An Empirical Bayes Approach to Frequency Estimation

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Abstract—In this paper we show that the classical problem of frequency estimation can be formulated and solved efficiently in an empirical Bayesian framework by assigning a uniform a priori probability distribution to the unknown frequency. We discover that the a posteriori covariance matrix of the signal model is the discrete-time counterpart of an operator whose eigenfunctions are the famous prolate spheroidal wave functions, introduced by Slepian and coworkers in the 1960's and widely studied in the signal processing literature although motivated by a different class of problems. The special structure of the covariance matrix is exploited to design an estimator for the hyperparameters of the prior distribution which is essentially linear, based on subspace identification. Bayesian analysis based on the estimated prior then shows that the estimated center-frequency is asymptotically coincident with the MAP estimate. This stochastic approach leads to consistent estimates, provides uncertainty bounds and may advantageously supersede standard parametric estimation methods which are based on iterative optimization algorithms of local nature. Simulations show that the approach is quite promising and seems to compare favorably with some classical methods.

Index Terms—Frequency estimation, Empirical Bayes, prolate spheroidal wave functions, modulated Sinc kernels, subspace methods, multiple frequency and DOA estimation.

I. INTRODUCTION

Frequency estimation is an old nonlinear problem encountered in many branches of science and engineering which has generated a huge literature. The survey of the literature up to 1993 in [1] contains more than 300 titles. Since the literature on this problem is so large it is impossible to present a reasonably complete summary in this introduction. For a general overview we shall just limit to refer to the books [2]– [4] and to the references therein.

The most classical frequency estimation method is via spectral analysis, based on the direct use of the periodogram which however tends to produce nonconsistent estimates and must rely on ad hoc recombinations of partial spectral estimates (see e.g. [5] and the comments in the introduction of Thomson's paper [6]). Research in this framework has nevertheless continued and we should here at least point to some recent interesting contributions such as [7]–[9].

Another rather popular class of methods is based on the so-called signal subspace decomposition. The forerunner of

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signal subspace decomposition method (SSDM) is Pisarenko harmonic decomposition, followed by MUSIC, ESPRIT, and multiple signal classification methods. A survey of these methods can be found in the book [2]. They are all based on linear algebra operations on the sample covariance matrix of the observed process and for this reason are quite popular. However in a way or another these methods rely on a rank estimation step and on a (unavoidably approximate) rankfactorization of the sample covariance. This feature, in our opinion may generate some uncertainty on their statistical properties, in particular consistency.

Accurate frequency estimation has been mostly approached in the literature by nonlinear optimization techniques, typically variants of Maximum Likelihood, of which a remarkable example is the early paper [10]. Unfortunately, because of nonconvexity, these methods are generally local and not guaranteed to yield a unique optimum. Convex relaxation algorithms based on atomic norm minimization have appeared recently [11], [12], [13] but these methods rely on heavy regularization which in principle cannot produce unbiased estimates. A thorough statistical analysis of these methods still seems to be missing.

New results

In this paper we follow a Bayesin approach. The underlying model is the classical sum of harmonic oscillations corrupted by additive white noise, whose frequencies are modeled as randomly varying parameters. Data are modeled as trajectories of a process whose frequency may deviate sightly about an unknown nominal value. It is then reasonable to model frequency as a random variable, a noisy versions of some nominal frequency.

Bayesian estimation techniques for this model have been proposed in various places, e.g. [14]–[19] based on various choices of the prior distribution. Here we propose an approach based on the *Empirical Bayes* philosophy, inferring from the observed data a family of parametric prior distributions on the unknown frequencies. This approach to frequency estimation seems to be new.

The parametric a priori density is chosen as a uniform distribution on a small frequency range of unknown width, which can be interpreted as an a priori confidence interval centered about some unknown nominal frequencies. The width and the relative center frequencies are the *hyperparameters* of the prior which are estimated from data. This simple model seems to be a reasonable model for a variety of applications. Frequency variations on a small bandwidth could describe an experiment where one is measuring the frequency shift of an oscillator (a function generator generating an AC waveform) with variable center frequency. That is, the central frequency is unknown (random in $[-\pi, \pi]$) and also there is an unknown frequency shift of [-W/2, W/2] radians/sample (which is also random and uniformly distributed). The random signal being observed under additive white Gaussian noise.¹

In this frame we show that the estimation of the hyperparameters can be approached by a simple efficient *subspace algorithm*. This in contrast with the standard marginal likelihood approach as considered for example in [20], [21]. Our work uses more deeply the structure of the data process and need not involve optimization, going well beyond the marginal likelihood approach. For a survey and some bibliography on Empirical Bayes methods we refer to [22, p. 262], [21], [23]–[26]. A general underlying motivation for the Empirical Bayes approach is that in some cases it has been proven to yield a mean squared error (MSE), which can even be smaller than maximum likelihood [27], [28].

Assuming a true model with a true unknown center frequency hyperparameter, one can prove consistency of the subspace estimation method which justifies our procedure in the framework of the traditional frequentist interpretation of the hyperparameter. Later on, we shall see that the (empirical) Bayesian MAP frequency estimate is very close to (and in fact may asymptotically coincide with) the subspace centerfrequency estimate.

Relation with Prolate Spheroidal Wave Functions

Imposing the class of parametric uniform priors leads to a simple probabilistic structure of the signal. One ends up by describing the observed signal as a special stationary process named bandlimited white noise which has a flat power spectrum within some finite bandwidth, whose generation was first studied in the conference papers [29], [30]. The remarkable fact is that the covariance operator of these processes has isomorphic properties to those uncovered in the 60's and 70's by D. Slepian and coworkers in a famous series of papers studying the energy concentration properties of time- and band- limited signals, a completely different problem in a completely deterministic context [31]-[34]. The monograph [35] is also a good reference on this topic. We believe that an important contribution of this paper is to point out this stochastic interpretation and show its usefulness in random signal analysis. In section III we make contact with the classical works of David Slepian and colleagues. In particular, here we elaborate on the bandpass analogues of Prolate Spheroidal Wave Functions whose properties were still unknown, as mentioned in a concluding remark in the paper [31].

We discover that the whole theory of bandlimited time/frequancy analysis of Slepian and co-workers, which for decades has only been used for deterministic signal analysis, can be transported to the stochastic setting allowing a deep understanding and a fine analysis of the structure of the covariance of stationary signals with harmonic components. This has dramatic consequences. For the first time our analysis allows a precise characterization of the finite-data approximation and truncation errors of the covariance kernel of the observed signal which is inherent in many covariance-based signal processing methods of the literature. Similar to Slepian's theory we discover that the eigenvalues of the covariance operator decay abruptly to infinitesimal values (practically zero) after staying constant up to a certain *a priori computable* number, which can be identified as the numerical rank of the matrix. One can in fact get a rather precise estimate of the rank of a finitely-truncated covariance matrix and work with approximations of known precision. This was never suspected before and in all current literature, the use of finite rank covariance approximations to finite data sets is assumed without much of no analysis of the quality of approximation.

In this setting we can rigorously justify the use of subspace methods based on finite rank purely-deterministic approximation of the process and its representation by state-space models.

The proposed stochastic model embraces (in a Bayesian framework) the theoretical covariance structure underlying many classical subspace methods used for frequency and DOA estimation such as MUSIC, ESPRIT and descendants. In a sense our theory and results shed light on the foundations and approximation inherent in these methods. In particular it allows a precise analysis of the finite-rank signal approximation which is rarely addressed in the literature. As a result of this analysis a neat general proof of consistency can be provided.

More specifically, because of the uniform frequency prior, the covariance of the observed process turns out to be a function of the modulated Sinc-type, which in the special case of nominal center frequency equal to zero, has been well studied in the afore-cited literature. The key property of the covariance operator in question is that its eigenvalues decay extremely fast to zero for indices greater than an *a priori* computable number (the so-called Slepian frequency [36]). This means that the eigenfunction expansion of the covariance kernel involves essentially only a finite number of terms. This key feature was already evident and well-studied in the classical deterministic literature when the center frequency is zero but for non zero center frequencies a thorough understanding of the behavior of these modulated Sinc operators was posed as an open problem in [31, p. 63]. Later it was shown to hold for continuous-time modulated Sinc kernels in [34], [37] but the discrete-time case was left open. In this paper, we provide a proof that modulated discrete-time kernels behave in a completely analogous way. This fact allows a direct and rather simple estimation of one hyperparameter of the prior. The resulting center frequency estimate is computed by a subspace algorithm followed by a simple averaging process which seems to yield very accurate and robust results, at least for a large enough sample size. This new estimation method is expounded for signals with multiple unknown frequencies.

Layout

The paper is organized as follows:

In Section II, we formulate the Bayesian framework for the frequency estimation problem. We first deal with signals with

¹We thank one reviewer for supplying this example.

one hidden sinusoidal component but the techniques and results are then extended to treat signals with multiple harmonic components of unknown frequencies by assigning them nonoverlapping rectangular (uniform) prior distributions. In this way the overall covariance kernel becomes the sum of the individual covariances of uncorrelated harmonic components. Our technique can still be applied and is somehow reminiscent of Multiple Kernel methods as in [38], [39].

Then in Section III, we discuss the special structure of the signal covariance which is a discrete-time counterpart of the *modulated Sinc kernel* class discussed in the literature. We prove the sharp decay property of the eigenvalues using techniques inspired by the continuous-time results from the literature. Then we illustrate our findings through a numerical example.

In Section IV we exploit the covariance structure to propose an extremely simple frequency estimate for signals with only one unknown frequency, which is only based on spectral data of the covariance. Note that because of non-ergodicity, consistent estimation of the covariance data is a non-trivial issue.

Section V attacks the main theme of the paper, namely estimation of multiple center-frequencies using a subspace method. By the finite rank property one can use a natural approximate state-space model of the data.

Consistency of the subspace estimator is then discussed in Section VI.

Section VII addresses the MAP Bayesian estimator of the random frequency ω based on the estimated prior discussed in Section V.

