Pure State Stabilization with Discrete–Time Quantum Feedback

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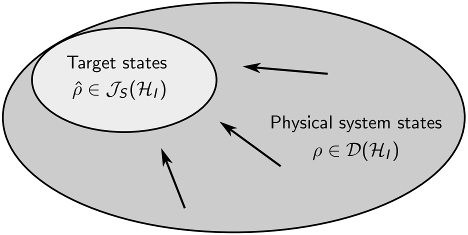
Motivation

Typical key tasks in Quantum Information are

- state preparation (entangled-state preparation, quantum register initialization,...)
- engineering of protected realization of QI.



These tasks tightly connect to stabilization control problems. This work will focus on these issues, providing a design strategy for engineering stable quantum subspaces.

Outline

Engineering stable quantum dynamics

Consider a given quantum physical system, accessible by measurements. Suppose that we can apply coherent control depending on the outcome of these measurements. How can we design a convenient control law that achieves global asymptotic stability of given target state?

In this work we

- characterize invariance, attractivity and global asymptotic stability of a pure state for discrete-time open quantum dynamics
- derive a canonical form with respect to the action of left multiplication by a unitary matrix
- derive an algorithm for stabilizing control law synthesis.

Quantum Systems evolution

Open Quantum Systems

The formalism of quantum operations is needed in the presence of coupling between subsystems, quantum measurements, interaction with surronding environment.

The most general, linear, and physically admissible dynamics of a quantum state are described by Trace Preserving (TP) Completely Positive (CP) maps.

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Kraus Maps or Operator-sum Representation

$$\mathcal{T}[\rho] = \sum_k M_k \rho M_k^\dagger$$

where ρ is a density operator and $\{M_k\}$ a family of operators satisfying $\sum_k M_k^\dagger M_k = I$.

Discrete-Time Quantum Dynamical Semigroups

Iteration of a given TPCP map results in the discrete time system

$$\rho(t+1) = \mathcal{T}[\rho(t)] = \sum_k M_k \rho(t) M_k^\dagger.$$

Discrete-Time Quantum Dynamical Semigroup

Given the initial conditions $\rho(0)$, forward composition law induces a semigroup structure:

$$\rho(t) = \mathcal{T}^t[\rho(0)].$$

Markovianity of the evolution is guaranteed from having $\{M_k\}$ independent from past states.

Invariance and Attractivity

Consider the decomposition

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R\,, \qquad \dim(\mathcal{H}_S) = 1.$$

This basis induces a block structure for operators

$$X = \left[\begin{array}{c|c} X_S & X_P \\ \hline X_Q & X_R \end{array}\right].$$

Analysis result

Necessary and sufficient conditions on

$$\{M_k\} = \left\{\left[\begin{array}{c|c} M_{k,S} & M_{k,P} \\ \hline M_{k,P} & M_{k,P} \end{array}\right]\right\}$$

for global asymptotic stability of the pure state ρ_S.

Invariance and Attractivity (cont.)

Global asymptotic stability (GAS) = invariance + attractivity.

Invariance

The pure state $\rho_S = \Pi_S$ is invariant if

$$\rho_S = \mathcal{T}[\rho_S].$$

Attractivity

The pure state $\rho_S = \Pi_S$ is attractive if for all $\rho \in \mathcal{D}(\mathcal{H}_I)$

$$\lim_{t\to\infty} \left\| \mathcal{T}^t[\rho] - \Pi_S \mathcal{T}^t[\rho] \Pi_S \right\| = 0.$$

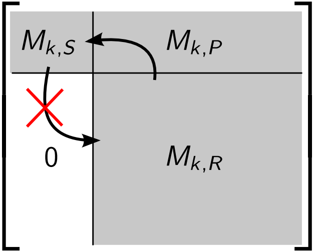
Conditions for Invariance

Invariance

Let $\mathcal{T}$ be described by the operators

$$M_k = \left[\begin{array}{c|c} M_{k,S} & M_{k,P} \\ \hline M_{k,Q} & M_{k,R} \end{array}\right].$$

Then ρ_S is invariant if and only if $M_{k,Q} = 0 \quad \forall k.$



Conditions for GAS

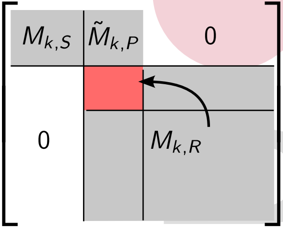
Global asymptotic stability

Consider the TPCP transformation $\mathcal{T}$ described by the operators

$$M_k = \left[\begin{array}{c|c} M_{k,S} & M_{k,P} \\ \hline 0 & M_{k,R} \end{array}\right].$$

