Linear-Quadratic-Gaussian Control Systems

GIORGIO PICCI

Department of Information Engineering,
Università di Padova, Italy

March 2013

Deterministic LQ problems

Consider the following quadratic optimization problem on the semi-infinite interval $t \geq 0$:

$$\text{minimize} \quad J(\xi, u) := \sum_{0}^{+\infty} \begin{bmatrix} x(t)^\top & u(t)^\top \end{bmatrix} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{1}$$

$$\text{subject to:} \quad x(t+1) = Ax(t) + Bu(t), \quad x(0) = \xi, \qquad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^p. \tag{2}$$

The minimization is to be performed with controls u for which the limit quadratic functional J exists [and is finite ?]. These controls will be called **admissible**. Here $Q = Q^\top, R = R^\top$ but the weights are otherwise arbitrary. We shall use the notation $w(x, u)$ for the quadratic form inside the summation and let

$$V(\xi) := \inf_u J(\xi, u), \qquad \xi = x(0).$$

This will be called **the value function**. Of course it may well happen that $V(\xi) = \pm\infty$ for some, or all ξ's.

Examples

Let $\quad y(t) = Cx(t) + Du(t)$

- *The LQ regulator problem*: minimize the ℓ^2 norm of the output:
  $$w(x,u) = \begin{bmatrix} x^\top & u^\top \end{bmatrix} \begin{bmatrix} C^\top C & C^\top D \\ D^\top C & D^\top D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

- *minimal energy transfer*: minimize $\quad \|u\|_2^2 - \|y\|_2^2$,
  $$w(x,u) = \begin{bmatrix} x^\top & u^\top \end{bmatrix} \begin{bmatrix} -C^\top C & -C^\top D \\ -D^\top C & I - D^\top D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

- *minimal energy transfer to electric networks*; let $y(t)$ voltage and $u(t)$ current flowing into the network; want to minimize $\sum_0^{+\infty} y(t)^\top u(t)$
  $$w(x,u) = \begin{bmatrix} x^\top & u^\top \end{bmatrix} \begin{bmatrix} 0 & C^\top \\ C & 1/2(D^\top + D) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}.$$

A class of optimization problems

Note that we have actually defined a whole class of variational problems since we have not specified the terminal conditions. The existence (and properties) of the solution depends on which terminal condition are attached to the infimization problem. In particular we may ask that $\lim_{t\to+\infty} x(t) := x(\infty)$ exists and belongs to some prespecified terminal subspace $\mathcal{T} \subset \mathbb{R}^n$. For example the problem

$$V_+(\xi) := \inf_u J(\xi, u), \qquad \text{subject to} \quad \lim_{t\to+\infty} x(t) = 0, \ \forall \xi$$

asks for an optimal control which asymptotically stabilizes the linear system $x(t+1) = Ax(t) + Bu(t)$. On the other extreme we may consider the *free terminal conditions* problem where the behaviour of $\lim_{t\to+\infty} x(t)$ is not constrained at all. Then whenever it is well-defined, the *free terminal cost*

$$V_f(\xi) := \inf_u J(\xi, u), \qquad \text{no conditions on} \quad \lim_{t\to+\infty} x(t)$$

will satisfy the inequality $V_f(\xi) \le V_{\mathcal{T}}(\xi)$ for all subspaces $\mathcal{T}$ (or, equiv. all V's).

Reachability Stabilizability and the Backward system

Recall the definition of (discrete-time) reachability and stabilizability.

Lemma 1 *If (A,B) is stabilizable; i.e. all unstable modes of A (including those corresponding to eigenvalues in the unit circle) are reachable, there are admissible controls for all ξ and any $\mathcal{T}$.*

Clearly, by stabilizability, there is a feedback law which makes the system asymptotically stable and a feedback control which for each ξ, drives $x(t)$ to zero exponentially fast so that $J(\xi, u) < \infty$. Since $0 \in \mathcal{T}$ there are admissible controls for all ξ and any $\mathcal{T}$.
Assume A is non-singular; then the system (2) is time-reversible and the Backward system

$$x(t-1) = A^{-1}x(t) - A^{-1}Bu(t-1)$$

is *backward-reachable* if and only if (2) is reachable. It is backward stabilizable if all modes of A^{-1} corresponding to eigenvalues in $\{|z| \geq 1\}$ are reachable. This is obviously the same as all modes of A corresponding to eigenvalues in $\{|z| \leq 1\}$ to be reachable.

Optimization over the past trajectories

There is a dual class of variational problems involving the behaviour of the system for negative times. For these problems to make sense we need to assume that the system (2) is time reversible, i.e. A is invertible, which we shall assume from now on. Consider the family of cost functionals

$$\bar{J}(\xi, u) := \sum_{-\infty}^{0} \begin{bmatrix} x(t)^{\top} & u(t)^{\top} \end{bmatrix} \begin{bmatrix} Q & S \\ S^{\top} & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

subject to: $\quad x(t+1) = Ax(t) + Bu(t)\,, \quad x(-\infty) \in \mathcal{T}, \quad x(0) = \xi\,.$

We then consider the family of problems

$$\bar{V}(\xi) := \inf_{u} \bar{J}(\xi, u), \qquad \text{subject to} \quad \lim_{t \to -\infty} x(t) \in \mathcal{T}, \quad x(0) = \xi$$

The infimization is to be performed with respect to controls $u_{(-\infty,-1]}$ steering the initial state $x(0) = \xi$ to $x(-\infty) \in \mathcal{T}$, for which the limit quadratic functional $\bar{J}$ exists and is finite. These controls are also called **admissible**.

If (A,B) is anti-stabilizable there are feedback laws which drive any initial state $x(0)=\xi$ backward in time, to $x(-\infty)=0$. Hence if (A,B) is anti-stabilizable there are admissible controls for all ξ and any terminal manifold $\mathcal{T}$.

The problem

$$\bar{V}_{+}(\xi):=\inf_{u}\bar{J}(\xi,u), \qquad \text{subject to} \qquad \lim_{t\to-\infty}x(t)=0,\ \forall\xi$$

asks for an optimal control which asymptotically **anti-stabilizes** the linear system $x(t+1)=Ax(t)+Bu(t)$. As we shall see, under certain condition this problem will have a unique solution. On the other extreme we may consider the *free initial conditions* problem where the behaviour of $\lim_{t\to-\infty}x(t)$ is not constrained at all. Then whenever it is well-defined, the *free backward terminal cost*

$$\bar{V}_{f}(\xi):=\inf_{u}\bar{J}(\xi,u), \qquad \text{no conditions on} \qquad \lim_{t\to-\infty}x(t)$$

will obviously satisfy the inequality $\bar{V}_f(\xi)\leq\bar{V}(\xi)\leq\bar{V}_{+}(\xi)$ for all $\bar{V}$'s.

Boundedness of the value function

The first question to settle is when the value functions V, $\bar{V}$ have a finite lower bound. For this we have the following basic lemma.

Lemma 2 *Assume that (A,B) is reachable. Then, if a value function $V(\xi)$ is finite for all $\xi \in \mathbb{R}^n$, it must satisfy the* **Inverse dissipation inequality (IDI)**

$$V(x(a)) \leq \sum_{t=a}^{b-1} w(x(t),u(t)) + V(x(b)), \qquad b-a \geq n. \tag{3}$$

for all $x(b) \in \mathbb{R}^n$ and $u_{[a,b)}$ driving the initial state $x(a)$ to $x(b)$. Moreover $V(0) = 0$.

Dually, assume that (A^{-1},B) is reachable. Then, if a $\bar{V}(\xi)$ is finite for all $\xi \in \mathbb{R}^n$ it satisfies the **dissipation inequality (DI)**

$$\bar{V}(x(b)) \leq \sum_{t=a}^{b-1} w(x(t),u(t)) + \bar{V}(x(a)), \qquad b-a \geq n. \tag{4}$$

for all $x(b) \in \mathbb{R}^n$ and $u_{[a,b)}$ driving the initial state $x(a)$ to $x(b)$. Moreover $\bar{V}(0) = 0$.