In the following Section VIII, the method is applied to several test examples. As can be seen, the results are very encouraging.

At last, Section IX concludes the paper.

Notation and conventions

Boldface symbols denote random quantities. For a square summable sequence y of complex numbers, we take the definition of the discrete-time Fourier transform (DTFT) to be the following

$$\mathcal{F}: \, \ell^2 \to L^2[-\pi,\pi]$$
$$y \mapsto \hat{y}(\omega) := \sum_{t \in \mathbb{Z}} y(t) e^{-it\omega},$$

where the convergence of the Fourier series is understood in L^2 norm. The inverse transform is given by

$$\mathcal{F}^{-1}: \, \hat{y} \mapsto y(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it\omega} \hat{y}(\omega) \mathrm{d}\omega.$$

The ℓ^2 norm of y is known as thefr energy of the signal.

The indicator function on a set $S\subset \Omega$ is defined as

$$\chi_S(\omega) = \begin{cases} 1 & \text{for } \omega \in S, \\ 0 & \text{for } \omega \in \Omega \setminus S. \end{cases}$$

II. SIGNAL MODEL

Consider the following signal model

$$\mathbf{y}(t) = \mathbf{x}(t) + \mathbf{w}(t), \quad t \in \mathbb{Z}$$
(1)

where t represents time, x is the sum of random oscillatory components (a quasi periodic process), that is

$$\mathbf{x}(t) := \sum_{\ell=1}^{\nu} \mathbf{a}_{\ell} \cos(\boldsymbol{\omega}_{\ell} t) + \mathbf{b}_{\ell} \sin(\boldsymbol{\omega}_{\ell} t), \qquad (2)$$

and w is additive white noise. The angular frequencies ω_{ℓ} are unknown but their number ν is fixed in advance. In addition we shall require that:

- the amplitude pairs a_k, b_k are zero-mean pairwise and mutually uncorrelated for all k and the two components a_k, b_k have equal variance: σ²_k = var[a_k] = var[b_k], k = 1,...,ν;
- each angular frequency ω_ℓ is a random variable taking values in the interval [0, π], independent of the amplitudes;
- The noise $\mathbf{w}(t)$ is assumed white, zero-mean Gaussian, stationary of variance $\sigma_{\mathbf{w}}^2$, independent of everything else.

We shall let $\boldsymbol{\omega} := \begin{bmatrix} \omega_1 & \dots & \omega_\nu \end{bmatrix}^\top$ and denote by **a**, **b** two similarly arranged amplitude vectors. Note that the model is linear in **a**, **b**, and hence estimation of the amplitudes and their variance is just a standard linear estimation problem when the frequencies are known. For this reason, in this paper we shall mostly concentrate on the problem of frequency estimation.

Let us now introduce the Empirical Bayesian framework. We shall impose that each component ω_{ℓ} of the random vector ω follows a uniform distribution on the frequency band $[\theta_{\ell} - W_{\ell}, \theta_{\ell} + W_{\ell}]$ such that the symmetrized sets w.r.t. the origin

$$S_{\ell} := [\theta_{\ell} - W_{\ell}, \theta_{\ell} + W_{\ell}] \cup [-\theta_{\ell} - W_{\ell}, -\theta_{\ell} + W_{\ell}], \quad \ell = 1, \dots, \nu$$

do not overlap. For simplicity we shall assume that the assigned bandwidth is the same for different frequencies, i.e., $W_1 = \cdots = W_{\nu} = W$. Here $0 \le \theta_{\ell} \le \pi$ is called a *center-frequency* and $0 \le W \le \pi$ the *bandwidth*. In the literature, both θ and W are called *hyperparameters* of the *a priori* distribution for the frequency ω .

The stated assumptions imply that for each fixed frequency value ω the ν components, say \mathbf{x}_{ℓ} , $\ell = 1, \ldots, \nu$ of the signal (2) are stationary uncorrelated processes. Hence the covariance function of the process \mathbf{y} for a fixed deterministic ω has the form

$$\Sigma(t,s \mid \omega) := \mathbb{E} \left\{ \mathbf{y}(t)\mathbf{y}(s) \mid \omega \right\} = K(t,s \mid \omega) + \sigma_{\mathbf{w}}^2 \,\delta(t,s)$$
(3)

where $\delta(t, s)$ is the Kronecker symbol, and

$$K(t,s \mid \omega) := \sum_{\ell=1}^{\nu} \mathbb{E} \left\{ \mathbf{x}_{\ell}(t) \mathbf{x}_{\ell}(s) \mid \omega \right\} = \sum_{\ell=1}^{\nu} K_{\ell}(t,s \mid \omega)$$

is the a priori conditional covariance of the signal x given $\omega = \omega$. To lighten the notation, we shall temporarily suppress the subscripts. The formulas below should be interpreted as holding for a generic index ℓ .

By the model assumptions, the following computation is straightforward:

$$K(t, s \mid \omega) = \mathbb{E} \left\{ \mathbf{a}^{2} \cos(\omega t) \cos(\omega s) + \mathbf{a} \mathbf{b} \cos(\omega t) \sin(\omega s) + \mathbf{a} \mathbf{b} \sin(\omega t) \cos(\omega s) + \mathbf{b}^{2} \sin(\omega t) \sin(\omega s) \right\}$$
$$= \sigma^{2} \cos \omega \tau \tag{4}$$

where $\tau := t - s$, and then computing the a posteriori covariance by integrating the function w.r.t. the uniform prior density, one gets

$$K(t,s) = \sigma^2 \mathbb{E} \left(\cos \omega \tau \right) = \sigma^2 \int_{\theta-W}^{\theta+W} \cos(\omega \tau) \frac{1}{2W} d\omega$$
$$= \sigma^2 \cos(\theta \tau) \frac{\sin W \tau}{W \tau}.$$
 (5)

Since the covariance function depends only on τ , the signal **x** is stationary, and so is **y**. In the following, we will write $K(\tau)$ in place of K(t, s).

For $\theta = 0$, the covariance function K is the well-known Sinc function, which is the inverse Fourier transform of a rectangular function, namely

$$\sigma^2 \frac{\sin W\tau}{W\tau} = \frac{\sigma^2}{2W} \int_{-W}^{W} e^{i\omega\tau} d\omega.$$
 (6)

It follows that a zero-frequency component of the process **x** must have a uniform spectral density $\frac{\pi\sigma^2}{W}\chi_{[-W,W]}(\omega)$. When $W = \pi$, the process is just a usual stationary white noise of variance σ^2 . For $W < \pi$, the process **x** is nontrivial, called a *bandlimited white noise* within the frequency band [-W, W]. In this case, it is a purely deterministic process with an absolutely continuous spectral distribution, since the logarithm of the density is obviously not integrable (see e.g., [40, p. 144]).

In this paper, we are primarily interested in the case $\theta_{\ell} \neq 0$, for which we make the assumption that $|\theta_{\ell}| > W$, so that each support set

$$S := [\theta - W, \theta + W] \cup [-\theta - W, -\theta + W]$$
(7)

is composed of two disjoint intervals symmetric with respect to the origin. Then the last expression in (4) can be rewritten as

$$\sigma^{2}\cos(\theta\tau)\frac{\sin W\tau}{W\tau} = \frac{\sigma^{2}}{4W}\int_{-\pi}^{\pi}\cos(\omega\tau)\chi_{S}(\omega)d\omega$$
$$= \frac{\pi\sigma^{2}}{2W}\int_{-\pi}^{\pi}e^{i\omega\tau}\chi_{S}(\omega)\frac{d\omega}{2\pi}$$
(8)

where χ_S is the indicator function of S, and the second equality holds due to the symmetry of the integrand. From the above relation, we see that the spectral density of the process x is now the sum of ν disjoint spectral terms, each of the form

$$\phi_{\mathbf{x}_{\ell}}(\omega) = \frac{\pi \sigma_{\ell}^2}{2W} \left(\chi_{[\theta_{\ell} - W, \ \theta_{\ell} + W]} + \chi_{[-\theta_{\ell} - W, \ -\theta_{\ell} + W]} \right).$$

The signal x can therefore be described as a sum of independent deterministic carriers, each of angular frequency θ_{ℓ} , amplitude-modulated by a bandlimited white noise process described before. For the same reason, the covariance function (4) has been called a *modulated sinc kernel* in [36], where it arises in a different context.

In practice we can only observe sample paths of finite length N from the process \mathbf{y} . For clarity of exposition, we shall now assume that $\nu = 1$ and neglect the subscript ℓ altogether. The generalization to multiple sinusoids, i.e., $\nu > 1$, will be obvious. Collect the observed random variables into a column vector, and in particular, let $\mathbf{X}_N := [\mathbf{x}(t), \mathbf{x}(t+1), \dots, \mathbf{x}(t+N-1)]^{\top}$. Then consider the $N \times N$ covariance matrix

$$\mathbf{K}_{N} := \mathbb{E}\{\mathbf{X}_{N}\mathbf{X}_{N}^{\top}\} = \begin{bmatrix} K(0) & K(1) & \cdots & K(N-1) \\ K(1) & K(0) & \cdots & K(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ K(N-1) & K(N-2) & \cdots & K(0) \end{bmatrix}.$$
(9)

This symmetric Toeplitz structure of the covariance matrix comes from the fact that the process is stationary and real-valued. Similarly, we can define the $N \times N$ covariance matrix of the process y, say Σ_N , and we have the relation

$$\Sigma_N = \mathbf{K}_N + \sigma_{\mathbf{w}}^2 I_N. \tag{10}$$

Analysis of the eigen-structure of \mathbf{K}_N will be of great importance to our frequency estimation problem, and that will be the content of the next section.

