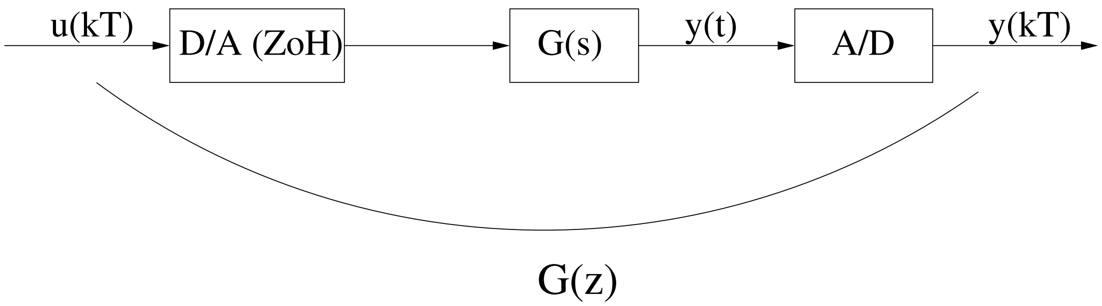
Real-time embedded controllers

Luigi Palopoli

Dealing with delays

DT models of sampled-data systems

- Using $\mathcal{Z}-$transform



- Supposed $G(s) = H(s)e^{-s\lambda}$,
- λ can be used to model computation delays
- The presence of a delay makes CT synthesis much more difficult (infinite dimension)

Example

- Example: $G(s) = e^{-s\lambda} \frac{a}{s+a}$ with $a > 0$

Example

- Example: $G(s) = e^{-s\lambda}\frac{a}{s+a}$ with $a > 0$
- Let $\lambda = lT - mT$, with $l \in \mathbb{N}$, $m \in [0,\ 1)$

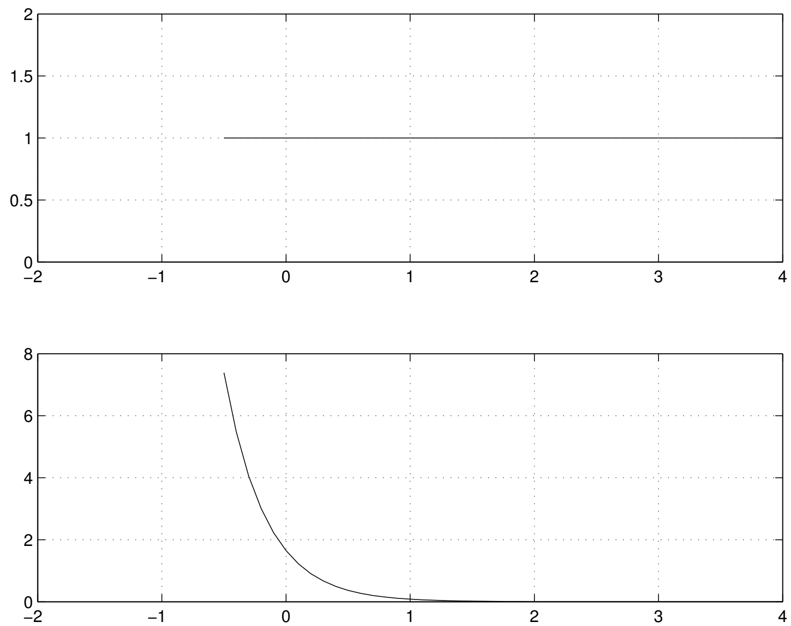
Example

- Example: $G(s) = e^{-s\lambda}\frac{a}{s+a}$ with $a > 0$
- Let $\lambda = lT - mT$, with $l \in \mathbb{N}$, $m \in [0,\ 1)$
- $G(z) = (1 - z^{-1})\mathcal{Z}[\mathcal{L}^{-1}[\frac{G(s)}{s}]]$

