# Silverman algorithm and the structure of discrete-time stochastic systems ${ }^{\text {Th }}$ 

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#### Abstract

It is a well-known, yet poorly understood fact that, contrary to the continuous-time case, the same discrete-time process $y$ can be represented by minimal linear models (see (1.1a), (1.1b) below) which may either have a non-singular or a singular $D$ matrix. In fact, models with $D=0$ have been commonly used in the statistical literature. On the other hand, for models with a singular $D$ matrix the Riccati difference equation of Kalman filtering involves in general the pseudo-inversion of a singular matrix. This "cheap filtering" problem, dual to the better known "cheap control" problem, has been studied for several decades in connection with the so-called "invariant directions" of the Riccati equation. For a singular $D$, a reduction of the order of the Riccati equation is in general possible. The reasons for such a reduction do not seem to be completely clear either. In this paper we provide an explanation of this phenomenon from the classical point of view of "zero flipping" among minimal spectral factors. Changing $D$ 's occurs whenever zeros are "flipped" from $z=\infty$ to their reciprocals at $z=0$. It is well known that for finite zeros, the zero-flipping process takes place by multiplication of the underlying spectral factor by a suitable rational all-pass matrix function. For infinite zeros, zero flipping is implemented by a dual version of the Silverman structure algorithm. Using this interpretation, we derive a new algorithm for filtering of non-regular processes, based on a reduced-order Riccati equation. We also obtain a precise characterization of the reduction of the order of the Riccati equation which is afforded by zeros either at $z=\infty$ or at the origin. This order reduction has traditionally been associated with the study of invariant directions,


[^0]a point of view which, as we show, does not capture the essence of the phenomenon. © 2002 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

Consider a linear discrete-time stochastic model

$$
\begin{align*}
& x(t+1)=A x(t)+B w(t),  \tag{1.1a}\\
& y(t)=C x(t)+D w(t) \tag{1.1b}
\end{align*}
$$

driven by a normalized $p$-dimensional white-noise process $w$, i.e., such that $E\{w(t)$ $\left.w(s)^{\mathrm{T}}\right\}=I_{p} \delta(t-s)$. The model (1.1a), (1.1b) represents a certain $m$-dimensional wide-sense stationary process $y$, which may be a measured signal used in estimation problems, or in identification, etc. Any stationary process $y$ admitting a representation of the form (1.1a), (1.1b) has a spectral density matrix $\Phi(z)$, which is an $m \times m$ rational function of $z$. Representations of the type (1.1a), (1.1b) are called stochastic realizations of the process $y$. In practice only "non-redundant", i.e., minimal stochastic realizations, where the dimension of the state vector $x$ is as small as possible, are of interest and, for such reason, in this paper we shall only deal with minimal stochastic realizations. Even assuming minimality, the representations (1.1a), (1.1b) are highly non-unique. In fact, a fundamental result of stochastic system theory [2,5] parametrizes the family of minimal stochastic realizations of a process with a given rational spectrum by the solutions of a certain linear matrix inequality, whose coefficients can be read off from a state-space realization of $\Phi(z)$. This matrix inequality reduces, in certain special (but important) instances, to an algebraic Riccati equation.

A general assumption, which we shall keep all through this paper, is that $y$ is a full-rank process. This is equivalent to the spectral density matrix $\Phi(z)$ being of full rank, i.e., an invertible matrix, almost everywhere in $z$. As a consequence, in the model (1.1a), (1.1b) the dimension of the process $w$ is always greater than or equal to that of $y$, i.e., $p \geqslant m$. Now, it is well known that the same discrete-time process $y$ can be represented by minimal realizations of the type (1.1a), (1.1b), which may either have a non-singular or a singular $D$ matrix. In fact, there may be realizations such as those used by Akaike in [1] and quite commonly encountered in the statistical literature, where one postulates $D=0$.

When the matrix $D$ in the representation (1.1a), (1.1b) is singular, the problem of estimating the state $x$ based on the (past) observations of $y$ is known as "cheap (or singular) filtering". This problem is dual to the better known "cheap control" problem, and has been discussed in the literature for several decades, see $[3,9,11]$, and references therein. It has been observed that, related to the singularity of $D$, there
is a possible reduction in the order of the Riccati equation. This reduction has been investigated in a series of papers by L. Silverman and co-authors (compare [20], and references therein), mostly in an optimal control context.

### 1.1. Motivations and outline of the paper

This paper is motivated by the observation that the analysis of [20] does not apply naturally to the stochastic setting and in particular to the Riccati equation of the stochastic realization problem. ${ }^{1}$ It seems fair to say that, in spite of the apparent simplicity of the model (1.1a), (1.1b) and of the extensive research over many years, there is still some mystery regarding the reason why some minimal realizations (1.1a), (1.1b) of the same process $y$ may have either a singular or a non-singular $D$ matrix. To the best of our knowledge, questions regarding the singularity of some (but in general not all) realizations of a discrete-time process have been around for several decades in the stochastic system literature, but have never been solved or explained completely. In this paper, we provide an explanation of this phenomenon in the traditional key of "zero flipping" among minimal spectral factors. The zeros which are "flipped" are zeros at $z=\infty$ being sent to their reciprocals at $z=$ 0 . Accordingly, one transforms spectral factors with a singular $D$ matrix into other spectral factors with a "less singular" (and eventually non-singular) $D$. This process, which for finite zeros takes place by right-multiplication by a suitable all-pass rational matrix function, is here implemented by a dual version of the Silverman structure algorithm. Using this interpretation, we point out that the reduction in the order of the ARE is related to the non-regularity of the process $y$ (see Definition 4.1 below) rather than to the singularity of the $D$ matrix in the given model. Moreover, we get a precise characterization of the amount of reduction of the order of the Riccati equation, which is afforded by zeros either at $z=\infty$ or at the origin, something which has traditionally been looked upon by studying "invariant directions".

In this respect, we should say that our paper takes a different, more algebraic point of view than the recent and very much related paper [12], where the problem is tackled entirely from a geometric angle. It is quite likely that some of our results are in close contact with the geometry of invariant directions as expounded in a very elegant way in the above paper. We shall leave further investigations on this topic to a future paper.

We outline the organization of the paper and its contributions. In Section 2 we discuss a dual (filtering) version of the Silverman structure algorithm and in Section 3 we recall some well-known results on spectral factorization.

Regularity of a stochastic process is defined and discussed in Section 4, where a new necessary and sufficient condition is derived. Condition 7 in Theorem 4.1 relates only to the spectral density $\Phi(z)$ of $y$, whereas all the existing conditions in the literature are based on particular spectral factors of $\Phi(z)$, i.e., on particular

[^1]representations of $y$. In Section 4 we also discuss the case of processes admitting models with $D=0$. We derive a characterization of this class of processes in terms of the spectral density $\Phi(z)$.

The main contribution of the present paper is in Section 5 where we propose a reduced-order algorithm for filtering of non-regular processes. We show that if $y$ is a non-regular process the optimal filter may be derived by solving a reduced-order Riccati equation. The structure of this equation and the amount of reduction are analyzed and related to the parameters $A, B, C, D$ of the given model and to the spectral density of the process $y$. Examples illustrating the implementation of the proposed algorithm are presented in Section 6.

## 2. Silverman algorithm and singular filtering

In this section we shall describe a dual (stochastic filtering) version of the Silverman structure algorithm. The steady-state version of this algorithm will provide a key tool in the analysis of non-regular processes which we shall take up later.

Suppose the output of the linear discrete-time stochastic model (1.1a), (1.1b) is observed starting at the initial time $t=0$, the initial state $x(0)$ having covariance matrix $P_{0}$. We do not assume here that the $D$ matrix of the model is right-invertible, so that we are dealing with a possibly "cheap" filtering problem.

The Kalman predictor $\hat{x}(t \mid t-1):=E[x(t) \mid y(s) ; 0 \leqslant s<t]$ can be computed by the following dual version of Silverman algorithm [20, pp. 333-337].

Let $P_{0}=H_{0} H_{0}^{\mathrm{T}}$ be a full-rank factorization of the initial state covariance matrix and set

$$
x(0)=H_{0} w(-1),
$$

where $w(-1)$ is a random vector of unit covariance. Since $x(0) \perp w(t), t \geqslant 0$, we have $w(-1) \perp w(t), t \geqslant 0$, as well. Substitute into the recursion (1.1a), (1.1b) at time $t=0$, to get

$$
\left[\begin{array}{l}
x(1)  \tag{2.1}\\
y(0)
\end{array}\right]=\left[\begin{array}{ll}
A H_{0} & B \\
C H_{0} & D
\end{array}\right]\left[\begin{array}{c}
w(-1) \\
w(0)
\end{array}\right] .
$$

Now, let $D$ have $r$ linearly independent columns, with $r \leqslant m$. Then, there exists an isometry $S_{0}$ such that

$$
\left[\begin{array}{ll}
A H_{0} & B \\
C H_{0} & D
\end{array}\right] S_{0}=\left[\begin{array}{cc}
L_{0} & K_{0} \\
0 & D_{0}
\end{array}\right],
$$

where $D_{0}$ and $L_{0}$ both have linearly independent columns; say, $D_{0}$ is $m \times r_{0}$, with $r \leqslant r_{0} \leqslant m$, and $L_{0}$ is $n \times \rho_{0}$, with $\rho_{0} \leqslant n$. It follows that (2.1) can be rewritten as

$$
\left[\begin{array}{l}
x(1)  \tag{2.2}\\
y(0)
\end{array}\right]=\left[\begin{array}{cc}
L_{0} & K_{0} \\
0 & D_{0}
\end{array}\right]\left[\begin{array}{l}
v(0) \\
e(0)
\end{array}\right],
$$

where

$$
\left[\begin{array}{l}
v(0)  \tag{2.3}\\
e(0)
\end{array}\right]=S_{0}^{\mathrm{T}}\left[\begin{array}{c}
w(-1) \\
w(0)
\end{array}\right]
$$

