A New Perspective on Gossip Iterations: from Symmetrization to Quantum Consensus*

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Abstract— **This paper extends the gossip algorithm, widely studied in the literature on distributed computing and control algorithms, to networks of quantum systems. In doing so, we reinterpret the classical algorithm and the average consensus task as a symmetrization problem with respect to the action of the permutation group. This allows us to extend in a natural way the gossip consensus algorithm to the quantum setting and prove its convergence properties to symmetric states while preserving the expectation of permutation-invariant global observables.**

I. Introduction

Among the recent trends in control and systems theory, the field of distributed control, estimation and optimization on networks has stimulated an impressive amount of research, see e.g. [1], [2], [3], [4], [5]. A basic task for distributed information processing is reaching *consensus* about the mean of some shared value or slack variable. For several applications, an asynchronous pairwise interaction setting is relevant, which has led to the study of so-called gossip algorithms [6].

The present paper extends this well-studied gossip algorithm to networks of quantum systems. Exploring the links between information processing tasks and stochastic dynamics on networks has recently opened new research directions towards "distributed" quantum information applications. Among these, we recall quantum computation [7], [8] in its potential implementation via dissipative means [9], and its connection to quantum random walks [10], [11]. Other applications include entanglement generation through stabilizing dissipative dynamics [12], [13], as well as most tasks in the control of open quantum systems [14].

A first attempt to bring consensus to the quantum context has been presented in [15]. It is based on a "cone geometry" approach, viewing open quantum dynamics as the non-commutative generalization of Markov chains that model consensus algorithms. The authors show how Birkhoff's Theorem and Hilbert's projective metric lead to a general convergence result and contraction ratio estimation. However, by describing the dynamics of the whole system of interest as governed by a single Markov transition mechanism, this formulation does not account for subsystem structure or network connections.

We here approach quantum consensus from an "operational", multi-agent control perspective including the basic classical ingredients: a network of subsystems, an interaction protocol with locality constraints, and a target consensus situation. In particular, we focus on a non-commutative equivalent of the well-known gossip algorithm[1]. To this aim, we reinterpret the gossip algorithm as a way to obtain *symmetrization* with respect to the permutation group of all subsystems in the network (Section II-A). From this reformulation, we construct the quantum equivalent of the gossip interaction (Section II-B). The main result of the paper is given in Section II-C: we show that the algorithm converges to a permutation-invariant state, and in doing so it "computes" the average of a class of global physical observables in a distributed way. Section III further explores the analogy between the classical and quantum setting from a more abstract, group theoretic viewpoint. In this reformulation, convergence of both methods is equivalent to symmetrization with respect to the *action* of a finite group, which is in turn implied by convergence of a suitably constructed Markov chain to the uniform distribution. A brief refresher about quantum systems and notations is given in the appendix.

In the light of this result, we believe that our work offers not only a generalization of the well-known consensus algorithm to quantum networks, but also a new viewpoint on the classical case, based on group symmetrization, that could be useful for a number of issues in distributed control. It is worth remarking that in the quantum control literature a wealth of methods for noise protection and dynamical error-correction are based on symmetrization techniques [16], [17], [18], [19], [20], [21].

II. Gossip Iterations from Permutations

The distributed computation context for consensus is formalized by assigning local agents (subsystems) to vertices $1, 2, ..., m \in V$ of a graph. Possible interactions between agents at time t are modeled by the edges $E(t) \subset \{(j,k) : j,k \in V\}$ of the graph. An undirected interaction graph identifies (j,k) with (k,j); we restrict ourselves to this case.

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[1]The lift to the non-commutative setting is motivated by the fact that general quantum observables cannot be measured simultaneously. This implies that their corresponding operators do not commute.

A. Classical Gossip Interactions

The so called *gossip algorithm* is usually described as follows [6]. Consider m subsystems, each one associated to a configuration variable $x_k \in \mathbb{R}^n$. The goal of *consensus* is reaching a situation where $x_1 = x_2 = ... = x_m$. In the gossip context, the subsystems evolve in discrete time through bilateral interactions. At each iteration, a single edge (j,k) is selected from the set $E(t)$ of available edges at that time. The associated agents move towards each other / their mean value, according to:

$$\begin{aligned} x_j(t+1) &= x_j(t) + \alpha(x_k(t) - x_j(t)) \\ &= (1-\beta)x_j(t) + \beta\,\tfrac{x_j(t)+x_k(t)}{2} \\ x_k(t+1) &= x_k(t) + \alpha(x_j(t) - x_k(t)) \\ &= (1-\beta)x_k(t) + \beta\,\tfrac{x_j(t)+x_k(t)}{2} \\ x_\ell(t+1) &= x_\ell(t) \quad \text{for all } \ell \notin \{j,k\}\,, \end{aligned} \tag{1}$$

where $\alpha \in (0,1)$ and $\beta = 2\alpha$.

