Stability and Robustness in Quantum Coherent Control

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Abstract

Even for the easy and apparently well known quantum "coherent control" problem, the standard analysis methods of system theory can give some deeper insight and natural formalization of two fundamental properties: stability and robustness. A weak Input to State Stability is demonstrated for unitary evolution. The Robustness idea is quantitatively formulated in a classical control theory framework and verified by comparing the analysis results with the evaluations found in literature.

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1 Introduction: a Simple Model

We consider an isolated n-dimensional quantum system with time evolution described by the following Schrödinger equation:

$$i\hbar|\dot{\psi}(t)\rangle \quad = \quad H(t)|\psi\rangle. \tag{1}$$

Here $|\psi(t)\rangle$ is a vector of unit norm in $\mathbb{C}^n$ representing the state of the system at time t. The unitary time evolution of the system is governed by the system Hamiltonian:

$$H(t) = H_0 + \sum_{j=1}^{m} H_j u_j(\theta, t). \tag{2}$$

The *internal Hamiltonian* $H_0 \in \mathbb{C}^{n\times n}$ is an Hermitian matrix describing the free evolution of the system. The *control Hamiltonian*

$$H_c(t) = \sum_{j=1}^{m} H_j u_j(\theta, t),$$

where $H_j \in \mathbb{C}^{n\times n}$ are also Hermitian matrices, accounts for the effects of the control inputs $u_1(\theta, t), ..., u_m(\theta, t)$ on the dynamics of the system. We assume that these control functions depend on a finite number of real parameters $\theta = (\theta_1, ..., \theta_p)$, $\theta_k \in \mathcal{T}$, with $\mathcal{T}$ being an open set in $\mathbb{R}^p$. This kind of assumption is reasonable if we think of the small set of parameters we can control in an experimental setting.

We consider the problem of steering the system from a given initial state $|\psi_0\rangle = |\psi(0)\rangle$ to a final state $|\psi_1\rangle$, where $|\psi_0\rangle$ and $|\psi_1\rangle$ are unit vectors in $\mathbb{C}^n$. We assume that the transition occurs (at t=T) when $\theta = \theta^*$, which we take as "nominal" value of the parameters. Clearly, if $\theta \neq \theta^*$, the transition will, in general, not occur.

2 Quantum Stability?

2.1 Classical stability concepts and Isolated Quantum Systems

Let us recall briefly two well established approaches to stability characterization. In classical system theory we can evaluate:

Lyapunov's Stability or stability with respect to the initial condition. Consider a continuous time system in the form $\dot{x} = f(x)$. A stationary point ($\bar{x}$ such that $f(\bar{x} = 0)$ is said to be *stable* if

$$\forall \varepsilon > 0 \exists \delta | |x(0)| < \delta \Rightarrow |x(t)| < \varepsilon \forall t > 0.$$

The stationary point is said *asymptotically stable* if it is stable and for $\lim_{t\to\infty} x(t) = \bar{x}$. This kind of stability is local for nonlinear systems.

Input-Output Stability or Bounded-Input-Bounded-Output stability. Considering the system as a map between signals sets, it is called *BIBO stable* if to every bounded input corresponds a bounded output. This is a global characteristic of the system.

Before trying to use these approaches in order to characterize the dynamical behavior of the model presented in the previous section, it is fundamental to remember that:

(a) The time evolution is driven exclusively by the Hamiltonian H that is an **Hermitian** matrix. This implies that:

1. Hamiltonian eigenvalues are real;
2. Evolution is unitary;
3. If we consider normalized states (i.e. states with unit norm) as initial condition for the evolution, the final states will also be unitary.

(b) The overall phase of a state vector has no physical meaning (i.e. it cannot be observed by a measure process).

Thus, both the classical approaches make a little sense in the isolated quantum dynamics. In fact, even extending the stationary point definition to comprehend the energy eigenvectors (the natural stationary states, thanks to the observation (b)), the unitary evolution ensures us that the norm of the vector connecting the initial condition of the evolution and a stationary state will be conserved. This implies that the norm of "errors" will be preserved. The system is always stable, but cannot be asymptotically stable. Another way to see it is to consider the eigenvalues for the bilinear system (**??**) with constant input functions. Since the Hamiltonian eigenvalues are real, the eigenvalues for $\frac{i}{\hbar}H$ are purely imaginary, and use the well known results from linear system theory.