Since $e(0)$ spans the same space as $y(0)$ and $v(0) \perp e(0)$, we have

$$
\begin{aligned}
\hat{x}(1 \mid 0) & :=E[x(1) \mid y(0)]=K_{0} e(0), \\
\tilde{x}(1 \mid 0) & :=x(1)-E[x(1) \mid y(0)]=L_{0} v(0) .
\end{aligned}
$$

Next, substitute the first line of (2.2) into the recursion (1.1a), (1.1b) at time $t=1$, to obtain

$$
\left[\begin{array}{l}
x(2)  \tag{2.4}\\
y(1)
\end{array}\right]=\left[\begin{array}{l}
A K_{0} e(0) \\
C K_{0} e(0)
\end{array}\right]+\left[\begin{array}{ll}
A L_{0} & B \\
C L_{0} & D
\end{array}\right]\left[\begin{array}{c}
v(0) \\
w(1)
\end{array}\right] .
$$

Note that

$$
\begin{aligned}
& \hat{x}(2 \mid 0):=E[x(2) \mid y(0)] \\
& \hat{y}(1 \mid 0):=E[y(1) \mid y(0)]=C K_{0} e(0), \\
&
\end{aligned}
$$

Hence, denoting $\tilde{x}(2 \mid 0):=x(2)-\hat{x}(2 \mid 0), \tilde{y}(1 \mid 0):=y(1)-\hat{y}(1 \mid 0)$, we have

$$
\left[\begin{array}{c}
\tilde{x}(2 \mid 0)  \tag{2.5}\\
\tilde{y}(1 \mid 0)
\end{array}\right]=\left[\begin{array}{ll}
A L_{0} & B \\
C L_{0} & D
\end{array}\right]\left[\begin{array}{c}
v(0) \\
w(1)
\end{array}\right] .
$$

Now, as in the previous step, introduce an isometric matrix $S_{1}$ such that

$$
\left[\begin{array}{ll}
A L_{0} & B \\
C L_{0} & D
\end{array}\right] S_{1}=\left[\begin{array}{cc}
L_{1} & K_{1} \\
0 & D_{1}
\end{array}\right],
$$

where both $D_{1}$ and $L_{1}$ have linearly independent columns; say, $D_{1}$ is $m \times r_{1}$, with $r \leqslant r_{1} \leqslant m$, while $L_{1}$ is $n \times \rho_{1}$, with $\rho_{1} \leqslant n$. It follows that (2.5) can be rewritten as

$$
\left[\begin{array}{c}
\tilde{x}(2 \mid 0)  \tag{2.6}\\
\tilde{y}(1 \mid 0)
\end{array}\right]=\left[\begin{array}{cc}
L_{1} & K_{1} \\
0 & D_{1}
\end{array}\right]\left[\begin{array}{l}
v(1) \\
e(1)
\end{array}\right],
$$

where

$$
\left[\begin{array}{l}
v(1)  \tag{2.7}\\
e(1)
\end{array}\right]=S_{1}^{\mathrm{T}}\left[\begin{array}{c}
v(0) \\
w(1)
\end{array}\right] .
$$

It is evident from (2.6) that $e(1)$ is the normalized innovation of the process $y$ at time $t=1$, i.e., a normalized version of the one-step prediction error $\tilde{y}(1 \mid 0)$, so that

$$
K_{1} e(1)=E[\tilde{x}(2 \mid 0) \mid e(1)]=E[x(2) \mid e(1)] .
$$

In other words, $K_{1}$ is the Kalman gain at time $t=1$ and we can update the early estimate $\hat{x}(2 \mid 0)$ to

$$
\hat{x}(2 \mid 1)=\hat{x}(2 \mid 0)+K_{1} e(1) .
$$

Moreover,

$$
\begin{aligned}
L_{1} v(1) & =\tilde{x}(2 \mid 0)-E[x(2) \mid e(1)] \\
& =x(2)-E[x(2) \mid e(0)]-E[x(2) \mid e(1)]=\tilde{x}(2 \mid 1)
\end{aligned}
$$

is the state prediction error at time $t=1$. From this expression, we can compute the prediction error covariance matrix at time $t=1$ as $P_{1}=L_{1} L_{1}^{\mathrm{T}}$ of rank $\rho_{1}$.

The procedure is iterated recursively. Assume that at time $t$ one has available the state estimate $\hat{x}(t \mid t-1)$ and the current measurement $y(t)$. From the recursion (1.1a), (1.1b) and the normalized expression $\tilde{x}(t \mid t-1)=L_{t-1} v(t-1)$ of the state prediction error, with $L_{t-1} \in \mathbb{R}^{n \times \rho_{t-1}}$ full column rank, we form the error model

$$
\left[\begin{array}{c}
\tilde{x}(t+1 \mid t-1)  \tag{2.8}\\
\tilde{y}(t \mid t-1)
\end{array}\right]=\left[\begin{array}{ll}
A L_{t-1} & B \\
C L_{t-1} & D
\end{array}\right]\left[\begin{array}{c}
v(t-1) \\
w(t)
\end{array}\right],
$$

where $\quad \tilde{x}(t+1 \mid t-1):=x(t+1)-\hat{x}(t+1 \mid t-1)=x(t+1)-A \hat{x}(t \mid t-1)$ and $\tilde{y}(t \mid t-1):=y(t)-\hat{y}(t \mid t-1)=y(t)-C \hat{x}(t \mid t-1)$. Then, acting by an isometry $S_{t}$ such that

$$
\left[\begin{array}{cc}
A L_{t-1} & B \\
C L_{t-1} & D
\end{array}\right] S_{t}=\left[\begin{array}{cc}
L_{t} & K_{t} \\
0 & D_{t}
\end{array}\right],
$$

where $D_{t}$ and $L_{t}$ both have linearly independent columns (say, $D_{t}$ is $m \times r_{t}$, with $r \leqslant r_{t} \leqslant m$, and $L_{t}$ is $n \times \rho_{t}$, with $\rho_{t} \leqslant n$ ), Eq. (2.8) can be rewritten as

$$
\left[\begin{array}{c}
\tilde{x}(t+1 \mid t-1)  \tag{2.9}\\
\tilde{y}(t \mid t-1)
\end{array}\right]=\left[\begin{array}{cc}
L_{t} & K_{t} \\
0 & D_{t}
\end{array}\right]\left[\begin{array}{l}
v(t) \\
e(t)
\end{array}\right],
$$

where

$$
\left[\begin{array}{l}
v(t)  \tag{2.10}\\
e(t)
\end{array}\right]=S_{t}^{\mathrm{T}}\left[\begin{array}{c}
v(t-1) \\
w(t)
\end{array}\right] .
$$

Here, $K_{t}$ is the Kalman gain at time $t$, since

$$
K_{t} e(t)=E[\tilde{x}(t+1 \mid t-1) \mid e(t)]=E[x(t+1) \mid e(t)] .
$$

From this, one can update the previous state estimate to get

$$
\hat{x}(t+1 \mid t)=A \hat{x}(t \mid t-1)+K_{t} e(t),
$$

and compute

$$
\begin{aligned}
L_{t} v(t) & =\tilde{x}(t+1 \mid t-1)-E[x(t+1) \mid e(t)] \\
& =x(t+1)-\sum_{s=0}^{t} E[x(t+1) \mid e(s)]=\tilde{x}(t+1 \mid t),
\end{aligned}
$$

which is the state prediction error at time $t$, whose covariance matrix is obtained as $P_{t}=L_{t} L_{t}^{\mathrm{T}}$, and has rank $\rho_{t}$.

If the stationary process $y$ or, equivalently, the spectral density matrix $\Phi(z)$ is of full rank, the variance matrix $\Lambda(t)=D_{t} D_{t}^{\mathrm{T}}$ of the non-stationary innovation process $\tilde{y}(t \mid t-1)$ becomes invertible after a finite number of steps, i.e., $r_{t}=m$ for $t$ sufficiently large. This follows from the well-known fact that $\Lambda(t)$ converges as $t \rightarrow \infty$
to the variance $\Lambda$ of the stationary innovation $y(t)-E[y(t) \mid y(s) ;-\infty<s<t]$, which is positive-definite if and only if $y$ is of full rank [22,23]. Likewise, the rank $\rho_{t}$ of the variance $P_{t}=L_{t} L_{t}^{\mathrm{T}}$ of the non-stationary state prediction error $\tilde{x}(t+1 \mid t)$ converges after a finite number of steps to the rank $n$ of $P_{-}=\lim _{t \rightarrow \infty} P_{t}$, the variance of the steady-state prediction error $x(t+1)-E[x(t+1) \mid y(s) ;-\infty<s \leqslant$ $t$ ], see Section 3.1. Hence our (non-stationary) cheap filtering algorithm eventually tends to a steady state.

## 3. Silverman algorithm and stochastic realization

### 3.1. Review of spectral factorization

The material in this section is standard and can be found in various places in the literature [6,7]. We shall just recall the basic facts in order to set notations.