Dissipative linear systems

After a change of sign, setting $S(x) \equiv -V(x)$ the inverse dissipation inequality (3) turns into:

$$S(x(b)) \leq \sum_{t=a}^{b-1} w(x(t), u(t)) + S(x(a)),$$

for all $u_{[a,b)}$ transferring $x(a)$ to $x(b)$ at time $b > a$. This is called the **dissipation inequality** since, when $S(x) \geq 0$ it can be interpreted as a **storage function** in the sense of Willems and the system $x(t+1) = Ax(t) + Bu(t)$ is *dissipative with supply rate* $w(x,u)$ see the papers [**?**]. Dissipativity is useful in stability of feedback systems since a dissipative system is automatically stable and the positivity condition makes $S(x)$ into a Lyapunov function; it is however a rather strong condition.

By Lemma 2, whenever $\bar{V}(\xi)$ is finite, then $W(\xi) := -\bar{V}(\xi)$ also satisfies the IDI. In particular we shall let

$$V_{-}(\xi) := -\bar{V}_{+} = -\inf_{u}\{\bar{J}(\xi, u), \quad \lim_{t \to -\infty} x(t) = 0\} \tag{5}$$

so that, whenever well defined, all W's satisfy $V_{-}(\xi) \leq W(\xi)$.

Proof of the inverse dissipation inequality

Let $a < b$ and let $u_{[a,+\infty,)}$ be an admissible control steering the initial state $x(a)$ to the terminal manifold $\mathcal{T}$ at time $t = +\infty$. This control exists by reachability. Then

$$V(x(a)) = \inf_{u_{[a,+\infty)}} J(x(a), u) \leq \sum_{t=a}^{b-1} w(x(t), u(t)) + \sum_{b}^{+\infty} w(x(t), u(t))$$

Let $b \geq a+n$ and fix a control $u_{[a,b)}$ which steers $x(a)$ to an arbitrary state $x(b)$ at time b. This control also exists by reachability. Then take the infimum of both members with respect to $u_{[b,+\infty)}$ to get (3). Note that, by time invariance, $V(x)$ does not depend on the particular time instant (either a or b) at which the initial state is considered.

Next consider $J(0,u)$; this is a quadratic homogeneous function of u so that $J(0,ku) = k^2 J(0,u)$; hence $V(0)$ cannot be negative otherwise by taking a scalar multiple of the optimal control we could make it negative and arbitrarily large. Therefore $V(0) \geq 0$ and hence $u \equiv 0$ is an optimal control since it makes $V(0)$ to achieve its minimal value $V(0) = 0$. $\square$

Proof of the dissipation inequality for $\bar{V}$

Let $a < b$ and let $u_{[-\infty,a)}$ be an admissible control steering the initial state manifold $\mathcal{T}$ at time $t = -\infty$ to $x(a)$. This control exists by reachability. Then

$$\bar{V}(x(b)) \leq \inf_{u_{[-\infty,a)}} \bar{J}(x(a),u) + \sum_{t=a}^{b-1} w(x(t),u(t))$$

where the control $u_{[a,b)}$ steers $x(a)$ to an arbitrary state $x(b)$ at time b. This control also exists by reachability if $b \geq a+n$. Take the infimum of both members with respect to $u_{[-\infty,a)}$ to get (4). Note that, by time invariance, $V(x)$ does not depend on the particular time instant (either a or b) at which the initial state is considered.

Next consider $\bar{J}(0,u)$; this is a quadratic homogeneous function of u so that $\bar{J}(0,ku) = k^2\bar{J}(0,u)$; hence $\bar{V}(0)$ cannot be negative otherwise by taking a scalar multiple of the optimal control we could make it negative and arbitrarily large. Therefore $\bar{V}(0) \geq 0$ and hence $u \equiv 0$ is an optimal control since it makes $\bar{V}(0)$ to achieve its minimal value $\bar{V}(0) = 0$. $\square$

Dynamic Programming

Not all bona-fide functions V, solutions of the IDI (3) are value functions attached to some LQ variational problem. The following is a well-known characterization of those which actually are.

Proposition 1 *A finite continuous function $V:\mathbb{R}^n\to\mathbb{R}$ is a value function for a deterministic LQ problem if and only if it satisfies the following **stationary Dynamic Programming Equation (DPE)***

$$V(x(a)) = \inf_{u_{[a,b)}} \{\sum_{t=a}^{b-1} w(x(t),u(t)) + V(x(b))\} \qquad b>a \tag{6}$$

where the infimum is over all controls steering the state $x(a)$ at time a to $x(b)$ at time b. If V is a function satisfying (6) *then it also satisfies the IDI* (3)*.*

Proof : Assuming V is a value function, the DPE (6) follows just by breaking the infimization of the functional $J(x(a),u)$ on the sub intervals $[a,b)\cup[b,+\infty)$ and taking last the infimum with respect to $u_{[a,b)}$.

Conversely, assume V is a continuous function satisfying (6) for arbitrary $b > a$ and let u be an admissible control. Since $\lim_{b\to+\infty} V(x(b)) = V(\lim_{b\to+\infty} x(b))$ is assigned by the boundary condition on $x(t)$ at time $+\infty$, it does not depend on u and the optimization on the infinite interval $[a, +\infty)$ is done on the functional $J(x(a), u)$ with respect to all admissible u's steering the state to certain final manifold. In particular, for the zero terminal condition problem $V(x) - V(0)$ is the value function but we already know that $V(0) = 0$. □

CONJECTURE: Is it true that $V(x) = 0$ for $x \in \mathcal{T}$?

The proof of the following proposition is rather technical and will be omitted.

Proposition 2 *If $V(\xi)$ is a finite value function, then it is a quadratic function of ξ; i.e. there exist a symmetric $n \times n$ matrix Π such that*

$$V(\xi) = \xi^\top \Pi \xi \,.$$

Dynamic Programming and the ARE

Let $V(\xi) = \xi^\top \Pi \xi$ be a finite value function and set $x \equiv x(t)$, $u \equiv u(t)$. The DPE written for $a = t$, $b = t+1$, yields

$$x^\top \Pi x = \inf_u \left\{ \begin{bmatrix} x^\top & u^\top \end{bmatrix} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + (x^\top A^\top + u^\top B^\top)\Pi(Ax+Bu) \right\} \equiv$$

$$x^\top \Pi x = \inf_u \left\{ x^\top A^\top \Pi A x + x^\top (S + A^\top \Pi B) u + u^\top (S^\top + B^\top \Pi A) x + u^\top (R + B^\top \Pi B) u + x^\top Q x \right\}$$

Assume that $R + B^\top \Pi B$ is positive definite (and hence non singular). Then the right hand side is minimized by $u(t) = K(\Pi) x(t) = (R + B^\top \Pi B)^{-1} (S^\top + B^\top \Pi A) x(t)$ and Π must satisfy

$$\Pi = A^\top \Pi A - (S + A^\top \Pi B)(R + B^\top \Pi B)^{-1}(S^\top + B^\top \Pi A) + Q\,.$$

an **Algebraic Riccati Equation** (ARE). This equation will be studied in more detail in due time. It is important since it provides an algebraic parametrization of all finite value functions.

About sufficient conditions.

Proposition 3 *Let $V(\xi) = \xi^\top \Pi \xi$ be a finite value function and assume that $R + B^\top \Pi B$ is positive definite. Then matrix Π satisfies the ARE. Conversely, if the ARE admits symmetric solutions Π such that $R + B^\top \Pi B$ is positive definite, then $V(\xi) = \xi^\top \Pi \xi$ is a finite value function; i.e.*

$$V(\xi) := \min_u J(\xi, u)\,, \qquad \xi = x(0)\,.$$

for some terminal condition $x(+\infty) \in \mathcal{T}$. The optimal control is unique and is given by the feedback law $u(t) = K(\Pi)x(t)$ defined above.

The second statement, in a more general framework, is called the "verification theorem" of dynamic programming. It is not quite a sufficient condition since we don't know if/when the ARE admits solutions (satisfying the positivity assumption).

To obtain sufficient conditions for the well-posedness of the variational problems we will have to follow a more general route.