Nevertheless, the Input-Output stability is always guaranteed directly by unitary evolution if we consider as the system output the system state.

Unitary evolution has a lot of other relevant consequences for the isolated system. Consider, for instance, two probabilistic mixtures of states $\rho = \sum_i^n p_i|\alpha_i >< \alpha_i|$, $\sigma = \sum_j^n q_i|\beta_j >< \beta_j|$, with $\sum_i p_i = \sum_j q_j = 1$. A natural "index" of the mixed state distinguishability is the Kullback-Liebler pseudo-distance, defined as:

$$\mathcal{D}(\rho\|\sigma) = Tr(\rho \log \rho - \rho \log \sigma).$$

It is not properly a metric (It is positive, but not symmetric and triangular inequality does not hold), but emerges as a relevant pseudo-distance in many aspects of quantum information theory.

Can be easily demonstrated that as long as evolution is unitary, driven by the same Hamiltonian (it follows directly from Schrödinger equation that $i\hbar\dot{\rho} = [H_{tot}, \rho], i\hbar\sigma = [H_{tot}, \sigma]$), holds that:

$$\frac{\partial}{\partial t}\mathcal{D}(\rho\|\sigma) = -\frac{i}{\hbar}Tr(H_{tot}, [\rho \log \sigma]) = 0.$$

In other words, also distinguishability between statistical mixtures of states is preserved under the same Hamiltonian evolution.

2.2 Input to State Continuity

The (simple) stability with respect to initial condition can be seen as continuity of the system trajectory (with the sup norm) with respect to the initial condition in a neighborhood of a stationary point. Another form of continuity can be easily proven: the final state of the evolution over a finite time interval is a continuous with respect to input variations.

To simplify the exposition, in the following demonstration it is considered the usual finite dimensional system driven by a single input function, although the extension to multi-input infinite dimensional systems does not present extra-difficulties. Consider the model (1-2) in the $m = 1$ case, which becomes:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{H_0 + u_1(t)H_1}{i\hbar}|\psi(t)\rangle;$$

suppose that:

- $[t_0, t_1]$ is the (fixed) time interval in which evolution take place;
- $|\psi_0\rangle = |\psi(t_o)\rangle$ is the initial state for the evolution;
- $u^*(t)$ is an effective control choice that leads $|\psi_0\rangle$ to the target state $|\psi_f\rangle = |\psi_{u^*}(t_1)\rangle$.

We want to analyze the effect of a "perturbed" input $u(t) := u^*(t) + \delta u(t)$ on the final state of the evolution, where we are assuming δu small in an opportune norm (i.e. $\|\delta u\| = \sup_{[t_0,t_1]} |\delta u(t)|$). The evolution is then determined by solving the following Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi_\delta(t)\rangle = H_0|\psi_\delta(t)\rangle + \; + u^*(t)H_1|\psi_\delta(t)\rangle + \delta u H_1|\psi_\delta(t)\rangle;$$

Comparing with the unperturbed one:

$$i\hbar\frac{\partial}{\partial t}(|\psi_\delta(t)\rangle - |\psi(t)\rangle) = H_0(|\psi_\delta(t)\rangle - |\psi(t)\rangle) + u^*(t)H_1(|\psi_\delta(t)\rangle + - |\psi(t)\rangle) + \delta u H_1|\psi_\delta(t)\rangle. \quad (3)$$

Define the "difference" vector as: $\Delta\psi(t) = |\psi(t)\rangle - |\psi_\delta\rangle$.
Reformulating the stability issue, we are wondering if $\forall\,\varepsilon > 0\ \exists\,\delta > 0$ such that, if:

$$\|\delta u(t)\| < \delta \Rightarrow \|\Delta\psi(t_1)\| < \varepsilon,$$

where with$\|\cdot\|$ we intend the sup norm.

Then rewrite (3) as:

$$\frac{\partial}{\partial t}\Delta\psi(t) = (H_0 + u^*(t)H_1)\,\Delta\psi(t) + H_1(|\psi_\delta(t)\rangle)\delta u(t). \quad (4)$$

In a compact form;

$$\frac{\partial}{\partial t}\Delta\psi(t) = H(t)\Delta\psi(t) + f(t). \quad (5)$$

In the hypothesis we have done $\Delta\psi(0) = 0$, H_1 is constant and $|\psi_\delta(t)\rangle$ is bounded: Forcing $\delta u(t)$ to be smaller than δ, we can obtain that $\|f(t)\|$ is arbitrarily small.

