Quantum State Preparation by Controlled Dissipation in Finite Time: From Classical to Quantum Controllers

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***Abstract*— A general way to implement state preparation for a register of qubits in a finite number of steps is proposed. Our splitting-subspace approach relies on control resources that are typically available in experimental implementations of quantum information (a universal set of unitary operations on the system, a controlled-not gate, and an ancillary resettable qubit), and can be seen as a "quantum-controller" implementation of a sequence of classical feedback loops. The realizability of the scheme with different control resources is discussed, along with an existing protocol used in state preparation and entanglement generation in experimental ion-trap systems.**

I. Introduction

State preparation problems are important for a variety of application of quantum information. If the preparation is to be achieved irrespective of the initial state of the system, from a control viewpoint they can be translated into stabilization problems. From a physical viewpoint, one is compelled to make the quantum evolution irreverisible, by introducing open-system features: a very natural way to address the problem is to consider measurements, and feedback control techniques. The emerging stabilization problems have been studied in some depth in the continuous-time case [16], [14], [9], [13], for either "output feedback" or strategies based on state reconstruction by quantum filtering. It is worth remarking that in this continuous-time scenario the desired state can be typically reached only asymptotically. Recently a linear-algebraic framework for analysis and synthesis has been extended to the discrete-time case. In particular, for a given indirect measurement and unconstrained unitary controls it has been shown in [3] that pure states are generically stabilizable. This is always true if the measurement is of projective type and, furthermore, if it is associated to a non degenerate observable, the desired state can be prepared in a single step (of measurement plus control) [4].

However, the needed control resources may be (and typically are) unavailable for many state-of-the-art experimental systems. Here we investigate how to prepare a desired state in finite time by means of a reasonable set of resources for a multipartite qubit system. In particular, the whole feedback loop will be "encoded" in a finite sequence of *coherent feedback* actions [6], [15], where the controller is a qubit itself and there is no-measurement involved. In addition, we will assume to be able to implement a universal set of gates (unitary control actions) on the target qubits. For certain experimental, and most notably ion traps, these control resources are not only achievable in principle, but have been experimentally demonstrated up to 5 qubits [2]. We will illustrate how the "stabilizer pumping" strategy proposed in [2] fits in our framework, and how control actions achieving stabilization of arbitrary targets can be designed.

Let us first recall a few basic notations and ideas about quantum systems and measurements. Consider a *finite-dimensional* quantum system of interest associated to a Hilbert space $\mathcal{H} \sim \mathbb{C}^d$. Observable quantities on the systems (or simply *observables*) are associated to self-adjoint operators on $\mathcal{H}$, here represented by Hermitian matrices $X = X^\dagger \in \mathfrak{H}(\mathcal{H})$, and the system *state* is associated to a density matrices $\rho \in \mathfrak{D}(\mathcal{H}) = \{\rho \in \mathfrak{H}(\mathcal{H}) | \rho \geq 0,\ \mathrm{tr}(\rho) = 1\}$. Unitary matrices are denoted by $U \in \mathcal{U}(\mathcal{H})$. The (real) spectrum of an observable represents the set of the possible outcomes. Suppose that we are interested in measuring X, which admits spectral decomposition $X = \sum_i x_i \Pi_i$. The basic postulates that describe the quantum (projective, or von Neumann's) measurements state that the probability of obtaining c_i on a system in the state ρ is $p_i = \mathrm{Tr}(\rho\Pi_i)$. Immediately after a measurement that gives c_i as an outcome the system state becomes: $\rho|_i = \frac{1}{\mathrm{tr}(\Pi_i\rho\Pi_i)}\Pi_i\rho\Pi_i$.