This can also be interpreted as a convex combination of two discrete operations, [keep your state] and [swap your state]. Namely:

$$\begin{aligned} (x_j(t+1),\ x_k(t+1)) &= (1-\alpha)\ (x_j(t),\ x_k(t)) \\ &\quad + \alpha\ (x_k(t),\ x_j(t)) \qquad (2) \\ x_\ell(t+1) &= x_\ell(t) \quad \text{for all } \ell \notin \{j,k\} \qquad (3) \end{aligned}$$

with $\alpha \in (0,1)$. As we shall see, this permutation oriented viewpoint turns out to have a natural quantum counterpart.

The way in which the active links are selected at each time leads to different evolutions for the whole system. We consider the following situations.

- *Random single interaction:* at each time t one link $(j(t),k(t))$ is selected at random, $(j(t),k(t))$ being a single-valued random variable onto the edge set $E(t)$.
- *Cyclic single interaction:* at each time t one link $(j(t),k(t))$ is selected deterministically by cycling through the elements of a time-invariant edge set E.

In either case, since the set of all pairwise swaps generates the whole permutation group [22], it is easy to see that the evolution up to time t can always be written as a convex combination of permutation operators on the initial subsystem states. Let $\mathfrak{P}$ denote the set of all permutations π of the integers $1,2,...m$ and let us pack all the x_i in a single vector $x = (x_1, x_2, ..., x_m)$. We define P_π as the unique matrix associated to π such that $P_\pi\, x = (x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(m)})$ for any $x_1, x_2, ..., x_m$. The result that gossip iterations – both random and cyclic – lead to consensus under sufficient graph connectivity assumptions [6], can be reformulated by saying that the evolution asymptotically drives the state towards the symmetric set

$$\mathcal{C} = \{x \in \mathbb{R}^{mn} :\ P_\pi\, x = x \ \text{ for all } \pi \in \mathfrak{P}\}\,. \tag{4}$$

In addition, one easily checks that gossip evolutions preserve the total average $\bar{x} = \frac{1}{m}\sum_{k=1}^m x_k$, so the state converges to $x_k = \bar{x}$ for all k. Gossip iterations thus allow for computation of the mean in a distributed, robust way.

B. Quantum Gossip Interactions

Let us now introduce a way to implement gossip-type interactions in a quantum setting. For a brief introduction on quantum system modeling, we refer the reader to the Appendix. Consider a multipartite system composed of m isomorphic quantum subsystems, labeled with indices $i = 1,\dots,m$, with associated Hilbert space $\mathcal{H}^m := \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_m \simeq \mathcal{H}^{\otimes m}$, with $\dim(\mathcal{H}_i) = \dim(\mathcal{H}) = n$ and $n \geqslant 2$. We shall refer to this multipartite system as to our *quantum network*. Recall that *states* for the quantum systems are associated to density operators ρ and physical observables to self-adjoint operators $X = X^\dagger$. Expectations of physical quantities are computed as $\mathbb{E}_\rho(X) = \mathrm{Tr}(\rho X)$.

Given an operator $\sigma \in \mathfrak{B}(\mathcal{H})$, we denote by $\sigma^{(i)}$ the *local* (i.e. acting non-trivially on a single subsystem) operator:

$$\sigma^{(i)} := \underbrace{I \otimes_{\dots} \otimes I}_{(i-1)\text{ times}} \otimes\, \sigma \otimes \underbrace{I \otimes_{\dots} \otimes I}_{(m-i)\text{ times}}\,.$$

The action of a permutation π of the quantum subsystems on the observables is obtained by the conjugate action of a unitary operator $U_\pi \in \mathfrak{U}(\mathcal{H}^m)$, uniquely defined by

$$U_\pi^\dagger (X_1 \otimes \ldots \otimes X_m) U_\pi = X_{\pi(1)} \otimes \ldots \otimes X_{\pi(m)}$$

for all bounded linear operators $X_1, \ldots X_m$ in $\mathfrak{B}(\mathcal{H})$. In the dual viewpoint (see the Appendix), where the permutation acts on the state ρ, the corresponding action is obtained by swapping U_π and $U_\pi^\dagger$. A state or observable is said to be *permutation invariant* if it commutes with all the subsystem permutations. It is worth noting that given any self-adjoint operator $Q \in \mathfrak{H}(\mathcal{H}^m)$ characterizing an observable on the network, we can define a permutation invariant observable X by considering:

$$X = \bar{\mathcal{E}}_{\mathfrak{P}}(Q) := \frac{1}{m!}\sum_{\pi \in \mathfrak{P}} U_\pi^\dagger Q U_\pi\,. \tag{5}$$

We will see in Section III that $\bar{\mathcal{E}}_{\mathfrak{P}}(\cdot)$ is the projection onto the fixed points of the conjugate unitary action of the permutation group on the linear operators on $\mathcal{H}^m$.