Introducing the Green's function for the system $\Phi(t_0, t)$, the general solution for $\Delta\psi(t)$ is of the form:

$$\Delta\psi(t) = \Phi(t_0, t)\Delta\psi(0) + \int_{t_0}^{t}\Phi(t_0, \sigma)f(\sigma)d\sigma, \quad (6)$$

where $\Delta\psi(0) = 0$,$\Phi(t_0, t)$ is bounded and for the second term it can be obtained, for every ϵ:

$$\|\int_{t_0}^{t}\Phi(t_0, \sigma)f(\sigma)d\sigma\| < \varepsilon,$$

opportunely choosing δ, with $\|\delta u(t)\| < \delta$. Thus we have shown that, taking $t = t_1$, the final state is a continuous function of the input.

3 Quantum Robustness

3.1 Pursuing a Coherent Definition

In classical control theory, plant uncertainty is described by a set $\mathcal{P}$ of possible plants [5]. This uncertainty can be either structured (parametrized by a finite number of scalar parameters or a discrete set of plants) or unstructured (disc-like uncertainty). A controller is said to be robust with respect to some property if this property holds for every plant in $\mathcal{P}$.

In the coherent quantum control, the expression "robustness of the control strategy" means that the control performance is insensible to errors in the control implementation. In [10], a control strategy is considered robust "*if significant local changes in the amplitude and the form of the pulse and of the chirp do not change significantly the final transfer probability.*" The pulse and the chirp, in the NMR setting, are the system inputs parameters. A quantitative definition of robustness is, however, missing.

Here we show that it is quite simple to reformulate this kind of property as a particular case of structured-like classical robustness. We need two basic ingredients:

- A set of systems $\mathcal{P}$ that models the uncertainty;
- A characteristic or a "performance index" to be maintained for all the systems in $\mathcal{P}$.

To specify the uncertainty system set, we can transfer the uncertainty from the control parameters to the internal Hamiltonian as in [11]. In fact, by defining $\delta u_i(\theta) = u_i(\theta) - u_i(\theta^*)$ we can write:

$$\begin{aligned} H(t) &= H_0 + \sum_{i=1}^{m} H_i \left(u_i(\theta^*) + \delta u_i(\theta) \right) \\ &= H_0 + \sum_{i=1}^{m} H_i \delta u_i(\theta) + \sum_{i=1}^{m} H_i u_i(\theta^*) \end{aligned}$$

$$= \quad (H_0 + \Delta H_u(\theta)) + \sum_{i=1}^{m} H_i u_i(\theta^*). \tag{7}$$

where $\Delta H_u(\theta) = \sum_{i=1}^{N} H_i \delta u_i(\theta)$. Such a cosmetic transformation shows that our control strategy uncertainty can be seen as a particular case of the plant uncertainty (with control inputs $u_i(\theta^*)$). The plant set $\mathcal{P}$ is here given by:

$$\mathcal{P} = \{(H_0 + \Delta H_u(\theta), H_1, ..., H_m)|\theta \in \mathcal{T}\}.$$

On the other hand, in order to formulate the most natural request for a "robust" control strategy we need to introduce the *error probability* for each control strategy. Consider the normalized final state for the time evolution, $|\psi(T,\theta)\rangle$. It can be written as $|\psi(T,\theta)\rangle = \langle\psi_1|\psi(T,\theta)\rangle|\psi_1\rangle + |\psi^{\perp}(\theta,T)\rangle$ with $|\psi^{\perp}(\theta,T)\rangle$ orthogonal to $|\psi_1\rangle$. If we imagine to perform a discrete measure[1] on an observable that has $|\psi_1\rangle$ as eigenstate, the probability to obtain the eigenvalue associated $|\psi_1\rangle$(that corresponds to the probability of finding the system in $|\psi_1\rangle$ immediately after the measure) is: $P_{|\psi_1\rangle} = |\langle\psi_1|\psi(T,\theta)\rangle|^2$. Then the *error probability* corresponding to the value θ is:

$$\begin{aligned} P_{err}(T,\theta) &= 1 - |\langle\psi_1|\psi(T,\theta)\rangle|^2 \\ &= \langle\psi^{\perp}(\theta,T)|\psi^{\perp}(\theta,T)\rangle, \end{aligned} \tag{8}$$

thanks to the fact that $|\psi(T,\theta)\rangle$ is normalized. By assumption, we have $P_{err}(T,\theta^*) = 0$.

