DEPARTMENT OF INFORMATION ENGINEERING
UNIVERSITY OF PADOVA

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Engineering Stable Discrete-Time Quantum Dynamics via a Canonical QR Decomposition

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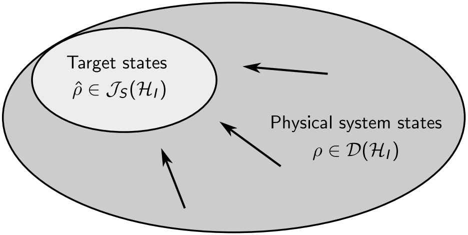
Motivation

Typical key tasks in Quantum Information are

- state preparation (entangled-state preparation, quantum register initialization,...)
- engineering of protected realization of QI.



These tasks tightly connect to stabilization control problems. This work will focus on these issues, providing a design strategy for engineering stable quantum subspaces.

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Outline

Engineering stable quantum dynamics

Consider a given quantum physical system, accessible by measurements. Suppose that we can apply coherent control on the basis of this measurements. How can we design a convenient control law that achieves global asymptotic stability of given target states?

In this work we

- characterize invariance, attractivity and global asymptotic stability for discrete-time open quantum dynamics
- derive a canonical form with respect to the action of left multiplication by a unitary matrix
- investigate the capabilities of feedback coherent control for quantum simulation
- derive an algorithm for stabilizing control law synthesis.

Previous works

The use of "noisy" dynamics to obtain a desired dynamical behavior has long been investigated in a variety of contexts.

- J. F. Poyatos, J. I. Cirac, and P. Zoller Quantum reservoir engineering with laser cooled trapped ions (1996).
- A. R. R. Carvalho, P. Milman, R. L. de Matos Filho, and L. Davidovich Decoherence, pointer engineering, and quantum state protection (2001).
- R. L. de Matos Filho and W. Vogel Engineering the hamiltonian of a trapped atom (1998).
- F. Verstraete, M. M. Wolf, and J. I. Cirac Quantum computation, quantum state engineering and quantum phase transitions driven by dissipation (2008).

Previous works (cont.)

Feedback subsystem stabilization in the quantum domain (continuos time case).

- F. Ticozzi and L. Viola. Quantum Markovian subsystems: Invariance, attractivity and control (2008).
- F. Ticozzi and L. Viola. Analysis and synthesis of attractive quantum Markovian dynamics (2009).

Motivation for discrete time modeling.

- L. Bouten, R. van Handel, and M. R. James. A discrete invitation to quantum filtering and feedback control (2009).

Proposed discrete time feedback control structure.

- S. Lloyd and L. Viola. Engineering quantum dynamics (2001).

Lyapunov tools for discrete time quantum channels.

- D. Burgarth and V. Giovannetti. The generalized Lyapunov theorem and its application to quantum channels (2007).

Quantum Systems evolution

Open Quantum Systems

The formalism of quantum operations is needed in the presence of coupling between subsystems, quantum measurements, interaction with surronding environment.

The most general, linear, and physically admissible dynamics of a quantum state are described by Trace Preserving (TP) Completely Positive (CP) maps.

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Kraus Maps or Operator-sum Representation

$$\mathcal{T}[\rho] = \sum_k M_k \rho M_k^\dagger$$

where ρ is a density operator and $\{M_k\}$ a family of operators satisfying $\sum_k M_k^\dagger M_k = I$.

Discrete-Time Quantum Dynamical Semigroups

Iteration of a given TPCP map results in the discrete time system

$$\rho(t+1) = \mathcal{T}\left[\rho(t)\right] = \sum_k M_k \rho(t) M_k^\dagger.$$

Discrete-Time Quantum Dynamical Semigroup

Given the initial conditions $\rho(0)$, forward composition law induces a semigroup structure:

$$\rho(t) = \mathcal{T}^t\left[\rho(0)\right].$$

Markovianity of the evolution follows from having $\{M_k\}$ independent from past states. This approach is consistent with more complicated markovianity assumptions.

Invariance and Attractivity

Consider the case

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$$

i.e. $\mathcal{S}$ is quantum subspace of the system $\mathcal{I}$.
This basis induces a block structure for matrices representing operators

$$X = \left[\begin{array}{c|c} X_S & X_P \\ \hline X_Q & X_R \end{array}\right].$$

State initialization

The system $\mathcal{I}$ with state ρ is initialized in $\mathcal{S}$ with state ρ_S if

$$\rho = \left[\begin{array}{c|c} \rho_S & 0 \\ \hline 0 & 0 \end{array}\right].$$

Let's call $\mathcal{J}_S(\mathcal{H}_I)$ the set of states in this form.

