Optimal estimation in networked control systems subject to random delay and packet loss

Luca Schenato

Abstract— **In this paper we study optimal estimation design for sampled linear systems where the sensors measurements are transmitted to the estimator site via a generic digital communication network. Sensor measurements are subject to random delay or might even be completely lost. We present two time-invariant estimator architectures and, surprisingly, we show that stability does not depend on packet delay but only on the packet loss probability. Finally, algorithms to compute critical packet loss probability and estimators performance in terms of their error covariance are given and applied to some numerical examples.**

Index Terms— **Optimal estimation, packet drop, random delay, smart-sensor, stability, Kalman filtering**

I. Introduction

Recent technological advances in MEMS, DSP capabilities, computing, and communication technology are revolutionizing our ability to build massively distributed networked control systems (NCS) [1]. These networks can offer access to an unprecedented quality and quantity of information which can revolutionize our ability in controlling of the environment, such as fine grane building environmental control, vehicular networks and traffic control, surveillance and coordinated robotics. However, they also pose challenging problems arising from the fact that sensors, actuators and controllers are not physically co-located and need to exchange information via a digital communication network. In particular, measurement and control packets are subject to random delay and loss. These problems are particularly evident in wireless communication networks which are rapidly replacing wired communication infrastructures in many engineering areas. This is happening because wireless systems are easier and cheaper to deploy and avoid cumbersome cabling and device positioning. Besides, new technologies like wireless sensor networks (WSNs), which are large networks of spatially distributed electronic devices – called nodes – capable of sensing, computation and wireless communication, will enable the development of applications previously unfeasible [2]. For example, WSNs have been already employed for animal habitat monitoring in inhospitable regions [3] and forest microclimate studies [4]. These are typical example of large scale fine grain sensor data-collection applications where information is collected and then analyzed off-line.

However, WSN are going to be employed also for real-time applications. For example consider a WSN deployed in a forest whose nodes are equipped with temperature and humidity sensors. The same network could be employed for monitoring climate variations (data-collection application) or for wild-fire detection and tracking (real-time application) [5]. Despite the fact that these two applications adopt the same infrastructure, they obviously have different packet delay and packet loss requirements. In fact, in data-collection applications it is only necessary to extract all data regardless of packet delay, while in real-time control applications both delay and packet loss are relevant. Unfortunately, the design of communication protocols for communication networks has to deal with unavoidable tradeoffs between packet loss and packet delay. In fact, communication protocols that aim at reducing packet loss require retransmission of lost packets and packet delivery acknowledgment, which increase traffic and consequently delay. Viceversa, reducing time delay requires dropping of packets to mitigate traffic and packet collisions. Therefore, it is not trivial to design communication protocols for control systems since both delay and packet loss negatively impact estimation and closed loop performance of controlled systems. Currently, communications protocols and networked control systems are designed separately. In particular, protocols are design based on conservative heuristics which specify what the maximum time delay and maximum packet loss should be, but with no clear understanding of their impact on the overall application performance. On the application side, control systems are not specifically designed to exploit information about packet loss and delay statistics of the communication protocols they will run on. From these observations few questions arise. For example, how should we design estimators for networked systems that take into account simultaneous random delay and packet loss? How can we estimate their performance? When is the closed loop system stable? How can we choose between a communication protocol with a large packet delay and a small packet loss and a protocol with a small packet delay and a large packet loss, in terms of best performance of a specific real-time application? These are the questions that motivate this work.

II. Previous Work and Contribution

Classical control has mainly focused on systems with constant delay [6] or with small delay perturbation known as jitter [7]. Recently several groups have looked at networked control systems with large random delay or packet loss. The survey paper [8] nicely reviews several results in this area. These results can be divided into two main groups: the first group focuses on variable delay but no packet drop, while the second group focuses on packet loss but no delay.

Within the first group, some authors derived stability conditions in terms of LMIs for closed loop continuous time linear systems with stochastic sampling time [9][10], and Nesic at al. [11] obtained Lyapunov-like stability conditions for continuous time nonlinear systems with unknown but bounded sampling time. These works simply determine stability for a given closed loop system, and there is no controller synthesis specifically designed to take into account

This research is supported in part by the European Community Research Information Society Technologies under Grant No. RECSYS IST-2001-32515, and by the Italian Ministry of Education, University and Research (MIUR).

L. Schenato is with Faculty of the Department of Information Engineering, University of Padova, Via Gradenigo, 6/b, Padova, Italy schenato@dei.unipd.it

delay. With this respect, Yue et al. [12] proposed an LMI approach for the design of stabilizing controllers for bounded delay, while Nilsson at al. [13] extended LQG optimal control design to sampled linear systems subject to stochastic measurement and control packet delay, and showed how the optimal controller gains are time-delay dependent. The previous results rely on the major assumption that there is no packet loss or there are at most m consecutive packet drops.