The transfer function $W(z)=C(z I-A)^{-1} B+D$ of any state-space representation of the process $y$ of the type (1.1a), (1.1b) is a spectral factor of $\Phi(z)$, i.e.,

$$
\Phi(z)=W(z) W\left(z^{-1}\right)^{\mathrm{T}} .
$$

Note that two $p$-dimensional normalized white-noise processes $w_{1}, w_{2}$ differing in multiplication by a constant $p \times p$ orthogonal matrix are indistinguishable (as sec-ond-order processes). Hence it is natural to consider two realizations (1.1a), (1.1b) with input noises differing by a constant orthogonal transformation as the same object. For this reason we will not distinguish among spectral factors differing by rightmultiplication by a constant $p \times p$ orthogonal matrix.

From now on, we shall fix our attention to causal realizations, where the matrix $A$ has all eigenvalues strictly inside of the unit circle of the complex plane. The transfer function of each model (1.1a), (1.1b) is then an analytic spectral factor of $\Phi(z)$, since it has no poles outside of the open unit disk, including the point $z=\infty$. To each such spectral factor of minimal degree (called a minimal spectral factor) we let correspond an equivalence class of minimal realizations (1.1a), (1.1b), defined modulo a change of basis in the state space, an arbitrary $p \times p$ constant orthogonal transformation of the white-noise process $w$ and, in the non-square case, the choice of some components of the noise process; see [14-16] for details. In this sense, the minimal causal realizations of $y$ are essentially in a one-to-one correspondence with the (equivalence classes of) minimal analytic spectral factors $W(z)$.

If we decompose $\Phi(z)=\Phi\left(z^{-1}\right)^{\mathrm{T}}$ into the analytic and co-analytic (with respect to the unit circle) components

$$
\begin{equation*}
\Phi(z)=\Phi_{+}(z)+\Phi_{+}\left(z^{-1}\right)^{\mathrm{T}}, \tag{3.1}
\end{equation*}
$$

then $\Phi_{+}(z)$ has a minimal realization whose parameters can be formally expressed as a function of the parameters of the model (1.1a), (1.1b) as

$$
\begin{equation*}
\Phi_{+}(z)=C(z I-A)^{-1} \bar{C}^{\mathrm{T}}+\frac{1}{2} \Lambda_{0}, \tag{3.2}
\end{equation*}
$$

where $\bar{C}^{\mathrm{T}}=A P C^{\mathrm{T}}+B D^{\mathrm{T}}, P$ being the state covariance $P:=E\left\{x(t) x(t)^{\mathrm{T}}\right\}$, solution of the Lyapunov equation $P=A P A^{\mathrm{T}}+B B^{\mathrm{T}}$.

Although seemingly dependent on $P$ and on the $B, D$ matrices of the particular model, the matrix $\bar{C}$ must obviously be an invariant of the process (in the given basis). The matrix $\Lambda_{0}$ in (3.2) is just the output covariance at lag zero, i.e.,

$$
\Lambda_{0}:=E\left\{y(t) y(t)^{\mathrm{T}}\right\}=C P C^{\mathrm{T}}+D D^{\mathrm{T}} .
$$

Well-known examples of minimal analytic spectral factors are the outer, also called minimum-phase, and the maximum-phase spectral factors, denoted $W_{-}(z)$ and $W_{+}(z)$, respectively. Both $W_{-}(z)$ and $W_{+}(z)$ are analytic in $\{z:|z| \geqslant 1\}$ including infinity, but while the outer factor has all zeros inside of the closed unit disk, $W_{+}(z)$ has all zeros outside of the open unit disk. The following result is standard [24,14].

Theorem 3.1. All minimal analytic rational spectral factors can be obtained by post-multiplying the minimum-phase factor $W_{-}(z)$ by a rational inner matrix function $Q(z)$, or by post-multiplying the maximum-phase factor $W_{+}(z)$ by a co-analytic rational inner matrix function $\bar{Q}(z)$, i.e., a rational matrix function analytic in $\{z$ : $|z|<1\}$, such that

$$
\bar{Q}(z) \bar{Q}\left(z^{-1}\right)^{\mathrm{T}}=I .
$$

Since the McMillan degree of minimal spectral factors has to be kept constant in the multiplication by the inner function, cancellation of zeros of $W_{-}(z)$ with poles of $Q(z)$ or cancellation of zeros of $W_{+}(z)$ with poles of $\bar{Q}(z)$ has to take place. Hence, some zeros are replaced by their reciprocal image with respect to the unit circle. This phenomenon is called "zero flipping" in the engineering literature. Zero flipping is closely related to solving a linear matrix inequality, as summarized in the following theorem; for the proof of which we refer e.g. to [7].

Theorem 3.2. Let $\left(A, \bar{C}^{\mathrm{T}}, C, \frac{1}{2} \Lambda_{0}\right)$ be a minimal realization of the analytic component $\Phi_{+}(z)$ of the spectral density matrix $\Phi(z)$. Then there is a one-to-one correspondence between minimal analytic spectral factors of $\Phi(z)$ and symmetric $n \times n$ matrices $P$ solving the Linear Matrix Inequality

$$
M(P):=\left[\begin{array}{ll}
P-A P A^{\mathrm{T}} & \bar{C}^{\mathrm{T}}-A P C^{\mathrm{T}}  \tag{3.3}\\
\bar{C}-C P A^{\mathrm{T}} & \Lambda_{0}-C P C^{\mathrm{T}}
\end{array}\right] \geqslant 0 .
$$

In fact, corresponding to each solution $P=P^{\mathrm{T}}$ of (3.3), consider the unique (modulo orthogonal transformations) full column rank matrix factor $\left[\begin{array}{l}B \\ D\end{array}\right]$ of $M(P)$,

$$
M(P)=\left[\begin{array}{l}
B  \tag{3.4}\\
D
\end{array}\right]\left[\begin{array}{ll}
B^{\mathrm{T}} & D^{\mathrm{T}}
\end{array}\right]
$$

and define the rational matrix $W(z)$ parametrized in the form

$$
\begin{equation*}
W(z)=C(z I-A)^{-1} B+D . \tag{3.5}
\end{equation*}
$$

Then (3.5) is a minimal realization of a minimal analytic spectral factor of $\Phi(z)$.
Conversely, to each minimal analytic spectral factor $W(z)$ there corresponds, by suitably choosing a basis in the state space, a minimal realization of the form (3.5) for some $B, D$ matrices. Then, the solution $P=P^{\mathrm{T}}$ of the Lyapunov equation $P-A P A^{\mathrm{T}}=B B^{\mathrm{T}}$ satisfies the matrix equation (3.4) and hence the Linear Matrix Inequality (3.3).

Moreover, all symmetric solutions $P$ of (3.3) are necessarily positive-definite.
It can be shown [5,7] that the set of solutions to the LMI (3.3),

$$
\mathscr{P}:=\left\{P \mid P=P^{\mathrm{T}}, M(P) \geqslant 0\right\},
$$

is closed, bounded and convex. Moreover, there are two special elements $P_{-}, P_{+} \in$ $\mathscr{P}$ such that

$$
P_{-} \leqslant P \leqslant P_{+} \quad \text { for all } P \in \mathscr{P},
$$

where $P_{1} \leqslant P_{2}$ means that $P_{2}-P_{1} \geqslant 0$, i.e., the difference $P_{2}-P_{1}$ is a positive semidefinite matrix. To such minimal and maximal solutions of the LMI there correspond minimum-rank matrix factors

$$
\left[\begin{array}{l}
B_{-} \\
D_{-}
\end{array}\right] \text {and }\left[\begin{array}{l}
B_{+} \\
D_{+}
\end{array}\right]
$$

in the factorization (3.4), which yield the minimum- and maximum-phase spectral factors

$$
\begin{align*}
& W_{-}(z)=C(z I-A)^{-1} B_{-}+D_{-}, \\
& W_{+}(z)=C(z I-A)^{-1} B_{+}+D_{+} . \tag{3.6}
\end{align*}
$$

If $\Lambda_{0}-C P C^{\mathrm{T}}>0$, a simple calculation yields that $M(P) \geqslant 0$ if and only if $P$ satisfies the algebraic Riccati inequality

$$
\begin{equation*}
P-A P A^{\mathrm{T}}-\left(\bar{C}^{\mathrm{T}}-A P C^{\mathrm{T}}\right)\left(\Lambda_{0}-C P C^{\mathrm{T}}\right)^{-1}\left(\bar{C}-C P A^{\mathrm{T}}\right) \geqslant 0 . \tag{3.7}
\end{equation*}
$$

In particular, if $P$ satisfies the algebraic Riccati equation

$$
\begin{equation*}
P=A P A^{\mathrm{T}}+\left(\bar{C}^{\mathrm{T}}-A P C^{\mathrm{T}}\right)\left(\Lambda_{0}-C P C^{\mathrm{T}}\right)^{-1}\left(\bar{C}-C P A^{\mathrm{T}}\right) \tag{3.8}
\end{equation*}
$$

the corresponding spectral factor $W(z)$ is square $m \times m$. The solutions $P=P^{\mathrm{T}}$ of (3.3) corresponding to square spectral factors form a subfamily $\mathscr{P}_{0} \subset \mathscr{P}$. If $P \notin \mathscr{P}_{0}$, then $W(z)$ is rectangular $m \times p$, with $p>m$.

### 3.2. Zero flipping at infinity

We want to analyze the minimal realizations $(A, B, C, D)$ of a process $y$ where $D$ is a singular matrix, the word "singular" meaning that $D$ is not of full row rank, i.e., $D$ does not possess a right-inverse. The following (quite obvious) lemma serves the purpose of linking singularity of $D$ to the presence of zeros at infinity. The proof will be skipped.

Lemma 3.1. A proper rational matrix $W(z)=C(z I-A)^{-1} B+D$ has zeros at $z=\infty$ if and only if $D$ is singular.