III. PROPERTIES OF THE COVARIANCE MATRIX

In this section, we show that the covariance matrix (9) also arises in a quadratic form which is the essential instrument for solving the energy concentration problem for discrete-time deterministic signals. In order to state the problem, we first need to set up some notations. Let J be a set that is a union of a finite number of pair-wise disjoint closed subintervals of $[-\pi, \pi]$, e.g., a union of sets like S in (7). Define the bandlimiting operator

$$\mathfrak{B}: \ell^2 \to \ell^2, \quad y \mapsto \mathcal{F}^{-1}[\chi_J \mathcal{F}(y)]$$
 (11)

that corresponds to a bandpass filter with prescribed bandwidth $\{\omega \in J\}$. Fix a positive integer N and let

$$I := \{0, 1, \dots, N - 1\}.$$
 (12)

Define similarly the time-limiting operator

$$\mathfrak{T}: \ell^2 \to \ell^2, \quad y \mapsto \chi_I y, \tag{13}$$

where χ_I is the indicator function in the time domain \mathbb{Z} .

The energy concentration problem that will be discussed in this section is

$$\sup_{y \in \ell^2} \frac{\|\mathfrak{BT}y\|_{\ell^2}^2}{\|y\|_{\ell^2}^2}.$$
(14)

Notice that the supremum can only be attained at a timelimited y, because the objective value of $\tilde{y} := \mathfrak{T}y$ is equal to $\|\mathfrak{B}\tilde{y}\|_{\ell^2}^2 / \|\tilde{y}\|_{\ell^2}^2$ which is not less than that of y. Therefore, it is equivalent to consider the problem

$$\sup_{\substack{y \in \ell^2 \\ \text{upp}(y) \subset I}} \frac{\|\mathfrak{B}y\|_{\ell^2}^2}{\|y\|_{\ell^2}^2},\tag{15}$$

where $\operatorname{supp}(\cdot)$ denotes the support of a function. In other words, the aim is to find a time-limited signal whose energy is most concentrated in the frequency band J.

A. The eigenvalue problem

The impulse response of the ideal bandpass filter $\chi_J(\omega)$ is just the inverse Fourier transform

$$\rho(t) := \frac{1}{2\pi} \int_{J} e^{it\omega} \mathrm{d}\omega \quad t \in \mathbb{Z}.$$
 (16)

Observe that the function ρ has the symmetry $\rho(-t) = \rho(t)^*$ where z^* means the complex conjugate (transpose) of $z \in \mathbb{C}$.

According to the definitions (11) and (13), we have

$$\mathfrak{BT} y = \mathcal{F}^{-1} \left[\chi_J(\omega) \mathcal{F}(\mathfrak{T} y) \right]$$
$$= \mathcal{F}^{-1} \left[\chi_J(\omega) \sum_{t=0}^{N-1} y(t) e^{-it\omega} \right]$$
$$= \rho * \mathfrak{T} y$$
(17)

where * denotes convolution. It follows that

$$\|\mathfrak{BT}y\|_{\ell^{2}}^{2} = \sum_{t \in \mathbb{Z}} \left| \sum_{k=0}^{N-1} \rho(t-k)y(k) \right|^{2}$$

= $\sum_{j=0}^{N-1} y(j)^{*} \sum_{k=0}^{N-1} y(k) \sum_{t \in \mathbb{Z}} \rho(t-j)^{*} \rho(t-k).$ (18)

The last summation can be rewritten

$$\sum_{t\in\mathbb{Z}}\rho(t-j)^*\rho(t-k) = \sum_{t\in\mathbb{Z}}\rho(j-t)\rho(t-k)$$

= $(\rho*x)(j),$ (19)

where the sequence $x(t) := \rho(t - k)$ has Fourier transform $\hat{x}(\omega) = e^{-ik\omega}\chi_J(\omega)$. The Fourier transform of $\rho * x$ is simply again $e^{-ik\omega}\chi_J(\omega)$. Hence the above sum is equal to $\rho(j-k)$, and we arrive at

$$\|\mathfrak{BT}y\|_{\ell^2}^2 = \sum_{j=0}^{N-1} y(j)^* \sum_{k=0}^{N-1} y(k) \,\rho(j-k)$$

= $\mathbf{y}^* \mathbf{Ry}$, (20)

where $\mathbf{y} = [y(0), y(1), \dots, y(N-1)]^{\top}$ is a slight abuse of notation, and

$$\mathbf{R} = \begin{bmatrix} \rho(0) & \rho(-1) & \cdots & \rho(-N+1) \\ \rho(1) & \rho(0) & \cdots & \rho(-N+2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(N-1) & \rho(N-2) & \cdots & \rho(0) \end{bmatrix}.$$
 (21)

The matrix **R** has a Hermitian Toeplitz structure, and it is also positive definite because the quadratic form determines the energy of $\mathfrak{BT}y$. Notice that when the set J is symmetric w.r.t. the origin such as S in (7), then the integral in (16) reduces to $\int_J \cos(t\omega) d\omega$. In that case, ρ is an even function of time, and the matrix **R** is real symmetric.

Now the objective functional in the energy concentration problem (14) is in fact equal to the Rayleigh quotient associated to \mathbf{R} . By the min-max theorem, the maximum of the objective is equal to the largest eigenvalue of \mathbf{R} , and it is attained when \mathbf{y} is the corresponding eigenvector. It is obvious that the eigenvalues of \mathbf{R} do not exceed 1, simply because both \mathfrak{B} and \mathfrak{T} are projection operators. *Remark* 1. Although it does not particularly interest us here, it is worth mentioning that the energy concentration problem (14) has a "dual" problem obtained by interchanging the two operators \mathfrak{B} and \mathfrak{T} , namely

$$\sup_{y \in \ell^2} \frac{\|\mathfrak{TB}y\|_{\ell^2}^2}{\|y\|_{\ell^2}^2}.$$
(22)

The problem (22) is equivalent to determining the supremum of $\|\mathfrak{T}y\|_{\ell^2}^2$ over all band-limited signals subject to the constraint $\|y\|_{\ell^2} = 1$. By a standard variational argument using the Lagrange multiplier, one can conclude that the maximum of the dual objective is equal to the largest eigenvalue of a linear integral operator with a (modified) Dirichlet kernel. Moreover, following the lines in [41, Section 5], it is not difficult to show that the eigenvalues of such an integral operator are identical to those of **R**, and the corresponding eigenfunctions are related via the Fourier transform.

B. Asymptotic distribution of the eigenvalues

We shall now allow the dimension of \mathbf{R} to increase. In other words, the integer N introduced by the set I in (12) is considered as a variable tending to infiity. Let $\lambda_j(N)$ be the *j*-th eigenvalue (arranged in nonincreasing order) of \mathbf{R} . We know from the previous subsection that $0 < \lambda_j(N) \le 1$ for all $j = 1, \ldots, N$. It also follows easily that

$$\sum_{j=1}^{N} \lambda_j(N) = \operatorname{tr} \mathbf{R} = N\rho(0) = \frac{\mathrm{m}(J)}{2\pi}N, \qquad (23)$$

where the notation $m(\cdot)$ denotes the Lebesgue measure of a set. Now for a real number $0 < \gamma < 1$, define $M(\gamma, N)$ to be the number of eigenvalues of **R** that are no less than γ . Again we have included the explicit dependence on the dimensional variable N. The next result is a first-order description of the asymptotic eigenvalue distribution of the matrix **R**. The proof borrows techniques from [37] and can be found in the appendix.

Theorem 1. It holds that

$$\lim_{N \to \infty} \frac{M(\gamma, N)}{N} = \frac{\mathrm{m}(J)}{2\pi}$$
(24)

independent of γ .

A more precise formula for the asymptotic expansion of the quantity $M(\gamma, N)$ is given in [34] for the continuous-time case. The second term in the asymptotic expansion is shown to be proportional to $\log N$. Slepian's asymptotic expressions for the eigenvalues, valid for $\theta = 0$, are also reported in [6, p. 1059]. Although we believe that analogous discrete-time estimates should hold, a formal proof is yet to be worked out. For our problem of frequency estimation, Corollary 1 below is anyway sufficient.

By choosing γ arbitrarily close to 1, an immediate consequence of the above theorem and formula (23) is the following.

Corollary 1. For $N \to \infty$, the matrix **R** has rank

$$n = N\mathrm{m}(J)/2\pi,\tag{25}$$

and all the nonzero eigenvalues tend to 1.

The convergence is very fast since, as it is shown in the proof of the Theorem, the matrix \mathbf{R} has only o(N) eigenvalues that are between 0 and 1, and for large sample size they can be reasonably neglected.

For $\nu = 1$ the covariance matrix in (9) is just a scalar multiple of **R** via $\mathbf{K}_N = \frac{\pi \sigma^2}{2W} \mathbf{R}_N$ (here for notational consistency we have added the subscript _N to **R**). Clearly, the constant factor only rescales the eigenvalues. In particular, the assertion on the rank in Corollary 1 holds for **K**. Below we show some simulations of how the eigenvalues decay.

Fig. 1 shows the behavior of the eigenvalues μ_k of the sinc kernel for $N = 1000, W/2\pi = 0.02, \sigma^2 = 1$ which yields a rank approximately equal to 40. We can clearly see that for n < 40 the eigenvalues are all equal to the same constant while for n > 40 the μ_k 's very quickly decrease to zero.

The behavior of the eigenvalues of **R** is the same except that the normalization makes the μ_k all practically equal to one for k < n. In order to get the same normalization we just need to substitute μ_k with $2W\mu_k/2\pi$.

As for the modulated sinc kernel, Fig. 2 shows the eigenvalues of a matrix **K** with the same values of N, W, and σ^2 . One sees that the eigenvalues have exactly the same behavior as those of the Sinc kernel. Only the value of n such that for k > n, $\mu_k \simeq 0$ is now $4NW/2\pi = 80$, i.e., twice the value of n for the sinc kernel. Moreover, the amplitudes of the eigenvalues for k < n are half of those of the sinc kernel, for equal values of W. This follows the from the symmetry of the spectrum and matches also the experimental findings of [36].

In order to get the largest eigenvalues of the modulated sinc kernel equal to one, a different normalization should be made by substituting μ_k with $4W\mu_k/2\pi$. This agrees with the matrix rescaling described above.