In the second group of results, there has been a considerably effort to apply optimal control and estimation to discrete time systems where measurements and control packets can be dropped with some probability, but have otherwise no delay. This framework is equivalent of saying that all packets have either no delay or infinite delay. For example, in [14][15][16] the authors proposed compensation techniques for i.i.d Bernoulli packet-drop communication networks and derived stability conditions for closed loop discrete time system. Elia et al. [17] proposed a stochastic perturbation approach for general MIMO LTI discrete time systems and showed that the optimal controller design is equivalent to solving a convex LMI optimization problem. Sinopoli at al. [18] looked specifically at minimum variance estimation design for packet-drop networks and showed that the optimal estimator is necessarily time-varying, and these results have been extended to LQG controller design in [19] and [20].

The previous two groups of results suffer from some limitations. In fact, even with retransmission mechanisms present in all current digital communication networks, and in particular in the wireless ones, it is impossible to guarantee that all packets are correctly delivered to the destination. On the hand, in wireless sensor networks which implement multi-hop communication, delay is not negligible and is subject to large variations. Therefore, none of the modelings considered so far, i.e. random delay but no packet loss and packet loss but no delay, fully represent control systems interconnected by digital communication networks. Very little work has been done to take into account simultaneous packet drop and packet delay, leading to somewhat conservative results as they are based on worst-case scenarios [21] [22].

In this paper we propose a probabilistic framework to analyze estimation where observation packets are subject to arbitrary random delay and packet loss. This allows packets to arrive in burst or even out of order at the receiver side, as long as the measurements are time-stamped at the sensor side. We present two alternative estimator architectures which constrain the estimator gains to be constant rather than stochastic as in the true optimal estimator [25]. In particular we show how to compute the optimal constant gains if the packet arrival statistic is stationary and known. We derive necessary and sufficient condition for stability of the estimator. Surprisingly we show that stability does not depend on packet delay but only on a critical packet loss probability which is a function of the unstable eigenvalues of the system to be estimated. We also provide quantitative measures for the expected error covariance of such estimators which turns out to be the solution of modified algebraic Riccati equations and Lyapunov equations. These measures can be used to compare different communication protocols for real-time control applications. Very importantly, these results do not depend on the specific implementation of the digital communication network (fieldbuses, Bluetooth, ZigBee, Wi-Fi, etc ..) as long as the packet arrival statistics are known, i.i.d and stationary.

In the interest of space, proofs of theorems in the next sections are omitted and are available in a longer version of this paper in [23].

III. Problem formulation

Consider the following discrete time linear stochastic plant:

$$x_{t+1} = Ax_t + w_t \quad (1)$$
$$y_t = Cx_t + v_t, \quad (2)$$

where $t \in \mathbb{N} = \{0, 1, 2, \ldots\}$, $x, w \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $y \in \mathbb{R}^m$, $C \in \mathbb{R}^{m\times n}$, (x_0, w_t, v_t) are Gaussian, uncorrelated, white, with mean $(\bar{x}_0, 0, 0)$ and covariance (P_0, Q, R) respectively. We also assume that the pair (A, C) is observable, $(A, Q^{1/2})$ is reachable, and $R > 0$[1]. Measurements are time-stamped, encapsulated into

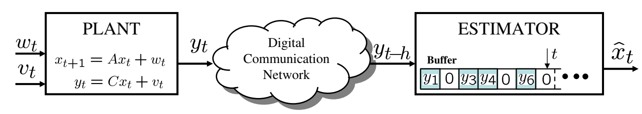

Fig. 1. Networked systems modeling. Sampled observations at the plant site are transmitted to the estimator site via a digital communication network. Due to retransmission and packet loss, observation packets arrive at the estimator site with possibly random delay.

packets, and then transmitted through a digital communication network (DCN), whose goal is to deliver packets from a source to a destination (see Fig. 1). Time-stamping of measurements is necessary to reorder packets at the receiver side as they can arrive out of order. Modern DCNs are in general very complex and can greatly differ in their architecture and implementation depending on the medium used (wired, wireless, hybrid), and on the applications they are meant to serve (real-time monitoring, data extraction, media-related, etc ..). In our work we model a DCN as a module between the plant and the estimator which delivers observation measurements to the estimator with possibly random delays. This model allows also for packets with infinite delay which corresponds to a packet loss. We assume that all observation packets correctly delivered to the estimator site are stored in an infinite buffer, as shown in Fig. 1. The arrival process is modeled via the random variable γ_k^t defined as follows:

$$\gamma_k^t = \begin{cases} 1 & \textit{if } y_k \text{ arrived before or at time } t, \ t \geq k \\ 0 & \textit{otherwise} \end{cases} \quad (3)$$

From this definition it follows that $(\gamma_k^t = 1) \Rightarrow (\gamma_k^{t+h} = 1, \forall h \in \mathbb{N})$, which simply states that if packet y_k is present in the receiver buffer at time t, then it will be present for all future times. We also define the packet delay $\tau_k \in \{\mathbb{N}, \infty\}$ for observation y_k as follows:

$$\tau_k = \begin{cases} \infty & \textit{if } \gamma_k^t = 0, \forall t \geq k \\ t_k - k & \textit{otherwise, } t_k \stackrel{\Delta}{=} \min\{t \mid \gamma_k^t = 1\} \end{cases} \quad (4)$$

[1]These assumptions can be relaxed to (A, C) detectable, $(A, Q^{1/2})$ stabilizable, and $R \geq 0$, however the proofs of the following theorems would be more convoluted, therefore we decided to adopt the stronger hypotheses.