If a quantum system is obtained by composition of two physically distinguishable *subsystems* associated to Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, the corresponding mathematical description is carried out in the tensor product space, $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ [11], and observables and density operators remain associated with Hermitian and positive-semidefinite, trace-one operators on $\mathcal{H}_{12}$, respectively. If we get information about a quantum subsystem by measuring another one which is correlated to the former, we can obtain a reduced description of the effect of the measurement on the subsystem of interest by using the formalism of *generalized measurements*. To a set of k possible outcomes is associated a set of operators $\{M_k\}$ such that $\sum_k M_k^\dagger M_k = I$, where I is the identity operator. The probability of obtaining the k-th outcome is thus computed as $p_k = \mathrm{tr}(M_k^\dagger M_k\rho)$, ρ being the (reduced) state on the subsystem of interest which after the outcome is recorded is updated to $\rho|_k = \frac{1}{\mathrm{tr}(M_k\rho M_k^\dagger)}M_k\rho M_k^\dagger$.

If now take the average over the possible outcomes, we obtain a Trace-Preserving, Completely Positive (TPCP)

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linear transformation of the state in the form of a Kraus map [5]:

$$\mathbb{E}(\rho) = \sum_k M_k \rho M_k^\dagger.$$

The standard, projective-measurement rules are recovered by choosing $M_k = \Pi_k$.

II. Feedback Stabilization

Suppose that a generalized measurement operation can be performed on the system at times $t = 1, 2, \ldots$, resulting in an open system, discrete-time dynamics described by a given Kraus map, with associated Kraus operators $\{M_k\}$. Suppose moreover that we are allowed to enact arbitrarily control action on the state of the system, i.e. $\rho'' = U\rho U^\dagger$, $U \in \mathcal{U}(\mathcal{H})$, and the controls are fast with respect to the measurement time scale, or the measurement and the control act in distinct "time slots".

We can then in principle implement a Markovian feedback law, consisting in a map from the set of measurement outcomes to the set of unitary matrices, $U(k) : k \mapsto U_k \in \mathcal{U}(\mathcal{H}_I)$. The measurement-control loop is then iterated: If we average over the measurement results at each step, this yields a different TPCP map, which describes the evolution of the state *immediately after* each application of the controls:

$$\rho(t+1) = \sum_k U_k M_k \rho(t) M_k^\dagger U_k^\dagger.$$

Controllability and stabilizability for this class of discrete-time, closed-loop dynamics have been studied in detail in [8], [4], [3], [1]. In particular, from the results of [3], [1] it is immediate to see that if the following resources are available:
1) Arbitrary unitary control actions U_k on the system;
2) A *non-degenerate projective measurement*, associated to a resolution of the identity $\{\Pi_k = |\phi_k\rangle\langle\phi_k|\}_{k=1}^d$,
then the system can be prepared in *any desired pure state in one step*. Let $\rho_d = |\psi\rangle\langle\psi|$ be the target pure state: this is simply achieved by choosing control operations U_k such that

$$U_k|\phi_k\rangle = |\psi\rangle.$$

This is indeed an abstract description of the most commonly used experimental strategies to prepare a given state: first measure the system projecting to some known state, and then enact some open-loop, controlled transition to steer it to the desired state. However, in many relevant cases this is prevented by the unavailability of suitable measurement procedures (measurement may be destructive, slow, inaccurate or not having the needed resolution). In what follows, focusing on multi-qubit systems, we illustrate a way to tackle the same pure-state preparation by replacing the single, full resolution measurement above with ability of using an extra qubit as a resettable quantum controller [6], [7].

III. The Splitting Subspace Approach

Let us consider an N-qubit system $\mathcal{H}_Q = \bigotimes_j \mathcal{H}_j \sim \mathbb{C}^{2^N}$, with $\{|\phi_j\rangle\}_{j=1}^{2^N}$ denoting the standard basis of $\mathcal{H}_Q$, $|\phi_1\rangle = |0\ldots 00\rangle$, $|\phi_2\rangle = |0\ldots 01\rangle, |\phi_3\rangle = |0\ldots 10\rangle, \ldots$.