Consider a minimal spectral factor $W(z)=C(z I-A)^{-1} B+D$ with a singular $D$. The "flipping" of zeros at infinity to zeros at $z=0$ is accomplished by using a stationary version of the dual Silverman algorithm described in Section 2.

Assume the matrix $D$ has $p_{0}$ linearly independent columns, with $0 \leqslant p_{0} \leqslant m$. Let $Q_{0}$ be an orthogonal matrix such that $D Q_{0}=\left[D_{01} \mid 0\right]$, with $D_{01} \in \mathbb{R}^{m \times p_{0}}$ being full column rank. Let us partition $B Q_{0}=\left[B_{01} \mid B_{02}\right]$ conformably, obtaining the following block structure,

$$
\begin{equation*}
W_{0}(z):=W(z) Q_{0}=C(z I-A)^{-1}\left[B_{01} \mid B_{02}\right]+\left[D_{01} \mid 0\right], \tag{3.9}
\end{equation*}
$$

and let

$$
\hat{W}_{1}(z):=W_{0}(z)\left[\begin{array}{cc}
I_{p_{0}} & 0  \tag{3.10}\\
0 & z I_{p-p_{0}}
\end{array}\right] .
$$

Clearly, $\hat{W}_{1}(z)$ is also a spectral factor of $\Phi(z)$. Since

$$
\begin{aligned}
\hat{W}_{1}(z)= & {\left[D_{01}+C B_{01} z^{-1}+C A B_{01} z^{-2}+\cdots\right.} \\
& \left.\mid C B_{02}+C A B_{02} z^{-1}+C A^{2} B_{02} z^{-2}+\cdots\right] \\
= & C(z I-A)^{-1}\left[B_{01} \mid A B_{02}\right]+\left[D_{01} \mid C B_{02}\right],
\end{aligned}
$$

this spectral factor has necessarily McMillan degree $n$ and, hence, is minimal. At this point, either [ $D_{01} \mid C B_{02}$ ] is right-invertible or we may iterate the above procedure by introducing another orthogonal matrix $Q_{1}$ such that

$$
\left[D_{01} \mid C B_{02}\right] Q_{1}=\left[D_{11} \mid 0\right]
$$

with $D_{11} \in \mathbb{R}^{m \times p_{1}}$ of full column rank $p_{1} \geqslant p_{0}$, and define the minimal spectral factor

$$
\begin{equation*}
W_{1}(z):=\hat{W}_{1}(z) Q_{1}=C(z I-A)^{-1}\left[B_{11} \mid B_{12}\right]+\left[D_{11} \mid 0\right], \tag{3.11}
\end{equation*}
$$

where $\left[B_{11} \mid B_{12}\right]=\left[B_{01} \mid A B_{02}\right] Q_{1}$.
Since $y$ is a full-rank process, $W(z)$ as a rational function has full row rank $m$ and hence, after a finite number of steps of the above procedure, we get a minimal spectral factor

$$
W_{l}(z):=W(z) Q(z), \quad Q(z)=Q_{0} \prod_{i=0}^{l-1}\left[\begin{array}{cc}
I_{p_{i}} & 0  \tag{3.12}\\
0 & z I_{p-p_{i}}
\end{array}\right] Q_{i+1}
$$

such that $W_{l}(\infty)$ is right-invertible, i.e., $W_{l}(z)$ has no zeros at infinity. Equivalently, $W_{l}(z)$ has a realization of the form

$$
\begin{equation*}
W_{l}(z)=C(z I-A)^{-1}\left[B_{l 1} \mid B_{l 2}\right]+\left[D_{l 1} \mid 0\right], \tag{3.13}
\end{equation*}
$$

with $D_{l 1}$ square and invertible.

In the following, we shall denote the transfer function $W_{l}(z)$ obtained at the last step of Silverman algorithm by the symbol $W_{S}(z)$.

The indices $p_{0} \leqslant p_{1} \leqslant \cdots \leqslant p_{l}=m$ are related to the zero structure at infinity of the rational matrix $W(z)$, see [4,18,21], and references therein. In fact, there are two square biproper rational matrices $P(z)$ and $R(z)$, i.e., finite and non-singular at $z=\infty$ together with their inverses, such that

$$
P(z) W(z) R(z)=\left[\begin{array}{ccccc}
I_{q_{0}} & 0 & \cdots & 0 & 0  \tag{3.14}\\
0 & z^{-1} I_{q_{1}} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & z^{-l} I_{q_{l}} & 0
\end{array}\right]
$$

where $q_{0}:=p_{0}, q_{i}:=p_{i}-p_{i-1}$ for $i=1, \ldots, l$, and the last column of zero blocks only appears in the non-square case.

The geometric multiplicity of the zero at $z=\infty$ is $\sum_{i=1}^{l} q_{i}=m-\operatorname{rank} D$, while its algebraic multiplicity is given by $\sum_{i=1}^{l} i q_{i}=\sum_{i=0}^{l}\left(m-p_{i}\right)$.

In conclusion, the steady-state Silverman algorithm transforms a spectral factor $W(z)$ with a given zero structure at infinity to another, $W_{S}(z)$, which has no zeros at infinity. In fact, it can be shown (but we shall not do this here) that all zeros at infinity of $W(z)$ are replaced by corresponding zeros at $z=0$ of $W_{S}(z)$ of the same multiplicity.

## 4. Regularity

One of the questions which naturally arises in the discrete-time context is for what kind of processes all minimal realizations have a non-singular $D$ matrix. The following definition is from [5].

Definition 4.1. The process $y$ is regular if all its minimal realizations have a rightinvertible $D$ matrix.

Right-invertibility of $D$ is obviously equivalent to $D D^{\mathrm{T}}$ being non-singular and it can be seen that a regular process has none of the typical discrete-time pathologies which we did mention in the introduction. All system-theoretic properties of a regular process are exactly the same as of a continuous-time process with a spectral density matrix strictly positive-definite at infinity. The following theorem collects some equivalent characterizations of the property of regularity.

Theorem 4.1. Let y be a stationary process with a full-rank rational spectral density matrix $\Phi(z)$. Then the following are equivalent:

1. The process $y$ is regular.
2. For all solutions $P=P^{\mathrm{T}}$ of the LMI (3.3), $\Lambda_{0}-C P C^{\mathrm{T}}>0$.
3. $\Lambda_{0}-C P_{+} C^{\mathrm{T}}>0$, where $P_{+}=P_{+}^{\mathrm{T}}$ is the maximal solution of the LMI (3.3) or, equivalently, $D_{+}=W_{+}(\infty)$ is non-singular.
4. There exists a minimal spectral factor of $\Phi(z)$ having zeros neither at $z=0$ nor at $z=\infty$.
5. All minimal spectral factors of $\Phi(z)$ have zeros neither at $z=0$ nor at $z=\infty$.
6. The numerator matrix $\Gamma_{-}=A-B_{-} D_{-}^{-1} C$ of the minimum-phase spectral factor $W_{-}(z)$ is non-singular or, equivalently, $\lim _{z \rightarrow 0} W_{-}(z)^{-1}$ is finite.
7. $\Phi(z)$ has zeros neither at infinity nor at zero; more precisely, $\lim _{z \rightarrow \infty} \Phi(z)^{-1}$ is finite or, equivalently, $\lim _{z \rightarrow 0} \Phi(z)^{-1}$ is finite.

While Conditions 1-6 are more or less known, see [12,17], Condition 7 seems to be new. It states that the inverse $\Phi(z)^{-1}$ of the spectrum of a full-rank regular process is proper, i.e., has poles neither at $z=\infty$ (nor at $z=0$ ). Below we provide a proof.

Proof. Assume regularity. Then any minimal square spectral factor $W(z)=C(z I-$ $A)^{-1} B+D$ has a directly computable inverse given by

$$
W(z)^{-1}=D^{-1}-D^{-1} C(z I-\Gamma)^{-1} B D^{-1}
$$

where $\Gamma=A-B D^{-1} C$ has no zero eigenvalues, i.e., it is non-singular, by Condition 5. Therefore,

$$
\lim _{z \rightarrow 0} W(z)^{-1}=D^{-1}+D^{-1} C \Gamma^{-1} B D^{-1}
$$

It follows that

$$
\lim _{z \rightarrow \infty} \Phi(z)^{-1}=\lim _{z \rightarrow \infty} W\left(z^{-1}\right)^{-\mathrm{T}} W(z)^{-1}=\left(D^{-1}+D^{-1} C \Gamma^{-1} B D^{-1}\right)^{\mathrm{T}} D^{-1}
$$

which is obviously finite.
Conversely, assume $y$ is not a regular process, so that the numerator matrix $\Gamma_{-}$of the minimum-phase spectral factor $W_{-}(z)$ is singular, by Condition 6.

Recall that, by the full-rank assumption, $D_{-}=W_{-}(\infty)$ is non-singular. Hence, by minimality of the triple $\left(A, B_{-}, C\right)$, the limit

$$
\lim _{z \rightarrow 0} W_{-}(z)^{-1}=\lim _{z \rightarrow 0} D_{-}^{-1}-D_{-}^{-1} C\left(z I-\Gamma_{-}\right)^{-1} B_{-} D_{-}^{-1}
$$

cannot be finite, as the resolvent $\left(z I-\Gamma_{-}\right)^{-1}$ has a pole at $z=0$. Therefore

$$
\lim _{z \rightarrow \infty} \Phi(z)^{-1}=\lim _{z \rightarrow \infty} W_{-}\left(z^{-1}\right)^{-\mathrm{T}} W_{-}(z)^{-1}=\lim _{z \rightarrow 0} W_{-}(z)^{-\mathrm{T}} D_{-}^{-1}
$$

cannot be finite either. Hence $y$ is regular if Condition 7 holds.
Condition 7 has the following intuitive version: For a full-rank process y to be regular, the spectral density matrix $\Phi(z)$ must have only finite zeros, and none of them can be at the origin.