Example

- Example: $G(s) = e^{-s\lambda}\frac{a}{s+a}$ with $a > 0$
- Let $\lambda = lT - mT$, with $l \in \mathbb{N}$, $m \in [0,\ 1)$
- $G(z) = (1 - z^{-1})\mathcal{Z}[\mathcal{L}^{-1}[\frac{G(s)}{s}]]$
- $G(z) = (1 - z^{-1})\mathcal{Z}[\mathcal{L}^{-1}[e^{-lsT}(\frac{e^{msT}}{s} - \frac{e^{msT}}{s+a})] =$
  $(1 - z^{-1})z^{-l}\mathcal{Z}[\mathcal{L}^{-1}[\frac{e^{msT}}{s} - \frac{e^{msT}}{s+a}]$

Example



- $\mathcal{Z}\left[\frac{e^{smT}}{s}\right] = \frac{z}{z-1}$
- $\mathcal{Z}\left[\frac{e^{smT}}{s+a}\right] = \frac{ze^{-amT}}{z-e^{-aT}}$
- $G(z) = (1 - e^{-amT})\frac{z+\alpha}{z^l(z-e^{-aT})}$, where $\alpha = \frac{e^{-amT}-e^{-aT}}{1-e^{-amT}}$

Matlab Control toolbox code

```
10cm Td = 1.5
a= 1
T = 1
sysc = tf(a, [1 a],'td',Td);
sysD = c2d(sysc,T);
```

Effects of delays

- Due to the delay
  - l poles arose in the origin
  - a zero arose at $-\alpha$

$\rightarrow$ $\alpha \rightarrow +0$ when $m \rightarrow 1$ (small delay)

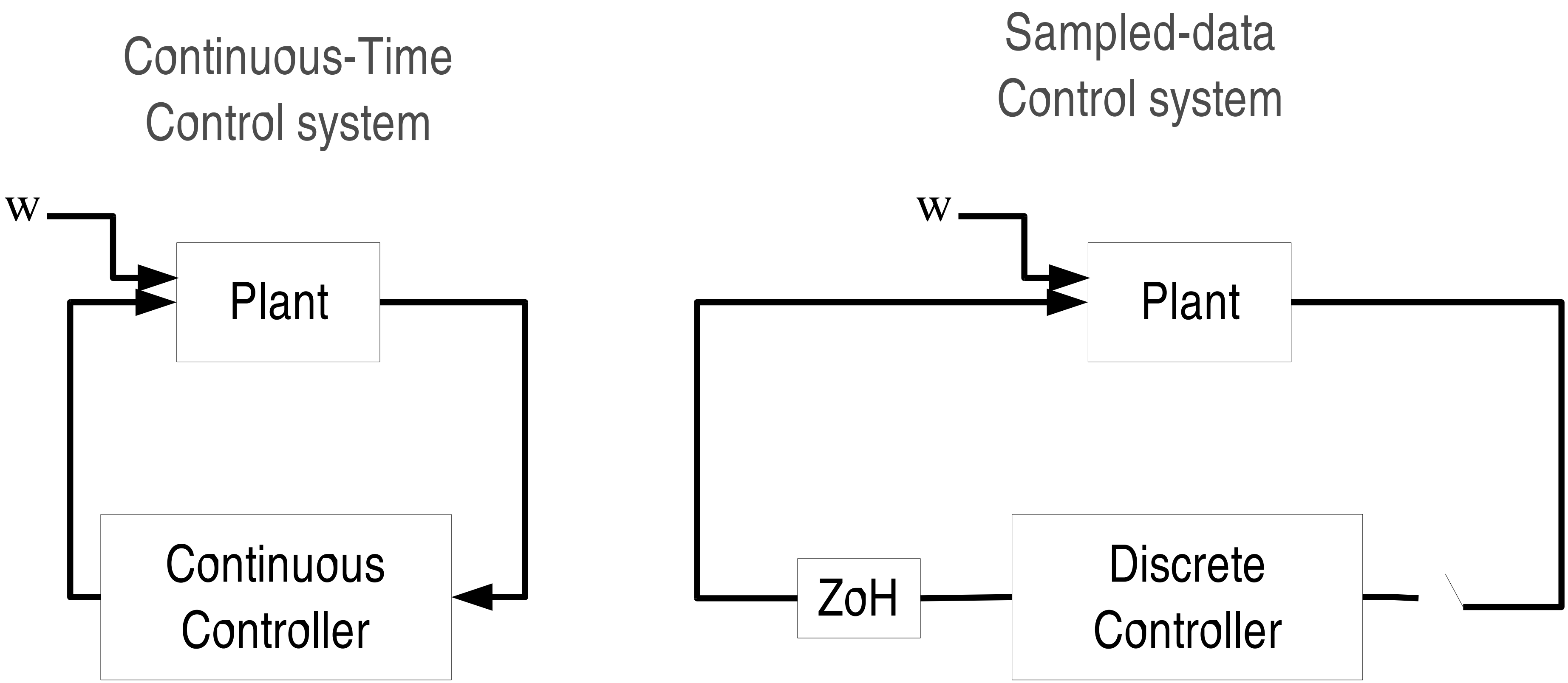
$\rightarrow$ $\alpha \rightarrow +\infty$ when $m \rightarrow 0$ (large delay)

