of a certain linear matrix inequality which in certain cases reduces to an algebraic Riccati equation. Concerning the notion of "pole dynamics" used in this context, we will say that two minimal realizations described by the quadruples ( $A, B, C, D$ ) and ( $F, G, H, J$ ) "have the same pole dynamics" if $A=F$ and $C=H$.

### 4.1. Pole Dynamics and Uniform Choice of Basis

We shall comment briefly on the significance of "having the same pole dynamics," in a stochastic setup. This algebraic condition is intimately related to the notion of a uniform choice of basis in the family of minimal splitting
subspaces $\mathscr{X}$, which will be recalled below. This notion will also be useful for understanding stochastic balancing, to be discussed in Section 5.

We shall need to introduce the forward and backward predictor spaces, $X_{-}$and $X_{+}$, defined as

$$
\begin{equation*}
X_{-}:=\bar{E}^{H^{-}} H^{+}, \quad X_{+}:=\bar{E}^{H^{+}} H^{-} \tag{4.1}
\end{equation*}
$$

These two subspaces are minimal splitting and are the smallest and the largest element in the family $\mathscr{X}$ with respect to a natural partial order (denoted $<$ ), defined in terms of the (cosines of the) angles that each $X$ makes with the future space $H^{+}$; se [21, Section 6].

Consider a minimal acausal realization (3.1). By minimality the components of the $n$-vector $x(0)$ must form a basis in the relative splitting subspace $X$. Now, we recall from [20, 21] that two families of bases, say $\{x(0)\}$ and $\{\bar{x}(0)\}$, for the family $\mathscr{X}$ [i.e., each vector $x(0)$ is a basis in one $X$ and similarly for cach $\bar{x}(0)$ ] are called uniformly ordered (or, for short, uniform) respectively in the forward or in the backward sense, if whenever $X_{1} \prec X_{2}$ and $x_{i}(0)$ are bases in $X_{i}(i=1,2)$, there holds

$$
\begin{equation*}
E^{X_{1}} x_{2}(0)=x_{1}(0) \tag{4.2}
\end{equation*}
$$

or

$$
\begin{equation*}
E^{X_{2}} \bar{x}_{1}(0)=\bar{x}_{2}(0) \tag{4.3}
\end{equation*}
$$

respectively, the vectors $\bar{x}_{i}(0)$ being bases in the corresponding subspaces $X_{i}$ ( $i=1,2$ ).

It readily follows from the definition that for any basis $x(0)$ in a forwarduniform family, and, respectively, for any $\bar{x}(0)$ in a backward-uniform family, we have the "invariant projection" property

$$
\begin{align*}
& E^{x_{-}} x(0)=x_{-}(0)  \tag{4.4a}\\
& E^{X_{+}} \bar{x}(0)=\bar{x}_{+}(0) \tag{4.4b}
\end{align*}
$$

where $x_{-}(0)$ is the basis relative to the forward predictor space $X_{-}$in the first family and $\bar{x}_{+}(0)$ is the basis relative to the backward predictor space $X_{+}$in the second family.

A forward-uniform choice of bases in $\mathscr{X}$ establishes a lattice isomorphism between $\mathscr{X}$ and the corresponding family of state covariance matrices $\mathscr{P}:=$ $\left\{P=E x(0) x(0)^{\prime} \mid x(0)\right.$ a basis in $\left.X\right\}$, the latter set being endowed with the
natural partial order of positive semidefinite matrices. This is equivalent to saying that $X_{1} \prec X_{2} \Leftrightarrow P_{1} \leqslant P_{2}$. In a backward-uniform choice the ordering of the corresponding state covariance matrices $\overline{\mathscr{P}}:=\left\{\bar{P}=E \bar{x}(0) \bar{x}(0)^{\prime} \mid \bar{x}(0)\right.$ a basis in $X\}$ is reversed, namely, $X_{1} \prec X_{2} \Leftrightarrow \bar{P}_{2} \leqslant \bar{P}_{1}$.

In particular, the bases $x_{-}(0)$ in $X_{-}$and $x_{+}(0)$ in $X_{+}$in a forwarduniform choice will have the smallest and, respectively, the largest state covariance matrices $P_{-}$and $P_{+}$in $\mathscr{P}$. For a backward-uniform family it will instead happen that $\bar{P}_{-}$is maximal and $\bar{P}_{+}$is minimal.

The most useful properties of uniform bases are summarized below.
Proposition 4.1 [21]. If $\{x(0)\}$ is a forward-uniform family of bases in $\mathscr{Z}$, then for each $x(0)$, the corresponding dual basis $\bar{x}(0)$, uniquely defined in $X$ by the condition

$$
\begin{equation*}
\operatorname{Ex}(0) \bar{x}(0)^{\prime}=I \tag{4.5}
\end{equation*}
$$

defines a backward-uniform family in $\mathscr{X}$.
Proposition 4.2 [21]. All causal realizations corresponding to a forward-uniform family of bases are described by the same ( $A, C$ ) pairs. Likewise, all anticausal realizations corresponding to a backward-uniform family of bases are described by the same $(\bar{A}, \bar{C})$.

The only backward-uniform family of bases we shall encounter in the following will be made of dual bases of a forward family.

Note that dual pairs $(x(0), \bar{x}(0))$ are related by the transformation

$$
\begin{equation*}
\bar{x}(0)=P^{-1} x(0) \tag{4.6}
\end{equation*}
$$

so that the covariance matrices of dual pairs are related by matrix inversion, i.e., letting $P=\operatorname{Ex}(0) x(0)^{\prime}$,

$$
\begin{equation*}
\bar{P}=E \bar{x}(0) \bar{x}(0)^{\prime}=P^{-1} \tag{4.7}
\end{equation*}
$$

as follows immediately from (4.5).
Proposition 4.2 ties together uniform bases and invariance of the pole dynamics ( $A, C$ ), at least for special causality structures of the corresponding realizations. By invoking the causal-anticausal decomposition (1.3) the result can actually be shown to hold also in the general case of an arbitrary causality structure. Namely: All realizations corresponding to a uniform family of bases are described by the same ( $A, C$ ) pairs.

As this observation will not be needed anywhere else in the following, we shall content ourselves with mentioning it without providing any proof. A
complete justification would require the introduction of some more apparatus and distract the reader from the main objectives of the paper.

In a uniform family the bases $x_{-}(0)$ and $x_{+}(0)$ are essentially the "forward and backward steady-state Kalman filter" estimates of any $x(0)$. The first vector is computed as $x_{-}(0):=E^{X_{-}} x(0)$, while the second can be readily obtained by first going to the dual basis, as follows.

First, compute $\bar{x}_{+}(0)$ by projecting $\bar{x}(0):=P^{-1} x(0)$ onto $X_{+}$, i.e., $\bar{x}_{+}(0):=E^{X_{1}} P^{-1} x(0)$, and then go back to the "primal," i.e., let $x_{+}(0)-$ $P_{+}^{-1} \bar{x}_{+}(0)$. Note that the projection of $x(0)$ onto $X_{+}$is not invariant in a forward uniform basis.

Once $x_{+}(0)$ and (by a dual argument) $\bar{x}_{-}(0)$ are computed, they can be used to generate two dual families of uniform bases in $\mathscr{X}$, say $z(0)$ and $\bar{z}(0)$, by setting $z(0):=E^{X} x_{+}(0)$ and $\bar{z}(0):=E^{X} \bar{x}_{-}(0)$. It is immediate to check that $z(0)$ and $\bar{z}(0)$ are indeed dual bases and are related by the transformation $\bar{z}(0)=P^{-1} z(0)$, where $P=E z(0) z(0)^{\prime}$. Hence,

Proposition 4.3. An arbitrary basis $x(0)$ in a minimal splitting subspace $X$ can be uniquely extended to the whole family $\mathscr{X}$ in a uniform way (in either the forward or the backward sense).

### 4.2. A Generalized Positive-Real Lemma

In this subsection we shall describe the family $\mathscr{W}_{A, C}$ of all minimal spectral factors of a spectral density matrix $\Phi(s)$ having a fixed arbitrary pole dynamics described by the (observable) pair ( $A, C$ ). As announced, the main result will be a parametrization of $\mathscr{W}_{A, C}$ in terms of the solution set of a certain linear matrix inequality, much in the spirit of the well-known parametrization obtained for analytic spectral factors. There are some nontrivial difficulties to be solved in the process of obtaining this generalization, and some of them will be discussed at the end of the section.