Regularity is quite restrictive. For instance, scalar processes admitting an AR representation

$$
y(t)+\sum_{k=1}^{n} a_{k} y(t-k)=b_{0} w(t),
$$

with $w$ normalized white noise and $a_{n} \neq 0$, cannot be regular if $n>0$. Instead, MA processes described by models of the form

$$
y(t)=\sum_{k=0}^{n} b_{k} w(t-k)
$$

are regular. In fact, in the former case the spectral density function is

$$
\Phi(z)=b_{0}^{2} /\left\{\left(1+\sum_{k=1}^{n} a_{k} z^{-k}\right)\left(1+\sum_{k=1}^{n} a_{k} z^{k}\right)\right\},
$$

with a zero at $z=\infty$ of multiplicity $n$, while in the second case we get

$$
\Phi(z)=\left(\sum_{k=0}^{n} b_{k} z^{-k}\right)\left(\sum_{k=0}^{n} b_{k} z^{k}\right),
$$

whose inverse is bounded as $z \rightarrow \infty$.

### 4.1. The totally non-regular case

The "most degenerate" case with respect to regularity happens when there is a minimal realization of $y$ which has $D=0$. As already observed, this situation is of some interest since such models are occasionally used in the statistical literature.

Theorem 4.2. Let $\Phi(z)$ be a full-rank rational spectral density matrix. The following are equivalent:

1. $\Phi(\infty)=0$.
2. There exists a minimal spectral factor $W(z)$, with $W(\infty)=0$.

Proof. Let (3.2) be a minimal realization of $\Phi_{+}(z)$, let $P_{-}=P_{-}^{\mathrm{T}}$ be the minimal solution of the LMI (3.3) and $B_{-}, D_{-}$be such that

$$
\begin{align*}
& P_{-}=A P_{-} A^{\mathrm{T}}+B_{-} B_{-}^{\mathrm{T}}  \tag{4.1a}\\
& \bar{C}=C P_{-} A^{\mathrm{T}}+D_{-} B_{-}^{\mathrm{T}}  \tag{4.1b}\\
& \Lambda_{0}=C P_{-} C^{\mathrm{T}}+D_{-} D_{-}^{\mathrm{T}} \tag{4.1c}
\end{align*}
$$

Note that $D_{-}$is non-singular by the full-rank assumption.
Consider, now, the corresponding minimum-phase spectral factor $W_{-}(z)=$ $C(z I-A)^{-1} B_{-}+D_{-}$and assume $\Phi(\infty)=0$. From $\Phi(z)=W_{-}(z) W_{-}\left(z^{-1}\right)^{\mathrm{T}}$, we get

$$
\begin{equation*}
0=\lim _{z \rightarrow \infty} \Phi(z)=D_{-}\left\{D_{-}+\lim _{z \rightarrow \infty}\left[C\left(z^{-1} I-A\right)^{-1} B_{-}\right]\right\}^{\mathrm{T}} \tag{4.2}
\end{equation*}
$$

Thus, by non-singularity of $D_{-}$and minimality, $A$ is non-singular, yielding

$$
\begin{equation*}
D_{-}=C A^{-1} B_{-} . \tag{4.3}
\end{equation*}
$$

Substituting in (4.1b) and taking into account (4.1a), we also obtain

$$
\begin{align*}
\bar{C} & =C P_{-} A^{\mathrm{T}}+C A^{-1} B_{-} B_{-}^{\mathrm{T}} \\
& =C P_{-} A^{\mathrm{T}}+C A^{-1}\left(P_{-}-A P_{-} A^{\mathrm{T}}\right)=C A^{-1} P_{-} \tag{4.4}
\end{align*}
$$

Now, let $P_{0}:=A^{-1} P_{-} A^{-\mathrm{T}}$. We have:

1. $P_{0}-A P_{0} A^{\mathrm{T}}=A^{-1}\left[P_{-}-A P_{-} A^{\mathrm{T}}\right] A^{-\mathrm{T}}=A^{-1} B_{-} B_{-}^{\mathrm{T}} A^{-\mathrm{T}} \geqslant 0$,
2. $\bar{C}=C A^{-1} P_{-}=C P_{0} A^{\mathrm{T}}$,
3. $\Lambda_{0}-C P_{0} C^{\mathrm{T}}=C P_{-} C^{\mathrm{T}}+C A^{-1} B_{-} B_{-}^{\mathrm{T}} A^{-\mathrm{T}} C^{\mathrm{T}}-C P_{0} C^{\mathrm{T}}=0$.

Therefore, $P_{0}=P_{0}^{\mathrm{T}}$ solves the LMI (3.3) and the corresponding minimal analytic spectral factor

$$
\begin{equation*}
W_{0}(z)=C(z I-A)^{-1} B_{0}, \tag{4.5}
\end{equation*}
$$

with $B_{0}=A^{-1} B_{-}$, is such that $D_{0}=W_{0}(\infty)=0$.
Conversely, let

$$
\begin{equation*}
W(z)=C(z I-A)^{-1} B \tag{4.6}
\end{equation*}
$$

be a minimal realization of a minimal spectral factor of $\Phi(z)$, with $W(\infty)=0$. We first prove that $A$ is necessarily non-singular. Suppose, by contradiction, that $A \in$ $\mathbb{R}^{n \times n}$ is singular. Without loss of generality, we may assume that

$$
A=\left[\begin{array}{cc}
0 & A_{12}  \tag{4.7}\\
0 & A_{2}
\end{array}\right]
$$

where $A_{2} \in \mathbb{R}^{(n-1) \times(n-1)}$. Let us partition $B$ and $C$ conformably so that

$$
\begin{align*}
W(z) & =\left[C_{1} \mid C_{2}\right]\left(z I-\left[\begin{array}{cc}
0 & A_{12} \\
0 & A_{2}
\end{array}\right]\right)^{-1}\left[\begin{array}{l}
B_{1} \\
B_{2}
\end{array}\right] \\
& =\frac{C_{1} B_{1}}{z}+\frac{C_{1} A_{12}}{z}\left(z I-A_{2}\right)^{-1} B_{2}+C_{2}\left(z I-A_{2}\right)^{-1} B_{2} \tag{4.8}
\end{align*}
$$

It is clear that

$$
\begin{equation*}
W_{1}(z):=z W(z)=C_{1} B_{1}+C_{1} A_{12}\left(z I-A_{2}\right)^{-1} B_{2}+z C_{2}\left(z I-A_{2}\right)^{-1} B_{2} \tag{4.9}
\end{equation*}
$$

is also a spectral factor of $\Phi(z)$. Moreover, since

$$
\begin{equation*}
z\left(z I-A_{2}\right)^{-1}=A_{2}\left(z I-A_{2}\right)^{-1}+I, \tag{4.10}
\end{equation*}
$$

we have

$$
\begin{equation*}
W_{1}(z)=C_{1} B_{1}+C_{2} B_{2}+\left(C_{1} A_{12}+C_{2} A_{2}\right)\left(z I-A_{2}\right)^{-1} B_{2}, \tag{4.11}
\end{equation*}
$$

so that

$$
\begin{equation*}
\operatorname{deg}\left[W_{1}\right]<\operatorname{deg}[W], \tag{4.12}
\end{equation*}
$$

which contradicts the minimality of $W(z)$ as a spectral factor of $\Phi(z)$. Thus, by non-singularity of $A$, we conclude that

$$
\begin{equation*}
\Phi(\infty)=W(\infty) W(0)^{\mathrm{T}}=0 \cdot\left(-C A^{-1} B\right)^{\mathrm{T}}=0 . \tag{4.13}
\end{equation*}
$$

## 5. Steady-state filtering of non-regular processes: order-reduction of the ARE

In this section we shall consider steady-state estimation of the state $x(t)$ of the model

$$
\begin{align*}
& x(t+1)=A x(t)+B w(t)  \tag{5.1a}\\
& y(t)=C x(t)+D w(t) \tag{5.1b}
\end{align*}
$$

of a non-regular observation process $y(t)$.
The non-regular filtering problem stated above encompasses (but is more general than) the singular filtering problem (where $D$ is singular). Singular problems are usually addressed by writing the ARE with a Moore-Penrose pseudo-inverse ${ }^{\sharp}$ in place of the usual inverse

$$
\begin{equation*}
X=A X A^{\mathrm{T}}-\left(A X C^{\mathrm{T}}+B D^{\mathrm{T}}\right)\left(C X C^{\mathrm{T}}+D D^{\mathrm{T}}\right)^{\sharp}\left(A X C^{\mathrm{T}}+B D^{\mathrm{T}}\right)^{\mathrm{T}}+B B^{\mathrm{T}} . \tag{5.2}
\end{equation*}
$$

This formulation however hardly gives insight into the problem and may lead to substantially heavier computations than what is actually needed.

In this section we shall show that the size of the ARE (5.2) associated to a non-regular observation process is always fictitiously large and that the problem complexity may be conveniently reduced even if $D$ is non-singular. We would like to stress that the order-reduction is a consequence of the non-regularity of the process $y$, rather than of the singularity of $D D^{\mathrm{T}}$; in fact, the reduction does not depend on the particular realization (5.1a), (5.1b) of $y$ but only on the process $y$ itself. For this reason, the order-reduction procedure may be applied even in the standard (non-singular) filtering case.

