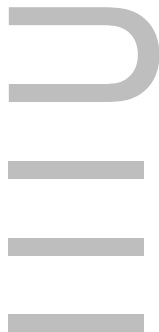


On a canonical QR decomposition and feedback control of discrete-time quantum dynamics

Francesco Ticozzi, *Saverio Bolognani*

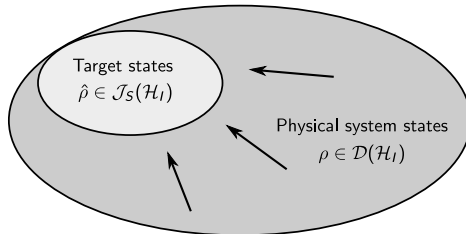
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Typical key tasks in Quantum Information are

- **state preparation** (entangled-state preparation, quantum register initialization,...)
- **engineering of protected realization of QI.**



These tasks tightly connect to stabilization control problems. This work will focus on these issues, providing a design strategy for engineering stable quantum pure states.

Engineering stable quantum dynamics

Consider a given quantum physical system, accessible by measurements. Suppose that we can apply coherent control depending on the outcome of these measurements. How can we design a convenient control law that achieves global asymptotic stability of given target state?

In this work we

- characterize **invariance**, **attractivity** and **global asymptotic stability** of a pure state for discrete-time open quantum dynamics
- derive a **canonical form** with respect to the action of left multiplication by a unitary matrix
- derive an **algorithm** for stabilizing control law synthesis.

Open Quantum Systems

The formalism of **quantum operations** is needed in the presence of coupling between subsystems, quantum measurements, interaction with surrounding environment.

The most general, linear, and physically admissible dynamics of a quantum state are described by **Trace Preserving (TP) Completely Positive (CP) maps**.

Open Quantum Systems

The formalism of **quantum operations** is needed in the presence of coupling between subsystems, quantum measurements, interaction with surrounding environment.

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Kraus Maps or Operator-sum Representation

$$\mathcal{T}[\rho] = \sum_k M_k \rho M_k^\dagger$$

where ρ is a density operator and $\{M_k\}$ a family of operators satisfying $\sum_k M_k^\dagger M_k = I$.

Iteration of a given TPCP map results in the discrete time system

$$\rho(t+1) = \mathcal{T}[\rho(t)] = \sum_k M_k \rho(t) M_k^\dagger.$$

Discrete-Time Quantum Dynamical Semigroup

Given the initial conditions $\rho(0)$, forward composition law induces a **semigroup structure**:

$$\rho(t) = \mathcal{T}^t[\rho(0)].$$

Markovianity of the evolution is guaranteed from having $\{M_k\}$ independent from past states.

Invariance and Attractivity

6

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Introduction

 Analysis
 results

 Control
 structure

 Synthesis
 results

Examples

Conclusions

Consider the decomposition

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R, \quad \dim(\mathcal{H}_S) = 1.$$

This basis induces a **block structure** for operators

$$X = \left[\begin{array}{c|c} X_S & X_P \\ \hline X_Q & X_R \end{array} \right].$$

Analysis result

Necessary and sufficient conditions on

$$\{M_k\} = \left\{ \left[\begin{array}{c|c} M_{k,S} & M_{k,P} \\ \hline M_{k,P} & M_{k,P} \end{array} \right] \right\}$$

for **global asymptotic stability** of the pure state ρ_S .

Global asymptotic stability (GAS) = invariance + attractivity.

Invariance

The pure state $\rho_S = \Pi_S$ is **invariant** if

$$\rho_S = \mathcal{T}[\rho_S].$$

Attractivity

The pure state $\rho_S = \Pi_S$ is **attractive** if for all $\rho \in \mathcal{D}(\mathcal{H}_I)$

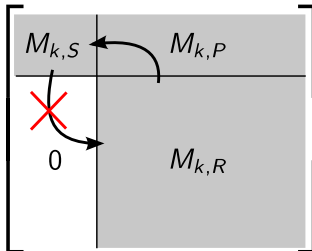
$$\lim_{t \rightarrow \infty} \|\mathcal{T}^t[\rho] - \Pi_S \mathcal{T}^t[\rho] \Pi_S\| = 0.$$

Invariance

Let \mathcal{T} be described by the operators

$$M_k = \left[\begin{array}{c|c} M_{k,S} & M_{k,P} \\ \hline M_{k,Q} & M_{k,R} \end{array} \right].$$

Then ρ_S is invariant if and only if $M_{k,Q} = 0 \quad \forall k$.