Theorem 4.1. Let $W(s)$ be a minimal spectral factor with minimal realization

$$
W(s)=\left(\begin{array}{l|l}
A & B \\
\hline C & D
\end{array}\right)
$$

of the $m \times m$ spectral density matrix $\Phi(s)$, i.e., let $\Phi(s)=W(s) W(-s)^{\prime}$. Then the family of all minimal spectral factors $V(s)$ of $\Phi(s)$ with pole dynamics described by the matrix pair ( $A, C$ ), i.e.,

$$
V(s)=\left(\begin{array}{c|c}
A & G \\
\hline C & J
\end{array}\right)
$$

is parametrized by the set $\mathscr{P}$ of symmetric solutions, $\Sigma=\Sigma^{\prime}$, of the following linear matrix incquality:

$$
\mathscr{L}(\Sigma):=\left[\begin{array}{c|c}
A \Sigma+\Sigma A^{\prime}+B B^{\prime} & \Sigma C^{\prime}+B D^{\prime}  \tag{4.8}\\
\hline C \Sigma+D B^{\prime} & D D^{\prime}
\end{array}\right] \geqslant 0
$$

in the sense that, for each such $V(s)$,

$$
\left[\begin{array}{l}
G  \tag{4.9}\\
J
\end{array}\right]\left[\begin{array}{ll}
G^{\prime} & J^{\prime}
\end{array}\right]=\mathscr{L}(\Sigma)
$$

for some $\Sigma \in \mathscr{P}$, and, conversely, given any $\Sigma \in \mathscr{P}$, a full-rank factorization of $\mathscr{L}(\Sigma)$ as in (4.9) yields a minimal spectral factor $V(s)$. This correspondence is one-to-one, provided spectral factors which differ by right multiplication by a constant orthogonal matrix are identified.

If $R:=\Phi(\infty)$ is nonsingular (and hence $>0$ ), the parameter set $\mathscr{P}$ coincides with the set of symmetric solutions of the algebraic Riccati inequality

$$
\begin{equation*}
\Lambda(\Sigma):=A \Sigma+\Sigma A^{\prime}-\left(\Sigma C^{\prime}+B D^{\prime}\right) R^{-1}\left(C \Sigma+D B^{\prime}\right) \geqslant 0 \tag{4.10}
\end{equation*}
$$

In this case, if the state-space realizations of $W(s)$ and $V(s)$ are written in standard form, i.e.,

$$
\left[\begin{array}{c}
B  \tag{4.11}\\
D
\end{array}\right]=\left[\begin{array}{c|c}
B_{1} & B_{2} \\
\hline D_{1} & 0
\end{array}\right], \quad\left[\begin{array}{c}
G \\
J
\end{array}\right]=\left[\begin{array}{c|c}
G_{1} & G_{2} \\
\hline J_{1} & 0
\end{array}\right]
$$

with $D_{1}$ and $J_{1}$ square $m \times m$ and nonsingular, then the parametrization is described by the equations

$$
\begin{align*}
G_{1} & =\left(\Sigma C^{\prime} D_{1}^{-T}+B_{1}\right)\left(J_{1}^{-1} D_{1}\right)^{\prime}  \tag{4.12a}\\
G_{2} G_{2}^{\prime} & =\Lambda(\Sigma)  \tag{4.12b}\\
J_{1} J_{1}^{\prime} & =D_{1} D_{1}^{\prime} \tag{4.12c}
\end{align*}
$$

establishing a correspondence

$$
\Sigma \leftrightarrow\left[\begin{array}{c|c}
G_{1} & G_{2} \\
\hline J_{1} & 0
\end{array}\right],
$$

which is one-to-one provided that matrices $G_{2}$ and $J_{1}$ differing only by right multiplication by square orthogonal matrices are identified.

The proof of this theorem requires a preliminary observation relating the minimality of realizations of the spectral matrix $\Phi(s)$ to the minimality of realizations of the factors. Although the result is probably well known, for completeness we shall provide a proof.

Lemma 4.1. Consider the rational matrix $W(s)$ with state-space realization

$$
W(s)=\left[\begin{array}{c|c}
A & B \\
\hline C & D
\end{array}\right] .
$$

Then the spectral density matrix $\Phi(s):=W(s) W(-s)^{\prime}$ has the Hamiltonian realization

$$
\Phi(s) \equiv\left[\begin{array}{cc|c}
A & B B^{\prime} & B D^{\prime} \\
0 & -A^{\prime} & -C^{\prime} \\
\hline C & D B^{\prime} & D D^{\prime}
\end{array}\right]=:\left[\begin{array}{c|c}
\mathscr{A} & \mathscr{B} \\
\hline \mathscr{C} & \mathscr{D}
\end{array}\right]
$$

Assume the realization $\left[\begin{array}{l|l}\Lambda & B \\ \hline C & D\end{array}\right]$ is minimal. Then $W(s)$ is a minimal spectral factor of $\Phi(s)$ if and only if $\left[\mathscr{A} \left\lvert\, \mathscr{\mathscr { B }} \begin{array}{l}\mathscr{C} \\ \hline \mathscr{D}\end{array}\right.\right]$ is a minimal realization of $\Phi(s)$.

Proof. It is immediate to check that

$$
\begin{aligned}
& \mathscr{C}(s I-\mathscr{A})^{-1} \mathscr{B}+\mathscr{D} \\
&= {\left[\begin{array}{ll}
C & D B^{\prime}
\end{array}\right]\left[\begin{array}{cc}
s I-A & -B B^{\prime} \\
0 & s I+A^{\prime}
\end{array}\right]^{-1}\left[\begin{array}{c}
B D^{\prime} \\
-C^{\prime}
\end{array}\right]+D D^{\prime} } \\
&= {\left[\begin{array}{ll}
C & D B^{\prime}
\end{array}\right]\left[\begin{array}{cc}
(s I-A)^{-1} & (s I-A) \\
{ }^{1} B B^{\prime}\left(s I+A^{\prime}\right)^{1} \\
0 & \left(s I+A^{\prime}\right)^{-1}
\end{array}\right] } \\
& \cdot\left[\begin{array}{c}
B D^{\prime} \\
-C^{\prime}
\end{array}\right]+D D^{\prime} \\
&= C(s I-A)^{-1} B D^{\prime}+C(s I-A)^{-1} B B^{\prime}\left(-s I-A^{\prime}\right)^{-1} C^{\prime} \\
&+D B^{\prime}\left(-s I-A^{\prime}\right)^{-1} C^{\prime}+D D^{\prime} \\
&= {\left[C(s I-A)^{-1} B+D\right]\left[B^{\prime}\left(-s I-A^{\prime}\right){ }^{1} C^{\prime}+D^{\prime}\right] } \\
&= W(s) W(-s)^{\prime} .
\end{aligned}
$$

The statement about minimality is then a direct consequence of the definition of a minimal spectral factor.

Proof of Theorem 4.1. Let $\Sigma \in \mathscr{P}$ be a symmetric solution of the LMI (4.8), and let $\left[\frac{G}{J}\right]$ be a full-column-rank matrix satisfying the factorization equation (4.9). Notice that all such full-rank "square roots" of $\mathscr{L}(\Sigma)$ differ by right multiplication by orthogonal matrices, so that there is just one equivalence class of factors attached to each $\Sigma$.

It is then a standard calculation to check that the rational matrix

$$
V(s) \equiv\left[\begin{array}{c|c}
A & G \\
\hline C & J
\end{array}\right]
$$

is a spectral factor of $\Phi(s)$. In fact, in view of the identity

$$
\left[C(s I-A)^{-1} \mid I\right]\left[\begin{array}{c|c}
A \Sigma+\Sigma A^{\prime} & \Sigma C^{\prime} \\
\hline C \Sigma & 0
\end{array}\right]\left[\begin{array}{c}
\left(-s I-A^{\prime}\right)^{-1} C^{\prime} \\
I
\end{array}\right]=0
$$

we have

$$
\begin{aligned}
V(s) V(-s)^{\prime}= & {\left[C(s I-A)^{-1} \mid I\right]\left[\frac{G}{J}\right]\left[G^{\prime} \mid J^{\prime}\right]\left[\frac{\left(-s I-A^{\prime}\right)^{-1} C^{\prime}}{I}\right] } \\
= & {\left[C(s I-A)^{-1} \mid I\right]\left[\left.\frac{A \Sigma+\Sigma A^{\prime}+B B^{\prime} \mid \Sigma C^{\prime}+B D^{\prime}}{C \Sigma+D B^{\prime}} \right\rvert\, \frac{D D^{\prime}}{}\right] } \\
& \cdot\left[\frac{\left(-s I-A^{\prime}\right)^{-1} C^{\prime}}{I}\right] \\
= & {\left[C(s I-A)^{-1} \mid I\right]\left[\frac{B}{D}\right]\left[B^{\prime} \mid D^{\prime}\right] \cdot\left[\frac{\left(-s I-A^{\prime}\right)^{-1} C^{\prime}}{I}\right] } \\
= & W(s) W(-s)^{\prime} .
\end{aligned}
$$

Minimiality of $V(s)$ follows immediately, as the McMillan degree $\delta[V]$ is exactly half of the degree of $\Phi$.