where t_k is the arrival time of observation y_k at the estimator site . Since the packet delay can be random, observation measurements can arrive out of order at the estimator site (see Fig. 2, $t = 5$). Also it is possible that between two consecutive sampling periods no packet (see Fig. 2, $t = 4$) or multiple packets (see Fig. 2, $t = 6$) are delivered. In our work we do not consider quantization distortion due to data encoding/decoding since we assume that observation noise is much larger then quantization noise, as it is the case in most DCNs where each packet allocates hundreds of bits for measurement data[2]. Also we do not consider channel noise since we assume that if any bit error incurred during packet transmission is detected at the receiver, then the packet is dropped. If observation y_k is not yet arrived at the estimator

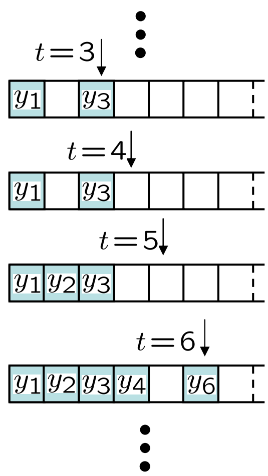

Fig. 2. Packet arrival sequence and buffering at the estimator location. Shaded squares correspond to observation packets that have been successfully received by the estimator. Cursor indicates current time.

at time t, we assume that a zero is stored in the k-slot of the buffer, as shown in Fig. 2[3]. More formally, the value stored in the k-slot of the estimator buffer at time t can be written as follows:

$$\tilde{y}_k^t = \gamma_k^t y_k = \gamma_k^t C x_k + \gamma_k^t v_k \tag{5}$$

Our goal to compute the optimal mean square estimator $\hat{x}_{t|t}$ which is given by:

$$\hat{x}_{t|t} \triangleq \mathbb{E}[x_t \mid \tilde{\mathbf{y}}_t, \boldsymbol{\gamma}_t, \bar{x}_0, P_0] \tag{6}$$

where $\tilde{\mathbf{y}}_t = (\tilde{y}_1^t, \tilde{y}_2^t, \ldots, \tilde{y}_t^t)$ and $\boldsymbol{\gamma}_t = (\gamma_1^t, \gamma_2^t, \ldots, \gamma_t^t)$. It is important to remark that the estimator above has the information weather a packet has been delivered or not, and it is not equivalent to computing $\hat{x}_{t|t} \neq \check{x}_{t|t} \triangleq \mathbb{E}[x_t \mid \tilde{\mathbf{y}}_t, \bar{x}_0, P_0]$. The latter estimator would in fact consider the zero entries of the buffer as true measurements and not as dummy variables, thus providing a lower performance. It is also useful to design the estimator error and error covariance as follows:

$$e_{t|t} \triangleq x_t - \hat{x}_{t|t} \tag{7}$$

$$P_{t|t} \triangleq \mathbb{E}[\, e_{t|t} e_{t|t}^T \mid \tilde{\mathbf{y}}_t, \boldsymbol{\gamma}_t, \bar{x}_0, P_0] \tag{8}$$

The estimate $\hat{x}_{t|t}$ is optimal in the sense that it minimizes the error covariance, i.e. given any estimator $\tilde{x}_{t|t} = f(\tilde{\mathbf{y}}_t, \boldsymbol{\gamma}_t)$, where f is a measurable function, we always have

$$\mathbb{E}[(x_t - \tilde{x}_{t|t})(x_t - \tilde{x}_{t|t})^T \mid \tilde{\mathbf{y}}_t, \boldsymbol{\gamma}_t, \bar{x}_0, P_0] \geq P_{t|t}.$$

Another property of the mean square optimal estimator is that $\hat{x}_{t|t}$ and its error $e_{t|t} \triangleq x_t - \hat{x}_{t|t}$ are uncorrelated, i.e. $\mathbb{E}[e_{t|t}\, \hat{x}_{t|t}^T] = 0$. This is a fundamental property since it gives rise to the separation principle for the LQG optimal control, which is of the most widely used tool in control system design [24] [20].

IV. Optimal filtering with constant gains

In this section we will study minimum error covariance filters with constant gains under stationary i.i.d arrival processes.

Assumption: The packet arrival process at the estimator site is stationary and i.i.d. with the following probability function:

$$\mathbb{P}[\tau_t \leq h] = \lambda_h \tag{9}$$

where $t \geq 0$, and $0 \leq \lambda_h \leq 1$ is a non-decreasing in $h = 0, 1, 2, \ldots$, and τ_t was defined in Equation (4).

Equation (9) corresponds to the probability that a packet sampled h time steps ago has arrived at the estimator. Obviously, λ_h must be non-increasing since $\lambda_h = \mathbb{P}[\tau_t \leq h-1] + \mathbb{P}[\tau_t = h] = \lambda_{h-1} + \mathbb{P}[\tau_t = h]$.

Also, we define the packet loss probability as follows:

$$\lambda_{loss} \triangleq 1 - \sup\{\lambda_h | h \geq 0\} \tag{10}$$

The arrival process defined by Equation (9) can be also be defined with respect to the probability density of packet delay. In fact, by definition we have $\mathbb{P}[\tau_k = 0] = \lambda_0$, $\mathbb{P}[\tau_k = h] = \lambda_h - \lambda_{h-1}$ for $h \geq 1$, and $P[\tau_k = \infty] = \lambda_{loss}$.