As in the classical case, a graph can be used to model the possible interactions among subsystems. In a controlled quantum network, one can typically engineer unitary transformations affecting neighboring subsystems. In particular, transformations that implement the "identity" evolution and the swapping of two subsystems are realistic. Let us denote $U_{(j,k)}$ the operator that swaps subsystems j and k. Following the analogy with the classical case (3), we define a *quantum gossip interaction* as the map:

$$\rho(t{+}1) = \mathcal{E}_{j,k}(\rho(t)) = (1\text{-}\alpha)\,\rho(t) + \alpha\, U_{(j,k)}\rho(t)U_{(j,k)}^\dagger\,, \tag{6}$$

with $\alpha \in (0,1)$. Such iteration is written in the so called Operator Sum Representation (OSR) and therefore it defines a *quantum channel* i.e. a completely positive (CP) and trace preserving (TP) map from density operators to density operators $\mathcal{E}_{j,k} : \mathfrak{D}(\mathcal{H}^m) \to \mathfrak{D}(\mathcal{H}^m)$ (see the appendix). Under the dynamics (6), the resulting evolution from $\rho(t_0)$

to $\rho(t_f)$ for any $t_f \geq t_0$ is still described by a CPTP map. Specifically, it admits an OSR in the form

$$\mathcal{E}_{t_0}^{t_f}(\rho) = \sum_{\pi \in \mathfrak{P}} p_\pi \, U_\pi \rho U_\pi^\dagger, \tag{7}$$

where the $p_\pi \in [0,1]$ depend on the choice of gossip edges between t_0 and t_f, and satisfy $\sum_k p_k = 1$. Such a map can thus be thought of as a probabilistic mixture of unitary evolutions. In addition to being CPTP, this map is also *unital* i.e. it preserves the identity.

C. Convergence to Consensus for the quantum algorithm

We study convergence under the two types of gossip dynamics introduced above: cyclic interaction, and trajectory-wise for the random interaction. We begin by characterizing the fixed points of maps like (7).

Proposition 1: (see e.g. [17]) Let $\{V_i\}_{i=1}^K$ an OSR of a *unital* CP map $\mathcal{E}(\cdot)$ and define:

$$\mathcal{A}_{\mathcal{E}} = \{X \in \mathfrak{B}(\mathcal{H}^m) \, : \, XV_i = V_i X \;\; \text{for } i = 1, \ldots, K\} \, . \tag{8}$$

Then $\bar{X} \in \mathfrak{B}(\mathcal{H}^m)$ is a fixed point of $\mathcal{E}$, i.e. $\mathcal{E}(\bar{X}) = \bar{X}$, if and only if $\bar{X} \in \mathcal{A}_{\mathcal{E}}$. □

Lemma 1: Let $U_{(j,k)}$ denote the pairwise swap operation of subsystems (j,k) on $\mathcal{H}^m$. If the edge set E defines a connected graph, then the set of fixed points of any CP unital map of the form

$$\mathcal{E}(X) = q_0 \, X + \sum_{(j,k) \in E} q_{j,k} \, U_{(j,k)}^\dagger X U_{(j,k)} \;\; , \tag{9}$$

with $q_0 + \sum q_{j,k} = 1 \, , \quad q_0, \, \{q_{j,k}\} > 0$, coincides with the set of permutation-invariant operators.

Proof: According to Proposition 1 the fixed points are the X satisfying $XU_{(j,k)} = U_{(j,k)}X$, or equivalently $U_{(j,k)}^\dagger X U_{(j,k)} = X$, for all (j,k). The latter expresses that X is invariant with respect to pairwise swaps on all the graph edges. It is well known that sequences of pairwise swaps on the edges of a connected graph generate the full set of permutations on the set of nodes [22], which gives the conclusion. □

The following lemma shows how the contribution of the identity, i.e. the trivial permutation, in the CP map plays a crucial role in the convergence.

Lemma 2: If a CPTP map $\mathcal{E}$ admits an OSR with a term $V_1 = \sqrt{\alpha}\, I, \;\; \alpha > 0$, then viewing it as a linear map on $\mathfrak{B}(\mathcal{H}^m)$ its eigenvalues all have a modulus < 1, up to eigenvalues that precisely equal 1.