Invariance and Attractivity (cont.)

Invariance

The subsystem $\mathcal{S}$ supported on the subspace $\mathcal{H}_S$ of $\mathcal{H}_I$ is invariant if the evolution of any $\rho \in \mathcal{J}_S(\mathcal{H}_I)$ obeys

$$\rho(t) = \left[\begin{array}{c|c} \mathcal{T}_S^t[\rho_S] & 0 \\ \hline 0 & 0 \end{array}\right] \in \mathcal{J}_S(\mathcal{H}_I).$$

Attractivity

The subsystem $\mathcal{S}$ supported on the subspace $\mathcal{H}_S$ of $\mathcal{H}_I$ is attractive if for all $\rho \in \mathcal{D}(\mathcal{H}_I)$

$$\lim_{t\to\infty} \left\|\mathcal{T}^t[\rho] - \Pi_S \mathcal{T}^t[\rho]\Pi_S\right\| = 0$$

where Π_S is the projection operator over the subspace $\mathcal{H}_S$.

Characterization of GAS

Let's derive conditions on

$$\{M_k\} = \left\{ \left[\begin{array}{c|c} M_{k,S} & M_{k,P} \\ \hline M_{k,P} & M_{k,P} \end{array} \right] \right\}$$

to characterize invariance, attractivity, and global asymptotic stability (invariance + attractivity) of the TPCP map $\mathcal{T}[\cdot]$.

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This

- allows the interpretation of the dynamics of ρ through "probability fluxes"
- suggests a control synthesis technique

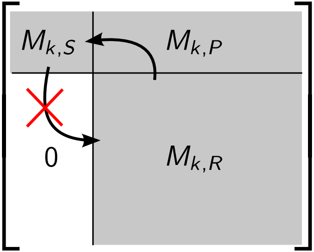
Conditions for Invariance

Invariance

Let $\mathcal{T}$ be described by the operators

$$M_k = \left[\begin{array}{c|c} M_{k,S} & M_{k,P} \\ \hline M_{k,Q} & M_{k,R} \end{array}\right].$$

Then $\mathcal{J}_S(\mathcal{H}_I)$ is invariant if and only if $M_{k,Q} = 0 \quad \forall k$.



Lyapunov / La Salle approach

Consider as a Lyapunov function the error probability

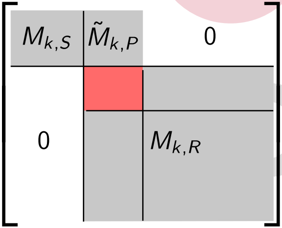
$$V(\rho) = \text{Trace}\left(\Pi_R \rho\right).$$

La Salle's Invariance Principle states that trajectories converge to the largest invariant set in

$$E = \{\rho | \Delta V(\rho) = 0\}$$

which in this case corresponds to

$$E = \left\{ \rho \mid \text{supp}(\rho_R) \subseteq \bigcap_k \ker(M_{k,P}) \right\}$$



Conditions for GAS

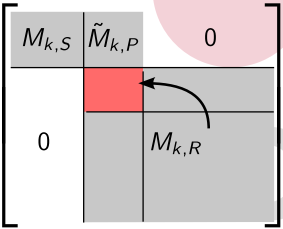
Global asymptotic stability

Consider the TPCP transformation $\mathcal{T}$ described by the operators

$$M_k = \left[\begin{array}{c|c} M_{k,S} & M_{k,P} \\ \hline 0 & M_{k,R} \end{array}\right].$$

Then $\mathcal{J}_S(\mathcal{H}_I)$ is GAS if and only if there are no invariant states with support on

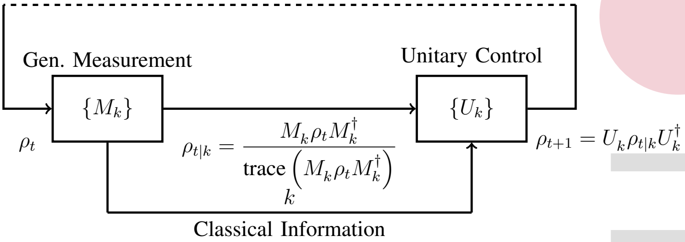
$$\bigcap_k \ker(M_{k,P}).$$



The key role is played by the blocks $M_{k,P}$, implementing "probability flows" towards $\mathcal{H}_S$.