Finally, we define the maximum delay of arrived packets as follows:

$$\tau_{max} \triangleq \begin{cases} \min\{H | \lambda_H = \lambda_{H+1}\} & \textit{if } \exists H \text{ s.t. } \lambda_h = \lambda_H, \forall h \geq H \\ \infty & \textit{otherwise} \end{cases} \tag{11}$$

Fig. 3 shows some typical scenarios that can be modeled. Scenario (A) corresponds to a deterministic process where all packets are successfully delivered to the estimator with a constant delay. This scenario is typical of wired systems. Scenario (B) models a DCN that guarantees delivery of all packets within a finite time window τ_{max}, but the delay is not deterministic. This is a common scenario in drive-by-wire systems. Scenario (C) represents a DCN which drops packets that are older than τ_{max} and consequently a fraction $\lambda_{loss} > 0$ of observations is lost. This scenario is often encountered in wireless sensor networks. Scenario (D) corresponds to a DCN with no packet loss but with unbounded random packet delay. One example of such a scenario is a DCN that continues to retransmit a packet till it not delivered and the transmission channel is such that the packet is not delivered

[2] For example, ATM communication protocols adopts packets with 384-bit data field, Ethernet IEEE 802.3 packets allows for at least 368 bits for data payload, Bluetooth for 499 bits [8] and IEEE 802.15.4 for up to 1000 bits. This assumption might not hold for multimedia signal like audio and video signals, which however are not in the scope of this work.

[3] In practice, any arbitrary value can be stored in the buffer slots corresponding to the packets which have not arrived, since as it will be shown later, the optimal estimator does not use those values as they do not convey any information about the state x_t. Our choice of storing a zero simply reduces some mathematical burden.

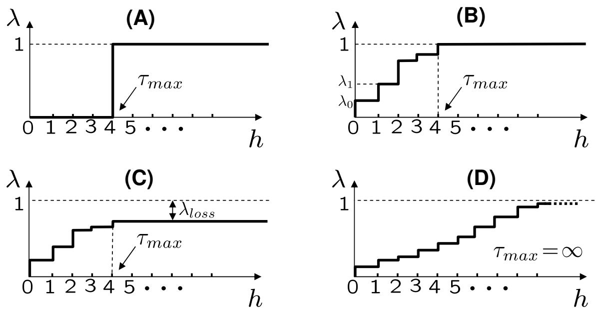

Fig. 3. Probability function of arrival process $\lambda_h = \mathbb{P}[\tau_k \le h]$ for different scenarios: deterministic packet arrival with fixed delay (A); bounded random packet delay with no packet loss (B); bounded random packet delay with packet loss (C); unbounded random packet delay with no packet loss (D).

correctly with a probability ϵ. Simple calculations show that in this case $\lambda_h = 1 - \epsilon^h$.

In the rest of the paper we will use the following definition of stability for an estimator.

Definition: Let $\tilde{x}_{t|t} = f(\tilde{\mathbf{y}}_t, \boldsymbol{\gamma}_t)$ be an estimator, and $\tilde{e}_{t|t} = x_t - \tilde{x}_{t|t}$ and $\tilde{P}_{t|t} = \mathbb{E}[\tilde{e}_{t|t}\tilde{e}_{t|t}^T | \tilde{\mathbf{y}}_t, \boldsymbol{\gamma}_t]$ its error and error covariance, respectively. We say that the estimator is mean-square stable stable if and only if $\lim_{t\to\infty} \mathbb{E}[\tilde{e}_{t|t}] = 0$ and $\mathbb{E}[\tilde{P}_{t|t}] \le M$ for some $M > 0$ and for all $t \ge 1$.

The previous definition can be rephrased in terms of the moments of the estimator error. In fact the conditions above are equivalent to $\lim_{t\to\infty} \mathbb{E}[||\tilde{e}_{t|t}||] = 0$ and $\mathbb{E}[\,||\tilde{e}_{t|t}||^2] \le \mathrm{trace}(M)$.

Let us consider the following static-gain estimator $\tilde{x}_{t|t} = \tilde{x}^t_{t|t}$ with finite-buffer of dimension N, where $\tilde{x}^t_{t|t}$ is computed as follows:

$$\begin{aligned}
\tilde{x}^t_{t-k|t-k} &= A\tilde{x}^t_{t-k-1|t-k-1} + \\
&+ \gamma^t_{t-k} K_k(\tilde{y}^t_{t-k} - CA\tilde{x}^t_{t-k-1|t-k-1}) \quad (12)\\
\tilde{x}^t_{t-N|t-N} &= \tilde{x}^{t-1}_{t-N|t-N} \quad (13)\\
\tilde{x}^t_{-k|-k} &= \bar{x}_0, \quad \gamma^t_{-k} = 0, \quad \tilde{y}^t_{-k} = 0 \quad (14)
\end{aligned}$$

for $k = N-1, \ldots, 0$, where the last line include some dummy variables necessary to initialize the estimator for $t = 1, \ldots, N$.