Sample rate selection

Effects of sample rate selection

- Shannon theorem: not obvious!!!
- smoothness of responses
- Noise rejection
- Robustness

Block-diagrams



Continuous-time system

- Let the closed loop equation of the system be:

$$\dot{x} = Fx + G_1 \dot{w}$$

where $w(t)$ is a wiener Process.

Continuous-time system

- Let the closed loop equation of the system be:

$$\dot{x} = Fx + G_1\dot{w}$$

  where $w(t)$ is a wiener Process.

- A random process $w(t)$ is said a Wiener process if:
  - $w(0) = 0$
  - $w(t) - w(s)$ is a Gaussian process with mean 0 and variance $(t-s)R_w$
  - for all times $0 < t_1 < t_2 < \dots t_n$, $w(t_1)$, $w(t_2) - w(t_1)$, ..., $w(t_n) - w(t_{n-1})$ are independent

Continuous-time system

- Let the closed loop equation of the system be:

$$\dot{x} = Fx + G_1\dot{w}$$

  where $w(t)$ is a wiener Process.

- A random process $w(t)$ is said a Wiener process if:
  - $w(0) = 0$
  - $w(t) - w(s)$ is a Gaussian process with mean 0 and variance $(t-s)R_w$
  - for all times $0 < t_1 < t_2 < \ldots t_n$, $w(t_1)$, $w(t_2) - w(t_1)$, $\ldots$, $w(t_n) - w(t_{n-1})$ are independent
- Property: $\dot{w}$ is gaussian with mean 0 and variance R_w

Continuous-time system

- Let the closed loop equation of the system be:

$$\dot{x} = Fx + G_1 \dot{w}$$

  where $w(t)$ is a wiener Process.

- A random process $w(t)$ is said a Wiener process if:
  - $w(0) = 0$
  - $w(t) - w(s)$ is a Gaussian process with mean 0 and variance $(t-s)R_w$
  - for all times $0 < t_1 < t_2 < \ldots t_n$, $w(t_1)$, $w(t_2) - w(t_1)$, $\ldots$, $w(t_n) - w(t_{n-1})$ are independent
- Property: $\dot{w}$ is gaussian with mean 0 and variance R_w
- Assume that increments dw are incorrelated with x and with w

Continuous-time system

- Let the closed loop equation of the system be:

$$\dot{x} = Fx + G_1 \dot{w}$$

  where $w(t)$ is a wiener Process.

- A random process $w(t)$ is said a Wiener process if:
  - $w(0) = 0$
  - $w(t) - w(s)$ is a Gaussian process with mean 0 and variance $(t - s)R_w$
  - for all times $0 < t_1 < t_2 < \ldots t_n$, $w(t_1)$, $w(t_2) - w(t_1)$, $\ldots$, $w(t_n) - w(t_{n-1})$ are independent
- Property: $\dot{w}$ is gaussian with mean 0 and variance R_w
- Assume that increments dw are incorrelated with x and with w

Continuous-time system -I

- Property: $\dot{w}$ is gaussian with mean 0 and variance R_w
- Assume that increments dw are incorrelated with x and with w

CT system (Mean)

- If the initial state has nonzero mean then

  $$\frac{dm}{dt} = Fm$$

  $$m(0) = m_0$$

  where $m(t)$ is the mean of $x(t)$.