The positivity Theorem

Theorem 1 *Assume that* (A,B) *is reachable and controllable. Then all solutions of the IDI are finite if and only if, for all controls* u *driving the initial state* $x(0)=0$ *to* $x(T)=0$ *one has*

$$\sum_{0}^{T} w(x(t),u(t)) \geq 0, \qquad \textit{for all} \quad T \geq 0. \tag{7}$$

More precisely, if and only if (7) *holds, all solutions of the IDI are bounded. They form a convex set with a minimal and a maximal element; in fact, any solution* V *of the IDI satisfies the inequality*

$$V_{-}(\xi) \leq V(\xi) \leq V_{+}(\xi), \qquad \textit{for all} \quad \xi \in \mathbb{R}^n. \tag{8}$$

We shall call (7) the **positivity condition**.

Proof of Theorem 1

Write (3) with $a = 0$ and $b = T$. Assume $V(\xi)$ is finite so that $V(0) = 0$ (Lemma 2); then taking a control u which drives the initial state $x(0) = 0$ to $x(T) = 0$ one sees that (7) must hold.

Conversely assume that (7) holds and consider the following sequence of control actions transferring the initial state $x(-a) = 0$ at some fixed bounded time $-a$ to $x(b+a) = 0$ at time $b+a$

$$[x(-a) = 0] \xrightarrow{u_1} [x(0) = \xi] \xrightarrow{u} [x(b) = \eta] \xrightarrow{u_2} [x(b+a) = 0].$$

By time-invariance the control function steering $x(b) = \eta$ to $x(b+a) = 0$ can be chosen independent of b. Hence the last contribution $\sum_{b+1}^{b+a} w(x(t), u_2(t)) = \sum_0^{a-1} w(x(t), u_2(t)) := Q_2(\eta, a) < \infty$ does not depend on b. Likewise for the first contribution. Hence from (7) we get

$$-Q_1(\xi, a) - Q_2(\eta, a) \leq \sum_0^b w(x(t), u(t)), \qquad x(0) = \xi$$

where the left hand side is finite for all ξ. Taking the limit for $b \to +\infty$ on both sides and then taking the infimum w.r.t admissible controls u shows that $V(\xi)$ is indeed bounded from below. This proves the first statement.

Proof of Theorem 1 cont.d

Convexity is obvious. To prove the second statement, let's consider any (bounded) function satisfying the IDI (3). Take $x(b) = 0$ so that

$$V(x(a)) \leq \sum_{a}^{b} w(x,u)$$

where $x(a) = \xi$ is arbitrary and the control u steers $x(a) = \xi$ to zero at time b. Letting $b \to +\infty$ one gets $V(\xi) \leq V_{+}(\xi)$. Similarly, let V be any solution of the IDI. Taking $a < 0,\;\; b = 0$ and a general $x(0) = \xi \in \mathbb{R}^n$, the IDI (3) gives,

$$-\sum_{a}^{0} w(x,u) \leq V(\xi)$$

Let $a \to -\infty$ and choose admissible controls $u_{(-\infty,0)}$ driving $x(-\infty) = 0$ to $x(0) = \xi$. Since the second member does not depend on u, we have

$$\sup_{u_{(-\infty,0)}} \{-\sum_{-\infty}^{0} w(x,u)\} \leq V(\xi)$$

which, since $\sup\{-f(x)\} = -\inf\{f(x)\}$ yields $V_{-}(\xi) \leq V(\xi)$. $\square$

Positivity and the spectral function

Consider the matrix function

$$\Phi(z) := \begin{bmatrix} B^\top (z^{-1}I - A^\top)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}$$

which is sometimes called the *Popov function* or the *spectral function* of the LQ problem defined before. Clearly $\Phi(z)$ is para-Hermitian. We have

Theorem 2 *The positivity condition* (7) *holds if and only if the spectral function is nonnegative definite on the unit circle; i.e.*

$$\Phi(e^{j\theta}) \geq 0, \qquad \textit{for all} \quad \theta \in [-\pi, \pi] \tag{9}$$

Proof : The statement follows from Parseval theorem for the Fourier transform. Just consider inputs u equal to zero for negative times, transferring

$x(0) = 0$ to $x(T) = 0$ at an arbitrarily large time T and equal to zero thereafter. Since before $t = 0$ and after time T the state will also stay identically equal to zero we have

$$\sum_{0}^{T} w(x(t), u(t)) = \sum_{-\infty}^{+\infty} w(x(t), u(t)) = \int_{-\pi}^{+\pi} \hat{u}(e^{j\theta})^* \Phi(e^{j\theta}) \hat{u}(e^{j\theta}) \geq 0$$

where $\hat{u}$ is the Fourier transform of u and the star denotes conjugate transpose. It can be shown that the family of such $\hat{u}$'s is rich enough to insure that (9) holds. $\square$

Positivity and the control LMI

Lemma 3 *Let*

$$N(P) = \begin{bmatrix} A^\top PA - P & A^\top PB \\ B^\top PA & B^\top PB \end{bmatrix}$$

for some $n \times n$ *symmetric matrix* P*. Then we have, identically in* z

$$\begin{bmatrix} B^\top (z^{-1}I - A^\top)^{-1} & I \end{bmatrix} N(P) \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix} \equiv 0$$

Proof : Use the identity

$$(z^{-1}I - A^\top)P(zI - A) = P - A^\top PA - A^\top P(zI - A) - (z^{-1}I - A^\top)PA$$

and left multiply by $B^\top (z^{-1}I - A^\top)^{-1}$ and right multiply by $(zI - A)^{-1}B$ to get

$$\begin{aligned} B^\top (z^{-1}I - A)^{-\top}(P - A^\top PA)(zI - A)^{-1}B = B^\top (z^{-1}I - A^\top)^{-1}A^\top PB + \\ + B^\top PA(zI - A)^{-1}B + B^\top PB \end{aligned}$$

which is what we needed to show.

Positivity and spectral factorization

Theorem 3 *If there exists $P = P^\top$ such that the Control Linear Matrix Inequality (* **CLMI** *)*

$$L(P) := \begin{bmatrix} A^\top PA - P + Q & S + A^\top PB \\ S^\top + B^\top PA & R + B^\top PB \end{bmatrix} \geq 0 \qquad (10)$$

is satisfied, the spectral function $\Phi(z)$ is positive semidefinite on the unit circle.
Conversely, assume reachability of (A, B), then if any of the value functions V is well defined, there exists a symmetric solution of the LMI (10) *such that $V(\xi) = \xi^\top P\xi$.*

Hence the solvability of the CLMI is a necessary and sufficient condition for the well-posedness of the LQ optimal control problems stated at the beginning.

Connection with dissipative systems

Note that $N(-P) = -N(P)$ serves the same purpose of $N(P)$ and, by renaming $X := -P$ we could form a Linear Matrix Inequality

$$M(X) := \begin{bmatrix} X - A^\top XA + Q & S - A^\top XB \\ S^\top - B^\top XA & R - B^\top XB \end{bmatrix} \geq 0$$

which looks in a sense dual of the LMI of stochastic realization obtained via the transformation $A \leftrightarrow A^\top, B \leftrightarrow C^\top, S \leftrightarrow \bar{C}^\top$. Note however that here Q should correspond to the zero matrix in the stochastic setting. Under this change of sign, setting $S(x) \equiv -V(x)$ (Proposition 2) the dissipation inequality (3) turns into

$$S(x(b)) \leq \sum_{t=a}^{b-1} w(x(t), u(t)) + S(x(a)),$$

for all $u_{[a,b)}$ transferring $x(a)$ to $x(b)$ at time $b > a$. When $S(x) = x^\top Xx \geq 0$ it can be interpreted as a **storage function** in the sense of Willems. Hence the system $x(t+1) = Ax(t) + Bu(t)$ is *dissipative with supply rate* $w(x,u)$ if and only if the LMI $M(X) \geq 0$ has a symmetric **positive definite** solution.