In was shown in [25] and [23], the optimal choice of the gains is given by the following theorem:

Theorem 1: Let us consider the stochastic linear system given in Equations (1)-(2), where (A, C) is observable, $(A, Q^{1/2})$ is controllable, and $R > 0$. Also consider the arrival process defined by Equations (9)-(11), and the set of estimators with constant gains $\{K_k\}_{k=0}^N$ defined in Equations (12)-(14). If A is not strictly stable and $\lambda_{loss} \ge 1 - \lambda_c$, where λ_c depends on A, C, then there exist no stable estimator with constant gains. Otherwise, let N such that $\lambda_N > \lambda_c$ and consider the optimal gains $\{K^N_k\}_{k=0}^N$ defined as follows:

$$\begin{aligned}
K^N_k &= V^N_k C^T (C V^N_k C^T + R)^{-1}, \; k = 0, \ldots, N \quad (15)\\
V^N_{N-1} &= \Phi_{\lambda_{N-1}}(V^N_{N-1}) \quad (16)\\
V^N_k &= \Phi_{\lambda_k}(V^N_{k+1}), \quad k = N-1, \ldots, 0 \quad (17)\\
\Phi_\lambda(P) &= APA^T + Q - \lambda\, APC^T (CPC^T + R)^{-1} CPA^T \quad (18)
\end{aligned}$$

Also let us define $\overline{P}^t_{k+1|k} = \mathbb{E}[(x_{k+1} - \tilde{x}^t_{k+1|k})(x_{k+1} - \tilde{x}^t_{k+1|k})^T]$, then $\lim_{t\to\infty} \overline{P}^t_{t-k+1|t-k} = V^N_k$, independently of initial conditions $(P_0, \bar{x}_0)$. For any other choice of gains $\{K_k\}_{k=0}^N$ for which the following equations exist:

$$\begin{aligned}
T^N_N &= \mathcal{L}_{\lambda_N}(K_N, T^N_N) \quad (19)\\
T^N_k &= \mathcal{L}_{\lambda_k}(K_k, T^N_{k+1}), \; k = N-1, \ldots, 0 \quad (20)\\
\mathcal{L}_\lambda(K, P) &= \lambda A(I - KC)P(I - KC)^T A^T + \\
&+ (1-\lambda)APA^T + Q + \lambda AKRK^T A^T \quad (21)
\end{aligned}$$

then $\lim_{t\to\infty} \overline{P}^t_{t-k+1|t-k} = T^N_k$, and $V^N_k \le T^N_k$ for $k = 0, \ldots, N$. Also $V^{N+1}_0 \le V^N_0$. Finally, if $\tau_{max} < \infty$, then $V^N_0 = V^{\tau_{max}}_0$ for all $N \ge \tau_{max}$.

The previous theorems shows that the optimal gains can be obtained by finding the fixed point of a modified algebraic Ricatti Equation (16) and then iterating N time an operator with the same structure but with different λ_k. The theorem also demonstrates that a stable estimator with static gains exists if and only if the optimal estimator with static gains exists, therefore the optimal estimator design implicitly solves the problem of finding stable estimators. If the system to be estimated is unstable, then the estimator is stable if and only if the packet loss probability λ_{loss} is sufficiently small. This is a remarkable result since it implies that stability of estimators does not depend on the packet delay τ_{max} as long as most most of the packets eventually arrive. Another important result is that the performance of the estimator, i.e. its steady state error covariance $\lim_{t\to\infty} P_{t+1|t} = \lim_{t\to\infty} \mathbb{E}[e_{t+1|t} e^T_{t+1|t}] = V^N_0$, improves as the buffer length N is increased. However, if the maximum packet delay is finite $\tau_{max} < \infty$, then the performance of the estimator does not improve for $N > \tau_{max}$.

V. Optimal Estimator for co-located Smart Sensors

In this section we describe an alternative coding at the sensor location which improves the overall performance of the estimator at the controller side. This scheme was independently proposed in [26] and [27] where it was suggested to compute and transmit the state estimate rather than the raw measurement at the sensor. As will be shown shortly, this approach gives an estimator with a better performance, however it is applicable only if some computational resources are available on the sensor, commonly known as ”smart sensor”, and when all entries of the observation vector y_t are collected from sensors which are collocated. For example, this scenario is rarely the case in applications running over sensor networks where sensor are distributed and have very limited computation resources [28].

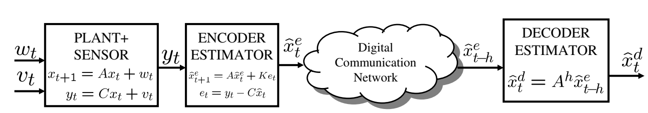

Fig. 4. Smart sensor with state estimator at encoder before transmission.