Assume the following control resources are available:
a) A universal set of unitary control actions $\{U\}$ on the target qubits;
b) An auxiliary *control qubit*, with Hilbert space $\mathcal{H}_c$, that can be reset to a known pure state, say $|1\rangle$;
c) Controlled-not unitaries $C_{out} = I_2 \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1|,\ C_{in} = |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes \sigma_x \in \mathcal{U}(\mathcal{H}_c \otimes \mathcal{H}_j)$ between the control qubit and one of the first target qubits.
We next show that a simple approach can be developed in order to design "circuits" preparing in a finite number of steps the target pure state. The first step is to provide a characterization of the target state in terms of a family of *splitting subspaces*.

Lemma 3.1: **(Splitting subspaces)** Any $|\psi\rangle \in \mathcal{H}_Q$ can be described as the unique unit-norm vector in the intersection of N subspaces $\mathcal{S}_k$ of dimension 2^{N-1}, i.e.

$$\text{span}\{|\psi\rangle\} = \bigcap_k \mathcal{S}_k.$$

Proof. It is sufficient to provide an explicit way to construct the $\mathcal{S}_k$: start by relabeling $|\psi_1\rangle := |\psi\rangle$ and complete it with 2^N vectors so that $\{|\psi_i\rangle\}_{i=1}$ is an orthonormal basis[1]. Then define $\mathcal{S}_1 = \text{span}\{|\phi_k\rangle,\ k = 1, ..., 2^{N-1}\}$, $\mathcal{S}_2 = \text{span}\{|\phi_k\rangle,\ k = 1, ..., 2^{N-2}, 2^{N-1}+1, \ldots, 2^{N-1}+2^{N-2}\}$, and so on for the next subspaces, as depicted in Figure 1. Formally, by defining the matrices $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$$\Pi^{(k)} = I_2^{\otimes k-1} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_2^{\otimes n-k},$$

in the $|\psi_k\rangle$-basis, we can define the splitting subspace as

$$\mathcal{S}_k = \text{range}(\Pi^{(k)}).$$

By construction, that the only vector in the intersection of the subspaces is $|\psi_1\rangle$. $\square$

It is worth noting that for *stabilizer states* [10], the splitting subspaces can be taken to be the 1-eigenspaces of the associated stabilizer operators. However, as showed in the Lemma above, there is no need to restrict to this subset of pure states.

The next step is to show that the system state can be prepared in a splitting subspace $\mathcal{S}$ in one step given the available control resources. If we could perform a projective measurement of $\Pi_S, \Pi_S^\perp$, the orthogonal projection onto $\mathcal{S}$ and its orthogonal complement, it would be be sufficient to use a feedback law that does nothing if Π_S is measured, and for the other outcome applies a $U_\perp$ such that $U_\perp \Pi_S^\perp U_\perp^\dagger = \Pi_S$. Notice, again, that this control action would be highly not unique (at least there is freedom on the unitary mapping from $\mathcal{S}^\perp$ to $\mathcal{S}$, which is an element of $SU(2^{N-1})$). The average total evolution would be of the form:

$$\rho(t+1) = \mathcal{E}_S(\rho_t) = \Pi_S \rho(t) \Pi_S + U_\perp \Pi_S^\perp \rho(t) \Pi_S^\perp U_\perp^\dagger, \quad (1)$$

[1] Of course, there is a lot of freedom in doing so, equivalent to choosing an element of $SU(2^N - 1)$, that can be potentially exploited in reducing the complexity of the unitary operations in what follows.