Figure 1. Eigenvalues of the sinc kernel covariance matrix, rank ≈ 40

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In the case of one hidden frequency, we have rank $\mathbf{K}_N \approx \frac{2W}{\pi}N$ according to Corollary 1. We can see that the bandwidth W can be inferred from the rank information of the signal



Figure 2. Eigenvalues of the modulated sinc kernel covariance matrix, rank ≈ 80

covariance matrix K. Since our measurements come from the process y, we start by estimating its covariance matrix Σ .

A well-known difficulty in frequency estimation is that stationary random processes with periodic components, even when the frequencies are exactly known, are not ergodic. Nonergodicity means in particular that, when the sample size goes to infinity, the limit of the process sample covariance is *sample* dependent, that is, the limit sample covariance depends on the random amplitudes of its elementary oscillatory components (see e.g., [42, pp. 105-109]). This lack of ergodicity is even more serious when the frequency is random. For this reason, one-sample-path estimation runs into difficulty and the standard approach in many practical situations is to consider estimation from cross-sectional or panel data (also called *snapshots*), as described in e.g., [43] and e.g. done in DOA estimation. Cross-sectional frequency data can be the result of parallel measurements by multiple sensors which is quite common for example in testing of turbo, and in general rotating machines, but also in many directional signal processing and biomedical applications.

For the reasons above, we shall need to assume that our observed data consist of L strings of sample observations (snapshots), assumed for simplicity all of length N:

$$y_k(t) = a_k \cos(\omega_k t) + b_k \sin(\omega_k t) + w_k(t), \qquad (26)$$

where k = 1, ..., L, t = 1, ..., N, (a_k, b_k) are sample determinations of the random variables (\mathbf{a}, \mathbf{b}) , and the frequencies ω_k are sample determinations of the random variable $\boldsymbol{\omega}$ which is uniformly distributed on the fixed interval $[\theta - W, \theta + W]$. We assume that noises of different cross sections are independent. Furthermore, we assume that the random samples $[a_k, b_k, \omega_k]$ come from i.i.d. copies of $[\mathbf{a}, \mathbf{b}, \boldsymbol{\omega}]$, then the covariance matrix can be estimated by first subtracting the sample mean from the data, i.e.

$$\tilde{y}_k(t) := y_k(t) - \frac{1}{N} \sum_{t=1}^N y_k(t)$$

and then doing a cross-sectional average

$$\hat{\boldsymbol{\Sigma}}_{N,L} := \frac{1}{L} \sum_{k=1}^{L} \boldsymbol{\mathcal{Y}}_k \boldsymbol{\mathcal{Y}}_k^\top, \qquad (27)$$

where $\mathcal{Y}_k = \begin{bmatrix} \tilde{y}_k(1) & \cdots & \tilde{y}_k(N) \end{bmatrix}^\top$ is a column *N*-vector of centered data. The procedure is asymptotically equivalent (for $L \to \infty$) to first computing the standard (biased) covariance estimator within each sample path [2, Chapter 2],

$$\hat{\sigma}_k(\tau) := \frac{1}{N-\tau} \sum_{t=1}^{N-\tau} \tilde{y}_k(t+\tau) \tilde{y}_k(t),$$

constructing the sample Toeplitz estimate

$$\hat{\Sigma}_k := \operatorname{SymToep}\{\hat{\sigma}_k(0), \dots, \hat{\sigma}_k(N-1)\}$$
(28)

and then doing cross sectional average w.r.t. k to obtain $\hat{\Sigma}_{N,L}$ which is still symmetric-Toeplitz (here the subscript N just refers to the dimension which is fixed). By the strong law of large numbers, we have

$$\hat{\Sigma}_{N,L} \to \Sigma_N \text{ as } L \to \infty$$
 (29)

almost surely. Let $\hat{\lambda}_N$ be the smallest eigenvalue of $\hat{\Sigma}_N$. Then given (10) and Theorem 1, we have

$$\lim_{L,N\to\infty}\hat{\lambda}_N = \sigma_{\mathbf{w}}^2.$$
(30)

The limit here and those similar ones in the following are understood as first letting $L \to \infty$ and then $N \to \infty$. In this sense we are able to build a consistent estimator of the signal covariance matrix \mathbf{K}_N :

$$\hat{\mathbf{K}}_N := \hat{\mathbf{\Sigma}}_N - \hat{\lambda}_N I_N, \tag{31}$$

A consistent estimator of the signal variance is given by

$$\hat{\sigma}_{\mathbf{x}}^2 := \hat{\sigma}_{\mathbf{y}}(0) - \hat{\lambda}_N, \qquad (32)$$

since we have $\sigma_{\mathbf{y}}(0) = \sigma_{\mathbf{x}}^2 + \sigma_{\mathbf{w}}^2$ by (3).

Next, for $\varepsilon > 0$ close to zero, the *numerical rank* of \mathbf{K}_N can be estimated using Theorem 1 as

$$\operatorname{rank}(\mathbf{K}_N) \simeq M(\varepsilon, N) = \frac{2W}{\pi}N$$
 (33)

with an approximation error which roughly grows as $O(\log N)$. In particular we have

$$\hat{W} := \frac{\pi}{2} \frac{\operatorname{rank}(\mathbf{K}_N)}{N} \to W \tag{34}$$

when N is large. Unfortunately this estimator of W depends heavily on the estimate of the numerical rank whose computation is delicate and is not very reliable unless N is very large. We shall comment on this in the next subsection.

The relation for the scalar covariance of lag 1

$$\sigma_{\mathbf{x}}(1) = \sigma^2 \cos \theta \, \frac{\sin W}{W},\tag{35}$$

could then be used to get a rough estimate of the center frequency:

$$\hat{\theta} := \arccos\left(\frac{\hat{\sigma}_{\mathbf{x}}(1)}{\hat{\sigma}^2} \frac{\hat{W}}{\sin\hat{W}}\right),\tag{36}$$

where $\hat{\sigma}_{\mathbf{x}}(1)$ is an estimator of $\sigma_{\mathbf{x}}(1) = \mathbb{E}\mathbf{x}(t+1)\mathbf{x}(t)$.

In the next section we shall describe a more general reliable estimator based on the subspace philosophy.

Remark 2. The independence of the cross sections, although often assumed in the literature, may seem quite strong. A more natural assumption could be to require that the strings (26) are sample observations of length N from an *exchangeable* sequence of N-dimensional random vectors, $\{\mathbf{y}_k\}_{k=1}^L$. For reasons of space this alternative viewpoint will not be further pursued here.

V. A SUBSPACE APPROACH TO HYPERPARAMETER ESTIMATION

Consider now the general measurement model (1), with the signal x consisting of multiple sinusoids as in (2) satisfying all assumptions listed in Sec. II. For simplicity, we shall assume that the amplitude variances are the same, $\sigma_1^2 = \cdots = \sigma_{\nu}^2 = \sigma^2$. The covariance of y can then be computed similarly to that in Sec. II. We have

$$\Sigma(\tau) = K(\tau) + \sigma_{\mathbf{w}}^{2}\delta(\tau, 0)$$

$$= \sum_{\ell=1}^{\nu} \sigma^{2} \mathbb{E} \left(\cos \omega_{\ell} \tau \right) + \sigma_{\mathbf{w}}^{2}\delta(\tau, 0)$$

$$= \sigma^{2} \frac{\sin W\tau}{W\tau} \sum_{\ell=1}^{\nu} \cos \theta_{\ell} \tau + \sigma_{\mathbf{w}}^{2}\delta(\tau, 0)$$

$$= \frac{\pi \sigma^{2}}{2W} \int_{-\pi}^{\pi} e^{i\omega\tau} \sum_{\ell=1}^{\nu} \chi_{S_{\ell}}(\omega) \frac{\mathrm{d}\omega}{2\pi} + \sigma_{\mathbf{w}}^{2}\delta(\tau, 0).$$
(37)

Under the assumptions listed in Sec. II, the sum $\sum_{\ell=1}^{\nu} \chi_{S_{\ell}}(\omega)$ is the indicator function on the set $S := \bigcup_{\ell=1}^{\nu} S_{\ell}$. From the integral expression for the covariance function, we see immediately that Corollary 1 is applicable, and the asymptotic rank of \mathbf{K}_N is now $\frac{2\nu W}{\pi} N$. A rank estimator for the bandwidth W similar to (34) can be used since we have assumed that the supporting intervals for different frequencies have the same bandwidth. A more general situation with different W's can also be dealt with but it yields complicated formulas and will not be discussed here. Next, we will concentrate on the estimation of the center frequency vector $\theta := [\theta_1, \dots, \theta_{\nu}]^{\top}$. *Remark* 3. When the amplitudes $\sigma_1^2, \ldots, \sigma_{\nu}^2$ are different, the spectral density of our signal is a sum of nonoverlapping rectangular functions and can always be written as a weighted sum of indicator functions. The assertion on the rank in Corollary 1 must still hold and a proof could be given based on Szegö's eingenvalue distribution theorem for Toeplitz matrices (see e.g., [44]).