Proof: If $\mathcal{E}$ is a CPTP map it is a contraction in trace norm [7], [23], so its eigenvalues λ_k belong to the closed unit disk. By virtue of the Kraus-Stinespring representation theorem (see e.g. [24]), also $\mathcal{F} = \frac{1}{1-\alpha}(\mathcal{E} - \alpha I)$ is CPTP and thus has eigenvalues μ_k in the closed unit disk. Therefore the eigenvalues $\lambda_k = (1-\alpha)\mu_k + \alpha$ of $\mathcal{E} = (1-\alpha)\mathcal{F} + \alpha I$ in fact belong to the circle of radius $(1-\alpha)$ centered at α, which is strictly inside the unit circle except for a tangency point at $1 \in \mathbb{C}$. □

By combining the above properties we get the main result, i.e. a convergence result for quantum gossip that ensures as in the classical case a distributed computation of the mean for a class of operators.

Theorem 1: Assume that there exists a $T > 0$ such that the union of graphs associated to possible interactions in the time interval $[t, t+T]$ is connected for all t. Then the quantum gossip algorithm (6) ensures global convergence to the *permutation invariant* state

$$\rho_* = \frac{1}{m!} \sum_{\pi \in \mathfrak{P}} U_\pi \rho_0 U_\pi^\dagger \, ; \tag{10}$$

- deterministically, when the edges on which a gossip interaction occurs at a given time are selected by periodically cycling, in any predefined way, through the set of edges;
- in probability, when the edges on which a gossip interaction occurs at a given time are selected randomly from a bounded probability distribution. [2]

In addition, consider

$$S = \frac{1}{m} \sum_i^m \sigma^{(i)}, \tag{11}$$

where σ is any self-adjoint operator on $\mathcal{H}$. Then

$$\lim_{t\to\infty} \mathrm{Tr}(\sigma^{(\ell)}\rho(t)) = \lim_{t\to\infty} \mathrm{Tr}(S\rho(t)) = \mathrm{Tr}(S\rho_0) \tag{12}$$

holds for all $\ell \in \{1, \ldots, m\}$ and for all ρ_0.

Proof: Recall that, for any CPTP map $\mathcal{E}$ the dual dynamics $\mathcal{E}^\dagger$ is the unital CP map such that

$$\mathrm{Tr}(X\mathcal{E}(\rho)) = \mathrm{Tr}(\mathcal{E}^\dagger(X)\rho)$$

for any X, ρ. Using the fact that the pairwise permutation operators are self-adjoint, one easily sees that any S of the form (11) is invariant under the dual map of (6). This readily yields

$$\mathrm{Tr}[\mathcal{E}_{j,k}(\rho)S] = \mathrm{Tr}[\rho\mathcal{E}_{j,k}^\dagger(S)] = \mathrm{Tr}[\rho S] \;\; \forall \;\; \rho \, , \tag{13}$$

proving the second equality in (12).

For a *cyclic evolution*, we consider $\mathcal{E}_C$ the map that concatenates the gossip evolutions over one cycle. Thanks to the presence of the identity in each gossip interaction step, all the pairwise swaps are still present with a weight different from zero in the OSR of $\mathcal{E}_C$. Therefore the necessary part of Lemma 1 holds (any fixed point must be permutation-invariant); the sufficient part holds trivially. Now consider the dynamics associated to $\mathcal{E}_C$ as a linear, time-invariant map acting on the space of hermitian matrices. From Lemma 2 and the fact that the time-invariant linear map leaves $\mathfrak{D}(\mathcal{H}^m)$ invariant (excluding unstable Jordan blocks), we have that all the *modes* of the LTI system are asymptotically stable

[2] By a bounded probability distribution, we mean that there exists a constant $\gamma > 0$ such that any edge $(j,k) \in E(t)$ has a probability $q_{j,k}(t) > \gamma$ to be chosen. By convergence in probability, we mean that for any initial state ρ_0 and any $\delta, \varepsilon > 0$, there exists a time $T > 0$ such that

$$\mathbb{P}[\,\mathrm{Tr}((\rho(T) - \rho_*)^2) \, > \varepsilon \,] \, < \delta \, .$$

when $\rho(t)$ is computed according to the gossip algorithm.

except those corresponding to the fixed-point set, namely the permutation-invariant set: every initial state converges to a fixed point ρ_∞ in this set. Thus the permutation invariant set is globally asymptotically stable, and in fact exponentially stable since the map is linear. Let us now prove that ρ_∞ has the form (10). For all *permutation invariant* X, from (13) we have that:

$$\mathrm{Tr}[X\mathcal{E}_C^t(\rho_0)] = \mathrm{Tr}[X\rho_0] \quad \text{for any number of iterations } t\,. \tag{14}$$