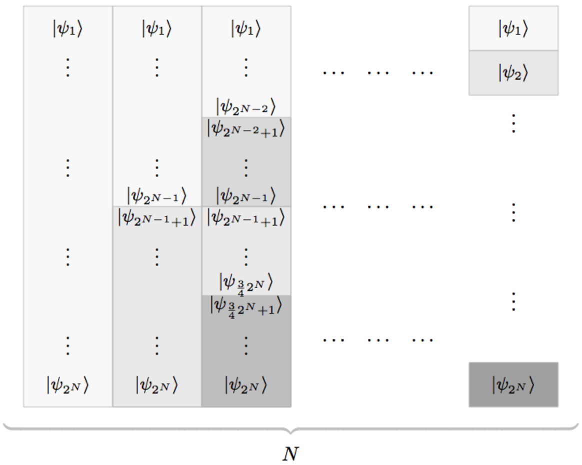

Fig. 1. Splitting subspaces.

and it is easy to see that the support of $\rho(t+1)$ is contained in $\mathcal{S}$:

$$\begin{aligned}\mathrm{tr}(\Pi_S\rho(t+1)) &= \mathrm{tr}(\Pi_S\rho(t)) + \mathrm{tr}(U_\perp^\dagger \Pi_S U_\perp \Pi_S^\perp \rho(t)\Pi_S^\perp) \\ &= \mathrm{tr}(\Pi_S\rho(t)) + \mathrm{tr}(\Pi_S^\perp\rho(t)) = 1,\end{aligned}$$

where we used the fact that we chose $U_\perp$ such that $U_\perp \Pi_S^\perp U_\perp^\dagger = \Pi_S$. The exact same TPCP map can be implemented using the extra qubit as a quantum controller: let us assume without loss of generality that it can interact with the first target qubit, and denote as $\{|\psi_k\rangle\}$ an orthonormal basis for $\mathcal{H}_Q$, such that the first 2^{N-1} elements are a basis for $\mathcal{S}$. Then proceed as it follows:

1) Perform the unitary $U_\psi^\dagger$ on the target qubits, where $U_\psi|\phi_k\rangle = |\psi_k\rangle$. This "maps" the information on whether or not the state of the system is in $\mathcal{S}$ *on the state of the first qubit*.
2) Perform C_{out} on the control and the first qubit, mapping the information of the first qubit on the state of the control qubit;
3) Perform C_{in} on the control and the first qubit, changing the state of the first qubit depending on the projection of the control qubit on $|0\rangle$;
4) Perform the change of basis U_ψ to return to the original basis and (if needed for successive steps) reset the control qubit.

The net effect on the target qubits of these operations is:

$$\begin{aligned}U_{tot} &= U_\psi C_{in} C_{out} U_\psi^\dagger \\ &= U_\psi C_{in} U_\psi^\dagger U_\psi C_{out} U_\psi^\dagger = \hat{C}_{in}\hat{C}_{out} \\ &= (|1\rangle\langle 1| \otimes I_{2^N} + |0\rangle\langle 0| \otimes U_\perp)(I_2 \otimes \Pi_S + \sigma_x \otimes \Pi_S^\perp),\end{aligned}$$

where $U_\perp$ satisfies the condition $U_\perp \Pi_S^\perp U_\perp^\dagger = \Pi_S$. Hence, by averaging over the control-qubit degrees of freedom by use of the partial trace [10], we get:

$$\begin{aligned}\rho(t+1) &= \mathrm{tr}_{\mathcal{H}_c}(U_{tot}(|1\rangle\langle 1| \otimes \rho(t))U_{tot}^\dagger) \\ &= \Pi_S\rho(t)\Pi_S + U_\perp \Pi_S^\perp \rho(t)\Pi_S^\perp U_\perp^\dagger = \mathcal{E}_{\mathcal{S}}(\rho_t).\end{aligned} \tag{2}$$

If we have $N-1$ splitting subspace constructed as in the proof of Lemma 3.1, the basis $\{|\psi_k\rangle\}$ can be chosen *at each step* to be a reordering of the first one. With this choice, the desired state-preparation result can be then easily proven.

Proposition 3.1: Any target state $\rho_d = |\psi\rangle\langle\psi|$ can be prepared using control resources a)-b)-c) in N feedback steps irrespective of the initial state of the qubit system ρ.