Rather than sensing the raw measurements y_t over the DCN the sensor compute the optimal state estimate as follows:

$$\begin{aligned}\hat{x}_t^e &= A\hat{x}_{t-1}^e + K_t^e(y_t - A\hat{x}_{t-1}^e) &(22)\\ K_t^e &= P_t^e C^T(CP_t^eC^T+R)^{-1} &(23)\\ P_{t+1}^e &= AP_t^eA^T+Q-AP_t^eC^T(CP_t^eC^T+R)^{-1}CP_t^eA^T &(24)\\ &= \Phi_1(P_t) &\\ P_0^e &= P_0, \quad \hat{x}_0^e=\bar{x}_0 &(25)\end{aligned}$$

These are the equations for the standard Kalman filter, i.e. the minimum error covariance estimator $\hat{x}_t^e = \mathbb{E}[x_t\,|\,y_t,\ldots,y_1]$ whose estimation error $e_t^e = x_{t+1} - A\hat{x}_t^e$ has covariance $cov(e_t^e) = \mathbb{E}[e_t^e e_t^{e^T}\,|\,y_t,\ldots,y_1] = P_t^e$. The state estimate computed by the sensor encoder is then transmitted over the DCN to the decoder estimator. Using the same notation of Equation (5) the value stored at the buffer can be written as follows:

$$\tilde{y}_k^t = \gamma_k^t \hat{x}_k^e \tag{26}$$

Let us define the delay of the most recent packet arrived at the decoder estimator as $\kappa_t = t - \max\{k\,|\,\gamma_k^t = 1\}$ if $\exists \gamma_k^t = 1$, or $\kappa_t = t$ otherwise. The estimate of current state at the decoder estimator $\hat{x}_t^d$ is computed as follows:

$$\hat{x}_t^d = A^{\kappa_t}\tilde{y}_{t-\kappa_t}^t = A^{\kappa_t}\hat{x}_{t-\kappa_t}^e \tag{27}$$

Note that the decoder estimate is equivalent to $\hat{x}_t^d = \mathbb{E}[x_t\,|\,y_{t-\kappa_t},\ldots,y_1]$ and that the its error $e_t^d = x_{t+1} - A\hat{x}_t^d$ has covariance:

$$\begin{aligned}cov(e_t^d) &= \mathbb{E}[e_t^d e_t^{d^T}\,|\,y_{t-\kappa_t},\ldots,y_1]\\ &= \Phi_0^{t-\kappa_t}(P_{t-\kappa_t}^e) = \Phi_0^{t-\kappa_t}\circ\Phi_1^{\kappa_t}(P_0),\end{aligned}$$

where the superscript of $\Phi_\lambda^n(P)$ indicates $\Phi_\lambda \circ \cdots \circ \Phi_\lambda(P)$ composed n-times. Therefore, the decoder estimator error at any time step t is equivalent to the optimal estimator that one would obtain if all observations up to time $t-\kappa_t$ were successfully delivered. This estimation architecture is clearly superior to the estimation architecture proposed in the previous section. Besides having a better performance, the estimator proposed in this section requires very limited computational requirements at the receiver side, in fact it suffices to store the most recent packet arrived at the receiver and then to compute the best state estimate at current time by pre-multiplying the packet data with a matrix which depend on the packet delay.

However, if the packet arrival statistics are stationary and i.i.d, then it is possible to give stability criteria and to compute the expected error covariance as shown in the following theorem:

Theorem 2: Let us consider the stochastic linear system given in Equations (1)-(2), where (A,C) is observable, $(A,Q^{1/2})$ is controllable, and $R>0$. Also consider the arrival process defined by Equations (9)-(11), and the estimator architecture given by Equations (22)-(27). Then the estimator is stable if and only if A is stable, or $\lambda_{loss} \geq \frac{1}{\max_i |\sigma_i^u(A)|^2}$, where $\sigma_i^u(A)$ are the unstable eigenvalues of the matrix A. If the estimator is stable then the covariance of the estimation error defined as $e_t^d = x_{t+1} - Ax_t^d$ has the following property:

$$\lim_{t\to\infty}\mathbb{E}[e_t^d e_t^{d^T}] = D^\infty = \lim_{N\to\infty} D_0^N \tag{28}$$

where the matrix D_0^N is computed as follows:

$$\begin{aligned}D_N^N &= (1-\lambda_N)AD_N^NA^T+(1-\lambda_N)Q+\lambda_N P_\infty^e &(29)\\ D_k^N &= (1-\lambda_k)AD_{k+1}^NA^T+(1-\lambda_k)Q+\lambda_k P_\infty^e &(30)\end{aligned}$$

for $k = N-1,\ldots,0$ where P_{kln} is the unique positive definite fixed point of the Ricatti Equation $P_\infty^e = \Phi_1(P_\infty^e)$. If $\tau_{max} < \infty$, then $D^\infty = D_0^{\tau_{max}} = D_0^N$, for all $N \geq \tau_{max}$.

VI. Numerical Examples

Here we illustrate the use of the tools developed in the previous sections with the aid of some numerical examples.

Let us consider the following probability function of packet delay:

$$\lambda_h = \begin{cases} 0.05\,h, & h = 0,\ldots,15\\ 0.75, & h > 15\end{cases} \tag{31}$$

Let us consider the following discrete time system:

$$A = \begin{bmatrix} 1.00 & 0.05\\ 0.05 & 1.00\end{bmatrix}, \quad \begin{matrix} C = [\,1 \quad 0\,]\\ R = 0.01\end{matrix}, \quad Q = \begin{bmatrix} 0 & 0\\ 0 & 0.01\end{bmatrix} \tag{32}$$

which corresponds to the discretization with sampling period $T = 0.05$ of the continuous time system $\ddot{x} - x = 0$. This system has one stable pole and one unstable pole, and it is the model for the discrete time dynamics of an inverted pendulum. The discrete time eigenvalues of the matrix A are $eig(A) = (1.05, 0.95)$, which give the critical probability $\lambda_c = 1 - 1/1.05^2 = 0.095$, as follows from Theorem 2 in [23]. According to Theorem 1 and 2 in this paper the estimator is stable if and only if $N \geq 2$, in fact $\lambda_1 = 0.05 < \lambda_c$ and $\lambda_2 = 0.01 > \lambda_c$.