Combining the latter with the fact that ρ_∞ is permutation-invariant, that the set of all permutations is self-adjoint, and using (5), we get for arbitrary $Q \in \mathfrak{H}(\mathcal{H}^m)$:

$$\begin{aligned}
\mathrm{Tr}[Q\rho_\infty] &= \mathrm{Tr}[Q\frac{1}{m!}\sum_{\pi\in\mathfrak{P}} U_\pi \rho_\infty U_\pi^\dagger] \\
&= \mathrm{Tr}[\frac{1}{m!}\sum_{\pi\in\mathfrak{P}} U_\pi Q U_\pi^\dagger \rho_\infty] \\
&= \mathrm{Tr}[\frac{1}{m!}\sum_{\pi\in\mathfrak{P}} U_\pi Q U_\pi^\dagger \rho_0] \\
&= \mathrm{Tr}[Q\,\frac{1}{m!}\sum_{\pi\in\mathfrak{P}} U_\pi \rho_0 U_\pi^\dagger]\,.
\end{aligned}$$

This implies that indeed $\rho_\infty = \rho_*$ as described in (10). Replacing Q by $\sigma^{(\ell)}$ and noting that $\frac{1}{m!}\sum_{\pi\in\mathfrak{P}} U_\pi \sigma^{(\ell)} U_\pi^\dagger = \frac{1}{m}\sum_{i=1}^m \sigma^{(i)}$, the second line directly yields the first equality in (12).

For the *random trajectory evolution*, we repeat a proof similar to the classical case. Since $\mathcal{E}$ for a single interaction is linear, *self-adjoint*, with eigenvalues in the closed unit disk, it is a contraction for the Frobenius norm distance $\mathrm{Tr}((\rho_A-\rho_B)^2)$ between any two states $\rho_A, \rho_B \in \mathfrak{D}(\mathcal{H}^m)$. Indeed, $\mathcal{E}$ has non-increasing orthonormal modes, so by writing any operator $X \in \mathfrak{H}(\mathcal{H}^m)$ in the modal basis we directly get $\mathrm{Tr}(\mathcal{E}(X)^\dagger\,\mathcal{E}(X)) \leq \mathrm{Tr}(X^\dagger X)$; taking $X = \rho_A - \rho_B$ yields the contraction. This is exactly analogous to the non-increasing Euclidean norm $x^T x = \|x\|^2$ under a classical consensus interaction with an *undirected* graph, and the related contraction of $\|x_A - x_B\|^2$. Now taking in particular $\rho_A = \rho$ and $\rho_B = \rho_*$, we get that the Frobenius distance from ρ to ρ_* can never increase. Moreover, by transitivity of the permutation operators, $\frac{1}{m!}\sum_{\pi\in\mathfrak{P}} U_\pi \rho U_\pi^\dagger = \frac{1}{m!}\sum_{\pi\in\mathfrak{P}} U_\pi \rho_0 U_\pi^\dagger = \rho_*$ for any ρ along the trajectory of the gossip algorithm. Now given the convergence under cyclic evolution, there must exist some $\lambda < 1$ and integer $M > 0$ such that

$$\mathrm{Tr}((\mathcal{E}_C^M(\rho) - \rho_*)^2) \leq \lambda \mathrm{Tr}((\rho-\rho_*)^2)$$

for any ρ for which $\frac{1}{m!}\sum_{\pi\in\mathfrak{P}} U_\pi \rho U_\pi^\dagger = \rho_*$. The proof then concludes by observing that the probability to obtain an edge sequence which includes successions of M cyclic evolutions a sufficiently large number of times to have ε-convergence, gets arbitrarily close to 1 if we wait long enough. $\square$

The theorem thus shows that the mean value of any (global) observable $S = \frac{1}{m}\sum_\ell \sigma^{(\ell)}$, with arbitrary σ, can be asymptotically retrieved from the state of any single subsystem after having applied one of the quantum gossip algorithms. The convergence speed for the random case can be quite low, and a faster map would be obtained by effectively taking a mixture of all possible updates at each time – some sort of synchronous quantum gossip.

Example: Consider a network of $m = 4$ two-level systems, i.e. $\mathcal{H} \cong \mathbb{C}^2$. Denoting $(|0\rangle, |1\rangle)$ an orthonormal basis for $\mathcal{H}$, take as initial state: $\rho_0 = |1,0,1,0\rangle\langle 1,0,1,0|$, which is pure, and is not permutation invariant. Depending on how well a particular quantum subsystems swap can succeed, we will have different evolutions of the gossip algorithm. However, as long as the union of swap-links forms in expectation a connected graph (i.e. a path, a 3-branch star, or anything containing one of those), Theorem 1 ensures that the state asymptotically converges to:

$$\begin{aligned}
&\lim_{t\longrightarrow\infty} \rho(t) = \rho_* = \frac{1}{3!}\sum_{\pi\in\mathfrak{P}} U_\pi \rho_0 U_\pi^\dagger = \\
&= \frac{1}{6}(|1,1,0,0\rangle\langle 1,1,0,0| + |1,0,1,0\rangle\langle 1,0,1,0| \\
&\quad + |1,0,0,1\rangle\langle 1,0,0,1| + |0,1,1,0\rangle\langle 0,1,1,0| \\
&\quad + |0,1,0,1\rangle\langle 0,1,0,1| + |0,0,1,1\rangle\langle 0,0,1,1|).
\end{aligned} \tag{15}$$