Then ρ_S is GAS if and only if there are no invariant states with support on

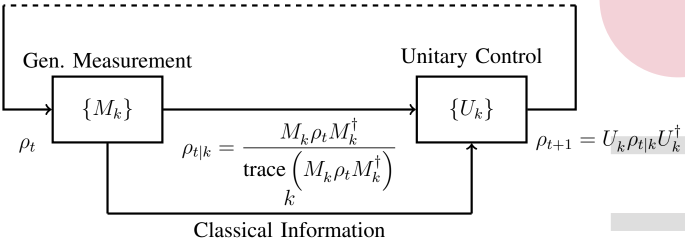
$$\bigcap_k \ker(M_{k,P}).$$



The key role is played by the blocks $M_{k,P}$, implementing "probability flows" towards $\mathcal{H}_S$.

Control structure

Consider the combination of discrete-time measurements and unitary control.



- Measurement: given generalized measurement.
- Unitary control: assume that we can implement any coherent transformation in finite time, between measurement.

Closed loop dynamics

According to the outcome k, the corresponding U_k is applied, yielding to the (average) closed loop evolution

$$\rho(t+1) = \sum_k U_k M_k \rho(t) M_k^\dagger U_k^\dagger = \sum_k N_k \rho(t) N_k^\dagger.$$

Simulating generalized measurements

Assume we are given a generalized measurement described by $\{M_k\}$, and we can apply the coherent transformation $\{U_k\}$ depending on the outcome of the measurement. Is it possible to implement any different dynamic described by $\{N_k\}$ by closing the loop?

Stabilization of a pure state

Suppose the operators $\{M_k\}$ are given, corresponding to a measurement performed on the quantum system with outcomes in $\{k\}$. Consider a given pure state ρ_S.

- does a stabilizing control law $\{U_k\}$ exist?
- is there a constructive algorithm to design it?

Stabilization of a pure state

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- is there a constructive algorithm to design it?

Constructive result

Consider a subspace orthogonal decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$, and a given generalized measurement $\{M_k\}$.
If asympotic stability of the pure state ρ_S can be achieved by a measurement-dependent unitary control $\{U_k\}$, then it can be achieved by building $\{U_k\}$ using the following iterative algorithm.

Canonical QR decomposition

QR decompositions $A = QR$ is not unique for complex valued, possibly singular, matrices. We defined a canonical QR decomposition.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} -\sqrt{2} & -\sqrt{2}/2 & -\sqrt{2}/2 & -\sqrt{2} \\ 0 & -\sqrt{2}/2 & +\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 \end{bmatrix} \times$$

$$R_{\text{canonical}} = \begin{bmatrix} +\sqrt{2} & +\sqrt{2}/2 & +\sqrt{2}/2 & +\sqrt{2} \\ 0 & +\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & +\sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \checkmark$$

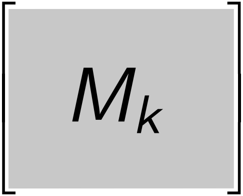
R is a canonical form

$\mathcal{F}(A) = R$ is a canonical form with respect to $\mathcal{U}(n)$ and its action on $\mathbb{C}^{n \times n}$ by left multiplication.

Sketch of the algorithm

Iterative application of basic operations to construct the controls U_k:

- canonical QR factorization
- change of basis
- "destabilization" of invariant states



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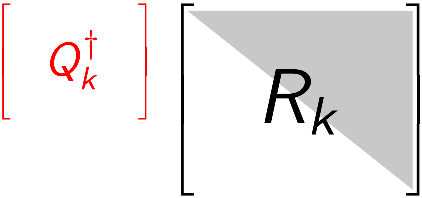

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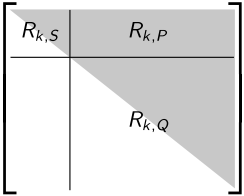

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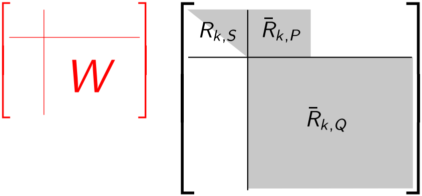

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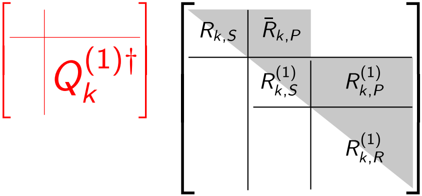

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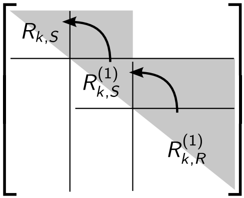
case A