CT system (continued)

- Introduce $P = E\{\tilde{x}\tilde{x}^T\}$, where $\tilde{x} = x - m$

CT system (continued)

- Introduce $P = E\{\tilde{x}\tilde{x}^T\}$, where $\tilde{x} = x - m$
- Intuitive way:

$$\frac{d\tilde{x}}{dt} = F\tilde{x} + G_1 dw$$

$$d\tilde{x} = F\tilde{x}dt + G_1 dw$$

CT system (continued)

- Introduce $P = E\{\tilde{x}\tilde{x}^T\}$, where $\tilde{x} = x - m$
- Intuitive way: $$\frac{d\tilde{x}}{dt} = F\tilde{x} + G_1 dw$$ $$d\tilde{x} = F\tilde{x}dt + G_1 dw$$
- Expressing variations of xx^T (from now on use x for $\tilde{x}$):

$$d\,xx^T = xx^T - (x + dx)(x + dx)^T = xdx^T + dxx^T + dxdx^T$$

CT system (continued)

- ...from which

$$dxx^T = x(Fx)^T dt + x(G_1 dw)^T dt + Fxx^T + G_1 dwx^T + (Fx)(Fx)^T dt^2 +$$
$$+(Fx)(G_1 dw)^T dt^2 + (G_1 dw)(Fx)^T dt^2 + (G_1 dw)(G_1 dw)^T$$

CT system (continued)

- ...from which

$$dxx^T = x(Fx)^T dt + x(G_1 dw)^T dt + Fxx^T + G_1 dwx^T + (Fx)(Fx)^T dt^2 +$$
$$+(Fx)(G_1 dw)^T dt^2 + (G_1 dw)(Fx)^T dt^2 + (G_1 dw)(G_1 dw)^T$$

- Taking expected values of both sides and considering that 1) w is incorrelated with x, 2) $E(dwdw^T) = R_w dt$ we get

$$\frac{dP}{dt} = PF^T + FP + G_1 R_w G_1^T$$

CT system (continued)

- ...from which

$$dxx^T = x(Fx)^T dt + x(G_1 dw)^T dt + Fxx^T + G_1 dwx^T + (Fx)(Fx)^T dt^2 +$$
$$+(Fx)(G_1 dw)^T dt^2 + (G_1 dw)(Fx)^T dt^2 + (G_1 dw)(G_1 dw)^T$$

- Taking expected values of both sides and considering that 1) w is incorrelated with x, 2) $E(dwdw^T) = R_w dt$ we get

$$\frac{dP}{dt} = PF^T + FP + G_1 R_w G_1^T$$

- Taking the limit $dt \to 0$:

$$\dot{P} = PF^T + FP + G_1 R_w G_1^T$$

Covariance

- Definition
  $r(s,t) = Cov\{x(t), x(s)\} = E\{(x(t) - m(t))^T (x(s) - m(s))\}$
- Let $s \geq t$, the system evolution is given by:
  $x(s) = e^{A(s-t)}x(t) + \int_s^t e^{A(s-\tau)}w(\tau)d\tau$
- Computing the expectation value and because $x(t)$ is incorrelated from $w(t)$ we get

$$r(s,t) = e^{A(s-t)}P(t), s \geq t$$

Example

- First order systems: $\frac{dx}{dt} = fx + g_1 w, f < 0$
  $var\{x(t_0)\} = r_0, mean(x(t_0)) = m(0) = m_0$
- Mean value:
  $$\frac{dm}{dt} = fm \rightarrow m(t) = m_0 e^{f(t-t_0)}$$
- Covariance function
  - Differential equation, let $r_1 = g_1^2 R_w$,
    $$\dot{P} = 2fP + g_1^2 R_w, P(t_0) = r_0 \rightarrow P(t) = e^{2f(t-t_0)} r_0 + \frac{r_1}{2f}(e^{2f(t-t_0)} - 1)$$
  - Assuming $f < 0$ and $m_0 = 0$, we get: $r(s,t) = \frac{r_1}{2f} e^{f|t-s|}$
- the process is asymptotically stationary: $r(\tau) = \frac{r_1}{2f} e^{f|\tau|}$
- the spectral density is $\phi(\omega) = \frac{r_1}{2\pi} \frac{1}{\omega^2 + a^2}$