Global asymptotic stability

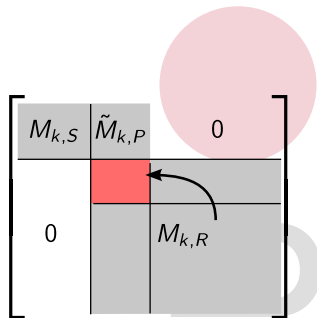
Consider the TPCP transformation \mathcal{T} described by the operators

$$M_k = \left[\begin{array}{c|c} M_{k,S} & M_{k,P} \\ \hline 0 & M_{k,R} \end{array} \right].$$

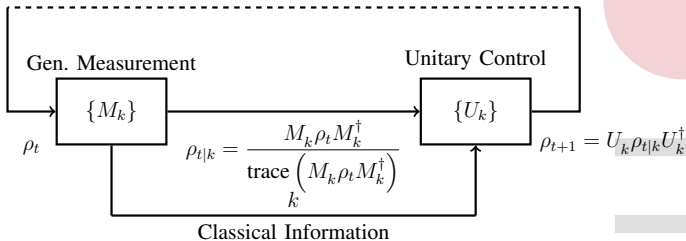
Then ρ_S is GAS if and only if there are no invariant states with support on

$$\bigcap_k \ker(M_{k,P}).$$

The key role is played by the blocks $M_{k,P}$, implementing “probability flows” towards \mathcal{H}_S .



Consider the combination of **discrete-time measurements** and **unitary control**.



- **Measurement:** given generalized measurement.
- **Unitary control:** assume that we can implement any coherent transformation in finite time, between measurement.

According to the outcome k , the corresponding U_k is applied, yielding to the (average) **closed loop evolution**

$$\rho(t+1) = \sum_k U_k M_k \rho(t) M_k^\dagger U_k^\dagger = \sum_k N_k \rho(t) N_k^\dagger.$$

Suppose the **operators** $\{M_k\}$ are given, corresponding to a measurement performed on the quantum system with outcomes in $\{k\}$. Consider a given **pure state** ρ_S .

- does a **stabilizing control law** $\{U_k\}$ exist?
- is there a **constructive algorithm** to design it?

Canonical QR decomposition

12

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Introduction

Analysis
results

Control
structure

Synthesis
results

Examples

Conclusions

QR decompositions $A = QR$ is not unique for complex valued, possibly singular, matrices. We defined a **canonical QR decomposition**.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} -\sqrt{2} & -\sqrt{2}/2 & -\sqrt{2}/2 & -\sqrt{2} \\ 0 & -\sqrt{2}/2 & +\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 \end{bmatrix} \quad \times$$

$$R_{\text{canonical}} = \begin{bmatrix} +\sqrt{2} & +\sqrt{2}/2 & +\sqrt{2}/2 & +\sqrt{2} \\ 0 & +\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & +\sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \checkmark$$

R is a canonical form

$\mathcal{F}(A) = R$ is a canonical form with respect to $\mathcal{U}(n)$ and its action on $\mathbb{C}^{n \times n}$ by left multiplication.

Condition for stabilizability of a pure state

In the case in which

$$[\rho_S, R_k] = 0 \quad \text{for all } k$$

then it is **impossible** to render ρ_S globally asymptotically stable with the given generalized measurement.

Condition for stabilizability of a pure state

In the case in which

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then it is **impossible** to render ρ_S globally asymptotically stable with the given generalized measurement.

Constructive result

If asymptotic stability of the pure state ρ_S can be achieved by a measurement-dependent unitary control $\{U_k\}$, then it can be achieved by building $\{U_k\}$ using the following iterative algorithm.

Iterative application of basic operations to construct the controls U_k :

- canonical QR factorization
- change of basis
- “destabilization” of invariant states

$$M_k$$



Iterative application of basic operations to construct the controls U_k :

- canonical QR factorization
- change of basis
- “destabilization” of invariant states

$$\left[\begin{array}{c} Q_k^\dagger \end{array} \right] \left[\begin{array}{c} R_k \end{array} \right]$$



Sketch of the algorithm

14

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Introduction

Analysis
results

Control
structure

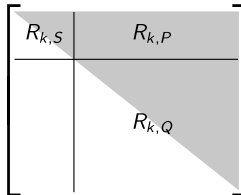
Synthesis
results

Examples

Conclusions

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Sketch of the algorithm

14

MTNS 2010
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Hungary

Introduction

Analysis
results

Control
structure

Synthesis
results

Examples

Conclusions

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$$\left[\begin{array}{c|c} & \\ \hline & W \end{array} \right] \quad \left[\begin{array}{c|c} R_{k,S} & \bar{R}_{k,P} \\ \hline & \bar{R}_{k,Q} \end{array} \right]$$