Control structure

Consider the combination of discrete-time measurements and unitary control.



- Measurement: for example implemented by
  1. an auxiliary measurement apparatus
  2. coherent manipulation
  3. projective measurement on the auxiliary system
- Unitary control: according to the outcome k of the measurement, a certain coherent transformation is applied.

Closed loop dynamics

The measurement step results in an open system, discrete-time dynamics described by the Kraus operators $\{M_k\}$.
The control law consists in a set $\{U_k\}$ of coherent transformations.
According to the outcome k, the corresponding U_k is applied, yielding to the (average) closed loop evolution

$$\rho(t+1) = \sum_k U_k M_k \rho(t) M_k^\dagger U_k^\dagger = \sum_k N_k \rho(t) N_k^\dagger.$$

Simulating generalized measurements

Assume we are able to perform a generalized measurement described by $\{M_k\}$. Is it possible to implement any different dynamic described by $\{N_k\}$ by closing the loop?

Canonical QR decomposition

Among all possible QR decompositions $A = QR$, consider the one satisfying $r_{ij} = 0$ for all columns j and for all $i > \rho_j$, where ρ_j is the rank of the first j columns of A, and with the first nonzero element of each row of R real and positive.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} -\sqrt{2} & -\sqrt{2}/2 & -\sqrt{2}/2 & -\sqrt{2} \\ 0 & -\sqrt{2}/2 & +\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 \end{bmatrix} \times$$

$$R_{\text{canonical}} = \begin{bmatrix} +\sqrt{2} & +\sqrt{2}/2 & +\sqrt{2}/2 & +\sqrt{2} \\ 0 & +\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & +\sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \checkmark$$

R is a canonical form

$\mathcal{F}(A) = R$ is a canonical form with respect to $\mathcal{U}(n)$ and its action on $\mathbb{C}^{n\times n}$ by left multiplication. That is

- $\mathcal{F}(A) \sim_{\mathcal{G}} A$
- $\mathcal{F}(A) = \mathcal{F}(B)$ if and only if $A \sim_{\mathcal{G}} B$

Simulating generalized measurements

Which measurement can be simulated?

A measurement with associated operators $\{N_k\}_{k=1}^m$ can be simulated by a certain choice of unitary controls from a measurement $\{M_k\}_{k=1}^m$, if and only if there exists a reordering $j(k)$ of the first m integers such that

$$\mathcal{F}(N_k) = \mathcal{F}\left(M_{j(k)}\right)$$

where $\mathcal{F}$ return the canonical R factor of the argument.

Notice that the control scheme we considered allows to modify only the conditioned states, not the probability of outcomes, since $\text{trace}(M_k^\dagger M_k \rho) = \text{trace}(M_k^\dagger U_k^\dagger U_k M_k \rho)$.

Stabilization of a quantum subspace

Suppose the operators $\{M_k\}$ are given, corresponding to a measurement performed on the quantum system with outcomes in $\{k\}$. Consider a given quantum subspace $\mathcal{S}$.

- does a stabilizing control law $\{U_k\}$ exist?
- is there a constructive algorithm to design it?

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Constructive result

Consider a subspace orthogonal decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$, and a given generalized measurement $\{M_k\}$.
If asympotic stability of the subspace $\mathcal{S}$ can be achieved by a measurement-dependent unitary control $\{U_k\}$, then it can be achieved by building $\{U_k\}$ using the following iterative algorithm.

Sketch of the algorithm

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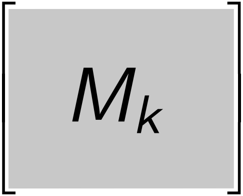
Introduction
Analysis results
Control structure
Synthesis results
Example
Conclusions

Iterative application of basic operations to construct the controls U_k:

- canonical QR factorization
- change of basis
- “destabilization” of invariant states



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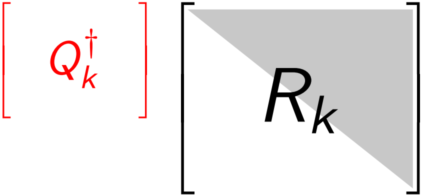

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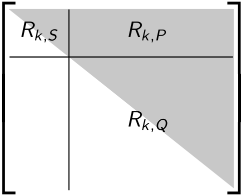