Proof of Theorem 3

Proof of sufficiency: let $L(P) \geq 0$ and let $W(z) = C(zI - A)^{-1}B + D$ where C, D (depending on P) are defined by the factorization

$$L(P) = \begin{bmatrix} C^\top \\ D^\top \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix},$$

then $W(z)$ satisfies the spectral factorization equation $\Phi(z) = W(z^{-1})^\top W(z)$ and therefore $\Phi(e^{j\theta}) \geq 0$.
Conversely, assume that $\Phi(e^{j\theta}) \geq 0$, and hence the positivity condition (7) holds (Theorem 2). This in turn means that all optimal value functions V are well defined, so that $V(\xi) = \xi^\top P \xi$ for some symmetric P (Proposition 2). We shall show that P solves the CLMI. To this end we shall need the following Lemma.

Lemma 4 *Let $\Pi = \Pi^\top$ be an arbitrary $n \times n$ symmetric matrix; then*

$$x(b)^\top \Pi x(b) - x(a)^\top \Pi x(a) + \sum_{t=a}^{b-1} w(x(t), u(t)) = \sum_{t=a}^{b-1} [\, x(t)^\top \; u(t)^\top \,] L(\Pi) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{11}$$

for all controls $u_{[a,b)}$ transferring the state $x(a)$ at time a to the state $x(b)$ at time $b > a$.

Proof : Consider the identity

$$\begin{aligned} x(t+1)^\top \Pi x(t+1) - x(t)^\top \Pi x(t) &= (u(t)^\top B^\top + x(t)^\top A^\top) \Pi (Ax(t) + Bu(t)) - x(t)^\top \Pi x(t) \\ &= [\, x(t)^\top \; u(t)^\top \,] N(\Pi) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \end{aligned}$$

The lemma follows by summing from $t = a$ to $t = b - 1$ and by adding $\sum_{t=a}^{b-1} w(x(t), u(t))$ on both sides. $\square$

Proof of Theorem 3

Now any finite value function must be a quadratic form $V(x) = x^\top Px$ satisfying the dissipation inequality (3) which means that, when $\Pi = P$, the left hand member of (11) must be non negative for all $x(a)$ and all controls steering $x(a)$ to $x(b)$. This implies that

$$\sum_{t=a}^{b-1} [x(t)^\top \; u(t)^\top] L(P) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \geq 0$$

for all $x(a)$ and all such admissible controls u. By reachability, the manifold of these controls and corresponding state trajectories are clearly rich enough to guarantee $L(P) \geq 0$. $\square$

On the solutions of the CLMI

A consequence of Theorems 1, 3 and Proposition 2 is the following:

Corollary 1 *Assume reachability of* (A, B)*. There is a 1:1 correspondence between quadratic functions* $V(\xi) = \xi^\top P \xi$ *solving the IDI and symmetric solutions* $P = P^\top$ *of the CLMI.*

It follows that the solution set $\mathcal{P}$ of the CLMI is a convex set with a minimal and maximal solutions P_- and P_+. In fact, from $V_-(\xi) = \xi^\top \Pi_- \xi$ and $V_+(\xi) = \xi^\top \Pi_+ \xi$, renaming $\Pi_+ := P_+ \,, \quad \Pi_- := P_-$, it follows from (8) that all $P \in \mathcal{P}$ satisfy the inequality

$$P_- \leq P \leq P_+ \,.$$

NB: The value functions form a distinct subset of all solutions of the IDI and likewise the corresponding set of symmetric matrices $\{\Pi\}$ forms a distinct subset of $\mathcal{P}$. As we shall see, this subset lies on the boundary of $\mathcal{P}$.

The quadratic solutions of the IDI

Proposition 4 *Assume that the positivity condition* (7) *holds. Then to each solution $P = P^\top$ of the CLMI there corresponds a spectral factorization*

$$\Phi(z) = W(z^{-1})^\top W(z) \tag{12}$$

with the spectral factor $W(z)$ described as $W(z) = C_P(zI - A)^{-1}B + D_P$ the matrices C_P, D_P being determined by the factorization of $L(P) = L(P)^\top \geq 0$,

$$L(P) = \begin{bmatrix} C_P^\top \\ D_P^\top \end{bmatrix} \begin{bmatrix} C_P & D_P \end{bmatrix}$$

modulo multiplication by an orthogonal matrix. Letting

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t)\,, \qquad x(t_0) = \xi\,, \\ y_P(t) &= C_P x(t) + D_P u(t)\,. \end{aligned} \tag{13}$$

The cost function corresponding to P can be represented as

$$J(\xi, u) = \sum_{-\infty}^{+\infty} y_P^\top(t) y_P(t) \tag{14}$$

The sum is computed by integrating the stable modes forward in time and the unstable modes backward in time. The modes of A on the unit circle must be non-observable since otherwise the sum would diverge. What is the meaning of $\xi^\top P\xi =$?

Non singularity and the Riccati Inequality

In the following we shall assume that $\mathcal{P}$ is non empty and that $R+B^\top PB$ is non singular for all solutions $P=P^\top$ of the CLMI. This condition is discussed in the literature but for the moment we shall not question it; it clearly implies that $R+B^\top PB>0$.
Under non-singularity we can block diagonalize $L(P)$ as

$$L(P) = \begin{bmatrix} I & K_P^\top \\ 0 & I \end{bmatrix} \begin{bmatrix} \Lambda(P) & 0 \\ 0 & R+B^\top PB \end{bmatrix} \begin{bmatrix} I & 0 \\ K_P & I \end{bmatrix} \tag{15}$$

where $K_P = (R+B^\top PB)^{-1}(S^\top + B^\top PA)$ and

$$\Lambda(P) := A^\top PA - P - (S+A^\top PB)(R+B^\top PB)^{-1}(S^\top + B^\top PA) + Q\,.$$

Theorem 4 *Assume that $R+B^\top PB$ is non singular, then $P\in\mathcal{P}$ if and only if it satisfies the* **Control Algebraic Riccati Inequality** $\Lambda(P)\geq 0$.

Under non-singularity the "boundary" of the solution set to the CLMI is defined by he Algebraic Riccati Equation $\Lambda(P)=0$ namely

$$P = A^\top PA - (S+A^\top PB)(R+B^\top PB)^{-1}(S^\top + B^\top PA) + Q\,, \qquad (ARE)$$

Value functions, the ARE and optimal controls

We know already from Proposition 3 that all value functions $V(x) = x^\top Px$ correspond 1:1 to symmetric solutions of the ARE. This can be seen also from the factorization (15).

Theorem 5 *Assume non singularity and that $\mathcal{P}$ is non empty, then all value functions $V(x) = x^\top Px$ correspond 1:1 to symmetric solutions P of the Algebraic Riccati equation. The optimal input u is generated by the* **feedback control law**

$$u(t) = -K_P x(t) .$$

where K_P is defined in (15)*.*

Proof: Letting $a = 0$ and $x(a) = \xi$, the dynamic programming equation (11) for the value function $V(x) = x^\top Px$, gives

$$
\begin{aligned}
0 = \inf_{u_{[0\,b)}} x(b)^\top Px(b) - \xi^\top P\xi + \inf_{u_{[0\,b)}} \sum_{t=0}^{b-1} w(x(t), u(t)) \\
= \inf_{u_{[0\,b)}} \sum_{t=0}^{b-1} \left[x(t)^\top \; u(t)^\top \right] L(P) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},
\end{aligned}
$$

the infimum being over all admissible controls steering $x(0) = \xi$ to $x(b)$ and for all $b > 0$. Using the block-diagonalization of $L(P)$, this leads to

$$
0 = \inf_{u_{[0\,b)}} \sum_{t=0}^{b-1} x(t)^\top \Lambda(P) x(t) + v(t)^\top (R + B^\top PB)\, v(t) \quad \forall b > 0
$$

where $v(t) = K_P x(t) + u(t)$. Since $\Lambda(P) \geq 0$ it is then clear that the infimum (which must be zero), is achieved iff $\Lambda(P) = 0$ and $v(t) \equiv 0$, that is when u is generated by the feedback control law $u(t) = -K_P x(t)$.