Sampled-data systems

- Sampled-data equation:

$$x((k+1)T) = \Phi(T)x(k) + e(kT)$$
$$\Phi(t) = \int_0^T e^{A\tau)} d\tau$$
$$e(kT) = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} G_1 w(\tau) d\tau$$

Sampled-data systems

- Sampled-data equation:

$$x((k+1)T) = \Phi(T)x(k) + e(kT)$$
$$\Phi(t) = \int_0^T e^{A\tau)} d\tau$$
$$e(kT) = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} G_1 w(\tau) d\tau$$

- Mean value Evolution:

$$m(k+1) = \Phi(T)m(k), m(0) = m_0$$

Sampled-data systems

- Sampled-data equation:

$$x((k+1)T) = \Phi(T)x(k) + e(kT)$$
$$\Phi(t) = \int_0^T e^{A\tau)} d\tau$$
$$e(kT) = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} G_1 w(\tau) d\tau$$

- Mean value Evolution:

$$m(k+1) = \Phi(T)m(k), m(0) = m_0$$

- Covariance evolution:

$$r_{xx}(k,h) = cov(x(k), x(h)) = E\{\tilde{x}(k)\tilde{x}(h)^T\}, \text{ where } \tilde{x} = x - m$$
$$\tilde{x}((k+1)T) = \Phi(T)\tilde{x}(kT) + e(kT)$$

Sampled-data systems

- Introduce $P(k) = cov(x(k), x(k))$ (T implied):

$$\tilde{x}(k+1)\tilde{x}^T(k+1) = \Phi\tilde{x}(k)\tilde{x}(k)^T\Phi^T + \Phi\tilde{x}(k)e(k)^T +$$
$$+e(k)(\Phi\tilde{x}(k))^T + e(k)e(k)^T$$

Taking the expectation and considering that e(k) and x(k) are independent...

$$P(k+1) = \Phi P(k)\Phi^T + E\{e(k)e(k)^T\}$$

Sampled-data systems

- Introduce $P(k) = cov(x(k), x(k))$ (T implied):

$$\tilde{x}(k+1)\tilde{x}^T(k+1) = \Phi\tilde{x}(k)\tilde{x}(k)^T\Phi^T + \Phi\tilde{x}(k)e(k)^T +$$
$$+e(k)(\Phi\tilde{x}(k))^T + e(k)e(k)^T$$

Taking the expectation and considering that e(k) and x(k) are independent...

$$P(k+1) = \Phi P(k)\Phi^T + E\{e(k)e(k)^T\}$$

- Considering that $e(kT) = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} G_1 \dot{w}(\tau) d\tau$, we get:

$$C_d = E(e(kT)e(kT)^T) = \int_0^T \Phi(\tau) G_1 R_w G_1^T \Phi(\tau) d\tau$$

Sampled-data systems

- Introduce $P(k) = cov(x(k), x(k))$ (T implied):

$$\tilde{x}(k+1)\tilde{x}^T(k+1) = \Phi\tilde{x}(k)\tilde{x}(k)^T\Phi^T + \Phi\tilde{x}(k)e(k)^T +$$
$$+e(k)(\Phi\tilde{x}(k))^T + e(k)e(k)^T$$

Taking the expectation and considering that e(k) and x(k) are independent...

$$P(k+1) = \Phi P(k)\Phi^T + E\{e(k)e(k)^T\}$$

- Considering that $e(kT) = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}G_1\dot{w}(\tau)d\tau$, we get:

$$C_d = E(e(kT)e(kT)^T) = \int_0^T \Phi(\tau)G_1R_wG_1^T\Phi(\tau)d\tau$$

- To compute the covariance:

$$\tilde{x}(k+1)\tilde{x}(k)^T = (\Phi\tilde{x}k + e(k))\tilde{x}(k)....$$

Because e(k) has zero mean and is independent from x(k)...