It turns out that the reduction of the order of the algebraic Riccati equation, $v$, is an invariant of the process, equal to the sum of the multiplicities of the zeros of the model (5.1a), (5.1b) located at $z=\infty$ and at $z=0$. A lower bound for $v$ based only on the rank of $D$ will be given in Proposition 5.1. The order-reduction $v$ will also be related to certain system theoretic properties of the matrices $\Gamma$ and $B_{2}$ (cf. (5.4)) that play a central role in stochastic realization theory [16] and smoothing estimation [8].

Let $W(z)=C(z I-A)^{-1} B+D$ be the transfer function of the given model (5.1a), (5.1b). In general $D$ will not be right-invertible but, by using the stationary

Silverman algorithm of Section 3, we can always obtain an equivalent ${ }^{2}$ model of $y$ described by the transfer function

$$
\begin{equation*}
W_{S}(z):=W(z) Q(z)=C(z I-A)^{-1}\left[B_{1} \mid B_{2}\right]+\left[D_{1} \mid 0\right] \tag{5.3}
\end{equation*}
$$

with $D_{1}$ non-singular. The function $Q(z)$ is a polynomial conjugate-inner function given by the expression (3.12). Let

$$
\begin{equation*}
\Gamma:=A-B_{1} D_{1}^{-1} C \tag{5.4}
\end{equation*}
$$

be the numerator matrix of $W_{S}(z)[12,13]$ and consider the orthogonal complement of the column space of $\left[\Gamma \mid B_{2}\right]$, which we may denote as Lker $\left[\Gamma \mid B_{2}\right]$. We have the following characterization of non-regular processes.

Lemma 5.1. The process $y$ is non-regular if and only if

$$
\begin{equation*}
\operatorname{Lker}\left[\Gamma \mid B_{2}\right] \neq\{0\} . \tag{5.5}
\end{equation*}
$$

Proof. Let $y$ be non-regular. By construction $W_{S}(\infty)$ is right-invertible, so that, by Condition 4 of Theorem 4.1, the minimal spectral factor $W_{S}(z)$ has an invariant zero at $z=0$. It is known that the invariant zeros $\left\{\lambda_{k}\right\}$ and the corresponding invariant zero-directions $[13,18]$ of $W_{S}(z)$ can be found by computing the left-kernel of the so-called system matrix of $W_{S}(z)$, i.e., by solving

$$
\left[\begin{array}{ll}
v^{\mathrm{T}} & u^{\mathrm{T}}
\end{array}\right]\left[\begin{array}{ccc}
A-\lambda I & B_{1} & B_{2}  \tag{5.6}\\
C & D_{1} & 0
\end{array}\right]=0
$$

In particular the invariant zero-directions associated with a zero at the origin are found by computing the left-kernel of the matrix

$$
\left[\begin{array}{ccc}
A & B_{1} & B_{2} \\
C & D_{1} & 0
\end{array}\right] .
$$

It is also well known from geometric control theory that the whole space of generalized invariant zero-directions, isomorphic to the quotient space $\mathscr{V}^{*} / \mathscr{R}^{*}$, can be identified, when $D_{1}$ is invertible, with the orthogonal complement of the reachability subspace $\left\langle\Gamma \mid B_{2}\right\rangle$ for the pair ( $\Gamma, B_{2}$ ). In this case, the subspace of invariant zero-directions corresponding to the eigenvalue $\lambda=0$ is precisely $\operatorname{Lker}\left[\Gamma \mid B_{2}\right]$.

Let $P_{0}=P_{0}^{\mathrm{T}}$ be the solution of the LMI (3.3) corresponding to the spectral factor $W_{S}(z)$ so that

$$
\begin{align*}
& P_{0}-A P_{0} A^{\mathrm{T}}=B_{1} B_{1}^{\mathrm{T}}+B_{2} B_{2}^{\mathrm{T}},  \tag{5.7}\\
& \bar{C}=\left(A P_{0} C^{\mathrm{T}}+B_{1} D_{1}^{\mathrm{T}}\right)^{\mathrm{T}}, \quad \Lambda_{0}=D_{1} D_{1}^{\mathrm{T}}+C P_{0} C^{\mathrm{T}}, \tag{5.8}
\end{align*}
$$

[^2]and let $P=P^{\mathrm{T}}$ be a solution of the LMI leading to a minimal square spectral factor with a non-singular $D$. One such solution of particular interest here is $P=P_{-}$, since the steady-state Kalman filter for the given model (1.1a), (1.1b) is uniquely determined once $P_{-}$is known. In fact the steady-state Kalman gain is given by
$$
K=\left(\bar{C}^{\mathrm{T}}-A P_{-} C^{\mathrm{T}}\right)\left(\Lambda_{0}-C P_{-} C^{\mathrm{T}}\right)^{-1} .
$$

However, the argument below will work for more general $P$ 's of the kind defined above.

Any such $P$ must satisfy

$$
\begin{align*}
& \Lambda_{0}-C P C^{\mathrm{T}}>0,  \tag{5.9a}\\
& P-A P A^{\mathrm{T}}-\left(\bar{C}^{\mathrm{T}}-A P C^{\mathrm{T}}\right)\left(\Lambda_{0}-C P C^{\mathrm{T}}\right)^{-1}\left(\bar{C}-C P A^{\mathrm{T}}\right)=0 . \tag{5.9b}
\end{align*}
$$

Define $\Delta:=P_{0}-P$. By subtracting (5.9b) from (5.7), we get the algebraic Riccati equation

$$
\begin{align*}
\Delta= & A \Delta A^{\mathrm{T}}-\left(A \Delta C^{\mathrm{T}}+B_{1} D_{1}^{\mathrm{T}}\right)\left(D_{1} D_{1}^{\mathrm{T}}+C \Delta C^{\mathrm{T}}\right)^{-1}\left(A \Delta C^{\mathrm{T}}+B_{1} D_{1}^{\mathrm{T}}\right)^{\mathrm{T}} \\
& +B_{1} B_{1}^{\mathrm{T}}+B_{2} B_{2}^{\mathrm{T}}, \tag{5.10}
\end{align*}
$$

which is the standard ARE satisfied by the steady-state error covariance matrix of the state estimate. Writing $A$ as $\Gamma+B_{1} D_{1}^{-1} C$, the ARE assumes the form

$$
\begin{equation*}
\Delta=\Gamma \Delta \Gamma^{\mathrm{T}}-\Gamma \Delta C^{\mathrm{T}}\left(D_{1} D_{1}^{\mathrm{T}}+C \Delta C^{\mathrm{T}}\right)^{-1} C \Delta \Gamma^{\mathrm{T}}+B_{2} B_{2}^{\mathrm{T}} . \tag{5.11}
\end{equation*}
$$

We now show that, if the process $y$ in non-regular, then the ARE (5.11) has a fictitiously large size. More precisely, in a suitable basis, any solution of (5.11) has the form

$$
\Delta=\left[\begin{array}{cc}
\tilde{\Delta}_{1} & 0  \tag{5.12}\\
0 & 0
\end{array}\right]
$$

where $\tilde{\Delta}_{1}$ is a solution of a reduced-order ARE (RARE). To this aim, let $\tilde{T}=\left[\begin{array}{l}U \\ V\end{array}\right]$ be an orthogonal $\left(\tilde{T} \tilde{T}^{\mathrm{T}}=\tilde{T}^{\mathrm{T}} \tilde{T}=I\right)$ matrix such that

$$
\begin{equation*}
\operatorname{row}-\operatorname{span}[V]=\operatorname{Lker}\left[\Gamma \mid B_{2}\right] . \tag{5.13}
\end{equation*}
$$

It is clear that

$$
\begin{align*}
& \tilde{T} \Gamma \tilde{T}^{\mathrm{T}}=\left[\begin{array}{cc}
\tilde{\Gamma}_{1} & \tilde{\Gamma}_{12} \\
0 & 0
\end{array}\right],  \tag{5.14}\\
& \tilde{T} B_{2}=\left[\begin{array}{c}
\tilde{B}_{21} \\
0
\end{array}\right], \tag{5.15}
\end{align*}
$$

so that, by multiplying Eq. (5.11) by $\tilde{T}$ on the left-hand side and by $\tilde{T}^{\mathrm{T}}$ on the right-hand side, it is easy to see that for any solution $\Delta$ of (5.11) we have

$$
\tilde{T} \Delta \tilde{T}^{\mathrm{T}}=\left[\begin{array}{cc}
\tilde{\Delta}_{1} & 0  \tag{5.16}\\
0 & 0
\end{array}\right],
$$

with $\tilde{\Delta}_{1}$ being a solution of the RARE

$$
\begin{equation*}
\tilde{\Delta}_{1}=\tilde{\Gamma}_{1} \tilde{\Delta}_{1} \tilde{\Gamma}_{1}^{\mathrm{T}}-\tilde{\Gamma}_{1} \tilde{\Delta}_{1} \tilde{C}_{1}^{\mathrm{T}}\left(D_{1} D_{1}^{\mathrm{T}}+\tilde{C}_{1} \tilde{\Delta}_{1} \tilde{C}_{1}^{\mathrm{T}}\right)^{-1} \tilde{C}_{1} \tilde{\Delta}_{1} \tilde{\Gamma}_{1}^{\mathrm{T}}+\tilde{B}_{21} \tilde{B}_{21}^{\mathrm{T}}, \tag{5.17}
\end{equation*}
$$

where we have partitioned $C \tilde{T}^{\mathrm{T}}$ as $C \tilde{T}^{\mathrm{T}}=\left[\tilde{C}_{1} \mid \tilde{C}_{2}\right]$.
Observe that Lemma 5.1 guarantees that, if $y$ is non-regular, $\tilde{\Delta}_{1}$ has a strictly smaller size than that of $\Delta$. In general, however, with this procedure we do not reach the maximal possible reduction of the order of the ARE, since it is not guaranteed that eventually

$$
\begin{equation*}
\operatorname{Lker}\left[\tilde{\Gamma}_{1} \mid \tilde{B}_{21}\right]=\{0\} . \tag{5.18}
\end{equation*}
$$

If (5.18) does not hold, one may further reduce the order of the RARE by using iteratively the same procedure.