This expression is clearly invariant under all the subsystem permutations. We can also check by direct computation that

$$\mathrm{Tr}(\sigma^{(\ell)}\rho_*) = \mathrm{Tr}(S\rho_*) = \mathrm{Tr}(S\rho_0)\,,$$

for any $\sigma = a|0\rangle\langle 0| + b|1\rangle\langle 1| + c(|0\rangle\langle 1| + |1\rangle\langle 0|) + id(|0\rangle\langle 1| - |1\rangle\langle 0|) \in \mathfrak{H}(\mathcal{H}^m)$, with $a,b,c,d \in \mathbb{R}$. Hence the expectation of σ at the end of the evolution is equal to the initial expectation of $S = \frac{1}{4}\left(\sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)} + \sigma^{(4)}\right)$.

III. Symmetrization as a Unifying Picture

To conclude, we present and analyze an alternative viewpoint that unifies classical and quantum gossip in a common framework, and allows for proving convergence by studying the asymptotic properties of a Markov chain over the elements of the permutation group.

Given a finite group $\mathcal{G}$ and an algebra X we consider the so called *action of the group*, namely a map $a : \mathcal{G} \times X \to X$ which is linear in X such that for each $g, h \in \mathcal{G}$, $x \in X$

- $a(h, a(g,x)) = a(hg, x)$,
- $a(e, x) = x$

where e is the identity element of the group.

The set $\mathfrak{P}$ of all the permutations of an m-elements set defines a finite group of order (i.e. number of elements) $|\mathfrak{P}| = m!$. Furthermore, $\mathbb{R}^d$ is an algebra for every $d \in \mathbb{N}_{>0}$. We can then define $a_{\mathbb{R}}(\pi, x) = P_\pi\, x$, such that an interaction of the classical gossip (1) rewrites as:

$$x(t+1) = \alpha\, a_{\mathbb{R}}(\pi_{j,k}, x) + (1-\alpha)\, a_{\mathbb{R}}(e, x) \tag{16}$$

with $\alpha \in (0,1)$ and $\pi_{j,k}$ denoting the permutation that performs the pairwise swap of j and k only. Henceforth we have that:

$$x(t) = \sum_{\pi \in \mathfrak{P}} p_\pi(t)\, a_{\mathbb{R}}(\pi, x(0)) \ , \tag{17}$$

with $p_\pi(t) \geqslant 0$ such that $\sum_{\pi \in \mathfrak{P}} p_\pi(t) = 1$, for every t.

On the other hand in the quantum case, since also $\mathfrak{B}(\mathcal{H}^m)$ is an algebra, we can define $a_{\mathfrak{B}}(\pi, x) = U_\pi\, x\, U_\pi^\dagger$ and from (6) we get again:

$$\rho(t) = \sum_{\pi \in \mathfrak{P}} p_\pi(t)\, a_{\mathfrak{B}}(\pi, \rho(0)) \ , \tag{18}$$

with $p_\pi(t) \geqslant 0$ such that $\sum_{\pi \in \mathfrak{P}} p_\pi(t) = 1$, for every t.

Thus in both cases, an evolution under the gossip algorithm can be described as a probabilistic mixture of the actions of the elements of the permutation group. If we pack all the $p_\pi(t)$ into the single vector $\mathsf{p}(t)$ and let $(j(t), k(t))$ be the edge activated at time t, *the gossip-induced dynamics is associated in both the classical and the quantum case to the same Markov chain on the weights* p*:*

$$\mathsf{p}(t+1) = M_{(j(t),k(t))}\, \mathsf{p}(t). \tag{19}$$

Note that here the "nodes" of the Markov chain are not the subsystems, but the elements π of the permutation group, so the Markov transition matrix M has dimension $m!$ times $m!$. Now the behavior of both the classical and the quantum gossip algorithms — and maybe of other variants — can be studied by analyzing the properties of the same transition matrices $M_{j(t),k(t)}$, for the specific initial state $\mathsf{p}(0)$ having all zero entries except an entry 1 at $\pi = e$. The convergence to average consensus of the gossip algorithm then means that the Markov chain, at least when starting at this particular $\mathsf{p}(0)$, converges to:

$$\lim_{t \to \infty} \mathsf{p}(t) = \frac{1}{m!} \mathbf{1} \tag{20}$$

where $\mathbf{1}$ is the vector whose entries are all equal to 1. Note that, at this level, convergence is independent of the specific algebra and action that are being considered.