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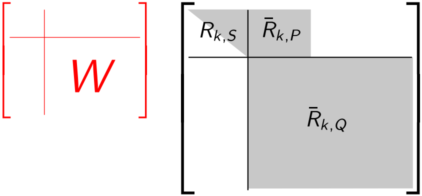

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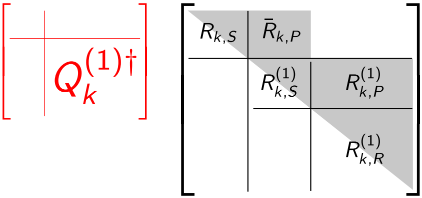

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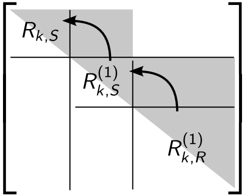
case A



Sketch of the algorithm

Iterative application of basic operations to construct the controls U_k:

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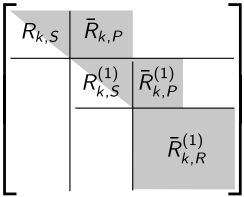
case B



Sketch of the algorithm

Iterative application of basic operations to construct the controls U_k:

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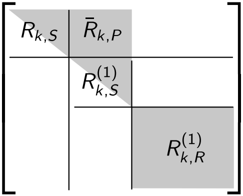
case C



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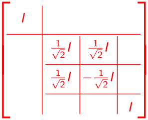

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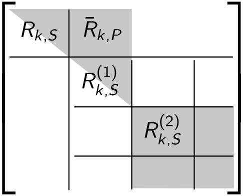

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Sketch of the algorithm

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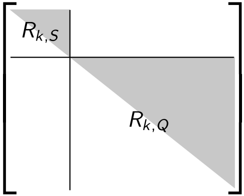
unfeasible



Toy problem

Consider the task of stabilizing the maximally entangled state of a two-qubit system:

$$\rho_d = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$

corresponding to

$$\rho_d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

in the Bell basis $\mathcal{B} = \left\{ \frac{|00\rangle+|11\rangle}{\sqrt{2}}, \frac{|00\rangle-|11\rangle}{\sqrt{2}}, \frac{|01\rangle+|10\rangle}{\sqrt{2}}, \frac{|01\rangle-|10\rangle}{\sqrt{2}} \right\}$.

Subspace stabilization problem

We have then successfully casted the problem of stabilizing the maximally entangled state into the problem of achieving asymptotic stability of a subspace $\mathcal{H}_S$.

Toy problem

Suppose that the measurement $\mathcal{T}[\rho] = \sum_{k=1}^{3} M_k \rho M_k^\dagger$ is available, with operators (represented in the computational basis):

$$M_1 = \frac{1}{\sqrt{4}}\left(\sigma_+ \otimes I\right), \quad M_2 = \frac{1}{\sqrt{4}}\left(I \otimes \sigma_+\right),$$
$$M_3 = \sqrt{I - M_1^\dagger M_1 - M_2^\dagger M_2}.$$

where $\sigma_+ = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

Physical meaning

These Kraus operators may be used to describe a discrete-time spontaneous emission process, where the event associated to $M_{1,2}$ corresponds to the decay of one qubit (with probability $\frac{1}{4}$ each), and we neglect the event of the two qubits decaying in the same time interval.

Toy problem

The algorithm then proceed by applying canonical QR decomposition to the operators M_k (expressed in the Bell basis). We obtain

$$R_1 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0.8660 & 0.2887 & 0 & 0 \\ 0 & 0.8165 & 0 & 0 \\ 0 & 0 & 0.8660 & 0 \\ 0 & 0 & 0 & 0.8660 \end{bmatrix}.$$

The problem is feasible

Indeed, no further step is needed: the stabilizing controls are $U_k = BQ_k^\dagger B^\dagger$, where B corresponds to the change of basis between the computational and Bell's basis, while Q_k are the unitary factors of the QR decomposition.

Conclusions and outlook

In this work we

- characterized invariance, attractivity and global asymptotic stability of TPCP maps
- derived a canonical form with respect to the action of left multiplication by a unitary matrix
- investigated the capabilities of feedback coherent control for quantum simulation
- derived an algorithm for the design of a stabilizing control law

Outlook

The proposed algorithm returns a stabilizing control law that does not take the structure of the quantum system into account. Further investigation is needed about the possibility of including constraints on the structure of the coherent controls.

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Thanks!

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