Environment-Assisted and Feedback-Assisted Stabilization of Quantum Stochastic Evolutions

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Abstract— **We consider a class of pure-state preparation problems for stochastic quantum dynamics, by means of Hamiltonian control, continuous measurement and quantum feedback, in the presence of a Markovian environment. We prove that, whenever suitable dissipative effects are induced either by the unmonitored environment or by continuous-time measurements, open-loop time-invariant control is in principle sufficient to achieve stabilization of the target state (in probability). When this is not sufficient, we show that state stabilization (in expectation) can be attained for a wide class of models by the addition of a switching, filtering-based feedback control Hamiltonian.**

I. Introduction

A pure state preparation problem for a quantum system, initially in an uncertain state, can be cast, in control-theoretic terms, as a *global state stabilization problem* for the underlying controlled dynamics. Such problem is of interest in a wide number of application of quantum control methods [4], ranging from thermodynamical cooling of quantum systems [18], [21] to quantum information processing [24], [31].

More specifically, we here consider the problem of global asymptotic stabilization of pure states of finite-dimensional systems interacting with a Markovian environment and undergoing continuous measurement processes. The most general description of these systems is provided by the so-called Stochastic Master Equation (SME), the quantum equivalent of a Kushner-Stratonovich equation for the probability density in the classical case [10], [19]. Such control design problem is doubtless challenging: the SME is a nonlinear, control-affine, matrix Stochastic Differential Equation (SDE) with state in the compact set of positive-semidefinite, unit-trace Hermitian matrices. In addition to this, some intrinsic symmetries in the action of noise prevent standard design methods to work. Stabilizing feedback laws relying on the real-time estimate of the system state (hence often called bayesian, or filtering-based techniques) have been presented in [9], [34], [30], [29], [26]. This approach has been shown to be robust with respect to (small) delays in the control loop [13], and to initialization of the filter [27]. Focusing on finite-dimensional systems, the approach of [29] is based on a (convex) numerical Lyapunov design which turns out to be viable only for low-dimensional systems, since the numerical design procedure suffers from scalability problems. A more general theoretical framework to solve the problem has been presented in [19], where the need for numerical methods is bypassed by resorting to switching controllers. However, the convergence to the target can be guaranteed only in expectation, and the control design method has to be tailored to each specific system.

In this paper, we aim to develop a constructive and flexible approach to stabilizing control design, by bringing into the picture two new elements: open-loop, time-invariant Hamiltonian control, and a Markovian environment. The key technical idea is to link the stability of the stochastic evolution under consideration to those of the corresponding Markov semigroup, and to show that one can prove stability not only in expectation but also *in probability*. In order to develop a system-theoretic analysis of our stochastic system, we take advantage of the linear algebraic approach that has been proposed in [24], [25] to develop a theory of their controlled invariants. Assuming we can engineer open-loop, time-independent Hamiltonians on the system, we provide necessary and sufficient conditions for pure state and subspace stochastic stabilizability (in probability) for a large class of systems, exploiting the dissipation effects induced either by the measurement or by the noise processes, without requiring any feedback control capability. When no suitable dissipation effect (neither monitored nor unmonitored) is available, we show that stabilization (in expectation) can be achieved by adding a feedback time-dependent Hamiltonian. The design of the feedback law we pursue follows [19], with some differences that are worth remarking: (i) it includes and relies on the effect of the open-loop Hamiltonian and the environment; (ii) it is parametrization free, and the method directly applies to a fairly wide class of models; (iii) the assumption we make are such that the role of the feedback part in the control strategy is "minimal", i.e. it is used only to enable state transitions that the open-loop control cannot produce. Further implications of these differences will be discussed in the conclusions.

II. Dynamical Models and Invariant Pure States

Let us first recall some basics of quantum mechanics, quantum filtering and stability of quantum Markovian semigroups we will use later on.