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14

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Introduction

Analysis
results

Control
structure

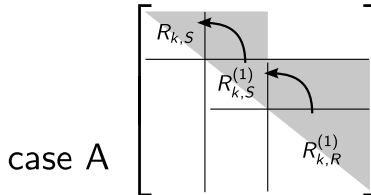
Synthesis
results

Examples

Conclusions

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Sketch of the algorithm

14

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Hungary

Introduction

Analysis
results

Control
structure

Synthesis
results

Examples

Conclusions

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case B

$$\begin{bmatrix} R_{k,S} & \bar{R}_{k,P} \\ R_{k,S}^{(1)} & \bar{R}_{k,P}^{(1)} \\ & \bar{R}_{k,R}^{(1)} \end{bmatrix}$$



Sketch of the algorithm

14

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Hungary

Introduction

Analysis
results

Control
structure

Synthesis
results

Examples

Conclusions

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case C

$$\begin{bmatrix} R_{k,S} & \bar{R}_{k,P} & & \\ & R_{k,S}^{(1)} & & \\ & & & R_{k,R}^{(1)} \\ & & & & & & & \end{bmatrix}$$

Sketch of the algorithm

14

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Hungary

Introduction

Analysis
results

Control
structure

Synthesis
results

Examples

Conclusions

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$$\left[\begin{array}{c|cc} I & & \\ \hline & \frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \\ & \frac{1}{\sqrt{2}}I & -\frac{1}{\sqrt{2}}I \\ & & & I \end{array} \right]$$

case C

$$\left[\begin{array}{c|cc} R_{k,S} & \bar{R}_{k,P} & \\ \hline & R_{k,S}^{(1)} & \\ & & R_{k,S}^{(2)} \\ & & & \end{array} \right]$$



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14

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Introduction

Analysis
results

Control
structure

Synthesis
results

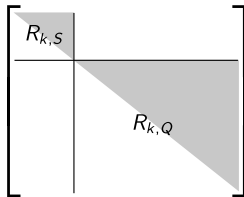
Examples

Conclusions

Iterative application of basic operations to construct the controls U_k :

- canonical QR factorization
- change of basis
- “destabilization” of invariant states

unfeasible



Example: Projective measurement

15

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Introduction

Analysis
results

Control
structure

Synthesis
results

Examples

Conclusions

Suppose that the M_k 's are rank one projectors (representing a non-degenerate von Neumann's measurement).

Projective measurement case

In the case in which a projective measurement is available, any pure state can be stabilized.

Example: Projective measurement

15

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Introduction

Analysis
results

Control
structure

Synthesis
results

Examples

Conclusions

Suppose that the M_k 's are rank one projectors (representing a non-degenerate von Neumann's measurement).

Projective measurement case

In the case in which a projective measurement is available, any pure state can be stabilized.

It is easy to check that all R_k 's have only the first row different from zero, and these rows form a basis. Therefore $R_{P,k}$ cannot be zero for all k 's.

Physical meaning

Physically, at any measurement step we obtain a known pure state, so we can drive it back to the target state ρ_S .

Example: a non stabilizable case

16

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Introduction

 Analysis
 results

 Control
 structure

 Synthesis
 results

Examples

Conclusions

Consider a system of dimension 2, and suppose that we want to stabilize

$$\rho_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

when these two measurements are available:

$$M_1 = \begin{bmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & 1 \end{bmatrix}.$$



Example: a non stabilizable case

16

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Introduction

 Analysis
 results

 Control
 structure

 Synthesis
 results

Examples

Conclusions

Consider a system of dimension 2, and suppose that we want to stabilize

$$\rho_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

when these two measurements are available:

$$M_1 = \begin{bmatrix} \sqrt{\rho} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} \sqrt{1-\rho} & 0 \\ 0 & 1 \end{bmatrix}.$$

Both M_1 and M_2 are already in their canonical form.

As $[\rho_0, M_1] = [\rho_0, M_2] = 0$, **the problem is not feasible.**

Example: Entanglement generation

17

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Introduction

 Analysis
 results

 Control
 structure

 Synthesis
 results

Examples

Conclusions

Consider the task of stabilizing the maximally entangled state of a two-qubit system:

$$\rho_d = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$

corresponding to

$$\rho_d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

in the Bell basis $\mathcal{B} = \left\{ \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right\}$.