Conversely, let

$$
V(s) \equiv\left[\begin{array}{c|c}
A & G \\
\hline C & J
\end{array}\right]
$$

be another minimal spectral factor of $\Phi(s)$. Then it follows from Lemma 4.1 that there are two minimal Hamiltonian realizations of the spectrum $\Phi(s)$, i.e.,

$$
\Phi(s)=\left[\begin{array}{cc|c}
A & B B^{\prime} & B D^{\prime} \\
0 & -A^{\prime} & -C^{\prime} \\
\hline C & D B^{\prime} & D D^{\prime}
\end{array}\right]=:\left[\begin{array}{c|c}
\mathscr{A} & \mathscr{B} \\
\hline \mathscr{C} & \mathscr{D}
\end{array}\right]
$$

and

$$
\Phi(s) \equiv\left[\begin{array}{cc|c}
A & G G^{\prime} & G J^{\prime} \\
0 & -A^{\prime} & -C^{\prime} \\
\hline C & J G^{\prime} & J J^{\prime}
\end{array}\right]=:\left[\begin{array}{c|c}
\mathscr{F} & \mathscr{G} \\
\hline \mathscr{H} & \mathscr{F}
\end{array}\right]
$$

so the two realizations must be related by a similarity transformation. In fact, the similarity transformation is unique. Hence there exists a unique invertible matrix $S$ such that

$$
\left[\begin{array}{c|c|c}
s & 0  \tag{4.13}\\
\hline 0 & I
\end{array}\right]\left[\begin{array}{c|c}
\mathscr{F} & \mathscr{G} \\
\hline \mathscr{H} & \mathscr{J}
\end{array}\right]=\left[\begin{array}{c|c}
\mathscr{H} & \mathscr{B} \\
\hline \mathscr{E} & \mathscr{D}
\end{array}\right]\left[\begin{array}{c|c}
S & 0 \\
\hline 0 & I
\end{array}\right] .
$$

The matrix $S$ is determined by the set of relations $\mathscr{H} \mathscr{F}^{k}=\mathscr{C} \mathscr{A}^{k} S, k=$ $0,1, \ldots$, and by the full-column-rank property of the "observability" matrices it can be obtained as

$$
S=\left[\begin{array}{c}
\mathscr{C} \\
\mathscr{C} \mathscr{A} \\
\mathscr{C} \mathscr{A}^{2} \\
\vdots
\end{array}\right]^{-L}\left[\begin{array}{c}
\mathscr{H} \\
\mathscr{H} \mathscr{F} \\
\mathscr{H}^{2} \\
\vdots
\end{array}\right],
$$

where the superscript $-L$ denotes left inverse. Note that the first column blocks of these compound matrices coincide, since

$$
\left[\begin{array}{c}
\mathscr{E} \\
\mathscr{E} \mathscr{A}^{2} \\
\mathscr{C} \mathscr{A}^{2} \\
\vdots
\end{array}\right]=\left[\begin{array}{c|c}
C & D B^{\prime} \\
C A & \vdots \\
C A^{2} & \\
\vdots &
\end{array}\right], \quad\left[\begin{array}{c}
\mathscr{H} \\
\mathscr{H} \mathscr{F} \\
\mathscr{H} \mathscr{F}^{2} \\
\vdots
\end{array}\right]=\left[\begin{array}{c|c}
C & J G^{\prime} \\
C A & \vdots \\
C A^{2} & \\
\vdots &
\end{array}\right]
$$

so that $S$ has the special form

$$
S=\left[\begin{array}{c|c}
I & S_{12} \\
\hline 0 & S_{2}
\end{array}\right],
$$

with $S_{2}$ invertible. Because of this particular structure of $S$ the similarity equations (4.13) yield $-S_{2} A^{\prime}=-A^{\prime} S_{2}$ and $-S_{2} C^{\prime}=-C^{\prime}$, i.e., $A=$ $S_{2}^{-T} A S_{2}^{\prime}$ and $C=C S_{2}^{\prime}$, respectively. We then obtain $C A^{k}=C A^{k} S_{2}^{\prime}, k=$ $0,1, \ldots$, and, recalling the observability of the pair ( $C, A$ ), we conclude that $S_{2}=I$. The remaining similarity equations become

$$
\begin{align*}
G G^{\prime} & =A S_{12}+S_{12} A^{\prime}+B B^{\prime}  \tag{4.14a}\\
\quad[ & \left.=A S_{12}^{\prime}+S_{12}^{\prime} A^{\prime}+B B^{\prime}(\text { by symmetry })\right] \\
G J^{\prime} & =B D^{\prime}+S_{12} C^{\prime}  \tag{4.14b}\\
{\left[J G^{\prime}\right.} & \left.=D B^{\prime}+C S_{12}^{\prime}(\text { by symmetry })\right] \\
J G^{\prime} & =D B^{\prime}+C S_{12}  \tag{4.14c}\\
J J^{\prime} & =D D^{\prime} \tag{4.14d}
\end{align*}
$$

Now by uniqueness of the similarity transformation, $S_{12}=S_{12}^{\prime}$. Denoting $S_{12}$ by $\Sigma$, we may rewrite Equations (4.14) in compact form as

$$
\left[\begin{array}{c}
G \\
\hline J
\end{array}\right]\left[G^{\prime} \mid J^{\prime}\right]=\left[\begin{array}{c|c}
A \Sigma+\Sigma A^{\prime}+B B^{\prime} & \Sigma C^{\prime}+B D^{\prime} \\
\hline C \Sigma+D B^{\prime} & D D^{\prime}
\end{array}\right]=: \mathscr{L}(\Sigma) \geqslant 0
$$

This shows that to every spectral factor $V(s)$ with the same pole structure as $W(s)$ there corresponds a unique solution $\Sigma$ of the LMI (4.8), satisfying the factorization equation (4.9).

The particularization to spectra which are nonsingular at infinity, i.e., $R:=\Phi(\infty)=D D^{\prime}>0$, follows from a standard block-diagonalization argument.

Remark 1. This result seems to be more general than the various "unstable" versions of the positive-real lemma, e.g., [10, Theorem 9.5], that have appeared in the literature so far. To our best knowledge all the existing versions of the positive-real lemma are stated in terms of a rational represen-
tation of $\Phi(s)$ deduced from an additive decomposition of the spectrum of the type

$$
\Phi(s)=Z(s)+Z(-s)^{\prime} \quad \text { with } \quad Z(s) \equiv\left[\begin{array}{c|c}
A & \bar{C} \\
\hline C & \frac{1}{2} R
\end{array}\right]
$$

However, as we shall see in an example below, not always does the spectrum $\Phi(s)$ admit an additive decomposition with $Z(s)$ having the same pole dynamics ( $A, C$ ) of an arbitrary minimal spectral factor $W(s)$. In other words, the equation $W(s) W(-s)^{\prime}=Z(s)+Z(-s)^{\prime}$ may fail to have a solution $Z(s)$ in the general case of an arbitrary pole location of $W(s)$. Therefore it is better to describe the spectral function $\Phi$ in terms of a reference factorization of the type $W(s) W(-s)^{\prime}$ as done in the statement of Theorem 4.1. For this reason Theorem 4.1 covers situations where the existing versions of the positive-real lemma are not applicable. Consider for instance the following example.

## Example. Let

$$
W(s):=1+\frac{2 s}{s^{2}-1} \equiv\left[\begin{array}{rr|r}
1 & 0 & 1 \\
0 & -1 & 1 \\
\hline 1 & 1 & 1
\end{array}\right]
$$

be a minimal spectral factor of

$$
\Phi(s)=W(s) W(-s)=1-\frac{4 s^{2}}{\left(s^{2}-1\right)^{2}}
$$

There is no additive decomposition $\Phi(s)=Z(s)+Z(-s)$ with $Z$ having the same poles as $W(s)$. Such a $Z$ should in fact have the form

$$
Z(s) \equiv\left[\begin{array}{rr|r}
1 & 0 & \bar{c}_{1} \\
0 & -1 & \bar{c}_{2} \\
\hline 1 & 1 & \frac{1}{2}
\end{array}\right]=\frac{1}{2}+\frac{\left(\bar{c}_{1}+\bar{c}_{2}\right) s+\left(\bar{c}_{1}-\bar{c}_{2}\right)}{s^{2}-1}
$$

so that

$$
Z(s)+Z(-s)=1+\frac{2\left(\bar{c}_{1}-\bar{c}_{2}\right)}{s^{2}-1} \neq 1-\frac{4 s^{2}}{\left(s^{2}-1\right)^{2}}=\Phi(s)
$$

for all $\bar{c}_{1}$ and $\bar{c}_{2}$. Therefore no such additive decomposition can exist. Thus, for this spectral function there is no "acausal" positive-real lemma parametrizing the spectral factors $V(s)$ having the particular choice of pole dynamics of the example. On the other hand it is relatively easy to compute say all square ( = scalar) factors of this type, which must look like

$$
V(s) \equiv\left[\begin{array}{rr|r}
1 & 0 & g_{1} \\
0 & -1 & g_{2} \\
\hline 1 & 1 & j
\end{array}\right]
$$

These factors are indeed four, corresponding to the four distinct solutions of the algebraic Riccati equation obtained from (4.10) with the equality sign.