The maximal order-reduction of (5.11) can in fact be performed in just one step. Theorem 5.1 below will describe the procedure and also clarify what is the maximal amount of reduction of the order of the ARE which one can get.

As a preliminary step, select a square orthogonal matrix $T$ such that

$$
\hat{\Gamma}:=T \Gamma T^{\mathrm{T}}=\left[\begin{array}{ccc}
\Gamma_{R} & \Gamma_{R I} & *  \tag{5.19}\\
0 & \Gamma_{I} & * \\
0 & 0 & N
\end{array}\right], \quad \hat{B}_{2}:=T B_{2}=\left[\begin{array}{c}
\bar{B}_{21} \\
0 \\
0
\end{array}\right]
$$

where the pair $\left(\Gamma_{R}, \bar{B}_{21}\right)$ is reachable, $\Gamma_{I}$ is invertible and $N \in \mathbb{R}^{\nu \times v}$ is nilpotent. Note that $N$ is the nilpotent part of the map induced by $\Gamma$ on the quotient space $\mathbb{R}^{n} /\left\langle\Gamma \mid B_{2}\right\rangle$. In other words, $N$ describes the invariant zero-dynamics at $z=0$ of $W_{S}(z)$.

Define

$$
\Gamma_{1}:=\left[\begin{array}{cc}
\Gamma_{R} & \Gamma_{R I}  \tag{5.20}\\
0 & \Gamma_{I}
\end{array}\right], \quad B_{21}:=\left[\begin{array}{c}
\bar{B}_{21} \\
0
\end{array}\right]
$$

and, as usual, partition $C T^{\mathrm{T}}$ as $C T^{\mathrm{T}}=\left[C_{1} \mid C_{2}\right]$ with $C_{1}$ of dimension $m \times(n-v)$.
Theorem 5.1. Let $T$ be an orthogonal matrix leading to the block diagonal form (5.19). There is a bijective correspondence between the symmetric solutions of the ARE (5.11) and those of the RARE

$$
\begin{equation*}
\Delta_{1}=\Gamma_{1} \Delta_{1} \Gamma_{1}^{\mathrm{T}}-\Gamma_{1} \Delta_{1} C_{1}^{\mathrm{T}}\left(D_{1} D_{1}^{\mathrm{T}}+C_{1} \Delta_{1} C_{1}^{\mathrm{T}}\right)^{-1} C_{1} \Delta_{1} \Gamma_{1}^{\mathrm{T}}+B_{21} B_{21}^{\mathrm{T}} \tag{5.21}
\end{equation*}
$$

given by

$$
\Delta=T^{\mathrm{T}}\left[\begin{array}{cc}
\Delta_{1} & 0  \tag{5.22}\\
0 & 0
\end{array}\right] T .
$$

The RARE (5.21) has order $n-v$, with $v$ being the algebraic multiplicity of the invariant zero at $z=0$ of $W_{S}(z)$. The ARE cannot be reduced further.

Proof. We have

$$
\hat{\Gamma}^{\nu}=\left[\begin{array}{cc}
\Gamma_{1}^{\nu} & *  \tag{5.23}\\
0 & 0
\end{array}\right],
$$

so that $\hat{\Gamma}^{\nu} T \Delta T^{\mathrm{T}}\left(\hat{\Gamma}^{\mathrm{T}}\right)^{\nu}$ has the form

$$
\hat{\Gamma}^{\nu} T \Delta T^{\mathrm{T}}\left(\hat{\Gamma}^{\mathrm{T}}\right)^{\nu}=\left[\begin{array}{ll}
* & 0  \tag{5.24}\\
0 & 0
\end{array}\right],
$$

and $\hat{\Gamma}^{\nu} T \Delta T^{\mathrm{T}}$ has the form

$$
\hat{\Gamma}^{\nu} T \Delta T^{\mathrm{T}}=\left[\begin{array}{l}
*  \tag{5.25}\\
0
\end{array}\right] .
$$

Moreover, for any $k \geqslant 0, \hat{\Gamma}^{k} T B_{2} B_{2}^{\mathrm{T}} T^{\mathrm{T}}\left(\hat{\Gamma}^{\mathrm{T}}\right)^{k}=\hat{\Gamma}^{k} \hat{B}_{2} \hat{B}_{2}^{\mathrm{T}}\left(\hat{\Gamma}^{\mathrm{T}}\right)^{k}$ has the form

$$
\hat{\Gamma}^{k} \hat{B}_{2} \hat{B}_{2}^{\mathrm{T}}\left(\hat{\Gamma}^{\mathrm{T}}\right)^{k}=\left[\begin{array}{ll}
* & 0  \tag{5.26}\\
0 & 0
\end{array}\right] .
$$

Hence, by multiplying Eq. (5.11) by $\hat{\Gamma}^{\nu-1} T$ on the left-hand side and by $T^{\mathrm{T}}\left(\hat{\Gamma}^{\mathrm{T}}\right)^{\nu-1}$ on the right-hand side, it is easy to see that also $\hat{\Gamma}^{\nu-1} T \Delta T^{\mathrm{T}}\left(\hat{\Gamma}^{\mathrm{T}}\right)^{\nu-1}$ has the form

$$
\hat{\Gamma}^{\nu-1} T \Delta T^{\mathrm{T}}\left(\hat{\Gamma}^{\mathrm{T}}\right)^{\nu-1}=\left[\begin{array}{ll}
* & 0  \tag{5.27}\\
0 & 0
\end{array}\right] .
$$

Similarly, by multiplying Eq. (5.11) only on the left-hand side by $\hat{\Gamma}^{\nu-1} T$, we see that $\hat{\Gamma}^{\nu-1} T \Delta T^{\mathrm{T}}$ has the form

$$
\hat{\Gamma}^{\nu-1} T \Delta T^{\mathrm{T}}=\left[\begin{array}{l}
*  \tag{5.28}\\
0
\end{array}\right]
$$

and, inductively, we show that for any $k \geqslant 0, \hat{\Gamma}^{k} T \Delta T^{\mathrm{T}}\left(\hat{\Gamma}^{\mathrm{T}}\right)^{k}$ has the form

$$
\hat{\Gamma}^{k} T \Delta T^{\mathrm{T}}\left(\hat{\Gamma}^{\mathrm{T}}\right)^{k}=\left[\begin{array}{ll}
* & 0  \tag{5.29}\\
0 & 0
\end{array}\right] .
$$

In conclusion, also in this case, we have

$$
T \Delta T^{\mathrm{T}}=\left[\begin{array}{cc}
\Delta_{1} & 0 \\
0 & 0
\end{array}\right]
$$

with $\Delta_{1}$ being a solution of the (maximally reduced) RARE (5.21). It can be easily checked that in this case (5.18) holds so that the ARE cannot be further reduced.

In conclusion, the computation of the steady-state Kalman filter gain for a nonregular realization of transfer function $W(z)$ can be performed in the following way:

1. Apply Silverman algorithm to $W(z)$ to get $W_{S}(z)$ with a non-singular $D$ matrix.
2. Compute the state covariance matrix $P_{0}$ of the transformed model by solving the Lyapunov equation (5.7).
3. Do an orthogonal change of basis on the realization of $W_{S}(z)$ (e.g. bring it to the real Schur form) to find $T$ and $\Gamma_{1}, C_{1}, B_{21}$.
4. Find the maximal solution $\Delta_{1, \text { MAX }}$ of the RARE (5.21).
5. Compute $\Delta_{\text {MAX }}$ using (5.22) and $P_{-}=P_{0}-\Delta_{\mathrm{MAX}}$ to get the steady-state Kalman gain $K$.

The reduction procedure is actually performed on the ARE relative to the model $W_{S}(z)$, obtained from the original model $W(z)$ by flipping the zeros at infinity. In the next proposition we provide a lower bound on the reduction of the ARE, which can be computed without performing the zero flipping and uses only the matrix $D$ of the original model.