Proof. Let $\{\mathcal{S}_i\}$ be splitting subspace description for $|\psi\rangle$ constructed as in the proof of Lemma 1, with $\mathcal{E}_{\mathcal{S}_i}$ the associated TPCP maps defined as in (1), and implement $\mathcal{E}_{\mathcal{S}_N} \circ \ldots \circ \mathcal{E}_{\mathcal{S}_1}(\rho)$. By construction, each $\mathcal{E}_{\mathcal{S}_i}$ maps $\left(\bigcap_{k=1}^{i-1}\mathcal{S}_k\right) \cap \mathcal{S}_i^\perp$ onto $\left(\bigcap_{k=1}^{i-1}\mathcal{S}_k\right) \cap \mathcal{S}_i$. Thus, at the N-th control step, any initial state is driven into $\left(\bigcap_{k=1}^{N}\mathcal{S}_k\right) = \mathrm{span}\{|\psi\rangle\}$. $\square$

We want to stress that, while the controlled-not operation entangling the control and the first qubits, as well as the resetting of the control qubit, are the same at each step, and hence in real-world implementation will take a fixed time to be enacted, *the U_ψ will be different* since it will include a reshuffling of the basis elements in order to obtain the "nested" subspace structure we used for the proof (it can be shown that this will need in general up to $(N-i)/2$ swaps of qubit at step i). This can add up to the total implementation time, making initial the choice of the (non-unique) basis $\{|\psi_k\rangle\}$ critical for efficient implementation.

A. Effective entangling operations

In real-world experimental settings, engineering exactly the controlled-not gate we assumed may still be hard to do. However, it is easy to see that the way of realizing the "feedback" map (1) with quantum controllers is highly not unique. Even if the controlled-not gates are achievable in principle, it is very likely that other choices may turn out to be more convenient to implement.

More precisely, it is easy to verify that *any* pair of entangling operators of the following form could achieve the desired task (possibly up to some extra unitary operation on the system qubits):

$$\tilde{C}_{out} = W\left(D_c \otimes (U_S\Pi_S) + O_c \otimes (U_S\Pi_S^\perp)\right), \tag{3}$$

$$\tilde{C}_{in} = (|1\rangle\langle 1| \otimes V_S + |0\rangle\langle 0| \otimes (V_S U_\perp))\, W^\dagger, \tag{4}$$

where D_c, O_c are unitary and diagonal (off-diagonal) in the standard basis of $\mathcal{H}_c$, while U_S and V_S are unitary operators for which *both* $\mathcal{S}$ and $\mathcal{S}^\perp$ are invariant, and finally W is an arbitrary unitary operator on the whole system.

This implies that $[U_S, \Pi_S] = [V_S, \Pi_S] = 0$, and similarly for $\Pi_S^\perp$. The total dynamics then reads:

$$\begin{aligned} \rho(t+1) &= \mathrm{tr}_{\mathcal{H}_c}(U_{tot}(|1\rangle\langle 1| \otimes \rho(t))U_{tot}^\dagger) \\ &= V_S \Pi_S U_S \rho(t) U_S^\dagger \Pi_S V_S^\dagger \\ &\quad + V_S U_\perp \Pi_S^\perp U_S \rho(t) U_S^\dagger \Pi_S^\perp U_\perp^\dagger V_S^\dagger = \tilde{\mathcal{E}}_{\mathcal{S}}(\rho). \end{aligned} \tag{5}$$

The presence of U_S, V_S could slow down, or even prevent, the desired state preparation since in general it is no longer true that $\left(\bigcap_{k=1}^{i-1} \mathcal{S}_k\right) \cap \mathcal{S}_i^\perp$ is mapped onto $\left(\bigcap_{k=1}^{i-1} \mathcal{S}_k\right) \cap \mathcal{S}_i$, a key step in the convergence proof. However, their effect can be eliminated by suitable (open loop) unitary gates on the system qubits alone.