The trace of the covariance of the estimator error with constant gains, V_0^N, and the estimator error for smart sensors, D_0^N are shown in Fig. 5. It is interesting to compare the performance of these estimators with the error covariance $P_\infty^e = \Phi_1(P_\infty^e)$, shown in the same figure, corresponding to the ideal case when there is no packet loss and no delay. In fact, P_∞^e gives an idea of the degradation due to the communication network. It is also relevant to evaluate the performance of an estimator with constant gains designed without exploiting the prior knowledge about the packet arrival statistics. A natural choice is to use the standard Kalman gain $K_\infty^e = P_\infty^e C^T(CP_\infty^eC^T+R)^{-1}$, i.e. $K_k = K_\infty^e, k = 0,\ldots,N$ rather than the optimal constant gains K_k^N defined in Theorem 1. The corresponding expected error covariance T_0^N can be obtained by Equations (19)-(21) and it is shown in Fig. 5. From this example it is clear that the tools developed in this paper can help to substantially reduce the degradation of performance when statistics of packet arrival are available.

VII. Conclusions

In this work we proposed a framework to optimally design and analyze the performance of estimators based on finite memory buffers and constant gains, and it was shown that if packet arrival is i.i.d., then the estimators are mean square stable if and only if the packet loss probability is below a critical value. Therefore, implicitly we also provided necessary and sufficient conditions about existence of stable estimators. Finally, we presented numerical algorithms for the computation of the expected estimator error covariance of all the proposed estimators.

The tools developed in this paper are useful both from a control system design perspective and from a communication design perspective. In fact, from a control perspective they can help to evaluate the tradeoffs between performance (error covariance), memory requirements (buffer length),

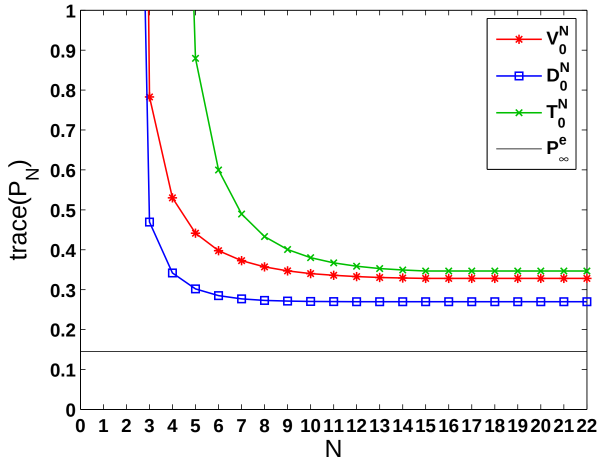

Fig. 5. Trace of the steady state error covariance for the optimal estimator with constant gains (V_0^N), for the optimal estimator with a smart sensor (D_0^N). The horizontal line P^e_∞ corresponds to the trace of the error covariance in the ideal scenario with no delay and no packet loss, i.e. $\lambda_h = 1$ for all h, while T_0^N is the actual steady state error when using the Kalman gain K^e_∞. The error covariances V_0^N, D_0^N are unbounded for $N < 2$, while the covariance P^e_∞ is unbounded for $N < 4$, and they are all cosnstant constant for $N \geq \tau_{max} = 15$.

and the hardware resources ("smart" sensor and fast matrix inversion). In particular, the knowledge of the packet arrival statistics can be used to find the optimal constant gains $\{K_k^N\}_{k=0}^N$ and thus improving performance. From a communication perspective, these tools can be used to aid communication protocol design for real-time applications. In fact, as mentioned in Section I, when designing communication protocols, in particular for wireless systems, there is tradeoff between packet loss and packet delay. At the moment, the choice between favoring reduction of overall packet delay or reduction of packet loss is based on heuristics and experience, and it is not tailored to the specific real-time applications. Therefore, being able to quantitatively measure performance of different protocols can improve cross-layer design of complex networked control systems.

A possible future avenue of research is the extension of this work to the design of optimal LQG-like controller design. This is not a trivial step as many important assumptions in standard LQG control, like the separation principle, do not always hold for NCSs [20]. Another research direction is the implementation and testing of these tools in real-time control applications for wireless sensor networks. A preliminary attempt has already been successfully applied to multiple target tracking [29], but extensive experimental work is still needed.

References

[1] D. Estrin, D. Culler, K. Pister, and G. Sukhatme, "Connecting the physical world with pervasive networks," *IEEE Pervasive Computing*, vol. 1, no. 1, pp. 59–69, January 2002.

[2] "Sensor networks and applications," *Proceedings of the IEEE, Special issue*, vol. 91, August 2003.