Efficient estimation of the hyperparameters can be based on maximum likelihood, assuming Gaussian additive noise. See [45, p. 429] for a general discussion of this point. The Gaussian likelihood function based on the k-th snapshot of Ndata can be written as (cf. [46])

$$l_k(\theta, W) = -\frac{N}{2} \log 2\pi - \frac{1}{2} \log \det \mathbf{\Sigma}(\theta, W) -\frac{1}{2} \mathcal{Y}_k^\top \mathbf{\Sigma}(\theta, W)^{-1} \mathcal{Y}_k,$$
(38)

where \mathcal{Y}_k is the vector introduced in (27), $\Sigma(\theta, W)$ is the theoretical covariance matrix of \mathcal{Y}_k , with entries given in (37) which do not depend on the index k. The first constant can be dropped from the objective function. By the independence of the sample paths, the log-likelihoods add to each other so that we end up with maximization of the function

$$l(\theta, W) = -\frac{L}{2} \log \det \Sigma(\theta, W) - \sum_{k=1}^{L} \frac{1}{2} \mathcal{Y}_{k}^{\top} \Sigma(\theta, W)^{-1} \mathcal{Y}_{k}$$
(39)

with respect to θ , W. This leads to the well-know unique maximizer, see e.g. [42, pp. 202–203], for the covariance matrix

$$\Sigma(\theta, W) = \Sigma_{N,L} \tag{40}$$

where $\Sigma_{N,L}$ is defined in (28). Such an equation should be solved for the unknown hyperparameters (θ, W) appearing in the known structure (37). Note that this equation can be interpreted as resulting from the well-known *method of moments* which is the theoretical basis of Subspace Methods [40, Chapt. 13]. Since the equation is nonlinear, one may think of setting up at the outset an iterative solution scheme. However, these numerical algorithms very often converge only locally. In fact, the likelihood function is nonconvex and contains many flat regions. Therefore, brute-force optimization seems to be a hard task.

We shall instead take advantage of the structure of the equation (40) to propose a *subspace-based* approach. For a fixed and large enough N, we may and shall here assume that the $N \times N$ covariance matrix \mathbf{K}_N of the process \mathbf{x} has exactly rank $n := \frac{2\nu W}{\pi}N$. As discussed in Subsection III-B, for N large this is a quite accurate approximation. In other words, we do a truncation in the spectral decomposition of the matrix \mathbf{K}_N , retaining the largest n eigenvalues, namely

$$\mathbf{K}_{N} = \frac{\pi \sigma^{2}}{2W} \mathbf{R}_{N} = \frac{\pi \sigma^{2}}{2W} \mathbf{Q}_{N} \mathbf{D}_{N} \mathbf{Q}_{N}^{\top}$$
$$\approx \frac{\pi \sigma^{2}}{2W} \mathbf{Q}_{N} \operatorname{diag}\{I_{n}, O_{N-n}\} \mathbf{Q}_{N}^{\top}, \qquad (41)$$

where O_m denotes the square all-zero matrix of size m. As before, the eigenvalues in the diagonal matrix \mathbf{D}_N are arranged in nonincreasing order.

Proposition 1. For N large enough, there are an $n \times n$ matrix A and an n-dimensional row vector c such that the random oscillatory signal \mathbf{x} can be represented by the system

$$\boldsymbol{\xi}(t+1) = A\boldsymbol{\xi}(t) \tag{42}$$

$$\mathbf{x}(t) = c\,\boldsymbol{\xi}(t) \tag{43}$$

where $\boldsymbol{\xi}(t) = [\xi_1(t), \xi_2(t), \dots, \xi_n(t)]^{\top}$ is an n-dimensional basis vector spanning the Hilbert space $\mathbf{H}(\mathbf{x})$ linearly generated by the N random variables of the set $\{\mathbf{x}(s) : t \ge s \ge t - N + 1\}$.

Proof. It is well-known that a rank-deficient covariance matrix (of rank n) must necessarily be the covariance of a purely deterministic process [40, p. 138, 276]. When the total support of the spectrum $S = \bigcup_{\ell=1}^{\nu} S_{\ell}$ is a proper subset of $[-\pi, \pi]$, **x** in (2) is a purely deterministic process which can be

represented by a deterministic linear recursion of order n or equivalently, by a n-dimensional state-space model. Any such state-space representation for the process \mathbf{x} is of the form (42), (43) where A can be chosen orthogonal so that $A^{\top} = A^{-1}$.

The output of (43) has the expression $\mathbf{x}(t) = cA^t \boldsymbol{\xi}(0)$, from which we can compute the covariance function of the process as

$$\sigma(t-s) = cA^t \mathbb{E}\boldsymbol{\xi}(0)\boldsymbol{\xi}(0)^\top (A^\top)^s c^\top = cA^t P A^{-s} c^\top.$$

The matrix $P := \mathbb{E}\boldsymbol{\xi}(0)\boldsymbol{\xi}(0)^{\top}$ satisfies a degenerate Lyapunov equation and commutes with A. Therefore, we have $\sigma(\tau) = cPA^{\tau}c^{\top}$. The spectral density of \mathbf{x} is a sum of Dirac deltas. To see this, we first notice that since A is orthogonal, its spectral decomposition can be written $A = T\Lambda T^*$ where T is unitary and $\Lambda = \text{diag}\{e^{i\varphi_1}, \ldots, e^{i\varphi_n}\}$ is a diagonal matrix of eigenvalues all having modulus 1. The eigenvalues should come in conjugate pairs $e^{\pm i\varphi}$ if $\varphi \neq 0, \pi$ due to the realness of A. The spectrum of the output process now follows:

$$\Phi_{\mathbf{x}}(\omega) = cP\mathcal{F}(A^{\tau})c^{\top} = cPT\mathcal{F}(\Lambda^{\tau})T^{*}c^{\top}$$

= $2\pi cPT \operatorname{diag}\{\delta(\omega - \varphi_{1}), \dots, \delta(\omega - \varphi_{n})\}T^{*}c^{\top},$
(44)

where the weights for the Dirac deltas are determined by the vectors cPT and T^*c^{\top} . See also [40, Eq. (8.129)].

Since the state-space realization will be constructed from the truncated covariance matrix (41), its spectrum should approximate the true one, i.e., the indicator function on S times a constant factor, in the sense that the supports of the Dirac deltas should be clustered in S. The center of each cluster, namely the average of the arguments φ_k inside one cluster, is an estimate of the center frequency. Such an idea is also justified by the fact that the (approximate) eigenvalues of \mathbf{K}_N do not depend on the center frequencies θ . Hence the whole dependence on θ must be in c and A.

Now the remaining point is how to obtain the parameters c, A in the realization from the measurements of \mathbf{y} . First, we estimate the rank of \mathbf{K}_N using the technique in Subsection VIII-B. Secondly, one can easily verify that the finite covariance matrix of \mathbf{x} in (43) can be written as $\mathbf{K}_N = H_N P H_N^{\top}$, where

$$H_k = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{k-1} \end{bmatrix}$$
(45)

for a positive integer k. This is in fact a rank n factorization of \mathbf{K}_N . Notice that such a factorization is unique modulo the choice of basis in the state space and one can always choose a basis such that P is a diagonal matrix. In that case, we can compare with (41) and choose $H_N = \mathbf{Q}_N(1 : N, 1 :$ n) and P just a constant multiple of the identity. Thirdly, due to additive structure of the covariance matrix (10), Σ_N has the same eigenvectors as \mathbf{K}_N . Therefore, we can estimate the covariance matrix Σ_N using the scheme (28), and extract the eigenvectors corresponding to the largest n eigenvalues to The vector c is simply the first row of H_N . The matrix A can be computed by a standard "shift-invariance" procedure of subspace identification. More precisely, for $k \leq N - 1$, consider the matrix H_k in (45) and its one row shifted counterpart $\downarrow H_k := H_N(2: k+1, :)$. The dynamic matrix A can be extracted by solving the equation $\downarrow H_k = H_kA$ in a least-squares sense. When A is constrained to be orthogonal, this is the well-known "orthogonal Procrustes problem". In [47, Subsec. 6.4.1], it is reported that such a problem is well-posed, and can be solved using SVD. A similar subspace method for oscillatory signals was proposed in [29], [48].

Given the cross sectional measurements (26) of size $L \times N$, we summarize our algorithm below:

- 1) Compute Σ_N , an estimate of the covariance matrix of y, using (28);
- 2) Estimate the rank n of the signal covariance matrix \mathbf{K}_N , and then estimate the bandwidth W by (34);
- 3) Do eigen-decomposition to $\hat{\Sigma}_N$, keep the largest *n* eigenvalues, and call the $N \times n$ matrix of corresponding eigenvectors H_N ;
- 4) Let k = N 1, and solve the orthogonal Procrustes problem $\downarrow H_k = H_k A$ for the orthogonal matrix A;
- 5) Compute the eigenvalues of A, and extract their phase angles (between $-\pi$ and π);
- 6) Run a clustering algorithm, e.g., k-means, on the phase angles, and take the centers of final clusters as estimates of the center frequencies.

In the last step of this subspace algorithm, the center of each cluster may be obtained by simply taking the average of all the points in the cluster. This yields the estimate

$$\hat{\theta}_{\ell} = \frac{1}{n_{\ell}} \sum_{k=1}^{n_{\ell}} \varphi_{k,\ell} \qquad \ell = 1, \dots, \nu$$
(46)

where n_{ℓ} is the number of phase points in each cluster of positive phases.

VI. CONSISTENCY

Subspace methods for finite-dimensional models are essentially an instance of the method of moments which is wellknown in Statistics to be generically consistent under very mild assumptions. However, here the true covariance matrix is infinite-dimensional and the basic consistency analysis of moment estimation for finitely parametrized models does not apply. In order to completely answer the convergence question of the subspace-based estimator to the true frequency hyperparameter, one should then combine the consistency property of subspace estimates which holds for the estimate of each finite dimensional approximate linear model (of fixed dimension), with the convergence, as the dimension of the covariance truncation tends to infinity, of the purely deterministic approximate process described previously to the a posteriori process which has a continuous spectrum. This is a rather technical issue essentially centering on symmetric Toeplitz spectral approximation which could not be reported in this paper and is treated in a companion publication [49].

Consistency follows from a result of [49] which establishes convergence (understood in a weak sense) of the line spectrum of the approximate model (42),(43) to the continuous spectrum of the infinite Toeplitz covariance matrix. This implies in particular that both the width and the centers of the discrete frequency clusters must converge to the width and center of the corresponding intervals supporting the continuous spectrum which are indeed the true frequency hyperparameters.

VII. BAYESIAN ESTIMATION

Assume now that we have a consistent estimate of the parameters of the prior, in particular of the center frequencies θ_{ℓ} . The question is what this estimate has to do with (say) the Bayesian Maximum A Posteriori (MAP) estimate² of the random angular frequency ω , computed from the relative posterior distribution. Is there any reason why the MAP estimate should coincide, at least asymptotically, with the center frequencies of the prior?