A finite-dimensional quantum system $\mathcal{Q}$, with Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$, is considered. Let $\mathfrak{B}(\mathcal{H})$

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represent the set of linear operators on $\mathcal{H}$, with $\mathfrak{H}(\mathcal{H})$ denoting the real subspace of Hermitian operators, which represent physical observables, and $\mathbb{I}$ being the identity operator. The adjoint of $A \in \mathfrak{B}(\mathcal{H})$ is denoted by $A^\dagger$. Our knowledge of the state of $\mathcal{Q}$ is condensed in a density operator, or *state*, ρ on $\mathcal{H}$, with $\rho \geq 0$ and $\mathrm{tr}(\rho) = 1$. Density operators form a compact and convex set, denoted $\mathfrak{D}(\mathcal{H}) \subset \mathfrak{H}(\mathcal{H})$.

The state dynamics for a given system while this is subjected to continuous observations can be derived from quantum filtering theory [6], [7]. In the following we only consider dynamics for a system interacting with a field in the vacuum state, which is in turn measured by homodyne detection [33], [5]. While for the sake of simplicity we consider only one measurement channel at a time, the results could be extended to multiple commuting measurement processes. The homodyne-detection measurement record Y_t can be represented as:

$$dY_t = \sqrt{\eta}\frac{1}{2}\mathrm{tr}(\rho_t(L_0 + L_0^\dagger))dt + dW_t, \tag{1}$$

where W_t is a standard real-valued Wiener process, L_0 is the linear operator associated to the system-field interaction and $0 \leq \eta \leq 1$ represents the efficiency of the measurement. We denote by $\mathcal{E}_t$ the filtration associated to $\{W_t, t \in \mathbb{R}^+\}$.

The dynamical equation for $\rho_t \in \mathfrak{D}(\mathcal{H})$ conditioned on the measurement record ($\{Y(s)\}_{t\geq s\geq 0}$) is the quantum filtering or Stochastic Master Equation (SME) à la Itô:

$$d\rho_t = \left\{-i[H,\rho_t] + \sum_{k=0}^{r}\mathcal{D}(L_k,\rho_t)\right\}dt + \mathcal{G}(L_0,\rho_t)dW_t, \tag{2}$$

where r is an integer, and

$$\begin{aligned}\mathcal{D}(L,\rho) &:= L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L),\\ \mathcal{G}(L,\rho) &:= \sqrt{\eta}(L\rho + \rho L^\dagger - \mathrm{tr}((L+L^\dagger)\rho)\rho).\end{aligned} \tag{3}$$

The Hermitian matrix H is associated to the Hamiltonian of the system, while $\mathcal{D}(L_0,\rho)$ and $\mathcal{G}(L_0,\rho)$ are the drift and diffusion parts introduced by the weak measurement of the operator $L_0 \in \mathfrak{B}(\mathcal{H})$. The additional drift terms $\mathcal{D}(L_k,\rho)$, determined by the *noise operators* $L_k \in \mathfrak{B}(\mathcal{H})$, $k = 1,\dots,r$, account for the non-unitary dynamics induced by interactions with an (unobservable) environment. The solution $\rho_t = \Phi_t(\rho_0)$ uniquely exists and is adapted to the filtration $\mathcal{E}_t$ and $\mathfrak{D}(\mathcal{H})$-invariant by construction, see [29], [6]. Considering (1) and (2) together, one can recognize the basic structure of a *Kushner-Stratonovich* equation: other correspondences and differences with the classical setting have been highlighted in e.g. [10], [19].

Let us assume that the operators H, $\{L_k\}$ of (2) are time invariant: the drift part is then linear, time-invariant and plays the role of the Kolmogorov's forward equation (or Fokker-Plank equation) associated to the SME (2):

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t)) = -i[H,\rho(t)] + \sum_{k=0}^{r}\mathcal{D}(L_k,\rho(t)). \tag{4}$$

In this Master Equation (ME), $\mathcal{L}$ is the generator of the Markov semigroup associated to (2), *i.e.* a Quantum Dynamical Semigroup (QDS) generator in the language of quantum theory [17], [11]. The resulting evolution is a continuous, one-parameter semigroup of Trace-Preserving, Completely-Positive (TPCP) maps $\{\mathcal{T}_t(\cdot)\}_{t\geq 0}$.