We require this probability not to exceed a fixed threshold $\epsilon \in [0,1)$ at a given T. All the ingredients of a classical robustness problem have now been specified.

In terms of our model, this concept of robustness can be qualitatively formulated as follows: A control strategy is robust when, for

[1]Quantum measure fundamental postulates can be founded in standard quantum mechanics textbooks, see e.g. [9],[8] or [2].

values of the parameters θ different from the nominal ones, the final state $|\psi(T,\theta)\rangle$ is close to the desired one $|\psi_1\rangle$. This robustness request is satisfied if $P_{err}(T,\theta)$ is small in the parameter set $\mathcal{T}$. Formally, introduce the *ϵ-robustness set* $\mathcal{R}_\epsilon$ as

$$\mathcal{R}_\epsilon = \{\theta \in \mathcal{T} | P_{err}(\theta, T) \leq \epsilon\}. \tag{9}$$

We give the following definition.

Definition 1 *A control strategy* $\{u_1(\theta^*,t), ..., u_m(\theta^*,t)\}$, $t \in [0,T]$ *is* ϵ-robust *with respect to parameters uncertainty if:*

$$\mathcal{R}_\epsilon = \mathcal{T}. \tag{10}$$

Notice that only the 0-robustness case ensures us an exact steering of the system state for all $\theta \in \mathcal{T}$.

Does this translation in "classical" control theory fit the evaluations and the robustness claims in relevant literature?

3.2 Robustness Evaluation in NMR

Robustness problems in unitary quantum control are mainly discussed in the NMR quantum control setting. We consider a two level quantum system, and the associated bi-dimensional Hilbert space. The time evolution is described by a scaled time Schrödinger equation in the form:

$$i\hbar \frac{\partial}{\partial s} |\psi(s)\rangle = TH(s)|\psi(s)\rangle, \tag{11}$$

where $s = t/T$ and

$$H(s) = \begin{pmatrix} -\Delta(s) & \Omega(s) \\ \Omega(s) & \Delta(s) \end{pmatrix}$$

is represented in the canonical (*diabatic*) base. The control functions:

$$\begin{cases} \Delta(s) = \Delta_0 \Phi(s) \\ \Omega(s) = \Omega_0 \Lambda(s) \end{cases},$$

are the inputs, with $\Phi(s), \Lambda(s)$ fixed envelops and Δ_0, $\Omega_0 \in \mathbb{R}^+$ amplitude parameters. In this picture we have $\Delta(s) = u_1(s, \Delta_0)$ and $\Omega(s) = u_2(s, \Omega_0)$. Thus $\theta = (\theta_1, \theta_2) = (\Delta_0, \Omega_0)$ are the parameters we are interested in. In the contest of particle-laser field interaction and the RWA (Rotating Wave Approximation [1]), these functions depend on the chirp (*detuning*) and the amplitude (*time-dependent Rabi frequency*) of the active pulse[2]. This model can be seen as a particular case of model (1-2), and it is suitable to describe control techniques based both on magnetic resonance and adiabatic passage.

Here we consider the *state-flip* problem. Given the initial normalized state of the system, the control aim is to steer the system to the orthonormal state in the state space. A simple way to obtain such a transfer is to use the magnetic resonance phenomena: Under properly tailored oscillating fields, the state vectors rotate between the two basis states [9, 7]. This kind of effect can be generated by the following fields-control functions:

$$\begin{cases} \Delta(s) = 0 \\ \Omega(s) = \Omega_0 \Lambda(s), \end{cases} \tag{12}$$

where $\Lambda(s)$ is the Ω-pulse envelope. This parametrization, and some easy calculations [9], lead to the following expression for the error probability:

$$P_{err}(T, \Omega_0, A_\Lambda) = \cos^2 \left[T\Omega_0 \int_{s_i}^{s_f} \Lambda(s) ds \right] = \cos^2 \Omega_0 T A_\Lambda, \tag{13}$$

with A_Λ the Ω-pulse envelope area.

Robustness for the control strategy should be carefully evaluated, since ϵ-robustness can be guaranteed only in small neighborhoods of the error probability zeros (13).

[2]To find some detailed information about the physical meaning of these parameters and about the resonance phenomenon see i.e. [12],[7].