IV. Case Study: Trapped Ions

A. Experimental Bell-State Pumping

In [2], a toolbox for simulating and controlling dynamics of a quantum system with up to five qubits has been presented. It is based on an trapped ions experimental system (in a linear trap) that implements a circuit-model quantum computer, combining multi-qubit gates with optical pumping to implement coherent operations as well as dissipative processes. We here review the proposed two-qubit *Bell-state pumping* procedure in the light of our splitting subspace approach. The same approach has been successfully employed to create GHZ states on up to 5 qubits.

Let us denote the four Bell-states as:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle),\ |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{6}$$

The system, initially in an unknown density ρ_S, is deterministically prepared in the Bell-state $|\Psi^-\rangle$ by realizing in two steps the quantum operation $\rho_S \mapsto |\Psi^-\rangle\langle\Psi^-|$. The Bell-states are *stabilizer states*[2]: for instance $|\Phi^+\rangle$ is associated to the commuting stabilizer operators Z_1Z_2 and X_1X_2, (it is the only two-qubit state being simultaneously an eigenstate of eigenvalue +1 of both, $Z_1Z_2|\Phi^+\rangle = +|\Phi^+\rangle$, $X_1X_2|\Phi^+\rangle = +|\Phi^+\rangle$), or equivalently X_1X_2 and Y_1Y_2. In fact, each of the four Bell-states (6) is uniquely determined as an eigenstate with eigenvalues ± 1 with respect to Y_1Y_2 and X_1X_2. The considered strategy engineers two maps under which the system qubit states are transferred from the +1 into the -1 eigenspace of Y_1Y_2 and X_1X_2.

In order to implement experimentally the first map, three unitary operations and a dissipative one have been used. All act on the two system qubits, denoted by subscript S, and an ancillary one, with subscript C. A key role is played by the Mølmer-Sørensen (MS) gates [12] $U_{X^2}(\theta) = \exp(-i\frac{\theta}{4}{S_x}^2)$ and $U_{Y^2}(\theta) = \exp(-i\frac{\theta}{4}{S_y}^2)$, where S_x and S_y denote collective spin operators: $S_x = \sum_{i=0}^n X_i$ and $S_y = \sum_{i=0}^n Y_i$. The main steps of this experimental quantum circuit are listed below:

[2]We shall use the standard notation for stabilizer states: X_k, Y_k, Z_k denote the operators acting as the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ on the k-th qubit, and the identity on the rest.

(i) Information about whether the system is in the +1 or -1 eigenspace of X_1X_2 is mapped by MS gate $U_{X^2}(\pi/2)$ onto the logical states $|0\rangle$ and $|1\rangle$ of the ancilla.
(ii) A controlled gate performs a conversion from the +1 eigenvalue of the stabilizer X_1X_2 to -1 by acting on the first system qubit[3]:

$$C(p) = |0\rangle\langle 0|_E \otimes Z_1 + |1\rangle\langle 1|_E \otimes I. \tag{7}$$

(iii) The MS gate $U_{X^2}(\pi/2)$ is re-applied, in order to move the state back to the initial basis representation.
(iv) The ancilla qubit is then reinitialized dissipatively in state $|1\rangle$.

Next, the whole circuit is repeated, but this time using $U_{Y^2}(\pi/2)$ gates (while the controlled-gate C_{in} remains the same). In order to show that this can be seen as an instance of of our splitting-subspace approach, we need to take a closer look at the structure of the MS gate.