We here recall some results on the stability of a pure state $\rho_d = |\psi\rangle\langle\psi|$ for QDS dynamics. Let $\mathcal{H}_S = \mathrm{span}\{|\psi\rangle\}$, and consider a decomposition: $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, inducing a block structure for matrices acting on $\mathcal{H}$:

$$X = \left(\begin{array}{c|c} X_S & X_P \\ \hline X_Q & X_R \end{array}\right). \tag{5}$$

Necessary and sufficient conditions for the blocks of H and L_k to ensure invariance of ρ_d are given by the following proposition.

Proposition 2.1: ([24]) Assume that $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, and let H, $\{L_k\}$ be the Hamiltonians and the error generators in (4). Then the set of states with support on $\mathcal{H}_S$ is invariant *iff* $\forall k$

$$\begin{aligned} L_k &= \left(\begin{array}{c|c} L_{k,S} & L_{k,P} \\ \hline 0 & L_{k,R} \end{array}\right),\\ iH_P &- \frac{1}{2}\sum_k L_{k,S}^\dagger L_{k,P} = 0. \end{aligned} \tag{6}$$

We say that ρ_d is "attractive" with respect to a family of TPCP maps $\{\mathcal{T}_t\}_{t\geq 0}$ if

$$\forall \rho \in \mathfrak{D}(\mathcal{H}), \quad \lim_{t\to\infty}\|\mathcal{T}_t(\rho) - \rho_d\| = 0. \tag{7}$$

A state that is invariant and attractive is Globally Asymptotically Stable (GAS). For QDS dynamics, uniqueness of an invariant state also guarantees that it is also GAS. The following result can be proven by observing that an invariant state for the SME needs to be invariant for both the associated QDS dynamics and the diffusion part.

Theorem 2.1: A pure state is an invariant subspace for the ME (4) if and only if $\mathfrak{I}_S(\mathcal{H})$ is invariant for the SME.

III. Environment-Assisted Stabilization of the SME

It has been shown in [25] that, if $\mathfrak{I}_S(\mathcal{H})$ is invariant for a ME and at least one of the $L_{P,k}$ blocks is non-zero (intuitively, "pumping" the dynamics towards the desired subspace), a certain subspace can be rendered stable for the ME via open-loop, time invariant control. Hence, it is easy to show that it would be at least stable in *expectation* for an underlying SME. However, we here prove that if $\mathcal{H}_S$ is GAS for the ME, it is also GAS *in probability* for the corresponding SME. Given this result, we can exploit the ideas of [25] to construct an open-loop time-invariant Hamiltonian that renders $\mathcal{H}_S$ GAS in probability for the SME. Let us start by showing that stability in expectation of a subspace entails its stability in probability.

Proposition 3.1: If $\mathfrak{I}_S(\mathcal{H})$ is GAS for the ME (4) then it is GAS in probability for the SME (2).

Proof: Consider the function $V_1(\rho) = \mathrm{tr}(\Pi_R\rho)$ as a Lyapunov function candidate. It is zero iff $\rho_R = 0$, i.e.,