Therefore, once the convergence of the Markov chain for the particular initial state is proven, it directly follows that both the classical and quantum gossip converge respectively toward:

$$\begin{aligned} x_\infty &= \frac{1}{m!} \sum_{\pi \in \mathfrak{P}} a_{\mathbb{R}}(\pi, x(0)) \ =: \bar{\mathcal{E}}_{\mathbb{R}}(x(0)) \,, \\ \rho_\infty &= \frac{1}{m!} \sum_{\pi \in \mathfrak{P}} a_{\mathfrak{B}}(\pi, \rho(0)) \ =: \bar{\mathcal{E}}_{\mathfrak{B}}(\rho(0)) \,. \end{aligned}$$

Using the properties of the group action and the fundamental laws of a group, it is easy to see that *independently of the algebra X on which the group acts*, the operator $\bar{\mathcal{E}}_X(\cdot)$ projects any state on the set of permutation invariant states. Indeed, let us consider any $\mu \in \mathfrak{P}$. Then

$$\begin{aligned} a_X(\mu, \bar{\mathcal{E}}_X(\cdot)) &= \frac{1}{m!} \sum_{\pi \in \mathfrak{P}} a_X(\mu\pi, \cdot) \\ &= \frac{1}{m!} \sum_{\nu \in \mathfrak{P}} a_X(\nu, \cdot) \ = \ \bar{\mathcal{E}}_X(\cdot) \,. \end{aligned}$$

IV. Conclusions and Research Directions

In this paper we develop a quantum version of the well-known gossip algorithm. Our approach follows the analogy with the classical setting as closely as possible, maintaining an operational viewpoint and working with a multipartite system (a quantum network). We propose a reformulation of the classical gossip-type algorithm in terms of the action of the permutation group, which leads to an immediate non-commutative counterpart. We then directly prove that the quantum version asymptotically prepares permutation invariant states while preserving the expectation of global permutation-invariant observables. The reformulation also leads to a unifying viewpoint on consensus problems as symmetrization problems, which in turn can be studied by resorting to a classical Markov dynamics on the group elements. Effective symmetrization can be asymptotically obtained if the auxiliary Markov dynamics converges to a uniform equilibrium.

A number of questions remain open. Among these an interesting point is to find further problems that can fit this symmetrization framework. Indeed, a number of quantum control problems are explicitly formulated in terms of symmetrization with respect to some unitary subgroup, see e.g. techniques related to *quantum dynamical decoupling* [16], [21]. It would also be interesting to assess the potential of devising continuous-time quantum consensus algorithms. This could build on some sort of "continuous swapping" Hamiltonian dynamics and lead to connections with problems of thermalization and quantum chaos in closed system dynamics [25].

Lastly, let us remark that in this paper we proposed a quantum algorithm in which the gossip-type interactions are selected in a classical way. The potential advantage of a fully quantum implementation, along with its connection to quantum random walks and Markov chain mixing properties [10], [11], is definitely worth further investigation.

Appendix

This paper considers finite-dimensional quantum systems. Their mathematical description starts by considering a finite dimensional complex Hilbert space $\mathcal{H} \simeq \mathbb{C}^d$. The (Dirac's) notation $|\psi\rangle$ denotes an element of $\mathcal{H}$ (called a *ket*), while $\langle\psi| = |\psi\rangle^\dagger$ is used for its dual (a *bra*), and $\langle\psi|\varphi\rangle$ for the associated inner product. We denote the set of linear operators on $\mathcal{H}$ by $\mathfrak{B}(\mathcal{H})$. The adjoint operator $X^\dagger \in \mathfrak{B}(\mathcal{H})$ of an operator $X \in \mathfrak{B}(\mathcal{H})$ is the unique operator that satisfies $(X|\psi\rangle)^\dagger\,|\chi\rangle = \langle\psi|\,(X^\dagger|\chi\rangle)$ for all $|\psi\rangle, |\chi\rangle \in \mathcal{H}$. We then denote $\mathfrak{H}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{H})$ the subset of self-adjoint operators, and $\mathfrak{U}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{H})$ the subset of unitary operators. The standard inner product in $\mathfrak{B}(\mathcal{H})$ is the Hilbert-Schmidt product $\langle X, Y\rangle = \mathrm{Tr}(X^\dagger Y)$, where Tr is the usual trace functional (which is canonically defined in a finite dimensional setting). We denote by I the identity operator. Working in a finite dimensional setting, we often consider vectors and operators as represented by complex matrices of suitable dimensions: $|\psi\rangle \in \mathcal{H} \simeq \mathbb{C}^d$ are represented by column vectors, so

$\langle\phi| \in \mathcal{H}^\dagger \simeq \mathbb{C}^d$ are row vectors; $X \in \mathfrak{B}(\mathcal{H}) \simeq \mathbb{C}^{d\times d}$ are $d \times d$ complex matrices, the adjoint $X^\dagger$ is the transpose conjugate of X, self-adjoint and unitary properties carry over to the associated matrices.