[3] R. Szewczyk, E. Osterweil, J. Polastre, M. Hamilton, A. M. Mainwaring, and D. Estrin, "Habitat monitoring with sensor networks," *Communication of the ACM*, vol. 47, no. 6, pp. 34–40, 2004.

[4] G. Tolle, J. Polastre, R. Szewczyk, N. Turner, K. Tu, P. Buonadonna, S. Burgess, D. Gay, W. Hong, T. Dawson, and D. Culler, "A macroscope in the redwoods," in *Proceedings of the Third ACM Conference on Embedded Networked Sensor Systems (SenSys05)*, San Diego, CA, USA, November 2005, pp. 51–63.

[5] C. Fok, G. Roman, and C. Lu, "Mobile agent middleware for sensor networks: An application case study," in *Proc. of the 4th Int. Conf. on Information Processing in Sensor Networks (IPSN'05)*, Los Angeles, California, USA, April 2005, pp. 382–387.

[6] K. Gu, V. Kharitonov, and J. Chen, *Stability of time-delay systems*, 1st ed. Birkhauser, 2003.

[7] A. Cervin, D. Henriksson, B. Lincoln, J. Eker, and K.-E. Årzén, "How does control timing affect performance?" *IEEE Control Systems Magazine*, vol. 23, no. 3, pp. 16–30, June 2003.

[8] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "Networked control systems: analysis and design," *Submitted for publication*, 2006.

[9] L. Montestruque and P. Antsaklis, "Stability of model-based networked control systems with time-varying transmission times," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1562–1572, September 2004.

[10] W. Zhang and M. S. Branicky, "Stability of networked control systems with time-varying transmission period," *Allerton Conf. of Communication, Control and Computing*, October 2001.

[11] D. Nesic and A. Teel, "Input-output stability properties of networked control systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1650–1667, October 2004.

[12] D. Yue, Q. Han, and C. Peng, "State feedback controller design for networked control systems," *IEEE Transactions on Circuits and Systems*, vol. 51, no. 11, pp. 640–644, November 2004.

[13] J. Nilsson, B. Bernhardsson, and B. Wittenmark, "Stochastic analysis and control of real-time systems with random time delays," *Automatica*, vol. 34, no. 1, pp. 57–64, January 1998.

[14] C. N. Hadjicostis and R. Touri, "Feedback control utilizing packet dropping network links," in *Proceedings of the IEEE Conference on Decision and Control*, vol. 2, Las Vegas, NV, Dec. 2002, pp. 1205–1210.

[15] Q. Ling and M. Lemmon, "Optimal dropout compensation in networked control systems," in *IEEE conference on decision and control*, Maui, HI, December 2003.

[16] D. Giorgiev and D. Tilbury, "Packet-based control," in *Proc. of the 2004 Amer. Contr. Conf.*, June 2004, pp. 329–336.

[17] N. Elia and J. Eisembeis, "Limitation of linear control over packet drop networks," in *Proceedings of IEEE Conference on Decision and Control*, vol. 5, Bahamas, December 2004, pp. 5152–5157.

[18] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. Sastry, "Kalman filtering with intermittent observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, September 2004.

[19] O. C. Imer, S. Yuksel, and T. Basar, "Optimal control of dynamical systems over unreliable communication links," *To appear in Automatica*, July 2006.

[20] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry, "Foundations of control and estimation over lossy networks," *To appear in Proceedings of IEEE*.

[21] N. Yu, L. Wang, T. Chu, and F. Hao, "An LMI approach to networked control systems with data packet dropout and transmission delay," *Journal of Hybrid Systems*, vol. 3, no. 11, November 2004.

[22] P. Naghshtrabizi and J. Hespanha, "Designing observer-based controller for network control systems," in *IEEE Conference on Decision and Control (CDC-ECC05)*, Sivilla, Spain, December 2005.

[23] L. Schenato, "Kalman filtering for networked control systems with random delay and packet loss," Univerity of Padova," submitted for publication, 2006. [Online]. Available: http://www.dei.unipd.it/˜schenato/PAPERS/KalmRand.pdf

[24] G. Chen, G. Chen, and S. Hsu, *Linear Stochastic Control Systems*. CRC Press, 1995.

[25] L. Schenato, "Kalman filtering for networked control systems with random delay and packet loss," in *Conference of Mathematical Theroy of Networks and Systems (MTNS'06)*, Kyoto, Japan, July 2006.

[26] Y. Xu and J. Hespanha, "Estimation under controlled and uncontrolled communications in networked control systems," in *To be presented at the IEEE Conference on Decision and Control*, Sevilla, Spain, December 2005.

[27] V. Gupta, D. Spanos, B. Hassibi, and R. M. Murray, "Optimal LQG control across a packet-dropping link," *Submitted to Systems and Control Letters*, 2005.

[28] B. Sinopoli, C. Sharp, L. Schenato, S. Schaffert, and S. Sastry, "Distributed control applications within sensor networks," *Proceedings of the IEEE, Special Issue on Distributed Sensor Networks*, vol. 91, no. 8, pp. 1235–1246, August 2003.

[29] S. Oh, L. Schenato, P. Chen, and S. Sastry, "Tracking and coordination of multiple agents using sensor networks: system design, algorithms and experiments," *To appear in Proceedings of IEEE*.