$\rho \in \mathfrak{I}_S(\mathcal{H})$, and positive for any $\rho \notin \mathfrak{I}_S(\mathcal{H})$. Moreover, taking into account the conditions for invariance of $\mathfrak{I}_S(\mathcal{H})$ of Proposition 2.1, direct computation yields

$$\mathcal{A}V_1(\rho) = -\mathrm{tr}\left(\left(\sum_k L_{k,P}^\dagger L_{k,P}\right)\rho_R\right)$$

where $\mathcal{A}(\cdot)$ is the infinitesimal generator of the SME (2). By the cyclic property of the trace, we observe that $\mathcal{A}V_1(\rho) \leq 0$ for any $\rho \in \mathfrak{D}(\mathcal{H})$. Thus, by the stochastic version of Lyapunov's theorem [16], it follows that $\mathfrak{I}_S(\mathcal{H})$ is stable in probability. To prove (by contradiction) that it is also attractive, suppose that for some $\rho_0 \in \mathfrak{D}(\mathcal{H})\backslash\mathfrak{I}_S(\mathcal{H})$, $\mathfrak{I}_S(\mathcal{H})$ is not attractive for (2), i.e. with finite probability $p > 0$:

$$\mathbb{P}\left(\lim_{t\to\infty}\mathfrak{d}(\Phi_t(\rho_0), \mathfrak{I}_S(\mathcal{H})) = 0\right) = 1 - p,$$

where the probability measure $\mathbb{P}$ is the one induced by the Wiener process in the SME and $\mathfrak{d}(\cdot,\cdot)$ is the minimum distance between a point and a set. Thus, by the properties of V_1, it must also be

$$\mathbb{P}\left(\lim_{t\to\infty} V_1(\Phi_t(\rho_0)) = 0\right) = 1 - p. \tag{8}$$

From the definition of the expectation,

$$\mathbb{E}\left[V_1(\Phi_t(\rho_0))\right] = \int_{\{V_1(\Phi_t(\rho_0))>0\}} V_1(\Phi_t(\rho_0))d\mathbb{P}.$$

Now, either the set $\{V_1(\Phi_t(\rho_0)) > 0\}$ has measure zero, contradicting (8), or there exists a (non-random) function $v(\rho_0) > 0$ such that the following inequality holds

$$\begin{aligned}
&\limsup_{t\to\infty}\mathbb{E}\left[V_1(\Phi_t(\rho_0))\right] \\
= \;& \limsup_{t\to\infty}\int_{\{V_1(\Phi_t(\rho_0))>0\}} V_1(\Phi_t(\rho_0))d\mathbb{P} \\
\geq \;& v(\rho_0)\limsup_{t\to\infty}\int_{\{V_1(\Phi_t(\rho_0))>0\}} d\mathbb{P} \\
= \;& v(\rho_0)\limsup_{t\to\infty}\left\{1 - \mathbb{P}\left(V_1(\Phi_t(\rho_0)) = 0\right)\right\}.
\end{aligned}$$

By "reverse" Fatou's Lemma, we obtain:

$$\begin{aligned}
&\limsup_{t\to\infty} E[V_1(\Phi_t(\rho_0))] \\
&\geq v(\rho_0)\left\{1 - \mathbb{P}\left(\limsup_{t\to\infty} V_1(\Phi_t(\rho_0)) = 0\right)\right\} = v(\rho_0)p.
\end{aligned}$$

By linearity of the expectation and the trace

$$\mathbb{E}[V_1(\Phi_t(\rho_0))] = V_1(\mathbb{E}[\Phi_t(\rho_0)]).$$

Finally, by continuity of V_1, there exists a constant k such that

$$\limsup_{t\to\infty}\mathfrak{d}(\mathbb{E}[\Phi_t(\rho_0)], \mathfrak{I}_S(\mathcal{H})) \geq \frac{v(\rho_0)p}{k} > 0.$$

Since the time evolution of $\mathbb{E}[\Phi_t(\rho_0)]$ is given by (4), this would imply that $\mathfrak{I}_S(\mathcal{H})$ is not attractive for (4). Thus, the assertion is proved. $\square$

The following theorem shows that having noise operators that are *not* block-diagonal (i.e. $L_p \neq 0$) is not only necessary but also sufficient for stabilization in open-loop of the SME.