In the Subsection VIII-C we shall provide experimental evidence that in our setting the inherent optimization problem leads to a MAP estimate of ω which is practically indistinguishable from the Empirical Bayes estimate of the center frequency θ . This fact is verified experimentally but should be also evident from the theoretical analysis which follows.

The MAP estimator of $\boldsymbol{\omega}$ is obtained by maximizing the log of the unnormalized posterior distribution of $\boldsymbol{\omega}$ given N observations³ $\mathbf{y} := \begin{bmatrix} y(t) & \dots & y(t-N) \end{bmatrix}^{\top}$, neglecting the denominator p(y) which does not depend on the parameters. The prior for one frequency is

$$p(\omega \mid \theta_{\ell}, W) = \frac{1}{2W} \chi_{[\theta_{\ell} - W, \ \theta_{\ell} + W]}$$

and since the intervals do not overlap we have independence and the overall prior of ω is the product of the priors for each ω_{ℓ} so that, recalling that the noise is Gaussian i.i.d. we have

$$\hat{\omega}^{\text{MAP}} = \underset{\omega \in [0,\pi]^{\nu}}{\operatorname{argmax}} \left\{ -\frac{1}{2\sigma_{\mathbf{w}}^{2}} \|\mathbf{y} - V(\omega)\mathbf{u}\|^{2} + \sum_{\ell} \log p(\omega \mid \theta_{\ell}, W) \right\}$$

with $V(\omega) = \begin{bmatrix} C(\omega) & S(\omega) \end{bmatrix}$ where

$$C(\omega) = \begin{bmatrix} \cos \omega_1 & \dots & \cos \omega_\nu \\ \vdots & \ddots & \vdots \\ \cos \omega_1 N & \dots & \cos \omega_\nu N \end{bmatrix} := \begin{bmatrix} \mathbf{c}_1(\omega_1) & \dots & \mathbf{c}_\nu(\omega_\nu) \\ \vdots & \ddots & \vdots \\ \sin \omega_1 N & \dots & \sin \omega_\nu N \end{bmatrix} := \begin{bmatrix} \mathbf{s}_1(\omega_1) & \dots & \mathbf{s}_\nu(\omega_\nu) \end{bmatrix}$$

²MAP is known to be the best estimate in a variety of norms.

³The estimation from multiple snapshots data can be dealt with in a similar way even in case of unequal measurement error variances.

and $\mathbf{u} = \begin{bmatrix} a_1 & \dots & a_{\nu} & b_1 & \dots & b_{\nu} \end{bmatrix}^{\top} := \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}^{\top}$ which could also be written in complex form as $\operatorname{Re}[\tilde{V}(\omega)\tilde{\mathbf{u}}]$ where $\tilde{V}(\omega)$ is the van der Monde matrix

$$\tilde{V}(\omega) = \begin{bmatrix} e^{j\omega_1} & \dots & e^{j\omega_\nu} \\ \vdots & \ddots & \vdots \\ e^{jN\omega_1} & \dots & e^{jN\omega_\nu} \end{bmatrix}$$

and $\tilde{\mathbf{u}} := \begin{bmatrix} a_1 - jb_1 & \dots & a_{\nu} - jb_{\nu} \end{bmatrix}^{\top}$. Since we are to compute real quantities this complex formulation does however not offer substantial simplifications.

Now the log of the prior is $-\infty$ outside of the intervals $J_{\ell} := [\theta_{\ell} - W, \ \theta_{\ell} + W]$ and equal to $\log \frac{1}{(2W)^{\nu}}$ inside (this is obviously true for each frequency and true for the whole prior). Hence the MAP estimator of ω can be found by solving the constrained minimization problem

$$\hat{\omega}^{\text{MAP}} = \underset{\omega}{\operatorname{argmin}} \left\{ \frac{1}{2\sigma_{\mathbf{w}}^2} \|\mathbf{y} - V(\omega)\mathbf{u}\|^2 + \nu \log(2W) \right\}$$

subject to : $\omega_{\ell} \in J_{\ell} \quad l = 1, \dots, \nu$ (47)

Suppose that $\hat{\theta}_{\ell}, \ell = 1, \dots, \nu$ and \hat{W} are our subspace estimates of the hyperparameters of the prior. Since these are consistent as discussed in the previous section, substituting these estimates for the true values leads to an asymptotically equivalent optimization problem. Here W appears as a nuisance parameter which shall be fixed to the estimated width \hat{W} . The Bayes MAP estimate of $\boldsymbol{\omega}$ can then in principle be compute by minimizing the quadratic criterion $\|\mathbf{y} - V(\boldsymbol{\omega})\mathbf{u}\|^2$ subject to the fixed deterministic constraint J: an hypercube in \mathbb{R}^{ν} centered in $\hat{\theta}$ of edge length $2\hat{W}$.

The minimization problem (47) can then equivalently be interpreted as the *Maximum Likelihood* estimation of a *deterministic angular frquency* ω ranging on the bounded compact set J. On this set the likelihood function is smooth and, according to standard statistical theory, the estimate must be consistent, that is converging for $N \to \infty$ to some "true value" ω_0 which has generated the observations, and asymptotically efficient.

For a finite data set problems of the type (47) have in general several local minima. However because of the bounded, compact, feasible set constraint $\omega \in J$, the solution must stay in a small neighborhood of the center frequency. Also, the squared norm term in (47) depends on ν sinusoidal functions of ω and hence, for small enough W 's there are no equivalent values of the frequency ω leading to the same value of the cost. The function has generically a unique minimum.

We now propose an algorithm for the problem (47) by using the a priori estimate $\hat{\theta}$ as a starting point for a gradient descent and solve the problem by a local search algorithm about $\hat{\theta}$. Since the subspace estimate, $\hat{\theta}$, asymptotically tends to the center frequency, for large N we are allowed to identify θ with $\hat{\theta}$.

As a first preliminary step, solve a least squares problem minimizing $\|\mathbf{y} - V(\hat{\theta})\mathbf{u}\|^2$ to get an estimate of the amplitude

vector \mathbf{u}^4 and use the estimated amplitude vector,

$$\hat{\mathbf{u}} = [V(\hat{\theta})^{\top}V(\hat{\theta})]^{-1}V(\hat{\theta})^{\top}\mathbf{y}$$

in place of \mathbf{u} in the formulas.

Let $\tilde{\mathbf{y}}(\hat{\theta}) := \mathbf{y} - V(\hat{\theta})\hat{\mathbf{u}}$ and introduce the deviation $\tilde{\omega} := \omega - \hat{\theta}$. The gradient of V with respect to ω computed at $\hat{\theta}$, is an array of 2ν rectangular $N \times \nu$ gradient matrices of the form

$$\nabla V(\hat{\theta}) = \left[\nabla_{\theta_1} \mathbf{c}_1(\hat{\theta}_1), \dots, \nabla_{\theta_\nu} \mathbf{c}_\nu(\hat{\theta}_\nu), \nabla_{\theta_1} \mathbf{s}_1(\hat{\theta}_1), \dots, \nabla_{\theta_\nu} \mathbf{s}_\nu(\hat{\theta}_\nu) \right]$$
(48)

where each matrix entry has only the k-th column nonzero, equal (in Matlab notation) to

$$\nabla_{\theta_k} \mathbf{c}_k(\hat{\theta}_k)[:,k] = -D_N \mathbf{s}_k(\hat{\theta}_k), \ \nabla_{\theta_k} \mathbf{s}_k(\hat{\theta}_k)[:,k] = D_N \mathbf{c}_k(\hat{\theta}_k),$$
(49)

where $D_N = \text{diag}\{1, 2, ..., N\}$. Hence $\nabla\{V(\hat{\theta})\hat{\mathbf{u}}\}$ turns out to be a linear combination of these 2ν , $N \times \nu$ matrices, properly combined by the corresponding components of the vector $\mathbf{u} \in \mathbb{R}^{2\nu}$. By this operation the zero columns are superseded and the linear combination leads to a $N \times \nu$ matrix made by linearly combining the 2ν nonzero column vectors in (48) to form a final matrix which we denote $\mathbf{M}(\hat{\theta})$. For $\nu = 1$ we have for example $\mathbf{u} = \begin{bmatrix} a & b \end{bmatrix}^{\top}$ and

$$M(\hat{\theta}) = D_N(-\mathbf{s}(\hat{\theta})a + \mathbf{c}(\hat{\theta})b) \in \mathbb{R}^{N \times 1}$$

With this gradient calculation established, we proceed to approximate (47) by a constrained local linear Least Squares minimization

$$\min_{\tilde{\omega}} \left\{ \| \tilde{\mathbf{y}} - \mathbf{M}(\hat{\theta}) \, \tilde{\omega} \|^2 \right\}$$

ubject to : $|\tilde{\omega}_{\ell}| \leq \hat{W}$ equivalent to $\omega_{\ell} \in J_{\ell}$, (50)

for $\ell = 1, \dots, \nu$. The solution can be refined iteratively by an algorithm of the form

$$\tilde{\boldsymbol{\omega}}(k+1) = [\mathbf{M}(\boldsymbol{\omega}(k))^{\top} \mathbf{M}(\boldsymbol{\omega}(k))]^{-1} \times \mathbf{M}(\boldsymbol{\omega}(k))^{\top} \tilde{\mathbf{y}}(\boldsymbol{\omega}(k)) \quad k = 1, 2, \dots$$
(51)

where at each step $\omega(k) := \tilde{\omega}(k) + \hat{\theta}$ is substituted back in place of $\omega(k-1)$ or, initially, of $\hat{\theta}$ in the expression of the gradient. The scheme is initialized for k = 0 setting $\omega(0) = \hat{\theta}$ and then stopping when the difference $\tilde{\omega}(k+1) - \tilde{\omega}(k) =$ $\omega(k+1) - \omega(k)$ becomes small enough. It requires to check at each step if $|\tilde{\omega}_{\ell}| \leq \hat{W}$ otherwise the estimator should be re-initialized. Alternatively, we may try to keep $\|\tilde{\omega}\|$ small by adding a ridge penalty term $\lambda(k) \|\tilde{\omega}(k)\|^2$ with $\lambda(k) \to 0$ for k large for consistency, to the least squares formulation. This may in fact also make the computation of the inverse better conditioned.