B. The MS gate

The MS entangling gate is based on pairwise two-ion interaction terms and can be parametrized by two angles θ and ϕ:

$$U_{MS}(\theta, \phi) = \exp\Big(-i\frac{\theta}{4}(\cos\phi S_x + \sin\phi S_y)^2\Big), \tag{8}$$

The sum in the collective spin operators $S_x = \sum_{i=0}^n X_i$ and $S_y = \sum_{i=0}^n Y_i$, is understood to be performed over all ions involved in the gate.
On the rest of the work we suppose $\phi = 0$, so (8) can be rewritten as:

$$U_{X^2}(\theta) = \exp\Big(-i\frac{\theta}{4}\Big(\sum_{i=0}^n X_i\Big)^2\Big). \tag{9}$$

In the experimental Bell-state pumping protocol we are examining, we need MS gates with a phase angle $\theta = \pi/2$. In this case the $U_{X^2}(\pi/2)$ operator can be decomposed in a more explicit form with respect to the $\mathcal{H}_C \otimes \mathcal{H}_S$ tensor decomposition:

$$U_{X^2}\Big(\frac{\pi}{2}\Big) = U_X'(I \otimes \Pi_{-1} + X \otimes \Pi_{+1}), \tag{10}$$

where U_X' is a unitary 8×8 matrix of the form:

$$U_X' = I \otimes \frac{1}{2}e^{-i\pi/8}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = I \otimes U_X''. \tag{11}$$

[3]In the experiment the gate is realized through the following sequence of operators ($C, 1$ denoting the control and the first system qubit, respectively): $C(p) = U_{Z_1}(\alpha)U_Y(\pi/2)U_{X^2}^{(C,1)}(-\alpha)U_Y(-\pi/2)$ with $U_{X^2}^{(C,1)}(-\alpha) = exp(i(\alpha/2)X_CX_1)$, and $U_{Z_1} = e^{i\alpha Z_1}$ and $p = \sin^2(\alpha)$.

The action of the unitary operator U''_X on the Bell states basis (6) is described as:

$$\begin{aligned}
U''_X|\Phi^+\rangle &= -\frac{1}{\sqrt{2}}e^{-i\pi/8}\cdot|\Psi^+\rangle \\
U''_X|\Phi^-\rangle &= \frac{1}{\sqrt{2}}e^{-i\pi/8}\cdot|\Phi^-\rangle \\
U''_X|\Psi^+\rangle &= -\frac{1}{\sqrt{2}}e^{-i\pi/8}\cdot|\Phi^+\rangle \\
U''_X|\Psi^-\rangle &= \frac{1}{\sqrt{2}}e^{-i\pi/8}\cdot|\Psi^-\rangle
\end{aligned}$$

Hence, U''_X is not harmful for our purpose since it swaps the Bell-states in +1 eigenspace of X_1X_2 and does not change, except for a constant, the other Bell-states in the -1 eigenspace. Explicitely, in the Bell U'' basis, $\mathcal{B}_{Bell} = \{|\Phi^+\rangle, |\Psi^+\rangle, |\Phi^-\rangle, |\Psi^-\rangle\}$, can be written as:

$$U''_{X,Bell} = \frac{1}{\sqrt{2}}e^{-i\pi/8}\left[\begin{array}{c|c} X & O \\ \hline O & I \end{array}\right], \tag{12}$$

where symbol O denotes a 2×2 matrix of zeros. We have thus obtained a decomposition of the MS gate $U_{X^2}(\pi/2)$ in a form that includes two terms:

1) The conditional operation $C^X_{out} = I\otimes\Pi_{-1} + X\otimes\Pi_{+1}$ that coherently transfer to the ancilla qubit the information about in which of the two subspaces the system state is;
2) An additional unitary U'_X that is not harmful for stabilization purposes, since $[U'_X, \Pi^{X_1X_2}_{\pm}] = 0$.

This means that the $U_{X^2}(\pi/2)$ gate can be decomposed precisely in the form of (3), and then it is a viable resource for implementing a splitting-subspace approach for preparation of the desired subspace.