Proposition 5.1. Let $\Gamma=A-B_{1} D_{1}^{-1} C$ be the numerator matrix of $W_{S}(z)$. Then

$$
\begin{equation*}
\operatorname{dim} \operatorname{Lker}\left[\Gamma \mid B_{2}\right] \geqslant m-\operatorname{rank} D \tag{5.30}
\end{equation*}
$$

Proof. Let $\bar{W}(z)=C(z I-A)^{-1}\left[\bar{B}_{1} \mid \bar{B}_{2}\right]+\left[\bar{D}_{1} \mid 0\right]=W(z) Q_{1}$, with $\bar{D}_{1}$ full column rank, be the spectral factor obtained at the first step of the Silverman procedure and let

$$
\mathscr{K}:=\operatorname{Lker}\left[\begin{array}{cc}
A & \bar{B}_{1}  \tag{5.31}\\
C & \bar{D}_{1}
\end{array}\right] .
$$

After the second step of the Silverman procedure we obtain the spectral factor

$$
\bar{W}_{1}(z)=C(z I-A)^{-1}\left[\bar{B}_{1} \mid A \bar{B}_{2}\right]+\left[\bar{D}_{1} \mid C \bar{B}_{2}\right]=\bar{W}(z)\left[\begin{array}{cc}
I & 0  \tag{5.32}\\
0 & z I
\end{array}\right] .
$$

After the third step of the Silverman procedure we obtain the spectral factor

$$
\begin{equation*}
\widetilde{W}(z)=C(z I-A)^{-1}\left[\widetilde{B}_{1} \mid \widetilde{B}_{2}\right]+\left[\widetilde{D}_{1} \mid 0\right]=\bar{W}_{1}(z) Q_{2} . \tag{5.33}
\end{equation*}
$$

It is clear that

$$
\begin{align*}
\mathscr{K} & =\operatorname{Lker}\left[\begin{array}{ccc}
A & \bar{B}_{1} & A \bar{B}_{2} \\
C & \bar{D}_{1} & C \bar{B}_{2}
\end{array}\right]=\operatorname{Lker}\left[\begin{array}{ccc}
A & \bar{B}_{1} & A \bar{B}_{2} \\
C & \bar{D}_{1} & C \bar{B}_{2}
\end{array}\right]\left[\begin{array}{cc}
I & 0 \\
0 & Q_{2}
\end{array}\right] \\
& =\operatorname{Lker}\left[\begin{array}{ccc}
A & \widetilde{B}_{1} & \widetilde{B}_{2} \\
C & \widetilde{D}_{1} & 0
\end{array}\right] . \tag{5.34}
\end{align*}
$$

After two more steps of the Silverman procedure we obtain the spectral factor

$$
\hat{W}(z)=C(z I-A)^{-1}\left[\hat{B}_{1} \mid \hat{B}_{2}\right]+\left[\hat{D}_{1} \mid 0\right]=\bar{W}_{1}(z)\left[\begin{array}{cc}
I & 0  \tag{5.35}\\
0 & z I
\end{array}\right] Q_{3}
$$

so that

$$
\begin{align*}
\mathscr{K} & =\operatorname{Lker}\left[\begin{array}{ccc}
A & \widetilde{B}_{1} & \widetilde{B}_{2} \\
C & \widetilde{D}_{1} & 0
\end{array}\right] \subseteq \operatorname{Lker}\left[\begin{array}{ccc}
A & \widetilde{B}_{1} & A \widetilde{B}_{2} \\
C & \widetilde{D}_{1} & C \widetilde{B}_{2}
\end{array}\right] \\
& =\operatorname{Lker}\left[\begin{array}{ccc}
A & \hat{B}_{1} & \hat{B}_{2} \\
C & \hat{D}_{1} & 0
\end{array}\right] . \tag{5.36}
\end{align*}
$$

In conclusion, at the end of the Silverman procedure we have the spectral factor (5.3) and

$$
\mathscr{K} \subseteq \operatorname{Lker}\left[\begin{array}{ccc}
A & B_{1} & B_{2}  \tag{5.37}\\
C & D_{1} & 0
\end{array}\right] .
$$

Since $D_{1}$ is invertible,

$$
\begin{align*}
\mathscr{K} & \subseteq \operatorname{Lker}\left[\begin{array}{ccc}
A & B_{1} & B_{2} \\
C & D_{1} & 0
\end{array}\right]=\operatorname{Lker}\left[\begin{array}{ccc}
A & B_{1} & B_{2} \\
C & D_{1} & 0
\end{array}\right]\left[\begin{array}{ccc}
I & 0 & 0 \\
-D_{1}^{-1} C & I & 0 \\
0 & 0 & I
\end{array}\right] \\
& =\operatorname{Lker}\left[\begin{array}{ccc}
\Gamma & B_{1} & B_{2} \\
0 & D_{1} & 0
\end{array}\right]=\left(\operatorname{Lker}\left[\Gamma \mid B_{2}\right]\right)\left[I \mid-B_{1} D_{1}^{-1}\right], \tag{5.38}
\end{align*}
$$

which completes the proof.
An obvious corollary of the above proposition is that the amount of reduction, $v$, is lower-bounded by the geometric multiplicity of the zero at infinity of $W(z)$, namely

$$
\begin{equation*}
v \geqslant m-\operatorname{rank} D . \tag{5.39}
\end{equation*}
$$

In particular, if $D$ is not right-invertible, it is guaranteed that the ARE can be reduced. However, as we have seen, as long as the process is non-regular, the reduction occurs even if the original $D$ matrix is right-invertible.

## 6. Examples

(A) Consider the problem of estimating the state of a system of the form (5.1a), (5.1b) with

$$
\begin{align*}
& A=\left[\begin{array}{cc}
-1 / 2 & 1 \\
-1 & 3 / 2
\end{array}\right], \quad B=\left[\begin{array}{ccc}
1 & 4 & 2 \\
1 / 2 & 3 & 1
\end{array}\right],  \tag{6.1}\\
& C=\left[\begin{array}{ll}
1 & 2 \\
1 & 1
\end{array}\right], \quad D=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] . \tag{6.2}
\end{align*}
$$

The matrix $D$ is not full row rank so that we are facing a singular filtering problem and the process $y$ is clearly non-regular. The standard Kalman filtering procedure would require the solution of the $\operatorname{ARE}$ (5.2) which, in this case, has order 2. We shall instead apply the algorithm illustrated in Section 5. The matrices $B_{1}, B_{2}$ and $D_{1}$ of the transfer function (5.3) are easily seen to be given by

$$
B_{1}=\left[\begin{array}{cc}
-1.067 & 0.919  \tag{6.3}\\
-0.65 & 0.263
\end{array}\right], \quad B_{2}=\left[\begin{array}{l}
-0.127 \\
-0.508
\end{array}\right], \quad D_{1}=\left[\begin{array}{cc}
-1.034 & 10.767 \\
0 & 7.616
\end{array}\right],
$$

so that

$$
P_{0}=\left[\begin{array}{ll}
3.1111 & 2.4444  \tag{6.4}\\
2.4444 & 2.7778
\end{array}\right], \quad \Gamma=\left[\begin{array}{ll}
-0.1935 & 0.2742 \\
-0.7742 & 1.0968
\end{array}\right] .
$$

We have

$$
T \Gamma T^{\mathrm{T}}=\left[\begin{array}{cc}
0.9032 & -1.0484  \tag{6.5}\\
0 & 0
\end{array}\right], \quad T B_{2}=\left[\begin{array}{c}
0.5236 \\
0
\end{array}\right]
$$

with

$$
T:=\left[\begin{array}{cc}
-0.2425 & -0.9701 \\
-0.9701 & 0.2425
\end{array}\right]
$$

so that $\Gamma_{1}=0.9032$ and $B_{21}=0.5236$. At this point, to compute $P_{-}$we only need to solve the RARE (5.17) which, in this case, is a scalar equation. The RARE has been obtained by linear operations with negligible computational burden, while the complexity required to solve the RARE is appreciably lighter than that required for the solution of the full-order ARE (5.2).

The solution $\Delta_{1, \mathrm{MAX}}$ turns out to be $\Delta_{1, \mathrm{MAX}}=0.8593$ so that

$$
\Delta_{\mathrm{MAX}}=T^{\mathrm{T}}\left[\begin{array}{cc}
\Delta_{1, \mathrm{MAX}} & 0  \tag{6.6}\\
0 & 0
\end{array}\right] T=\left[\begin{array}{ll}
0.0505 & 0.2022 \\
0.2022 & 0.8088
\end{array}\right]
$$

and

$$
P_{-}=P_{0}-\Delta_{\mathrm{MAX}}=\left[\begin{array}{ll}
3.0606 & 2.2423  \tag{6.7}\\
2.2423 & 1.9690
\end{array}\right]
$$

(B) In the case when $D$ is full row rank, the reduction procedure for filtering nonregular processes is even more direct. To illustrate this fact, consider the following example. Let

$$
\begin{align*}
& A=\left[\begin{array}{cc}
-0.5 & 1 \\
0 & 0.5
\end{array}\right], \quad B=\left[\begin{array}{ccc}
0.5 & 2.5 & 1 \\
0.5 & 1 & 0
\end{array}\right],  \tag{6.8}\\
& C=\left[\begin{array}{ll}
2 & 3 \\
1 & 2
\end{array}\right], \quad D=\left[\begin{array}{lll}
1 & 0 & 0 \\
1 & 1 & 0
\end{array}\right] . \tag{6.9}
\end{align*}
$$

In this case,

$$
D_{1}=\left[\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right]
$$

is clearly non-singular and we have

$$
\Gamma=A-B_{1} D_{1}^{-1} C=\left[\begin{array}{ll}
1 & 2  \tag{6.10}\\
0 & 0
\end{array}\right], \quad B_{2}=\left[\begin{array}{l}
1 \\
0
\end{array}\right] .
$$

Therefore, $\Gamma_{1}=1, B_{21}=1$ and the maximal solution of the scalar RARE (5.17) is $\Delta_{1, \mathrm{MAX}}=1.1708$ so that

$$
\Delta_{\mathrm{MAX}}=\left[\begin{array}{cc}
1.1708 & 0  \tag{6.11}\\
0 & 0
\end{array}\right]
$$

and

$$
P_{-}=P_{0}-\Delta_{\mathrm{MAX}}=\left[\begin{array}{ll}
7.2292 & 2.8667  \tag{6.12}\\
2.8667 & 1.6667
\end{array}\right]
$$

Also in this case, the same result could have been derived by the standard Kalman filtering algorithm based on solving an ARE of order 2, but this route would have required an appreciably heavier computational burden.

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[^1]:    ${ }^{1}$ In fact, past attempts in this direction [10] have been less than convincing.

[^2]:    2 "Equivalent" here means that $W_{S}(z)$ is also a minimal spectral factor of $\Phi(z)$. The state process of this model will in general be different from the original one.