In general there is an additive decomposition of $\Phi(s)$ with $Z(s)$ having the same pole dynamics of a spectral factor $W(s)$ if and only if the Lyapunov equation $A \Sigma+\Sigma A^{\prime}+B B^{\prime}=0$ is solvable.

Remark 2. The parametrization of all spectral factors $V(s)$ is described by the factorization equation (4.9). If $A$ has unmixed spectrum, there is a unique solution to the Lyapunov equation $A \Sigma+\Sigma A^{\prime}=G G^{\prime}-B B^{\prime}$ [the ( 1,1 )-block equation of (4.9)], and the solution $\Sigma \in \mathscr{P}$ parametrizing $V(s)$ is determined by the matrix $G$ alone. Necessarily, such a $\Sigma$ satisfies also the equation $\Sigma C^{\prime}=G J^{\prime}-B D^{\prime}$ [the $(1,2)$ block]. In case $A$ does not have unmixed spectrum, the Lyapunov equation has many solutions, but only one of these solutions solves also the second equation.

### 1.3. From Spectral Factors to Acausal Realizations

Now that the relation between all minimal spectral factors with the same pole dynamics has been described, we would like to study how the corresponding rcalizations of $d y$ are related. In the Fourier domain these realizations can be seen as "shaping filter" representations of the type $d \hat{y}=W d \hat{w}$, where $d \hat{w}$ is the Fourier transform of a suitable Wiener process associated to $W$. It is well known however that there is in gencral an inherent nonuniqueness of the process $d w$ asociated to nonsquare, or, more precisely, to non-left-invertible factors. Many $d w$ 's can serve as "input" processes in a
"shaping filter" representation of $d y$ in this case [21]. The general formulas describing the all-pass filters relating the input noise processes of different realizations involve "exogenous" Wiener processes uncorrelated with $d y$. We shall here restrict to a particular case where exogenous noise is not needed.

Assume $R:=\Phi(\infty)$ nonsingular, and consider the problem of describing the input processes $d v$ associated to the family of all minimal square acausal realizations of $d y$, with the same pole dynamics of a fixed "reference" realization ( $A, B, C, D, d w$ ), not necessarily square. The family of minimal realizations to be described will hence be internal minimal realizations of $d y$, in which case we have $\mathbf{H}(d v)=\mathbf{H}(d y)=\mathbf{H}(x)$ and the multiplicity of the joint process $[x, d y]^{\prime}$ is equal to $m$, the dimension of $d y$.

Let our reference minimal realization $(A, B, C, D, d w)$ be written in standard form, with $D$ partitioned as $D=\left[R^{1 / 2} 0\right]$ :

$$
\begin{align*}
& d z=A z d t+B_{1} d w_{1}+B_{2} d w_{2}  \tag{4.15a}\\
& d y=C z d t+R^{1 / 2} d w_{1} \tag{4.15b}
\end{align*}
$$

and correspondingly let ( $A, G, C, R^{1 / 2}, d v$ ) define an internal stochastic realization

$$
\begin{align*}
& d x=A x d t+G d v  \tag{4.16a}\\
& d y=C x d t+R^{1 / 2} d v \tag{4.16b}
\end{align*}
$$

of the same process $d y$, with the same pole dynamics ( $A, C$ ). Note that the driving Wiener process in (4.15), $d w=\left[d w_{1}^{\prime} d w_{2}^{\prime}\right]$ has been written in a partitioncd form congruent to the standard form of $D=\left[R^{1 / 2} 0\right]$, as explained above, so that both $d v$ and $d w_{1}$ are $m$-dimensional.

Now, by subtracting (4.16) from (4.15), we obtain the equations

$$
\begin{align*}
d(z-x) & =A(z-x) d t+B_{1} d w_{1}-G d v+B_{2} d w_{2}  \tag{4.17a}\\
0 & =C(z-x) d t+R^{1 / 2}\left(d w_{1}-d v\right) \tag{4.17b}
\end{align*}
$$

which, introducing the numerator matrix $\Gamma:=A-G R^{-1 / 2} C$, can be further elaborated into a stochastic realization

$$
\begin{align*}
d(z-x) & =\Gamma(z-x) d t+\left(B_{1}-C\right) d w_{1}+B_{2} d w_{2}  \tag{4.18a}\\
d v & =R^{-1 / 2} C(z-x) d t+d w_{1} \tag{4.18b}
\end{align*}
$$

of the stationary-increment process $d v$, in terms of the Wiener process $d w$, with state process $z-x$. In the Fourier domain this is equivalent to

$$
\begin{equation*}
d \hat{v}(i \omega)=\left([I, 0]+R^{-1 / 2} C(i \omega I-\Gamma)^{-1}\left[\left(B_{1}-G\right), B_{2}\right]\right) d \hat{\omega}(i \omega) \tag{4.19}
\end{equation*}
$$

relating the spectral measures $d \hat{w}$ and $d \hat{v}$. Now, for $d v$ to be a (wide-sense) Wiener process, it will be necessary and sufficient that the $m \times p$ rational matrix $L(s):=\left[\begin{array}{ll}I & 0\end{array}\right]+R^{-1 / 2} C(i \omega I-\Gamma)^{-1} L$, with $L:=\left[\left(B_{1}-G\right) B_{2}\right]$, be an all-pass function, i.e.,

$$
\begin{equation*}
L(s) L(-s)^{\prime}=I \tag{4.20}
\end{equation*}
$$

We shall now state a lemma, which in some sense is a dual to Lemma 2.1, giving a necessary and sufficient condition for this to happen.

Lemma 4.2. Let $(H, \Gamma)$ be a fixed observable pair, and let $\mathcal{Z}$ denote the set of symmetric solutions $Z=Z^{\prime}$ of the homogeneous Riccati inequality

$$
\begin{equation*}
\Gamma Z+Z \Gamma^{\prime}+Z H^{\prime} H Z \leqslant 0 \tag{4.21}
\end{equation*}
$$

Consider the $m \times p$ rutional matrix $L(s):=J+I I(s I-\Gamma)^{-1} L$, where $J=\left[\begin{array}{ll}I & 0\end{array}\right]$ and $L=\left[\begin{array}{ll}L_{1} & L_{2}\end{array}\right]$ are conformably partitioned. Then $L(s)$ is allpass, i.e., (4.20) holds, if and only if there is $Z \in \mathscr{Z}$ such that

$$
\begin{align*}
L_{1} & =-Z H^{\prime}  \tag{4.22a}\\
L_{2} L_{2}^{\prime} & =\left(\Gamma Z+Z \Gamma^{\prime}+Z H^{\prime} H Z\right) \tag{4.22b}
\end{align*}
$$

In fact, if the matrix $L$ is taken of full column rank, the set of all such all-pass functions $L(s)$ is in one-to-one correspondence with $\mathscr{Z}$.