Similarly, the MS gate $U_{Y^2}(\pi/2)$ can be decomposed in the form:

$$U_{Y^2}\left(\frac{\pi}{2}\right) = U'_Y(Z\otimes\Pi'_{-1} + X\otimes\Pi'_{+1}), \tag{13}$$

where in this case unitary operator U'_Y admits a decomposition:

$$U'_Y = Z\otimes\frac{1}{2}e^{-i\pi/8}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = Z\otimes U''_Y. \tag{14}$$

In conclusion, we have found a decomposition of the MS gate $U_{Y^2}(\pi/2)$ in the form (3) which involves a conditional unitary $C^Y_{out} = Z\otimes\Pi'_{-1} + X\otimes\Pi'_{+1}$, and an additional unitary U'_Y with the same roles as for the X_1X_2 case. This means that also $U_{Y^2}(\pi/2)$ is a potential resource for the splitting subspace method.

In the implementation of the protocol described above, the *second* application of the MS gates in step (*iii*) has essentially the effect of canceling the effect of U'_X, U'_Y on the system qubits. However, it is easy to see that for the first map there is no actual need for this MS gate. In fact, the net effect of the "residual" U'_X is to swap the state that were in the $+1$ eigenvalue of X_1X_2, but they end up being "pumped" in the correct subspace anyway.

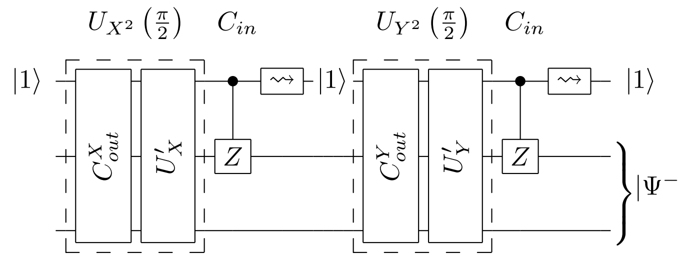

Fig. 2. Simplified version (a single MS gate is used in each control step) of the Bell-state pumping protocol proposed by Barreiro *et al.* [2].

In an analogous fashion, by direct calculation using (14), one can check that the action of U'_Y on the subspace that is prepared by the first map, namely the one generated by $|\Phi^-\rangle, |\Psi^-\rangle$, is:

$$\begin{aligned}
U''_Y|\Phi^-\rangle &= -\frac{1}{\sqrt{2}}e^{-i\pi/8}\cdot|\Psi^+\rangle, \\
U''_Y|\Psi^-\rangle &= \frac{1}{\sqrt{2}}e^{-i\pi/8}\cdot|\Psi^-\rangle.
\end{aligned}$$

The desired state $|\Psi^-\rangle$ is not perturbed: if now we apply the controlled-σ_z to the other case we get:

$$Z_1|\Psi^+\rangle = |\Psi^-\rangle,$$

so state preparation is achieved without need for step (iii), or, equivalently, without undoing U'_X, U'_Y. This simplified version of the control protocol, which in fact corresponds to the experimentally implemented one in [2] as noted in the supplementary material, is depicted in Figure 2.

V. Conclusions

We presented a general framework to design stabilizing controls that prepare *arbitrary pure states in finite time*: once a representation of the state via splitting subspaces is chosen, this can be achieved either by measurements and feedback [3], [4], or by resorting to *coherent controllers* [6]. The measurement and feedback steps are then substituted by suitable conditional operations C_{out}, C_{in}, respectively, enacting the necessary quantum-information flows directed out of the system towards the controller, and reverse. This implementation can be convenient in a number of situations where measurements are slow, inaccurate or destructive for the system itself. The method leaves freedom of choice in constructing a suitable basis for the system space, that can be exploited to minimize the complexity of control operations required, and allows for some flexibility in the needed conditional operations. A recently proposed algorithm for Bell-state state preparation in ion traps [2] is reviewed in the light of our splitting-subspace approach, demonstrating its applicability in experimental situations, and its potential utilization in design protocols that prepare arbitrary states in finite time.

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