Example: Entanglement generation

18

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Introduction

Analysis
results

Control
structure

Synthesis
results

Examples

Conclusions

Suppose that the measurement $\mathcal{T}[\rho] = \sum_{k=1}^3 M_k \rho M_k^\dagger$ is available, with operators (represented in the computational basis):

$$M_1 = \frac{1}{\sqrt{4}} (\sigma_+ \otimes I), \quad M_2 = \frac{1}{\sqrt{4}} (I \otimes \sigma_+),$$

$$M_3 = \sqrt{I - M_1^\dagger M_1 - M_2^\dagger M_2}.$$

where $\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Physical meaning

These Kraus operators may be used to describe a **discrete-time spontaneous emission process**, where the event associated to $M_{1,2}$ corresponds to the decay of one qubit (with probability $\frac{1}{4}$ each), and we neglect the event of the two qubits decaying in the same time interval.

Example: Entanglement generation

19

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Hungary

Introduction

Analysis
results

Control
structure

Synthesis
results

Examples

Conclusions

The algorithm then proceed by applying canonical QR decomposition to the operators M_k (expressed in the Bell basis). We obtain

$$R_1 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0.8660 & 0.2887 & 0 & 0 \\ 0 & 0.8165 & 0 & 0 \\ 0 & 0 & 0.8660 & 0 \\ 0 & 0 & 0 & 0.8660 \end{bmatrix}.$$

The problem is feasible

Indeed, no further step is needed: the stabilizing controls are $U_k = BQ_k^\dagger B^\dagger$, where B corresponds to the change of basis between the computational and Bell's basis, while Q_k are the unitary factors of the QR decomposition.

20

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Hungary

Introduction

Analysis
results

Control
structure

Synthesis
results

Examples

Conclusions

A potential limitation to the implementation of the resulting feedback strategy is the need for

- strong control capabilities
- perfect detection.



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- strong control capabilities
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Perturbed Kraus map

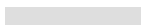
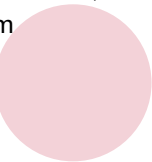
To account for small detection errors, imperfect knowledge of the model and other non-idealities, consider a **perturbed Kraus map**

$$\tilde{\mathcal{E}}[\cdot] = (1 - \epsilon)\mathcal{E}[\cdot] + \epsilon\mathcal{E}'[\cdot].$$

Is the new perturbed equilibrium $\rho^{(\epsilon)}$ **unique**, **attractive**, and a **continuous function of ϵ** ?

By expressing all density operators as d^2 -dimensional vectors, the TPCP maps that we are considering take the form

$$\bar{\rho}(t+1) = \begin{bmatrix} 1/\sqrt{d} \\ \rho_v(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ C & D \end{bmatrix} \begin{bmatrix} 1/\sqrt{d} \\ \rho_v(t) \end{bmatrix}$$



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and the perturbed map becomes

$$\bar{\rho}(t+1) = \left((1-\epsilon) \begin{bmatrix} 1 & 0 \\ C & D \end{bmatrix} + \epsilon \begin{bmatrix} 1 & 0 \\ C' & D' \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{d} \\ \rho_v(t) \end{bmatrix}.$$

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$$\bar{\rho}(t+1) = \begin{bmatrix} 1/\sqrt{d} \\ \rho_v(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ C & D \end{bmatrix} \begin{bmatrix} 1/\sqrt{d} \\ \rho_v(t) \end{bmatrix}$$

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Perturbed equilibrium

For a sufficiently small ϵ , the perturbed equilibrium state is

$$\rho^{(\epsilon)} = \frac{1}{\sqrt{d}} \left[(I - (1-\epsilon)D - \epsilon D')^{-1} ((1-\epsilon)C + \epsilon C') \right]$$

(a continuous function of ϵ).

In this work we

- characterized **invariance**, **attractivity** and **global asymptotic stability** of TPCP maps
- derived a **canonical form** with respect to the action of left multiplication by a unitary matrix
- derived an **algorithm** for the design of a stabilizing control law

Outlook

The proposed algorithm returns a stabilizing control law that does not take the structure of the quantum system into account. Further investigation is needed about the possibility of **including constraints on the structure of the coherent controls**.



Bolognani, S. and Ticozzi, F. (2009).

Engineering stable discrete-time quantum dynamics via a canonical QR decomposition.
arXiv preprint.



Ticozzi, F. and Bolognani, S. (2010).

On a canonical QR decomposition and feedback control of discrete-time quantum dynamics.
In *MTNS 2010*, Budapest, Hungary.

Thanks!

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R is a canonical form

$\mathcal{F}(A) = R$ is a canonical form with respect to $\mathcal{U}(n)$ and its action on $\mathbb{C}^{n \times n}$ by left multiplication. That is

- $\mathcal{F}(A) \sim_{\mathcal{G}} A$
- $\mathcal{F}(A) = \mathcal{F}(B)$ if and only if $A \sim_{\mathcal{G}} B$