$r_{xx}(k+1, k) = \Phi P(k)$ from which

$$r_{xx}(h, k) = \Phi^{h-k}P(k), h \geq k$$

Example

- First order systems: $\frac{dx}{dt} = fx + u + g_1\dot{w}, f > 0$
  $var\{x(t_0)\} = r_0, mean(x(t_0)) = m(0) = m_0$
- Design a discrete-time controller such that the equivalent CT system has a pole at $s = s_0$
- First order systems (ZoH for outputs):
  $x(k+1) = e^{fT}x(k) + \frac{e^{fT}-1}{f}u(k) + \int_{kT}^{(k+1)T} g_1\dot{w}(\tau)d\tau$
  $var\{0)\} = r_0, mean(x(0)) = m(0) = m_0$
- Choose $u(k) = kx(k)$ such that $e^{fT} + \frac{(e^{fT}-1)}{f} = e^{s_0 T}$
- Mean value:

$$m(k+1) = e^{s_0 T}m(k), m(0) = m_0 \rightarrow m(k) = e^{s_0(k-k_0)T}m_0$$

Example (continued)

- Covariance function
  - Computation of $P(k)$

$$
\begin{aligned}
P(k+1) &= e^{2s_0 T} P(k) + C_d \\
C_d &= \int_0^T e^{2f\tau} G_1^2 R_w d\tau = \frac{G_1^2 R_w}{2f}(e^{2fT} - 1) = \frac{r_1}{2f}(e^{2fT} - 1) \\
P(k) &= e^{2s_0(k-k_0)T} r_0 + C_d \frac{1 - e^{2s_0 T(k-k_0)}}{1 - e^{2s_0 T}}
\end{aligned}
$$

  - Steady state ($k \to \infty$):

$$
\begin{aligned}
m(k) &\to 0 \\
P(k) &\to \frac{C_d}{1 - e^{2s_0 T}} \\
r_{xx}(k,h) &= \frac{C_d e^{s_0(k-h)T}}{1 - e^{2s_0 T}}
\end{aligned}
$$

- Compare $r_{xx}(0)$ with what the value $r^c_{xx}(0)$ that we found for the continuous-time case:

$$
\frac{r_{xx}(0)}{r^c_{xx}(0)} = \frac{(e^{2fT} - 1)}{1 - e^{2s_0 T}}
$$

, which is increasing with period.

Outline

- Sample-rate selection
  - Shannon theorem
  - smoothness of responses
  - Noise rejection
  - Robustness

Robustness

- Consider a first order system: $\dot{x} = ax + bu$
- We consider a robustness problems
  - b is known with uncertainity $b = \tilde{b} + db$
- Sample with period T and design so that the closed loop poles are at $e^{s_0 T}$, $s_0 < 0$
- DT system:

$$x((k+1)T) = e^{aT} x(kT) + \frac{e^{aT} - 1}{a} bu(kT)$$

- Feedback: $u(kT) = \gamma x(kT)$ s.t.

$$e^{aT} + \gamma \frac{e^{aT} - 1}{a} b = e^{s_0 T}$$

Robustness with respect to db

- Stability $\left|e^{aT} + \gamma\frac{e^{aT}-1}{a}(b+db)\right| \leq 1$
- From which:

$$1 - e^{s_0 T} \geq \gamma\frac{e^{aT}-1}{a}db \geq -1 - e^{s_0 T}$$
$$\gamma\frac{e^{aT}-1}{a} = \frac{e^{s_0 T}-e^{aT}}{b}$$
$$\frac{1-e^{s_0 T}}{e^{s_0 T}-e^{aT}} \geq \frac{db}{b} \geq -\frac{1+e^{s_0 T}}{e^{s_0 T}-e^{aT}}$$

- The measure of the maximum relative deviation:

$$\mu_{db} = \left|\frac{2}{e^{s_0 T} - e^{aT}}\right|$$