Proof. Writing $L(s)$ as

$$
\left[\begin{array}{ll}
H(s I-\Gamma)^{-1} & I
\end{array}\right]\left[\begin{array}{l}
L \\
J
\end{array}\right]
$$

and computing $L(s) L(-s)^{\prime}$, one readily finds

$$
L(s) L(-s)^{\prime}=I+\left[\begin{array}{ll}
H & L_{1}^{\prime}
\end{array}\right]\left[\begin{array}{cc}
s I-\Gamma & -L L^{\prime} \\
0 & s I+\Gamma^{\prime}
\end{array}\right]^{-1}\left[\begin{array}{c}
L_{1} \\
-H^{\prime}
\end{array}\right]
$$

which shows that $L(s)$ solves (4.20) if and only if the (Hamiltonian) triple

$$
\mathscr{C}:=\left[\begin{array}{ll}
H & L_{1}^{\prime}
\end{array}\right], \mathscr{A}:=\left[\begin{array}{cc}
\Gamma & L L^{\prime}  \tag{4.23}\\
0 & -L^{\prime}
\end{array}\right], \mathscr{B}:=\left[\begin{array}{c}
L_{1} \\
-H^{\prime}
\end{array}\right]
$$

provides a (nonminimal) realization with state space of dimension $2 n$ of the $m \times m$ zero transfer function. Now, the same argument as in the proof of Lemma 2.1 leads to the equivalent statement

$$
\mathscr{X}_{c}=\mathscr{E}_{n 0}=\operatorname{im}\left[\begin{array}{l}
Z \\
I
\end{array}\right]
$$

for a unique square $n \times n$ matrix $Z$. Then the very definition of $\mathscr{X}_{n 0}$ as the maximal subspace such that

$$
\begin{align*}
& \mathscr{E} \mathscr{X}_{n 0}=0  \tag{4.24}\\
& \mathscr{A} \mathscr{X}_{n 0} \subset \mathscr{X}_{n 0} \tag{4.25}
\end{align*}
$$

leads to the set of equations

$$
\begin{align*}
{\left[\begin{array}{ll}
H & L_{1}^{\prime}
\end{array}\right]\left[\begin{array}{c}
Z \\
I
\end{array}\right] } & =0  \tag{4.26}\\
{\left[\begin{array}{cc}
\Gamma & L L^{\prime} \\
0 & -\Gamma^{\prime}
\end{array}\right]\left[\begin{array}{c}
Z \\
I
\end{array}\right] } & =\left[\begin{array}{c}
Z \\
I
\end{array}\right] \Lambda \tag{4.27}
\end{align*}
$$

where $\Lambda$ is some square $n \times n$ matrix. Writing down explicitly (4.27) yields (4.22) and (4.21). The symmetry of $Z$ also follows, for if $Z$ solves these equations, then $Z^{\prime}$ is also a solution. Thus, by uniqueness, we conclude that necessarily $Z=Z^{\prime}$, and this concludes the proof.

This lemma provides a procedure for generating the input processes for the family of all minimal internal acausal stochastic realizations of $d y$, with the same pole dynamics $(A, C)$ of the reference realization (4.15).

Theorem 4.2. Let dy be a stationary-increment process admitting the minimal realization (4.15). Denote for short $R^{-1 / 2} \mathrm{C}$ by $H$. Then the minimal internal realizations of $d y$, with the same pole dynamics of (4.15), are
precisely all stochastic systems of the form (4.16) where $G$ and do are obtained by the following transformation:
(1) $G=B_{1}+\Sigma H^{\prime}$;
(2) $d v$ is the output of the all-pass filter (4.18) with $\Gamma=A-G H$, the matrix $G$ being defined as above;
and where $\Sigma$ is any symmetric solution of the algebraic Riccati equation

$$
\begin{equation*}
A \Sigma+\Sigma A^{\prime}-\left(B_{1}+\Sigma H^{\prime}\right)\left(B_{1}^{\prime}+H \Sigma\right)+B B^{\prime}=0 \tag{4.28}
\end{equation*}
$$

In fact, the correspondence between internal realizations with pole dynamics described by the pair $(A, C)$ and the set of symmetric solutions of (4.28) is one-to-one, provided, as usual, that all matrices $\left[G^{\prime} R^{1 / 2}\right] U$, with $U$ an arbitrary $m \times m$ orthogonal matrix, are identified.

The transformation can be interpreted as output injection on the stochastic system (4.15), in the sense that, denoting the numerator matrix of (4.15) by $\Gamma_{1}:=A-B_{1} R^{-1 / 2} C=A-B_{1} H$, the family of all pairs $(\Gamma, G)$ describing internal realizations (4.16) of dy with fixed pole dynamics $(A, C)$ is described by the formulas

$$
\begin{equation*}
\Gamma=\Gamma_{1}-\Sigma H^{\prime} H, \quad G=B_{1}+\Sigma H^{\prime} \tag{4.29}
\end{equation*}
$$

where $\Sigma$ ranges on the set of symmetric solutions of the Riccati equation

$$
\begin{equation*}
\Gamma_{1} \Sigma+\Sigma \Gamma_{1}^{\prime}-\Sigma H^{\prime} H \Sigma+B_{2} B_{2}^{\prime}=0 \tag{4.30}
\end{equation*}
$$

equivalent to (4.28).
Proof. Sufficiency: Let $\Sigma$ be a symmetric solution of (4.28), and $G$ and $d v$ be as in the statement of the theorem. Let also $V_{\Sigma}(s):=R^{1 / 2}+C(s I-$ $A)^{-1} G$, and $L_{\Sigma}(s)$ be the transfer function of the system (4.18) corresponding to $\Sigma$. Then

$$
d \hat{v}=L_{\Sigma}(i \omega)\left[\begin{array}{l}
d \hat{w}_{1} \\
d \hat{w}_{2}
\end{array}\right]
$$

and by Lemma $4.2 L_{\Sigma}$ is all-pass, so that $d v$ is a wide-sense Wiener process.

Now, it is not hard to check that $W(s)=V_{\Sigma}(s) L_{\Sigma}(s)$. Since by assumption we have the representation

$$
d \hat{y}=W(i \omega)\left[\begin{array}{l}
d \hat{w}_{1} \\
d \hat{w}_{2}
\end{array}\right]
$$

it follows readily that $d \hat{y}=V_{\Sigma}(i w) d \hat{v}$, so that the output of the system with transfer function $V_{\Sigma}$ and input $d v$ is indeed the process $d y$. Therefore we have obtained an internal realization. The rest is easy.

## Remarks.

(1) If the reference realization (4.15) is already internal, i.e., $B_{2}=0$, then Theorem 4.2 is the exact dual of the "pole-flipping" result of Section 2, Theorem 2.1. In this case the output injection transformation (4.29) "flips" the zeros of the spectral factor $W(s)=R^{1 / 2}+C(s I-A)^{-1} B$ with respect to the imaginary axis. The eigenstructure of the numerator matrix $\Gamma$ of the factor $V_{\Sigma}(s)$ corresponding to a solution of the homogeneous equation

$$
\Gamma_{1} \Sigma+\Sigma \Gamma_{1}^{\prime}-\Sigma C^{\prime} R^{-1} C \Sigma=0
$$

is then described by the dual version of Theorem 2.2 obtained by exchanging $Q$ with $-\Sigma, A$ with $\Gamma^{\prime}$, and $B$ with $H^{\prime}$.
(2) Some analogy with "pole flipping" remains even in the nonhomogeneous case when $B_{2} \neq 0$. The "zero-flipping" picture of the internal case now holds on a restricted subspace, namely the largest $\Gamma_{1}^{\prime}$-invariant subspace orthogonal to the columns of $B_{2}$. This subspace, denoted $\left\langle\Gamma_{1} \mid B_{2}\right\rangle^{\perp}$ is in fact the state space of the zero dynamics of $W(s)$. On this invariant subspace the Riccati equation (4.30) induces a homogeneous equation. The exact mechanism of "zero-flipping" for nonsquare spectral factors is studied in detail in [22].
(3) The minimal solution $\Sigma_{-}$of the Riccati equation (4.30) corresponds to the numerator matrix $\Gamma_{-}$with an antistable zero structure [i.e., $\operatorname{Re} \lambda\left(\Gamma_{-}\right)>0$ ], and the maximal solution $\Sigma_{+}$to a stable zero structure [i.e., $\left.\operatorname{Re} \lambda\left(\Gamma_{+}\right)<0\right]$. This difference from the situation encountered in Section 2 (Theorem 2.3) is of course due to the sign diversity in the quadratic term of (4.30). Note that the coercivity of $\Phi(s)$ guarantees that no $\Gamma$ has eigenvalues (zeros) on the imaginary axis and that the gap $\Sigma_{+}-\Sigma_{-}$is positive definite.
(4) Assume $A$ has unmixed spectrum. Then all symmetric solutions of (4.30) can be written as $\Sigma=P-\Pi$, where $P$ is the (unique, by the unmixed
spectrum assumption) solution of the Lyapunov equation

$$
A P+P A^{\prime}+B B^{\prime}=0
$$

and $\Pi$ solves the Riccati equation

$$
\begin{equation*}
A \Pi+\Pi A^{\prime}+\left(\bar{C}^{\prime}-\Pi C^{\prime}\right) R^{-1}\left(\bar{C}^{\prime}-\Pi C^{\prime}\right)^{\prime}=0 \tag{4.31}
\end{equation*}
$$