More subtle techniques have been developed, using the *adiabatic* approximation for sufficiently slow and smooth evolution [3]. Without entering the technical discussion, it is instructive to evaluate our ϵ-robustness for such an adiabatic model.

The Allen-Eberly [1] parametrization allows to obtain an exact expression for the error probability and, in the $\Omega_0 = \Delta_0$ case, forces the state time evolution along the energy *level lines*, maintaining the energy eigenvalues $\varepsilon(\Omega(s), \Delta(s)) = c$, c constant for every s [10]. This kind of choice leads to good results in terms of error probability even quite far from the ideal $T \to \infty$ condition, as we are going to show. In terms of control functions, we consider:

$$\begin{cases} \Delta(s) = \Delta_0\sqrt{1 - \mathrm{sech}^2(s)} = \Delta_0 \tanh(s) \\ \Omega(s) = \Omega_0 \mathrm{sech}(s). \end{cases} \tag{14}$$

Then, the exact expression for the error probability is:

$$P_{err}(T, \Omega_0, \Delta_0) = \cosh^2\left(\pi T\sqrt{\Delta_0^2 - \Omega_0^2}\right)\mathrm{sech}^2\left(\pi\Delta_0 T\right), \tag{15}$$

for every regime, adiabatic or not. We can notice that, for large T and for $\Delta_0 \geq \Omega_0$, the error probability can be bounded by:

$$P_{err}(T, \Omega_0, \Delta_0) \leq 4e^{-2\pi T(\Delta_0 - \sqrt{\Delta_0^2 - \Omega_0^2})}. \tag{16}$$

Thus, for every Δ_0 and Ω_0, $\Delta_0 \geq \Omega_0$, the error probability decreases exponentially to zero in the adiabatic limit. The best choice for the parameters values is to take the largest $\Delta_0 = \Omega_0$. In the case $\Omega_0 > \Delta_0$, the error probability becomes:

$$P_{err}(T, \Omega_0, \Delta_0) = \cos^2\left(\pi T\sqrt{\Omega_0^2 - \Delta_0^2}\right)\mathrm{sech}^2\left(\pi\Delta_0 T\right). \tag{17}$$

This expression tends to zero with dumped oscillations, due to the term $\cos^2\left(\pi T\sqrt{\Omega_0^2 - \Delta_0^2}\right)$. Again, larger Δ_0 make P_{err} converge

[3]Demonstration of adiabatic theorem is given in [8], a description of the consequent techniques can be found in [12, 10].

faster. Thus, for each fixed ϵ, we can compute a T_ϵ such that the error probability $P_{err}(T, \Omega_0, \Delta_0) < \epsilon$ for every $T > T_\epsilon$. Indeed, it is easy to see that

$$T_\epsilon = \max\{-\frac{\ln \frac{\epsilon}{4}}{2\pi(\Delta_0 - \sqrt{\Delta_0^2 - \Omega_0^2})},\ -\frac{\ln \epsilon}{2\pi\Delta_0}\}.$$

This control strategy is therefore *intrinsically robust* for T sufficiently large. According to the Landau-Zener case, every choice of $\Omega_0 \neq 0$ and $\Delta_0 \neq 0$ drives the system to the target state. The level line condition ($\Delta_0 = \Omega_0$) and large Ω_0 give faster convergence to the desired state.

These results is coherent to the observations found in [10], due mainly to simulations verifying the analysis of the error probability based on DDP ([6, 4]) formula.

4 Conclusions

The present paper fills two significant holes in the coherent quantum control analysis.

Although a number of works can be found about the reachability-controllability properties (i.e.[3]), a systematic approach to the stability analysis is missing. Here we investigated the advantages and limits of considering a unitary evolution, reviewing the classical stability characterizations and showing a natural, weak (non converging) *Input-to-State stability* property.

In the relevant literature, *robustness with respect to control errors* has been claimed but not demonstrated, since a precise definition was missing. We have formulated a coherent definition, which allows precise robustness evaluation and, in some cases, control tuning once the uncertainty of the parameter has been specified. Throughout the paper we present some reenforcing examples of the main ideas presented.

Having in mind a real quantum control problem, however, decoherence effects should be considered. The intrinsic, conservative, simple stability due to the unitary evolution is no longer guaranteed, but the convergence feature appears, opening new prospectives both towards new sinthesis and analysis methods.

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