Remark 4. The reasoning above can be extended to include multiple snapshots of data in a straightforward manner. Since the conditional likelihood function for each snapshot multiplies given the hidden frequencies, the squared-norm term in the objective function of (47) becomes $\|\mathcal{Y} - V(\omega)\mathcal{U}\|_{\rm F}^2$, where \mathcal{Y} and \mathcal{U} are matrices whose columns are the data and the

⁴The estimate can also be justified based on a *noninformative prior* as in [16].

amplitude vectors, respectively, and the subscript $_{\rm F}$ denotes the Frobenius norm. A similar linearization scheme can be devised to solve the enlarged optimization problem.

VIII. SIMULATIONS

In this section, we provide simulation evidence showing that the subspace algorithm described at the end of Section V works quite well in the case of one or two hidden frequencies. Simulations comparing with the MAP estimate will also be shown.

In the second step of the subspace algorithm, in order to compute an estimate of the bandwidth W using (34) we need to estimate the asymptotic rank of the signal covariance matrix. It turns out that such a rank estimation task can be tricky if we are given (relatively) a small number of samples. This point will be discussed in the next subsection.

A. The bandwidth estimator

In the first example, we compare the decay property of eigenvalue sequence of the estimated covariance matrix with the theoretical behavior as shown in Figs. 1 and 2 in the case of two hidden frequencies. The measurements (26) are generated with **a**, **b** with uniform distribution U[-1.3813, 1.3813] and ω drawn from the uniform distribution in $[\theta - W, \theta + W]$ with the hyperparameters $\theta = [\theta_1, \theta_2] = 2\pi \times [0.3145, 0.4201]$ and $W = 2\pi \times 0.0465$. ⁵ The signal length N and the number of snapshots L are both equal to 100. The additive noise is i.i.d. Gaussian with variance $\sigma_{\mathbf{w}}^2$. The signal-to-noise ratio (SNR) defined as $20 \log_{10}(\sigma/\sigma_{\mathbf{w}})$ has a value of 15 dB. In Fig. 3, we report the eigenvalues of the estimated covariance matrix (28).



Figure 3. Eigenvalues of the theoretical and estimated covariance matrices with N = L = 100.

By comparison with the eigenvalues of the theoretical covariance matrix (red dashed line), we can see a significant distortion in the large eigenvalues due to the slow convergence

⁵These numbers come from one trial in the Monte-Carlo simulations.

of the estimator (28). However, the flat regions of two eigensequences still overlap nicely. Inspired by such an observation, we propose an ad-hoc scheme: replace $rank(\mathbf{K}_N)$ in (34) with the index maximizing the following ratio

$$\underset{k \in \{1,\dots,N-1\}}{\operatorname{argmax}} \frac{\lambda_k^2(\boldsymbol{\Sigma}_N)}{\lambda_{k+1}^2(\hat{\boldsymbol{\Sigma}}_N)},$$
(52)

where $\lambda_k(\Sigma_N)$ denotes the *k*-th eigenvalue of the estimated covariance matrix $\hat{\Sigma}_N$ arranged in nonincreasing order. Intuitively, the maximum should be attained at the beginning of the flat region in the eigen-plot.

Next, we do a Monte-Carlo simulation to test our idea. In each trial, the hyperparameters are generated randomly. More precisely, first the bandwidth W is drawn from the uniform distribution in $2\pi \times [0.01, 0.05]$, and then the center frequencies θ_1 and θ_2 are drawn from $U[W, \pi - W]$ such that $|\theta_1 - \theta_2| > 2W$ so that the supporting intervals for the two frequencies do not overlap. Given L independent measurement sequences of length N, the covariance matrix is estimated through (28), and then the rank is computed via (52), which gives an estimate of W by (34). The relative estimation error of \hat{W} is defined by the ratio $(\hat{W} - W)/W$. Notice that we have not taken the absolute value of the numerator because we want to show that the scheme (52) tends to overestimate the rank of the signal covariance matrix. This feature is important in practice since the estimated rank determines the eigen-truncation performed in the subspace algorithm (Step 3). Clearly, we want to retain the eigenvectors of the covariance matrix corresponding to large eigenvalues. Hence, an underestimation of the rank should be avoided since otherwise, useful information about the spectral content of the signal could be lost.

Each Monte-Carlo simulation consists of 1000 trials. In the first experiment, we fix N = L = 100 and estimate the bandwidth W, or equivalently the numerical rank of the signal covariance matrix, under different SNRs. In Fig. 4, the relative errors of \hat{W} are depicted using the boxplot. We see from the box on the right that a low SNR results in an underestimate of the rank which is undesirable for the subsequent estimation of the band centers. The overall error is not small mainly because we have a poor estimate of the covariance matrix given the number of available samples (see Fig. 3). However, we want to emphasize that the estimation of W is a separate problem, and a large error here does not propagate to the estimation of the center frequencies. As we will see in the next subsection, the center frequencies can be estimated quite accurately given a rough estimate of the bandwidth.

In the second experiment, we fix the SNR = 15 dB and estimate W as both N and L change while keeping N = L. The result is depicted in Fig. 5. One can see that as N = Lincreases, the estimates become more and more accurate.

We want to comment that preliminary results for the arccos estimator of one center frequency θ could be obtained from (36) but they will not be discussed in depth since much more reliable estimates will be obtained by the subspace method of Sec. V.



Figure 4. Relative estimation errors of the bandwidth W versus the SNR with N=L=100.



Figure 5. Relative estimation errors of the bandwidth W versus the signal length and the number of cross sections N = L while SNR = 15 dB is fixed.

B. The Subspace estimator of the band centers

Given the estimated rank of the signal covariance matrix in the previous subsection, we proceed to implement the subspace algorithm described at the end of Sec. V. Again we do a Monte-Carlo simulation of 1000 trials. The signal length N = 100 and the SNR = 15 dB are fixed, and we change the number of snapshots L. The data has already been generated in estimating W, and we only need to use the estimated covariance matrix.

The relative estimation errors of the center frequency is defined as $\|\hat{\theta} - \theta\| / \|\theta\|$, and their values in Monte-Carlo simulations are plotted in Fig. 6. It appears that apart from the outliers (the red crosses), the performance of the algorithm is quite good as the cumulative relative error is lower than 2%, even in the case of few snapshots (L = 50). The simulation result also seems to indicate that the algorithm works very well when the covariance estimate is sufficiently accurate.



Figure 6. Relative estimation errors of two hidden frequencies (θ_1, θ_2) using the Subspace method in Sec. V versus the number L of cross sections with N = 100 and SNR = 15 dB.

Fig. 7 shows the discrete spectrum of the output process (43) in one simulation trial in the case of L = 100. The horizontal axis is scaled to represent the frequency in Hz. In this particular trial, the true hyperparameters are $[\theta_1, \theta_2, W] = 2\pi \times [0.1499, 0.2524, 0.0155]$, and the estimated band centers are $\hat{\theta} = 2\pi \times [0.1503, 0.2532]$. The theoretical (asymptotic) rank of the signal covariance matrix is $\frac{2\nu W}{\pi}N \approx 12$, while the ratio scheme (52) produces a rank estimate equal to 20. One can see that the Dirac deltas indeed cluster around the true center frequencies inside the supporting interval.



Figure 7. Discrete spectrum estimate with two hidden frequencies. The true hyperparameters are $[\theta_1, \theta_2, W] = 2\pi \times [0.1499, 0.2524, 0.0155]$ and the estimated band centers are $\hat{\theta} = 2\pi \times [0.1503, 0.2532]$.

C. The Bayesian MAP estimator of the frequencies

Given the center frequencies and the bandwidth estimated from the Subspace procedure, we can now compute the Empirical Bayes MAP estimator using the algorithm described in Sec. VII. The data are the same as those used for covariance and hyperparameter estimation. Once again, we fix the signal length N = 100 and SNR = 15 dB, and do Monte-Carlo simulations of 1000 trials as the number L of cross sections changes. The relative errors of $\hat{\omega}^{MAP}$ with respect to the true center frequencies are shown in Fig. 8. It appears that the MAP estimate of the frequencies is close to the true band centers with a cumulative relative error below 6%. Moreover, the estimation accuracy improves as more snapshots of data are available. It is noticed that the cumulative error size is larger than that of the empirical Subspace method (Fig. 6) probably due to the linearization scheme in solving the original nonlinear least squares problem subject to interval constraints.



Figure 8. Relative estimation errors of two hidden frequencies (θ_1, θ_2) using the Bayesian MAP method in Sec. VII versus the number L of cross sections while N = 100 and SNR = 15 dB.

Remark 5. A quite reasonable conjecture, which unfortunately so far we have not been able to prove rigorously, is that for $N \to \infty$ and small enough W, the Bayesian estimate $\hat{\omega}^{MAP}$ should converges a.s. to the true center frequency θ .

The conjecture is based on the observation that both $\hat{\omega}^{MAP}$ and $\hat{\theta}$ are asymptotic maximizers of the likelihood function based on the same data. In fact, $\hat{\theta}$ asymptotically solves (40) which is the maximizing equation of the marginal likelihood function, marginalized by integrating with respect to the a priori distribution of ω and hence parametrized only in terms of the hyperparameters (θ, W) . We will leave a detailed discussion of this point to a future publication.