where $\bar{C}=D B^{\prime}+C P$ (this is an invariant quantity not depending on $\Sigma$ ). Observe that (4.31) looks like the standard algebraic Riccati equation encountered in stochastic realization theory $[2 ; 21$, Section 7] with the notable difference that $A$ is not asymptotically stable in general, so that $\Pi$ can no longer he interpreted as the state variance matrix of a minimal realization. In fact, both $P$ and $\Pi$ will be indefinite in the general case of acausal models. If $A$ is stable, one may recognize (4.28) as being precisely the algebraic Riccati equation describing the steady-state Kalman filter associated to the model (4.15). In this case $P=E x(t) x(t)^{\prime}$ and it is well known that the minimal and maximal solutions, usually denoted by $P_{-}$and $P_{+}$, of (4.31) are the state covariance matrices of the two "extrcme" realizations of $d y$ corresponding to the forward and backward steady-state Kalman filters of (4.15). These realizations are well known to correspond to the minimum-phase $[\operatorname{Re} \lambda(\Gamma)<0]$ and maximum-phase $[\operatorname{Re} \lambda(\Gamma)>0]$ spectral factors respectively. In particular it follows from this discussion that $\Sigma_{+}=P-P_{-}$and $\Sigma_{-}=P-P_{+}$, so that

$$
\Sigma_{+}-\Sigma_{-}=P_{+}-P_{-} .
$$

(5) There is a slight generalization of the identity (2.27) (compare [30]) which holds for the nonhomogeneous quadratic function $\Lambda_{\Gamma}(\Sigma): \Sigma \rightarrow \Gamma \Sigma+$ $\Sigma \Gamma^{\prime}-\Sigma H^{\prime} H \Sigma+B_{2} B_{2}^{\prime}$, namely

$$
\begin{equation*}
\Lambda_{\Gamma}(\Sigma)-\Lambda_{\Gamma}\left(\Sigma_{0}\right)=\bar{\Omega}_{\Gamma_{0}}\left(\Sigma-\Sigma_{0}\right), \tag{4.32}
\end{equation*}
$$

where $\bar{\Omega}_{\Gamma_{0}}(\Delta)$ is the homogeneous function $\Gamma_{0} \Delta+\Delta \Gamma_{0}^{\prime}-\Delta H^{\prime} H \Delta$. Letting $\mathscr{P}_{\Gamma}$ denote the set of symmetric solutions of the Riccati equation $\Lambda_{\Gamma}(\Sigma)=0$, from the identity (4.32) it follows, pretty much by the same argument used in Section 2, that

$$
\begin{equation*}
\mathscr{P}_{\Gamma}=\Sigma_{0}+\overline{\mathscr{Q}}_{\Gamma_{0}}, \tag{4.33}
\end{equation*}
$$

where $\overline{\mathscr{Q}}_{\Gamma_{0}}$ is the set of symmetric solutions of the homogeneous Riccati
equation $\bar{\Omega}_{\Gamma_{0}}(\Delta)=0, \Sigma_{0}$ is any particular solution of $\Lambda_{\Gamma}(\Sigma)=0$, and $\Gamma_{0}=\Gamma-\Sigma_{0} H^{\prime} H$. From this it is casy to obtain information on the maximal and minimal elements of $\mathscr{P}_{\Gamma}$ in the general case where $A$ is not necessarily stable (and in fact need not have unmixed spectrum). For instance letting $\Sigma_{0}$ equal to $\Sigma_{+}$and $\Sigma_{-}$, and arguing as in the proof of Theorem 2.3, it is seen that $\Sigma_{+}-\Sigma_{-}=P_{+}-P_{-}$, even in this more general situation.

## 5. STOCHASTIC BALANCING

Assume that a model of the form (3.1), perhaps coming from the description of a certain "physical" engineering problem, is given. As stressed in the introduction, we shall take the point of view of interpreting the model just as a particular realization of the output process $d y$. Without much loss of generality we shall also assume that the stochastic realization corresponding to the model is minimal, i.e., $X$ is a minimal Markovian splitting subspace. This roughly means that there are no degrees of freedom spent for unnecessary modeling of intermediate variables of the model which are actually white noise.

We shall start our discussion by attaching to each random variable $\xi$ in $X$ a pair of indices which quantify "how well" $\xi$ can be estimated on the basis of the past or future history of the output process. We shall then define a choice of basis in $X$ which has some "canonical" desirable properties in this respect. Initially our discussion will be completely coordinate-free.

For a random variable $\xi \in X$ we define the numbers

$$
\begin{equation*}
\eta_{1}(\xi):=\frac{\left\|E^{H^{+}} \xi\right\|^{2}}{\|\xi\|^{2}}, \quad \eta(\xi):=\frac{\left\|E^{H^{-}} \xi\right\|^{2}}{\|\xi\|^{2}} \tag{5.1}
\end{equation*}
$$

called respectively the future and the past relative efficiency of $\xi$. The numbers $\eta_{ \pm}(\xi)$ are nonnegative and $\leqslant 1$, and in the statistical literature are sometimes also referred to as the "percentage of explained variance" (of the random variable being estimated). Clearly, the larger $\eta_{ \pm}(\xi)$, the better (in the sense of smaller estimation error variance) will be the corresponding estimate $E^{H \pm} \xi$.

The relative efficiency indices have also a direct system-theoretic interpretation in terms of the observability and constructibility operators associated to $X[20,21]$, defined respectively as

$$
\begin{array}{ll}
\mathscr{O}: X \rightarrow H^{+}, & \mathscr{O} \xi:=E^{H^{+}} \xi, \\
\mathscr{C}: X \rightarrow H^{-}, & \mathscr{C} \xi:=E^{H^{-}} \xi . \tag{5.3}
\end{array}
$$

In terms of $\mathscr{O}$ and $\mathscr{C}$ the indices $\eta_{+}(\xi)$ and $\eta_{-}(\xi)$ may be interpreted as the relative "degree of observability" or the relative "degree of constructibility" of $\xi \in X$.

Recall that the observability and constructibility operators, introduced in geometric realization theory [20], play a somewhat similar role to the observability and reachability operators in deterministic systems theory in characterizing minimality of a state space. In fact the splitting property of a subspace $X$ can be shown to be equivalent to a factorization of the Hankel operator $\mathbb{H}:=\left.E^{H^{-}}\right|_{H^{+}}: H^{+} \rightarrow H^{-}$through the space $X$, as [20]

$$
\begin{equation*}
\mathbb{H}=\mathscr{C} \mathscr{O}^{*} \tag{5.4}
\end{equation*}
$$

a fundamental characterization of minimality being that $X$ is a minimal splitting subspace if and only if the factorization (5.4) is canonical, i.e., $\mathscr{E}$ and OO are both injective operators. Equivalently (in the finite-dimensional case), $\mathscr{O}^{*}=\left.E^{X}\right|_{H^{+}}$is surjective. Hence, for a minimal splitting subspace, both the constructibility and the observability Gramians, $\mathscr{C}^{*} \mathscr{C}$ and $\mathscr{O}^{*} \mathscr{O}$, are invertible maps $X \rightarrow X$.

It follows that in a minimal splitting subspace $X$ there are two distinct orthonormal bases of eigenvectors, say $\left(\xi_{1}^{+}, \ldots, \xi_{n}^{+}\right)$and $\left(\xi_{1}^{-}, \ldots, \xi_{n}^{-}\right)$in which the operators $\mathscr{O}^{*} \mathscr{O}$ and $\mathscr{C}^{*} \mathscr{E}$ diagonalize, i.e.,

$$
\begin{array}{ll}
\mathscr{G}^{*} \mathscr{O}=\operatorname{diag}\left\{\lambda_{1}^{+}, \ldots, \lambda_{n}^{+}\right\}, & 1 \geqslant \lambda_{1}^{+} \geqslant \cdots \geqslant \lambda_{n}^{+}>0 \\
\mathscr{C}^{*} \mathscr{C}=\operatorname{diag}\left\{\lambda_{1}^{-}, \ldots, \lambda_{n}^{-}\right\}, & 1 \geqslant \lambda_{1}^{-} \geqslant \cdots \geqslant \lambda_{n}^{-}>0 \tag{5.6}
\end{array}
$$

the statistical interpretation being that the states in $X$ can he ordered in two different ways: according to the magnitudes of their future or their past relative efficiency indices. It is in fact immediate from the definition (5.1) that, in the ordering according to the index $\boldsymbol{\eta}_{+}$, the " most observable" states are just those which lie parallel to the vector $\xi_{1}^{+}$, having maximal index $\eta_{+}(\xi)=\lambda_{1}^{+}$, while the "least observable" states $\xi$ are those parallel to the direction $\xi_{n}^{+}$, having the smallest possible relative efficiency $\eta_{+}(\xi)=\lambda_{n}^{+}$. Of course, a completely similar picture corresponds to the ordering induced by past relative efficiency.