In statistical quantum theory, the state of a quantum system is represented by a *density operator* ρ, that is any self-adjoint, positive semi-definite operator with trace one. We denote the convex set of these operators (the state space) by $\mathfrak{D}(\mathcal{H})$. The extreme points of this set, namely the rank-one operators $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle \in \mathcal{H}$ and $\langle\psi|\psi\rangle = 1$, are called *pure states*. All the predictions about physical observations on the quantum system are then computed through the Hilbert-Schmidt product $\mathrm{Tr}(\rho X)$ of ρ with "observables" $X \in \mathfrak{H}(\mathcal{H})$ that characterize the measurement. Hence, one can equivalently describe the effect of a quantum evolution either on its state $\rho \in \mathfrak{D}(\mathcal{H})$ (this is called Schrödinger's picture), or in the dual representation on its observables $X \in \mathfrak{H}(\mathcal{H})$ (Heisenberg's picture).

A general framework for quantum evolution is offered by *quantum channels* [24], [7], that is, linear, completely positive (CP) and trace preserving (TP) maps from states to states $\mathcal{E} : \mathfrak{D}(\mathcal{H}^m) \to \mathfrak{D}(\mathcal{H}^m)$. It can be shown [24] that such maps admit an *Operator Sum Representation* (OSR):

$$\mathcal{E}(\rho) = \sum_{k=1}^{K} A_k \rho A_k^\dagger \quad \text{with} \quad \sum_{k=1}^{K} A_k^\dagger A_k = I \tag{21}$$

where $K \leqslant \dim(\mathcal{H})^2$. Given a linear, CPTP map $\mathcal{E}$, we can define its dual map with respect to the Hilbert-Schmidt inner product, $\mathcal{E}^\dagger : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ through the relation:

$$\mathrm{Tr}[X\mathcal{E}(\rho)] = \mathrm{Tr}[\mathcal{E}^\dagger(X)\rho] \tag{22}$$

The dual map, modeling the quantum channel in the Heisenberg picture, is still linear and CP but instead of TP it is unital i.e. $\mathcal{E}^\dagger(I) = I$.

If two quantum systems[3], with associated Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, are taken together to form a larger bipartite quantum system, the Hilbert space $\mathcal{H}_{1,2}$ associated to the composite quantum system is the tensor product of the individual quantum subsystem Hilbert spaces, $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Let $\{|\psi_k\rangle\}_{k=1}^{d_1}$ and $\{|\phi_l\rangle\}_{l=1}^{d_2}$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then an orthonormal basis for $\mathcal{H}_{1,2}$ can be written as:

$$\{|\psi_k\rangle \otimes |\phi_l\rangle\}_{k,l=1}^{d_1,d_2}\ , \tag{23}$$

from which we get that $\dim(\mathcal{H}_{1,2}) = \dim(\mathcal{H}_1)\dim(\mathcal{H}_2) = d_1 d_2$. We use the short notation $|\psi,\phi\rangle := |\psi\rangle \otimes |\phi\rangle$ for any $|\psi\rangle \in \mathcal{H}_1$ and $|\phi\rangle \in \mathcal{H}_2$. The composite Hilbert space is naturally endowed with the inner-product $\langle u_1, u_2|v_1, v_2\rangle := \langle u_1|v_1\rangle\langle u_2|v_2\rangle$. A representation and basis for operators in $\mathfrak{B}(\mathcal{H}_{1,2})$ is derived from this vector counterpart in the standard way. In particular, given two operators $X_1 \in \mathfrak{B}(\mathcal{H}_1)$ and $X_2 \in \mathfrak{B}(\mathcal{H}_2)$, one defines $X_1 \otimes X_2 \in \mathfrak{B}(\mathcal{H}_{1,2})$ such that

$$\begin{aligned} &X_1 \otimes X_2(|u_1\rangle \otimes |u_2\rangle) = \\ &= X_1|u_1\rangle \otimes X_2|u_2\rangle \quad \forall |u_1\rangle \in \mathcal{H}_1\ ,\ |u_2\rangle \in \mathcal{H}_2\ . \end{aligned} \tag{24}$$

This rule for combining subsystems enriches quantum theory with the phenomenon of *entangled states*, associated to vectors in $|\xi\rangle \in \mathcal{H}_{1,2}$ that *cannot* be factorized as $|\xi\rangle = |\psi\rangle \otimes |\phi\rangle$ into separate subsystems states.

[3] The general case of $n > 2$ systems is easily obtained by iteration.

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