Letting $b \to +\infty$ and $x(+\infty) = 0$ the same argument applies to $V_+(\xi)$. From

$$0 = -\xi^\top P_+ \xi + \inf_{u_{[0+\infty)}} \sum_{t=0}^{+\infty} w(x(t), u(t))$$

$$= \inf_{u_{[0+\infty)}} \sum_{t=0}^{+\infty} [\, x(t)^\top \; u(t)^\top \,] \, L(P_+) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

the infimum being over all admissible controls steering $x(0) = \xi$ to zero, it follows that $u(t) = -K_{P_+} x(t)$ is the optimal control and $V_+(\xi) = \xi^\top P_+ \xi$. A dual argument applies also to V_-. $\square$

Structure of the optimal systems

Let P be a solution of the CLMI and define $\hat{G}(z) := K_P(zI-A)^{-1}B+I$; it follows from (15) that the spectral function admits the decomposition

$$\Phi(z) = B^\top(z^{-1}I-A^\top)^{-1}\Lambda(P)(zI-A)^{-1}B+\hat{G}(z^{-1})^\top(R+B^\top PB)\,\hat{G}(z) \quad (16)$$

By setting $D_P := (R+B^\top PB)^{1/2}$, $C_P := D_PK_P$ one has the **spectral factorization**

$$\Phi(z) = G_P^\top(z^{-1})G_P(z) \qquad \text{where} \qquad G_P(z) := C_P\,(zI-A)^{-1}B+D_P$$

so that all P solutions of the Riccati equation $\Lambda(P)=0$, and only these P's, correspond to **square spectral factors** of $\Phi(z)$ of dimension $m\times m$.
Note that $G_P(z)$ is the transfer function of a system with a ficticious output

$$\begin{aligned} x(t+1) &= Ax(t)+Bu(t)\,, \qquad x(t_0)=\xi\,, \\ z(t) &= C_Px(t)+D_Pu(t)\,. \end{aligned} \quad (17)$$

This system is subjected to the optimal feedback law corresponding to P (solution of the Riccati equation),

$$u_P(t) = -K_P x(t)$$

since from (17) one gets $z(t) = C_P x(t) - D_P K_P x(t) \equiv 0$ it follows that this feedback law is **output nulling**, that is, it makes $z(t) \equiv 0$ on the whole time axis, for any initial condition. Hence:

Theorem 6 *The closed loop dynamics of the optimal system corresponding to a solution P of the ARE, is governed by the autonomous evolution equation*

$$x(t+1) = \Gamma_P x(t)\,, \qquad x(t_0) = \xi \tag{18}$$

where the eigenvalues of the closed-loop transition matrix $\Gamma_P := A - BK_P$ are precisely the transmission zeros of the transfer function $G_P(z)$.

The spectrum of Γ_P

The spectrum of the closed loop matrix Γ_P determines the asymptotic state evolution of the optimal system. Intuitively, since $\Phi(z) = G_P^\top(z^{-1})G_P(z)$ the eigenvalues of Γ_P and their reciprocals together should be the zeros of $\Phi(z)$. This is proven in the following.

Let $\Delta(P) := R + B^\top PB$. Start from a state space realization of

$$\Phi(z) = (B^\top(z^{-1}I - A^\top)^{-1}K_P^\top + I)\Delta(P)(K_P(zI - A)^{-1}B + I)$$

$$\begin{bmatrix} x(t+1) \\ \xi(t-1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ K_P^\top \Delta(P) K_P & A^\top \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} B \\ K_P^\top \Delta(P) \end{bmatrix} u(t)$$

$$\eta(t) = \Delta(P)K_P x(t) + B^\top \xi(t) + \Delta(P)u(t)$$

and compute a realization for the inverse, namely

$$
\begin{bmatrix} x(t+1) \\ \xi(t-1) \end{bmatrix} = \begin{bmatrix} \Gamma_P & -B\Delta(P)^{-1}B^\top \\ 0 & \Gamma_P^\top \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} B\Delta(P)^{-1} \\ K_P^\top \end{bmatrix} \eta(t)
$$
$$
u(t) = -K_P x(t) - \Delta(P)^{-1}B^\top \xi(t) + \Delta(P)^{-1}\eta(t)
$$

because of reachability, this is a minimal [CHECK] realization of the inverse so that

Proposition 5 *If and only if $\Delta(P)$ is non singular, $\Phi(z)$ has generically full rank and has no zeros neither at $z=0$ nor at infinity.*
In this case the matrix Γ_P is non singular for all P solution of the ARE and

$$
\sigma(\Gamma_P) \cup \sigma(\Gamma_P^{-1}) = \text{ zeros of } \Phi(z)\,, \tag{19}
$$
$$
\sigma(\Gamma_P) \cap \sigma(\Gamma_P^{-1}) = \text{ zeros of } \Phi(z) \text{ on the unit circle}\,. \tag{20}
$$

counting multiplicity.

Proof: The first statement follows from (16): the first term is nonnegative while the second is non-singular a.e. and from [**?**, Theorem 4.1]. The rest is obvious. □

Therefore there are two feedback matrices Γ_+ and Γ_- whose spectrum is contained in the region $\mathcal{D}_+ := \{z; |z| \leq 1\}$ and $\mathcal{D}_- := \{z; |z| \geq 1\}$.

Theorem 7 *Γ_+ and Γ_- correspond to the extreme solutions P_+ and P_- of the ARE, that is*

$$\Gamma_+ = A - BK_+ = A - B(R + B^\top P_+ B)^{-1}(S^\top + B^\top P_+ A)\,, \tag{21}$$

$$\Gamma_- = A - BK_- = A - B(R + B^\top P_- B)^{-1}(S^\top + B^\top P_- A) \tag{22}$$

The proof is based on the following lemma (See Lemma 3.1 and 3.2 of [**?**]).

Lemma 5 *Let P be a solution of the ARE and let $X := P - P_-$ and $Y := P_+ - P$; then X and Y satisfy the equations*

$$X = \Gamma_-^\top X \Gamma_- - \Gamma_-^\top X B(R + B^\top P B)^{-1} B^\top X \Gamma_-$$
$$Y = \Gamma_+^\top Y \Gamma_+ + \Gamma_+^\top Y B(R + B^\top P B)^{-1} B^\top Y \Gamma_+$$

or, equivalently

$$X = \Gamma_-^{-\top} X \Gamma_-^{-1} + XB(R + B^\top P B)^{-1} B^\top X \tag{23}$$

$$Y = \Gamma_+^{-\top} Y \Gamma_+^{-1} - YB(R + B^\top P B)^{-1} B^\top Y \tag{24}$$

Proof of Theorem 7: For a matrix A denote:

$$\mathcal{L}_<(A)\,, \quad \mathcal{L}_1(A)\,, \quad \mathcal{L}_>(A)\,,$$

the (invariant) eigenspaces corresponding to eigenvalues, respectively inside the unit circle, on the unit circle and outside of the closed unit circle. It is proven in [**?**, Theorem 3.2] that

$$\mathcal{L}_1(\Gamma_-) \subset \mathrm{Ker}X \qquad \mathcal{L}_1(\Gamma_+) \subset \mathrm{Ker}Y$$

and since $\mathcal{L}_1(\Gamma_-) = \mathcal{L}_1(\Gamma_-^{-1})$ the latter is also included in $\mathrm{Ker}X$. Similarly for Γ_+. Now we know that Γ_- has no eigenvalues inside the unit circle and hence there is a unitary change of basis U such that

$$U^\top \Gamma_-^{-1} U = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$$

where A_1 has only eigenvalues of modulus one (the reciprocals of those of Γ_-) while A_2 has only eigenvalues in the open unit disk. From the invariance $\mathcal{L}_1(\Gamma_-^{-1}) \subset \mathrm{Ker}X$ it follows that $U^\top X U = \mathrm{diag}\{0, X_2\}$ so that equation (23) restricted to the asymptotically stable subspace of Γ_-^{-1} becomes

$$X_2 = A_2^\top X_2 A_2 + Q_2$$

where Q_2 is the lower diagonal block of $U^\top XB(R+B^\top PB)^{-1}B^\top XU$ which is symmetric and nonnegative definite. Then by standard Lyapunov theory $X_2 \geq 0$ and hence all solutions X of the Lyapunov-like equation $X = \Gamma_-^{-\top} X \Gamma_-^{-1} + Q$ are nonnegative definite; i.e. $X \geq 0$. Similary $\sigma(\Gamma_+^{-1}) \subset \{z; |z| \geq 1\}$ implies $Y \geq 0$ (because of the minus sign in the equation). Hence the maximal solution P_+ of the ARE is a stabilizing solution (not asymptotically in general) while the minimal solution is anti-stabilizing. □