Assumc for a moment that $H^{+} \cap H^{-}=\varnothing$ (which will bc the case if, say, the spectrum of the process is coercive [21]). Then a direction of "very observable" states in $X$, being at a small relative angle with the future subspace $H^{+}$, will generally form a "large" angle with the past subspace $H^{-}$ and hence give rise to projections onto $H^{-}$of small relative norm, i.e., to small $\eta_{-}(\xi)$. The opposite phenomenon is of course to be expected in case a
direction "very close" to $H^{-}$is selected. The idea of balancing in the stochastic framework has to do with a choice of basis which, roughly speaking, is meant to "balance" the two sets of efficiency indices. There is here a substantial difference from the deterministic case, however, in that we have now a whole family of minimal $X$ which needs to be considered simultaneously for the choice of a balanced basis. For this reason the stochastic procedure will necessarily be somehow less obvious and transparent than in the deterministic case.

In order to analyze the effects of choosing a particular basis, say $x(0)$, in a minimal splitting subspace $X$, we shall introduce the linear map $T_{x(0)}: \mathbb{R}^{n} \rightarrow$ $X$, defined by $T_{x(0)} a:=a^{\prime} x(0)$. Note that if $\mathbb{R}^{n}$ is equipped with the inner product $\langle a, b\rangle_{P}:=a^{\prime} P b$, where $P$ is the covariance matrix of $x(0)$, then $T_{x(0)}$ becomes an isometry. From this observation it is not hard to check that $T_{x(0)}$ has the following properties:

Lemma 5.1. Let $P$ be the covariance matrix of the basis $x(0)$ in $X$. Then,

$$
\begin{equation*}
T_{x(0)}^{-1}=P^{-1} T_{x(0)}^{*}, \quad T_{\bar{x}(0)}=T_{x(0)} P^{-1} \tag{5.7}
\end{equation*}
$$

where $\bar{x}(0)$ is the dual basis of $x(0)$.
Proof. Let $\xi=a^{\prime} x(0)$ and $\eta=b^{\prime} x(0)$ be arbitrary random variables in $X$. Then $\langle\xi, \eta\rangle=b^{\prime} P a=\left\langle b, T_{x(0)}^{*} T_{x(0)} a\right\rangle$, where the last inner product is ordinary Euclidean inner product in $\mathbb{R}^{n}$. It follows that $T_{x(0)}^{*} T_{x(0)} a=P a$ for all $a \in \mathbb{R}^{n}$, which proves the first identity. The second descends from expressing $\eta=\bar{b}^{\prime} \bar{x}(0)$ in the dual basis and using a similar argument, now with $\eta$ written as $\eta=T_{\bar{x}(0)} \bar{b}$, thus obtaining $T_{\bar{x}(0)}^{*} T_{x(0)} a=a$ and hence $T_{\bar{x}(0)}^{*}=T_{x(0)}^{-1}$. This proves the lemma.

Obviously the efficiency indices (5.1) can be expressed in terms of the coordinates $a, b$, once a specific basis has been chosen. In particular the expressions for the numerators will be quadratic forms described by certain symmetric positive definite matrices which we shall call, respectively, observability and constructibility Gramians (relative to that particular basis). Provided they are expressed in dual bases, the two Gramians have a particularly simple expression that will be given in the proposition below. Recall (Proposition 4.3) that a basis in an arbitrary $X$ can be extended, together with its dual, tot he whole family of minimal splitting subspaces $\mathscr{X}$ in such a way as to form a uniform basis.

Proposition 5.1. Let $x(0)$ be a basis in the minimal splitting subspace $X$, and $\bar{x}(0)$ be its dual basis. Then the constructibility and observability

Gramians relative to the bases $x(0)$ and $\bar{x}(0)$ respectively, are given by

$$
\begin{align*}
& \overline{\mathscr{C}^{*} \mathscr{C}}:=T_{x(0)}^{*} \mathscr{C}^{*} \mathscr{C} T_{x(0)}=P_{-},  \tag{5.8a}\\
& \overline{\mathscr{O}^{*} \mathscr{O}}:=T_{\bar{x}(0)}^{*} \mathscr{O}^{*} \mathscr{O} T_{\bar{x}(0)}=\bar{P}_{+}=P_{+}^{-1}, \tag{5.8b}
\end{align*}
$$

where $P_{-}$and $\bar{P}_{+}$are the covariance matrices of $x_{-}(0)$ and $\bar{x}_{+}(0)$ in the uniform basis induced by $x(0)$.

In particular the two Gramians do not depend on the particular minimal splitting subspace $X$ and are invariant over $\mathscr{X}$.

Proof. The formulas follow from the orthogonality of any minimal splitting subspace to the so-called "junk" spaces $N^{-}, N^{+}$(the subspace of $\mathrm{H}^{-}$orthogonal to the future and, respectively, the subspace of $\mathrm{H}^{+}$orthogonal to the past); se e.g., [21, Corollary 4.9]. This leads to the identities

$$
\begin{equation*}
\mathscr{C} \xi:=E^{I^{-}} \xi=E^{X_{-}} \xi, \quad \mathscr{O} \xi=E^{H^{+}} \xi=E^{X_{+}} \xi, \tag{5.9}
\end{equation*}
$$

the first of which, in view of (4.4a), can be rewritten as $\mathscr{C} \xi=a^{\prime} x_{-}(0)$ and immediately leads to (5.8a). The second follows by a similar computation, using the dual invariant projection property (4.4b).

Note that in the forward basis induced by $x(0)$, the expression of the observability Gramian would instead be

$$
\begin{equation*}
\overline{\mathscr{O}^{*} \mathscr{O}}:=T_{x(0)}^{*} \mathscr{O}^{*} \mathscr{O} T_{x(0)}=P P_{+}^{-1} P, \tag{5.10}
\end{equation*}
$$

which is no longer invariant.
The invariance of the two Gramians with respect to the particular state space of the realization, pointed out in the proposition above, clarifies the system-theoretic meaning of the notion of balanced realization, originally given by Desai and Pal [5] in terms of covariance matrices.

TheOrem 5.1. There is a choice of basis $\hat{x}(0):=\left[\hat{\xi}_{1}, \ldots, \hat{\xi}_{n}\right]^{\prime}$ in $X$, such that both the constructibility and observability Gramians are represented by a diagonal matrix. In fact, there is a diagonal matrix $\Sigma$, with positive entries

$$
\begin{equation*}
\Sigma=\operatorname{diag}\left\{\sigma_{1}, \ldots, \sigma_{n}\right\}, \quad 1 \geqslant \sigma_{1} \geqslant \cdots \geqslant \sigma_{n} \geqslant 0 \tag{5.11}
\end{equation*}
$$

such that, in the uniform basis induced by $\hat{x}(0)$ in $\mathscr{X}$, one has

$$
\begin{equation*}
\overline{\mathscr{C}^{*} \mathscr{C}}=\Sigma=\overline{\mathscr{C}^{*} \mathscr{O}}, \tag{5.12}
\end{equation*}
$$

where $\overline{\mathscr{C}^{*} \mathscr{C}}$ is the constructibility Gramian relative to the basis $\hat{x}(0)$ and $\widehat{\mathscr{O}^{*}}$ is the observability Gramian relative to the dual basis of $\hat{x}(0)$.

If the numbers $\sigma_{k}$ are all distinct, this choice of hasis is unique up to sign, i.e., for any other basis $\tilde{x}(0):=\left[\tilde{\xi}_{1}, \ldots, \tilde{\xi}_{n}\right]^{\prime}$ leading to a diagonal structure of the form (5.12), one has $\bar{\xi}_{k}= \pm \hat{\xi}_{k}, k=1, \ldots, n$.

Proof. Let $x(0)$ be any basis in $X$. The proof of this theorem consists in showing that there exists an $n \times n$ nonsingular matrix $T$ such that, in the new uniform basis induced by $\hat{x}(0):=T x(0)$, the covariance matrices $\hat{P}_{-}$and $\hat{P}_{+}^{-1}$ will have the diagonal structure of (5.12).

The following relations between transformed dual bases will be needed in the proof:

$$
\begin{equation*}
\hat{\bar{x}}(0)=\hat{P}^{-1} \hat{x}(0)=\hat{P}^{-1} T P \bar{x}(0)=T^{-*} \bar{x}(0) \tag{5.13}
\end{equation*}
$$

Note that the last equality descends from $\hat{P}=T P T^{*}$.
Now using (5.13), it is readily seen that in the new (uniform) basis, the two Gramians $P_{-}=T_{x(0)}^{*} \mathscr{C}^{*} \mathscr{C} T_{x(0)}$ and $\bar{P}_{+}=T_{\bar{x}(0)}^{*} \mathscr{O}^{*} \mathscr{O} T_{\bar{x}(0)}$ are transformed into

$$
\begin{equation*}
\hat{P}_{-}=T P_{-} T^{*}, \quad \hat{\bar{P}}_{+}=\hat{P}_{+}^{-1}=T^{-*} \bar{P}_{+} T^{-1} \tag{5.14}
\end{equation*}
$$

in other words, $P_{-}$and $P_{+}$transform in the same way under change of coordinates. The simultaneous diagonalization of $P_{-}$and $P_{+}^{-1}$ can then be obtained by a simple modification of the well-known result for deterministic systems (see e.g., [11]). The diagonalization procedure can be organized in the following algorithm.