Sketch of the algorithm

Iterative application of basic operations to construct the controls U_k:

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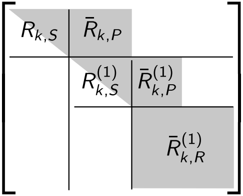
case B



Sketch of the algorithm

Iterative application of basic operations to construct the controls U_k:

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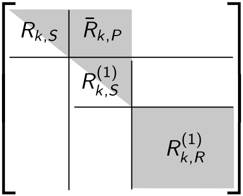
case C



Sketch of the algorithm

Iterative application of basic operations to construct the controls U_k:

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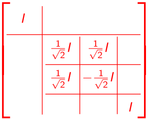

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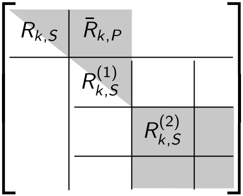

Sketch of the algorithm

Iterative application of basic operations to construct the controls U_k:

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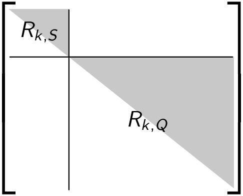
unfeasible



Sketch of the algorithm

Condition for stabilizablity of a pure state

In the case in which

$$[\rho_S, R_k] = 0 \quad \text{for all } k$$

then it is impossible to render ρ_S globally asymptotically stable with the given generalized measurement.

The proofs of both this result and the effectiveness of the algorithm in returning a stabilizing set of unitary matrices exploit the properties of the canonical QR decomposition.

Conclusions and outlook

In this work we

- characterized invariance, attractivity and global asymptotic stability of TPCP maps
- derived a canonical form with respect to the action of left multiplication by a unitary matrix
- derived an algorithm for the design of a stabilizing control law

Outlook

The proposed algorithm returns a stabilizing control law that does not take the structure of the quantum system into account. Further investigation is needed about the possibility of including constraints on the structure of the coherent controls.

Bolognani, S. and Ticozzi, F. (2009).
Engineering stable discrete-time quantum dynamics via a canonical QR decomposition.
arXiv preprint.

Bolognani, S. and Ticozzi, F. (2010).
Pure state stabilization with discrete-time quantum feedback.
In *ISCCSP 2010*, Larnaca, Cyprus.

Thanks!

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R is a canonical form

$\mathcal{F}(A) = R$ is a canonical form with respect to $\mathcal{U}(n)$ and its action on $\mathbb{C}^{n\times n}$ by left multiplication. That is

- $\mathcal{F}(A) \sim_{\mathcal{G}} A$
- $\mathcal{F}(A) = \mathcal{F}(B)$ if and only if $A \sim_{\mathcal{G}} B$

Toy problem

Consider the task of stabilizing the maximally entangled state of a two-qubit system:

$$\rho_d = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$

corresponding to

$$\rho_d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

in the Bell basis $\mathcal{B} = \left\{ \frac{|00\rangle+|11\rangle}{\sqrt{2}}, \frac{|00\rangle-|11\rangle}{\sqrt{2}}, \frac{|01\rangle+|10\rangle}{\sqrt{2}}, \frac{|01\rangle-|10\rangle}{\sqrt{2}} \right\}.$

Toy problem

Suppose that the measurement $\mathcal{T}[\rho] = \sum_{k=1}^{3} M_k \rho M_k^\dagger$ is available, with operators (represented in the computational basis):

$$M_1 = \frac{1}{\sqrt{4}} (\sigma_+ \otimes I), \quad M_2 = \frac{1}{\sqrt{4}} (I \otimes \sigma_+),$$
$$M_3 = \sqrt{I - M_1^\dagger M_1 - M_2^\dagger M_2}.$$

where $\sigma_+ = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

Physical meaning

These Kraus operators may be used to describe a discrete-time spontaneous emission process, where the event associated to $M_{1,2}$ corresponds to the decay of one qubit (with probability $\frac{1}{4}$ each), and we neglect the event of the two qubits decaying in the same time interval.

Toy problem

The algorithm then proceed by applying canonical QR decomposition to the operators M_k (expressed in the Bell basis). We obtain

$$R_1 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0.8660 & 0.2887 & 0 & 0 \\ 0 & 0.8165 & 0 & 0 \\ 0 & 0 & 0.8660 & 0 \\ 0 & 0 & 0 & 0.8660 \end{bmatrix}.$$

The problem is feasible

Indeed, no further step is needed: the stabilizing controls are $U_k = BQ_k^\dagger B^\dagger$, where B corresponds to the change of basis between the computational and Bell's basis, while Q_k are the unitary factors of the QR decomposition.