Proposition 6 *For every P solving the ARE, the subspace $\operatorname{Ker} X$ is Γ_--invariant and $\operatorname{Ker} Y$ is Γ_+-invariant .*

Proof : Let $x \in \operatorname{Ker} X$, then from (23)

$$Xx = 0 = \Gamma_-^{-\top} X \Gamma_-^{-1} x$$

which, since $\Gamma_-^{-\top}$ is non-singular, implies $X\Gamma_-^{-1}x = 0$; i.e. $\Gamma_-^{-1}x \in \operatorname{Ker} X$ so the subspace $\operatorname{Ker} X$ is mapped into itself by Γ_-^{-1}, that is $\operatorname{Ker} X$ is a Γ_-^{-1}-invariant subspace. However Γ_-^{-1} and Γ_- have the same invariant subspaces. The proof of the statement for $\operatorname{Ker} Y$ is analogous. □

Wimmer [**?**] proof: define

$$\tilde{\Gamma}_- := \Gamma_- - XB(R+B^\top PB)^{-1}B^\top X\Gamma_-$$

so that the ARE for X can be rewritten $X = \Gamma_-^\top X\tilde{\Gamma}_-$ or, equivalently,

$$\tilde{\Gamma}_-^{-\top} X = X\Gamma_-$$

since $\tilde{\Gamma}_- = (I - XB(R+B^\top PB)^{-1}B^\top X)\Gamma_-$ is also non-singular [??]. This seems to require R invertible which has no clear meaning.

Hence on $\mathrm{Ker}X$ we have $\Gamma_P = \Gamma_-$ while on $\mathrm{Ker}Y$ we have $\Gamma_P = \Gamma_+$. In fact, the general idea is to show that there are as many P's (or as many Γ_P) as there are *zero flipping* to reciprocal positions of the zeros of $\hat{G}(z)_+$ or $\hat{G}(z)_-$, keeping in mind that the zeros on the unit circle remain fixed in the flipping process.

the optimal cost corresponding to P should then be written as a quadratic form involving the temporal evolution of the state trajectory restricted to the unstable and stable subspaces of the form

$$J(\xi, u_P) = \sum_{-\infty}^{0} y_-(t)^\top y_-(t) + \sum_{0}^{+\infty} y_+(t)^\top y_+(t)$$

Since the (optimal) output is $y(t) = C_P \Gamma_P^{t-t_0} x(t_0)$ the optimal cost can be written

$$J(\xi, u_P) = \ \xi^\top \sum_{-\infty}^{+\infty} (\Gamma_P^\top)^t C_P^\top C_P \Gamma_P^t \, \xi \ := \xi^\top W \, \xi$$

where $W = W^\top$ is a solution of the Lyapunov-type equation

$$W = \Gamma_P^\top W \, \Gamma_P + C_P^\top C_P = \Gamma_P^\top W \, \Gamma_P + K_P^\top (R + B^\top P B) K_P$$

which for $W = P$ reduces to the CARE ($\tilde{Q} = 0$?) and hence has certainly the solution $W = P$ not necessarily positive semidefinite see [Wimmer].

All other solutions of the Riccati inequality correspond to rectangular spectral factors of dimension $(p+m) \times m$ where p is the rank of $\Lambda(P)$.

The structure of the solution set of the CARE

See Ran & Trentelman

For which problems does the CLMI (or CARE) have a solution?

The optimal regulator

For the optimal regulator problem the weight matrix in the cost functional is assumed to be positive semidefinite; i.e.

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \geq 0$$

A stronger assumption is $R > 0$ (positive definite control weight) which rules out a subclass of interesting problems called *cheap control problems*. If R is non singular we can block-diagonalize the weight matrix

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} = \begin{bmatrix} I & SR^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q - SR^{-1}S^\top & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 \\ R^{-1}S^\top & I \end{bmatrix}$$

and introduce the linear feedback $v(t) := u(t) + R^{-1}S^\top x(t)$, changing the problem into one with block-diagonal weight:

$$x(t+1) = Fx(t) + Bv(t)\,; \qquad \tilde{w}(x(t), v(t)) = \begin{bmatrix} x(t)^\top & v(t)^\top \end{bmatrix} \begin{bmatrix} \tilde{Q} & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

with $F = A - BR^{-1}S^\top$ and $\tilde{Q} = Q - SR^{-1}S^\top$. By positivity $\tilde{Q} = H^\top H \geq 0$ ($H \equiv \tilde{Q}^{1/2}$).

The non singular optimal regulator

Proposition 7 *Assume $R+B^\top XB$ is invertible; then the CARE of the regulator problem with block-diagonalized weights*

$$X = F^\top XF - F^\top XB(R+B^\top XB)^{-1}B^\top XF + \tilde{Q}$$

coincides with (and hence has the same solutions of) the CARE. In particular the solutions are the same; i.e. $X \equiv P$.
Moreover the closed-loop dynamics is also invariant; letting $\tilde{K}(X) = (R+B^\top XB)^{-1}B^\top XF$ one has:

$$F - B\tilde{K}(X) = A - BK_P$$

where $K_P = (R+B^\top PB)^{-1}(S^\top + B^\top PA)$. The statement remains true for the algebraic Riccati Inequality.
The Algebraic Riccati equation can also be written in symmetrized form *as*

$$X = (F-B\tilde{K})^\top X(F-B\tilde{K}) + \tilde{K}^\top R\tilde{K} + \tilde{Q}\,, \qquad \tilde{K} = (R+B^\top XB)^{-1}B^\top XF\,.$$

The proof is by algebraic verification.

Duality with the Kalman Filter

When R is invertible the theory of the optimal regulator with stability is completely dual of that of the (steady state) Kalman filter.

Look for solutions of the CARE which are ≥ 0 . The cost $J(\xi, u)$ is nonnegative and hence every $V(\xi) \geq 0$. In particular want the (maximal) solution V_{+}.

Theorem 8 **Existence :** *If (A, B) is stabilizable; equivalently, (F, B) is stabilizable, there is a nonnegative definite solution of the CARE. The corresponding value function is hence nonnegative. In other words, if (A, B) is stabilizable, the maximal solution of the CARE exists and is nonnegative definite.*

Proof : Assume non-singularity. Consider the finite horizon optimal control problem

$$\min_{u_{[t_0,t_1)}} J(\xi, u) := \sum_{t_0}^{t_1} \begin{bmatrix} x(t)^\top & u(t)^\top \end{bmatrix} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + x(t_1)^\top M x(t_1)$$

subject to: $x(t+1) = Ax(t) + Bu(t)\,, \quad x(t_0) = \xi\,, \qquad M = M^\top \geq 0 \in \mathbb{R}^{n\times n}.$

By dynamic programming a necessary condition is that $V(t,\xi) := x(t)^\top \Pi(t) x(t)$ with $\Pi(t)$ satisfying the Riccati D.E.

$$\Pi(t-1) = F^\top \Pi(t) F - F^\top \Pi(t) B(R + B^\top \Pi(t) B)^{-1} B^\top \Pi(t) F + \tilde{Q} \qquad \Pi(t_1) = M$$

whereby $V(\xi) = \xi^\top \Pi(t_0,t_1,M)\xi$. Can show that for $M = 0$ the function $\Pi(t_0,t_1,0) \equiv \Pi(t_1 - t_0, 0)$ is monotonic non decreasing with $t_1 - t_0$. When $t_1 - t_0 \uparrow +\infty$ either $\Pi(t_1 - t_0, 0) \uparrow +\infty$ or it must solve the Algebraic Riccati equation.
By stabilizability there is a gain matrix L, making $F - BL$ asymptotically stable. Using the control $u(t) = -Lx(t)$ we obtain $J(\xi, u) = \xi^\top U(t_0,t_1,M)\xi$ where $U(t,t_1,0)$ solves the linear equation

$$U(t-1) = (F - BL)^\top U(t)(F - BL) + LB(R + B^\top U(t) B)^{-1} B^\top L^\top + \tilde{Q} \qquad U(t_1) = 0$$

which by stability of $F - BL$ has a uniformly bounded solution. Since by optimality $\Pi(t_1 - t_0, 0) \leq U(t_0, t_1, 0)$, Π is also uniformly bounded and hence converges to a positive semidefinite solution of the CARE.