Algorithm 5.1 (Computing the balancing transformation $T$ )

1. Compute a square factorization of $P_{-}$, i.e., let $P_{-}=R R^{\prime}$, where $R$ is square nonsingular, e.g., a Cholesky factor.
2. Do a singular-value decomposition of $R^{\prime} \bar{P}_{+} R$, i.e., compute the factorization $R^{\prime} \vec{P}_{+} R=U \Sigma^{2} U^{\prime}$ where $U$ is an orthogonal matrix and $\Sigma^{2}$ is diagonal with positive entries ordered by magnitude in the decreasing sense.
3. Define $T:=\Sigma^{1 / 2} U^{\prime} R^{-1}$. The matrix $T$ is the desired basis transformation matrix.
4. Check: Compute

$$
\begin{aligned}
T P_{-} T^{*} & =\Sigma^{1 / 2} U^{\prime} R^{-1} P_{-} R^{-\prime} U \Sigma^{1 / 2}=\Sigma, \\
T^{-*} \bar{P}_{+} T^{-1} & =\Sigma^{-1 / 2} U^{\prime} R^{*} \bar{P}_{+} R U \Sigma^{-1 / 2}=\Sigma .
\end{aligned}
$$

The proof of uniqueness can be found in the literature. See e.g., [6, Theorem 1] and [7].

Note that in view of (5.8a), (5.8b), and (5.12), the numbers $\left\{\sigma_{1}^{2}, \ldots, \sigma_{n}^{2}\right\}$ can be computed directly as the (ordered) eigenvalues of the ratio $P_{-} P_{+}^{-1}$.

The following statement, which elucidates the meaning of the elements of $\Sigma$ as the (nonzero) singular values of the Hankel operator of the process $d y$, will be reported here for completeness. It has been known for a long time [6, 26]. The proof in the present setup is particularly simple.

Proposition 5.2. The entries of $\Sigma=\operatorname{diag}\left\{\sigma_{1}, \ldots, \sigma_{n}\right\}$ are invariants of the process $d y$, equal to the nonzero singular values of the Hankel operator $\mathbb{H}$.

Proof. One just needs to notice that $\left\{\sigma_{1}^{2}, \ldots, \sigma_{n}^{2}\right\}$ are the eigenvalues of the operator $\mathscr{C} * \mathscr{C O} \mathscr{O}^{*} \mathscr{O}$, since

$$
\begin{equation*}
P_{-} P_{+}^{-1}=T_{x(0)}^{*} \mathscr{C}^{*} \mathscr{C} T_{x(0)} T_{\bar{x}(0)}^{*} \mathscr{O} * \mathscr{O} T_{\bar{x}(0)}=T_{x(0)}^{*} \mathscr{C}^{*} \mathscr{\mathscr { C }} \mathscr{Q}^{*} \mathscr{O} T_{\bar{x}(0)} \tag{5.15}
\end{equation*}
$$

and by (5.7) $T_{\bar{x}(0)} T_{x(0)}^{*}=I$. On the other hand, the squares of the nonzero singular values of $\mathbb{H}$ are the nonzero eigenvalues of $\mathbb{H}^{*} \mathbb{-}$, and it follows from the factorization (5.4) that the nonzero eigenvalues of $\mathbb{H} \mathbb{H}^{*} \mathbb{H}$ are indeed equal to those of $\mathscr{C}^{*} \mathscr{C} \mathscr{O}^{*} \mathscr{O}$.

The Hankel singular values coincide with the canonical correlation coefficients of the past and future spaces of the process $d y$. Relations between this concept and stochastic realization theory are discussed in several places in the literature, sce c.g., [26].

In conclusion, in our setup the concept of stochastic balancing is seen as a natural generalization of the deterministic idea of balancing for stable systems. In the geometric setting presented in this paper, however, the "statibility" (better, causality) of the model does not enter at all, as the choice of a particular state vector $x(0)$ has obviously nothing to do with the choice of a particular causality structure of the corresponding realization. The particular causality structure of the model influences instead the computation of the "balancing" basis transformation $\hat{x}(0)=T x(0)$ of Theorem 5.1.

In order to compute the transformation matrix $T$, knowledge of the minimal and maximal state covariance matrices $P_{-}$and $P_{+}$is required. The computation is (at least conceptually) straightforward if the original model is already in causal form, as in this case to compute $P_{-}$and $P_{+}$one needs to find just the minimal and the maximal solutions of the algebraic Riccati equation (4.28).

Finding these matrices in the general situation will involve solving a combination of Lyapunov and Riccati equations. One possible procedure is illustrated below.

Algorithm 5.2 (Algorithm for computing $P_{-}$and $P_{+}$)
Input data: The matrices ( $A, B, C, D$ ), with $A$ satisfying (1.2), $R:=$ $D D^{\prime}$ invertible, and ( $A, B, C$ ) a minimal triple.

1. Compute the minimal solution $Q$ _ of the algebraic Riccati equation $A Q+Q A^{\prime}+Q B B^{\prime} Q=0$,
2. Do state feedback: $A_{-}:=A+B B^{\prime} Q_{-}$, [then $\left.\operatorname{Re} \lambda\left(A_{-}\right)<0\right]$ and $C_{-}:=C+D B^{\prime} Q_{-}$.
3. Solve the Lyapunov equation $A_{-} P+P A_{-}^{\prime}+B B^{\prime}=0$ to find the variance matrix $P$.
4. Find the maximal solution $\Sigma_{+}$of the Riccati equation $A_{-} \Sigma+\Sigma A_{-}^{\prime}$ $-\left(B D^{\prime}+\Sigma C_{-}^{\prime}\right)\left(D D^{\prime}\right)^{-1}\left(B D^{\prime}+\Sigma C_{-}^{\prime}\right)^{\prime}=0$, and compute $P_{-}$from $\Sigma_{+}=$ $P-P_{-}$.
5. Set $H_{-}:=R^{-1 / 2} C_{-}$, and compute $B_{-}:=B_{1}+\Sigma_{+} H_{-}^{\prime}$ and $\Gamma_{-}$ $:=A_{-}-B_{-} H_{-}$[output injection formulas (4.29)], obtaining a realization of the minimum-phase spectral factor $W_{\ldots}(s)$.
6. Solve the Lyapunov equation $\Gamma_{-}^{\prime} \Delta+\Delta \Gamma_{-}+H_{-}^{\prime} H_{-}=0$. The solution is $\Delta=\left(\Sigma_{+}-\Sigma_{-}\right)^{-1}$.
7. Compute $P_{+}=P_{-}+\Delta^{-1}$ (see the remarks following Theorem 4.2).

This scheme is mostly of pedagogical value and has no pretense of being a practical computational procedure. There are indeed various possible shortcuts to eliminate some of the intermediate steps in this rather long chain of computations. The details will be discussed elsewhere.

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[^0]:    ${ }^{1}$ As usual in the engineering literature, stationary-increment processes are really integrated versions of the signals of interest and are, strictly speaking, only defined up to a fixed additive random vector, say $y_{0}$. A precise choice of this additive term is immaterial, as only the increments of $y$ enter in the formulas. The equivalence class of processes $y$, with respect to choice of $y_{0}$, is denoted $d y$.

[^1]:    ${ }^{2}$ All random variables considered here are zero-mean and belong to the $L^{2}$ space of the underlying probability space $\{\Omega, \mathscr{A}, \mu\}$ of the process.

[^2]:    ${ }^{3}$ In the sequel, given two subspaces $A$ and $B$, we shall write $A \vee B$ to denote the closure of $\{\alpha+\beta \mid \alpha \in A, \beta \in B\}$. To stress that the sum is direct we write instead $A+B$ or, if it is an orthogonal direct sum, $A \oplus B$.

[^3]:    ${ }^{4}$ For example, if $A$ is not taken to be stable, an additive decomposition of the spectral density $\Phi(s)$ of the form (1.10) may not even exist, and the linear matrix inequality cannot even be written in terms of the parameters of the decomposition (1.10).

[^4]:    ${ }^{5}$ The "zero structure" or, better, the zero dynamics, of a deterministic state-space system is a concept discussed in many places in the geometric-control literature; see e.g., [35, 13, 24].