Remark 6. At the end of this section want to comment on the difference between our method and classical subspace methods for frequency estimation such as MUSIC, ESPRIT, etc. All classical methods are designed for oscillatory signals with deterministic frequencies and perform the eigen-truncation of the estimated covariance matrix at an index equal to 2ν where ν is the number of unknown frequencies (the factor 2 is due to complexification of the real signal). In contrast, we show that in the case of uniform random frequencies, we have a stochastic multiband signal, and the eigen-truncation should be done at the approximate index $4WN \times \nu$ corresponding

to the asymptotic rank of the signal covariance matrix. Based on this observation, it is not surprising that classical subspace methods do not apply to the current problem setup. Moreover, our Empirical Bayes procedure provides both the band centers and the bandwidth for the random frequencies, which can be interpreted as confidence intervals for the frequency estimation.

Remark 7. Concerning the Atomic Norm approach, we may just say that it views the observed sinusoidal signal as a deterministic linear combination of elementary exponential components with deterministic frequencies. For this reason (similarly to the previous remark), it does not seem possible to compare to our random-frequency signal model, although it has been extended to deal with deterministic multiband signals [12].

IX. CONCLUSIONS

We have formulated the problem of frequency estimation in an Empirical Bayesian framework by first imposing a natural uniform prior probability density on the unknown frequency. In this way the estimation of the hyperparameters of the a priori distribution can be accomplished by exploiting the special structure of the covariance matrix of the posterior process which has been long studied in the framework of energy concentration problems by the signal processing community. In this setting the solution can be based on essentially linear techniques of subspace identification. Using the estimated prior parameters one can adapt the prior to the data and this leads to Bayesian estimates which are asymptotically maximum likelihood and therefore the best possible in a variety of metrics. The simulation results using this Empirical Bayesian philosophy are very encouraging.

APPENDIX

In the proof of Theorem 1 we shall need two auxiliary lemmas. The first is just a simple technical fact.

Lemma 1. If two sequences of bounded real numbers $\{a_n\}, \{b_n\}$ are such that

$$\lim_{n \to \infty} (a_n - b_n) = 0, \tag{53}$$

then

and

$$\limsup_{n \to \infty} a_n = \limsup_{n \to \infty} b_n$$

$$\liminf a_n = \liminf b_n.$$

Proof. The argument is quite standard. Let $\bar{a} := \limsup_{n \to \infty} a_n$. Then there exits a subsequence $\{a_{n_k}\}$ converging to \bar{a} . Define $\hat{b} := \limsup_{k \to \infty} b_{n_k}$. Then there exists a sub-subsequence $\{b_{n_{k_j}}\}$ converging to \hat{b} . The condition (53) holds for the subsequence indexed by n_{k_j} , which implies that $\bar{a} = \hat{b}$. It then follows that $\bar{b} := \limsup_{n \to \infty} b_n \geq \hat{b} = \bar{a}$. A symmetric argument leads to $\bar{a} \geq \bar{b}$, and therefore $\bar{a} = \bar{b}$. The proof for the limit inferior is similar and hence omitted.

The next lemma concerns the sum of squared eigenvalues It follows that of **R**.

Lemma 2.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \lambda_j^2(N) = \frac{\mathrm{m}(J)}{2\pi} \,. \tag{54}$$

Proof. Since \mathbf{R} is Hermitian, we have

$$\sum_{j=1}^{N} \lambda_j^2(N) = \operatorname{tr} \mathbf{R}^2 = \operatorname{tr}(\mathbf{R}\mathbf{R}^*)$$

=
$$\sum_{j=-N+1}^{N-1} |\rho(j)|^2 (N - |j|).$$
 (55)

It follows that

$$\frac{1}{N}\sum_{j=1}^{N}\lambda_j^2(N) = \sum_{j=-N+1}^{N-1} |\rho(j)|^2 \left(1 - \frac{|j|}{N}\right).$$
 (56)

We can view the latter summation over \mathbb{Z} by adding zeros. Apparently, each term in the infinite sum is dominated by $|\rho(j)|^2$. Moreover, for each fixed j the term-wise limit as $N \rightarrow \infty$ is also $|\rho(j)|^2$. Applying Lebesgue's dominated convergence theorem for the counting measure on \mathbb{Z} , we can conclude that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \lambda_j^2(N) = \sum_{j \in \mathbb{Z}} |\rho(j)|^2$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\chi_J(\omega)|^2 d\omega = \frac{\mathrm{m}(J)}{2\pi},$$
(57)

where the second equality is the Parseval identity.

Proof of Theorem 1

Proof. We first show that the number of eigenvalues not close to 0 or 1 is o(N). To this end, define the function

$$\mathbb{J}(N) := \sum_{j=1}^{N} \lambda_j(N) \left(1 - \lambda_j(N)\right)$$
(58a)

$$=\sum_{j=1}^{N} \lambda_j(N) - \sum_{j=1}^{N} \lambda_j^2(N),$$
 (58b)

where each summand in (58a) is nonnegative. Then according to (23) and Lemma 2, we have

$$\lim_{N \to \infty} \frac{\mathbb{J}(N)}{N} = 0.$$
(59)

In other words, the function $\mathbb{J}(N)$ is o(N). Fix $0 < \delta <$ $\gamma < 1$, and the number of eigenvalues $\delta \leq \lambda_i(N) < \gamma$ is $M(\delta, n) - M(\gamma, N)$. Clearly, for these eigenvalues we have

$$\lambda_j(N)\left(1-\lambda_j(N)\right) > \delta(1-\gamma) := \nu > 0, \qquad (60)$$

which implies that

$$\mathbb{J}(N) \geq \sum_{\delta \leq \lambda_j(N) < \gamma} \lambda_j(N) \left(1 - \lambda_j(N)\right) \\
\geq \nu \left[M(\delta, N) - M(\gamma, N)\right] \geq 0.$$
(61)

$$\lim_{N \to \infty} \frac{M(\delta, N) - M(\gamma, N)}{N} = 0,$$
 (62)

which means that the quantity $M(\delta, N) - M(\gamma, N)$ is also o(N).

Next, define the quantities

$$M_{+} := \limsup_{N \to \infty} \frac{M(\gamma, N)}{N},$$

$$M_{-} := \liminf_{N \to \infty} \frac{M(\gamma, N)}{N}.$$
(63)

Applying Lemma 1 in this appendix to the relation (62), we know that both M_+ and M_- do not depend on $0 < \gamma < 1$. We want to establish that the two quantities coincide so that the ordinary limit in (24) exits and is equal to the common value. Observe that

$$\operatorname{tr} \mathbf{R} = \sum_{j=1}^{M(\gamma,N)} \lambda_j(N) + \underbrace{\sum_{\substack{M(\gamma,N)+1 \\ > 0}}^{N} \lambda_j(N)}_{>0} \ge \gamma M(\gamma,N),$$
(64)

and similarly

$$\sum_{j=1}^{N} \lambda_j^2(N) = \sum_{j=1}^{M(\gamma,N)} \lambda_j^2(N) + \underbrace{\sum_{\substack{M(\gamma,N)+1 \\ \text{here each } \lambda_j(N) < \gamma}}^{N}}_{\text{here each } \lambda_j(N) < \gamma}$$

$$< \sum_{j=1}^{M(\gamma,N)} 1 + \sum_{\substack{M(\gamma,N)+1 \\ M(\gamma,N)+1}}^{N} \gamma \lambda_j(N)$$

$$< M(\gamma,N) + \gamma \operatorname{tr} \mathbf{R}.$$
(65)

It follows that

$$\sum_{j=1}^{N} \lambda_j^2(N) - \gamma \operatorname{tr} \mathbf{R} \le M(\gamma, N) \le \frac{\operatorname{tr} \mathbf{R}}{\gamma}, \qquad (66)$$

and furthermore, we have

$$M_{+} \leq \limsup_{N \to \infty} \frac{\operatorname{tr} \mathbf{R}}{\gamma N} = \frac{\mathrm{m}(J)}{2\pi\gamma},$$
(67a)
$$M_{-} \geq \liminf_{N \to \infty} \frac{1}{N} \left(\sum_{j=1}^{N} \lambda_{j}^{2}(N) - \gamma \operatorname{tr} \mathbf{R} \right)$$
$$= (1-\gamma) \frac{\mathrm{m}(J)}{2\pi},$$
(67b)

where we have used Lemma 2 again in (67b). Letting $\gamma \rightarrow 1$ in (67a) and $\gamma \rightarrow 0$ in (67b), we obtain

$$\frac{\mathrm{m}(J)}{2\pi} \le M_{-} \le M_{+} \le \frac{\mathrm{m}(J)}{2\pi},\tag{68}$$

and the claim of the theorem follows.

The next proposition concerns the time average of one sample path of the noisy sinusoidal signal.

Proposition 2. Let

$$y(t) = x(t) + w(t)$$

= $a\cos(\omega t) + b\sin(\omega t) + w(t)$ (69)

be a sample path of the process (1), where t = 1, 2, ... Then for each fixed ω with $|\omega| < \pi$,

> 2 19

$$\frac{1}{N}\sum_{t=1}^{N}y(t+\tau)y(t) \to \frac{a^2+b^2}{2}\cos\omega\tau + \sigma_{\mathbf{w}}^2\delta(\tau,0)$$

as $N \to \infty$ with probability one.

Proof. We have

$$\frac{1}{N}\sum_{t=1}^{N}y(t+\tau)y(t) = \frac{1}{N}\sum_{t=1}^{N}[x(t+\tau)x(t) + x(t+\tau)w(t) + w(t+\tau)x(t) + w(t+\tau)w(t)]$$

and that the first time average converges to $\frac{a^2+b^2}{2}\cos\omega\tau$ is shown in [42, pp. 105-109] or [2, pp. 171-172]. That the average of each cross term in the middle tends to 0, follows since the process $\tilde{\mathbf{w}}(t) := e^{i\omega t} \mathbf{w}(t)$ is (complex) zero-mean i.i.d. and by the assumed uncorrelation so is also $\mathbf{a}e^{i\omega\tau}\tilde{\mathbf{w}}(t)$ and hence so is its real part, so that the law of large numbers holds for each cross term. The time average of the last term tends to $\sigma_{\mathbf{w}}^2 \delta(\tau, 0)$ again by the law of large numbers.

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