Duality with the Kalman Filter: Stability

A solution $P = P^\top$ of the CARE is **stabilizing** if $|\lambda[A - BK_P]| < 1$ equivalently, $|\lambda[F - B\tilde{K}_P]| < 1$.

Theorem 9 *If (F, H) is detectable, then the feedback matrix $A - BK_P$ associated to any nonnegative definite solution of the CARE is asymptotically stable. In other words, any $P = P^\top \geq 0$ solving the CARE is* stabilizing*.*

Proof : Assume there is an eigenvalue λ_0 of $F - B\tilde{K}_P$ with $|\lambda_0| \geq 1$. Let a be a corresponding eigenvector. Then

$$a^* P a = a^*(F - B\tilde{K})^\top P(F - B\tilde{K})\, a + a^* \tilde{K}^\top R \tilde{K}\, a + a^* \tilde{Q}\, a$$

yields

$$(1 - |\lambda_0|^2) a^* P a = a^* \tilde{K}^\top R \tilde{K}\, a + a^* \tilde{Q}\, a$$

where the left memeber is ≤ 0 and the right member is ≥ 0. Hence both must be equal to zero; in particular, since R is non singular, we must have $\tilde{K}\, a = 0$ so that $(F - B\tilde{K})\, a = F\, a = \lambda_0\, a$ with $|\lambda_0| \geq 1$ while $H\, a = 0$ which contradicts detectabilty of (F, H). $\square$.

Duality with the Kalman Filter: Uniqueness

Lemma 6 *Let $P(t)$ and $\bar{P}$ be solutions of the Difference RE and of the ARE respectively; then*

$$P(t+1) - \bar{P} = (A - B\bar{K})^\top (P(t) - \bar{P})(A - BK(t))$$

where $K(t) = K(P(t))$ and $\bar{K} = K(\bar{P})$.

Theorem 10 *A positive semidefinite (and hence stabilizing) solution of the CARE is necessarily unique.*

Proof : Assume there are two solutions P_1 and P_2 of the CARE both symmetric and positive semidefinite, and hence both stabilizing. Then by iterating the formula of the lemma

$$P_1 - P_2 = (A - B\bar{K}_1)^\top (P_1 - P_2)(A - B\bar{K}_2)$$

one gets

$$P_1 - P_2 = \left[(A - B\bar{K}_1)^\top\right]^k (P_1 - P_2) \left[A - B\bar{K}_2\right]^k$$

so that letting $k \to \infty$ it follows that $P_1 = P_2$. $\square$

Duality with the Kalman Filter: Main Theorem

Theorem 11 (Kalman) *If and only if (F, B) is stabilizable and (F, H) is detectable, the CARE has a unique nonnegative solution $P = P^\top$ which is stabilizing and necessarily the maximal solution.*

Proof : Exactly as for the Kalman Filter. $\square$

The role of Detectability

Consider the regulator problem with $\tilde{Q} = 0$ and $\lim_{t\to\infty} x(t) = 0$. We want to asymptotically stabilize the system using the least control energy. Assume (F, B) is stabilizable, but F not necessarily stable (no detectability). What can we say?

We shall provisionally assume that F is non singular. The CARE

$$X = F^\top XF - F^\top XB(R+B^\top XB)^{-1}B^\top XF = F^\top \left[X - XB(R+B^\top XB)^{-1}B^\top X\right] F$$

is homogeneous and has the solution $P = 0$ which yields the trivial control law $u(t) \equiv 0$ which in general is not stabilizing. Consider non zero solutions $P = P^\top$ and the corresponding feedback matrix

$$\Gamma_P = F - BK_P = F - B(R + B^\top PB)^{-1}B^\top PF$$

so that the homogeneous CARE is written as

$$P = F^\top P\Gamma_P, \qquad \text{i.e.} \quad P\Gamma_P = F^{-\top}P$$

Assuming P invertible we get the similarity relation $\Gamma_P = P^{-1}F^{-\top}P$
So far we don't know if there are any invertible solutions of the CARE. By the matrix inversion lemma, for any such P

$$P - PB(R + B^\top PB)^{-1}B^\top P = \left[P^{-1} + BR^{-1}B^\top\right]^{-1}$$

which, since F is invertible, turns the CARE into

$$P^{-1} = F^{-1}\left[P^{-1} + BR^{-1}B^\top\right]F^{-\top}$$

that is into the Lyapunov equation

$$FP^{-1}F^\top = P^{-1} + BR^{-1}B^\top,$$

Lemma 7 *If F is unmixing and (F, B) is reachable, this Lyapunov equation has a unique nonsingular solution. In case F is totally antistable ($|\lambda(F)| > 1$) P is positive definite. In this case Γ_P is asymptotically stable.*

Frequency domain conditions

With the optimal control we have $v(t) := u(t) + R^{-1}S^{\top}x(t) \equiv 0$, and hence (when R is invertible) the spectral function of the regulator problem **with the optimal control** can be written as

$$\Phi^{o}(z) = B^{\top}(z^{-1}I - \Gamma^{\top})^{-1}\tilde{Q}\,(zI - \Gamma)^{-1}B\,, \qquad \Gamma = F - B\tilde{K}(P_{+})$$

For $z = e^{j\theta}$ this is in fact the Parseval-Fourier Transform of the optimal value function

$$V_{+}^{o}(\xi) = \sum_{0}^{+\infty} x_{o}(t)^{\top}\tilde{Q}x_{o}(t)\,, \qquad x_{o}(t+1) = (F - B\tilde{K})x_{o}(t)\,, \quad x_{o}(0) = \xi$$

Since $(B^{\top}, \Gamma^{\top})$ is an observable/detectable pair (assuming (A, B) reachable/stabilizable) it is clear that unstable modes in $x_{o}(t)$ may exist when and only when $(\Gamma^{\top}, \tilde{Q}^{1/2})$ is not stabilizable.
Since $F - B\tilde{K} = [I - B(R + B^{\top}P_{+}B)^{-1}B^{\top}P_{+}]F$... same as detectability of $(F, \tilde{Q}^{1/2})$??

The linear stochastic regulator

Consider the following linear stochastic system on $t \geq t_0$

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + v(t), \qquad \mathbb{E}x(t_0) = m, \quad \operatorname{Var} x(t_0) = \Sigma_0 \\ y(t) &= Cx(t) + w(t) \end{aligned} \tag{25}$$

where w and v are uncorrelated white noise processes of covariances V and W respectively. We want to solve the quadratic optimization problem on the finite interval $t_0 \leq t \leq T$:

$$\text{minimize } \mathbb{E}J(x(t_0), u) := \mathbb{E}\{\sum_0^{T-1} \begin{bmatrix} x(t)^\top & u(t)^\top \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + x(T)^\top Q_T x(T)\}$$

Where $Q = Q^\top, R = R^\top$ are positive semidefinite. Without loss of generality $S = 0$.

The minimization is to be performed with controls $u(t)$ which only use the available information at time t. These controls will be called **admissible**. In case $y(t) = x(t)$ we have the *full information* situation and u is admissible if $u(t) = \varphi(t, x(t))$. In general we only have available $y^t = \{y(t_0), \ldots, y(t)\}$. We shall call these control functions y^t-measurable.

Minimization and expectation commute

Complete state information:

Lemma 8 *Assume that the function $J(x,y,u)$ has a unique minimum w.r.to u for all x,y. Denote $u^o(x,y)$ the minimizer. Then,*

$$\min_{u(x,y)} \mathbb{E}\, J(x,y,u) = \mathbb{E}\, J(x,y,u^o(x,y)) = \mathbb{E} \min_u J(x,y,u)$$

Partial state information:

Lemma 9 *Assume that the function $F(y,u) := \mathbb{E}\left[J(x,y,u) \mid y\right]$ has a unique minimum w.r.to u for all y's. Denote $u^o(y)$ the minimizer. Then,*

$$\min_{u(y)} \mathbb{E}\, J(x,y,u) = \mathbb{E}\, J(x,y,u^o(y)) = \mathbb{E}\{\min_u \mathbb{E}\left[J(x,y,u) \mid y\right]\}$$

NOTA BENE: In general $\min_{u(y)} \mathbb{E}\, J(x,y,u) \geq \min_{u(x,y)} \mathbb{E}\, J(x,y,u)$

Dynamic Programming: Complete state information

$$\mathbb{E}\{\sum_{t_0}^{T-1}\begin{bmatrix}x(t)^\top & u(t)^\top\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}+x(T)^\top Q_T x(T)\} =$$

$$= \quad \mathbb{E}\{\sum_{t_0}^{t-1}\begin{bmatrix}x(s)^\top & u(s)^\top\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}x(s)\\ u(s)\end{bmatrix}\}+$$

$$\mathbb{E}\{\sum_{t}^{T-1}\begin{bmatrix}x(s)^\top & u(s)^\top\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}x(s)\\ u(s)\end{bmatrix}+x(T)^\top Q_T x(T)\}$$

The first term does not depend on $u(t),\ldots,u(T-1)$. To minimize w.r.to these variable just need to minimize the second term. Define the **optimal cost to go** function

$$V(x,t) := \min_{u(t),\ldots,u(T-1)} \mathbb{E}\left[\sum_{t}^{T-1}\begin{bmatrix}x(s)^\top & u(s)^\top\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}x(s)\\ u(s)\end{bmatrix}+x(T)^\top Q_T x(T) \mid x(t)=x\right]$$

THE BELLMAN EQUATION

Then $V(x,t)$ satisfies the **Bellman equation**:

$$V(x,t) = \min_{u(t)} \mathbb{E}\left[\begin{bmatrix} x(t)^\top & u(t)^\top \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + V(x(t+1), t+1) \mid x(t) = x\right] \tag{26}$$

with terminal condition:

$$V(x,T) = \min_{u(T)} \mathbb{E}\left[x(T)^\top Q_T x(T) \mid x(T) = x\right] = x^\top Q_T x$$

Next, prove the following

Lemma 10 *Let* $x \sim \mathcal{N}(m, \Sigma)$*, and* $S = S^\top$*. Then*

$$\mathbb{E} x^\top S x = m^\top S m + \mathrm{Tr}\,(S\Sigma)\,.$$

(Actually you don't need Gaussianness).

Solution of the Bellman equation

Try with a quadratic function $V(x,t) = x^\top S(t)x + s(t)$. Using the previous lemma

$$\mathbb{E}\left[V(x(t+1),t+1) \mid x(t)\right] = [Ax(t)+Bu(t)]^\top S(t+1)[Ax(t)+Bu(t)] + \mathrm{Tr}\,(V\,S(t+1))$$

substitute into (26) and you obtain the **Riccati recursion**

$$S(t) = A^\top S(t+1)A - L(t)^\top [R + B^\top S(t+1)B]L(t) + Q \tag{27}$$

$$L(t) = [R + B^\top S(t+1)B]^{-1} B^\top S(t+1)A \qquad S(T) = Q_T \tag{28}$$

$$s(t) = s(t+1) + \mathrm{Tr}\,(V\,S(t+1)) \tag{29}$$

The minimum is obtained for

$$u(t) = -L(t)x(t)$$

and is equal to

$$\mathbb{E}\,V(x(t_0),t_0) = m^\top S(t_0)\,m + \mathrm{Tr}\,(S(t_0)\Sigma_0) + \sum_{t_0}^{T-1} \mathrm{Tr}\,(V\,S(t+1))\,.$$

Asymptotic behaviour

The Riccati difference equation and the optimal control law are the same as for the deterministic regulator problem. May consider the infinite horizon $T \to \infty$. The same conditions for the existence of a positive semidefinite solution of the ARE apply here as well. The closed loop system is asymptotically stable etc.. In this case however the optimal cost tends to infinity. We minimize the *average cost per unit time:*

$$\text{minimize} \lim_{T\to\infty} \frac{1}{T}\{\sum_{0}^{T-1} \begin{bmatrix} x(t)^\top & u(t)^\top \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + x(T)^\top Q_T x(T)\}$$

as $T - t_0 \to \infty$ of $S(t) \to S$ (constant) independent of Q_T and $L(t) \to L$. Asymptotically we minimize

$$\min_{u_{[0,\infty)}} \mathbb{E}\{\begin{bmatrix} x(t)^\top & u(t)^\top \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}\} = \mathrm{Tr}\,(V\,S)\,.$$

D.P.: Incomplete state information

Same splitting as before

$$\mathbb{E}\{\sum_{t_0}^{T-1}\begin{bmatrix}x(t)^\top & u(t)^\top\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}+x(T)^\top Q_T x(T)\} =$$

$$= \quad \mathbb{E}\{\sum_{t_0}^{t-1}\begin{bmatrix}x(s)^\top & u(s)^\top\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}x(s)\\ u(s)\end{bmatrix}\}+$$

$$\mathbb{E}\{\sum_{t}^{T-1}\begin{bmatrix}x(s)^\top & u(s)^\top\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}x(s)\\ u(s)\end{bmatrix}+x(T)^\top Q_T x(T)\}$$

The first term does not depend on $u(t),\dots,u(T-1)$. To minimize w.r.to these variable just need to minimize the second term.
But now $u(t)=\varphi(t,y^t)$! Now use Lemma 9: the **optimal cost to go** function is conditional on $\mathbf{y}^t$:

$$V(y^t,t) := \min_{u(t),\dots,u(T-1)} \mathbb{E}\left[\sum_{t}^{T-1}\begin{bmatrix}x(s)^\top & u(s)^\top\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}x(s)\\ u(s)\end{bmatrix}+x(T)^\top Q_T x(T) \mid \mathbf{y}^t = y^t\right]$$

The Bellman equation: Incomplete sate information

Then $V(y^t, t)$ satisfies the **Bellman equation**:

$$V(y^t, t) = \min_{u(t)} \mathbb{E}\left[\begin{bmatrix} x(t)^\top & u(t)^\top \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + V(x(t+1), t+1) \mid \mathbf{y}^t = y^t \right] \tag{30}$$

Conditional expectations given $\mathbf{y}^t$ depend only on the conditional expectation of the state $\hat{x}(t \mid t)$! $V(y^t, t) = W(\hat{x}(t), t)$. Hence

$$W(\hat{x}(t), t) = \min_{u(t)} \mathbb{E}\left[\begin{bmatrix} x(t)^\top & u(t)^\top \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + W(\hat{x}(t+1), t+1) \mid \hat{\mathbf{x}}(t) \right]$$

with terminal condition

$$W(\hat{x}(T), T) = \min_{u(T)} \mathbb{E}\left[x(T)^\top Q_T x(T) \mid y^T\right] = \mathbb{E}\left[x(T)^\top Q_T x(T) \mid y^T\right] =$$
$$\hat{x}(T)^\top Q_T \hat{x}(T) + \mathrm{Tr}\left(Q_T P(T)\right)$$

where $P(t) := P(t \mid t)$ is the state error covariance matrix.

Solution of the Bellman equation

Theorem 12 *The solution of the Bellman equation is a quadratic function*

$$W(\hat{x}(t),t) = \hat{x}(t)^\top S(t)\hat{x}(t) + s(t)$$

where $S(t)$ satisfies exactly the same Riccati recursion as in the full information case. The minimum is attained by the linear control law:

$$u(t) = -L(t)\hat{x}(t)$$

where $\hat{x}(t) = \hat{x}(t \mid t)$ is the Kalman filter estimate of the state. The optimal cost is

$$\mathbb{E}V(x(t_0),t_0) = m^\top S(t_0)\, m + \mathrm{Tr}\,(S(t_0)\Sigma_0) + \sum_{t_0}^{T-1} \mathrm{Tr}\,(V\, S(t+1)) + \\ + \sum_{t_0}^{T-1} \mathrm{Tr}\,(P(t)\, L(t)^\top [R + B^\top S(t+1)B] L(t))\,.$$

The last term in the optimal cost is due to the uncertainty of the state estimate.