Theorem 3.1: Let $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$. Then, there exists a time-invariant control Hamiltonian H_c such that $\mathfrak{I}_S(\mathcal{H})$ is GAS in probability for the SME (2) with total Hamiltonian $H' = H + H_c$ *iff* at least one of $L_{k,P}$ is not zero, and $L_{k,Q} = 0$, $\forall k$.

The proof is straightforward given the equivalent result for ME developed in [25], and of course Proposition 3.1. Notice that, whenever a suitable noise operator is not already present, it could in principle be enacted by engineering a suitable system-field interaction, or by the introduction of a new measurement channel. This has to be designed in order not to violate the invariance conditions for the subspace.

IV. Feedback-Assisted Stabilization of the SME

A. Assumptions and open-loop Hamiltonian

In the following, we will show that the addition of the time-dependent term, along with a suitable feedback law $u(\rho_t)$, will allow to render $\mathcal{H}_S$ GAS in expectation even when the open-loop method fails, namely when $L_{k,P} = 0$ for any k. However, in order to do so, we will restrict our attention to a slightly less general class of problems and measurements. Since any 2-level case can be seen as a spin-1/2 system, and these have been already discussed in [29], we focus on larger systems, $\dim(\mathcal{H}) \geq 3$. Let as assume that the total system Hamiltonian is of the form:

$$H_t = H + H_c + u(\rho_t)H_f, \tag{9}$$

where the $u(\rho_t)$ is a state-dependent control field and H_f is *fixed*, while we assume as before we are free to design the time-independent part H_c, in order to "destabilize" the undesired invariant sets for the dynamics. A trivial, necessary requirement for H_f, in order to obtain the desired stabilization, is to dynamically "connect" the target (one-dimensional) subspace $\mathcal{H}_S$ to $\mathcal{H}_R$. In order to simplify the design and analysis of the control strategy, we also assume $H_{f,S} = 0$ and $H_{f,R} = 0$.

Lastly, we restrict the class of measurements: we require L_0 to be a normal operator with non-degenerate spectrum.

Since we showed that the cases in which feedback is needed entails $L_{k,P} = 0$ for all k, and $L_{k,S}$ is a scalar, this technical assumption affects only the block $L_{0,R}$, and is needed for the proof of convergence of the feedback law.

The first step in our approach is to design an open-loop Hamiltonian H_c ensuring that there exist no invariant sets with support only in $\mathcal{H}_R$ for the ME. This result will be used in the closed-loop approach to prove that H_c "destabilizes" the subspace $\mathcal{H}_R$ for the SME as well.

It will be convenient to consider a refinement of the Hilbert space decomposition by further splitting $\mathcal{H}_R$ into orthogonal subspaces $\mathcal{H}_R = \mathcal{H}_C \oplus \mathcal{H}_Z$. Any choice of orthonormal basis of each space in the right-hand side of $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_C \oplus \mathcal{H}_Z$ induces a block decomposition for matrices representing operators on $\mathcal{H}$:

$$X = \left(\begin{array}{c|c|c} X_S & X_U & X_V \\ \hline X_T & X_C & X_W \\ \hline X_X & X_Y & X_Z \end{array}\right).$$

In particular, the choice of the $\mathcal{H}_C$ is made so the feedback Hamiltonian has a simple form: let us define $\mathcal{H}_C = \mathrm{span}\{H^\dagger_{f,P}\} \subset \mathcal{H}_R$, where $H^\dagger_{f,P}$ must be non-zero, and $\mathcal{H}_Z = \mathcal{H}_R \ominus \mathcal{H}_C$. Being the "span" of a single vector, $\mathcal{H}_C$ has dimension one, and hence X_S, X_U, X_T, X_C are in fact scalar. In the induced block decomposition, given the assumptions we made, we have:

$$H = \left(\begin{array}{c|c|c} H_S & 0 & 0 \\ \hline 0 & H_C & H_W \\ \hline 0 & H_W^\dagger & H_Z \end{array}\right),\ H_f = \left(\begin{array}{c|c|c} 0 & H_{f,U} & 0 \\ \hline H_{f,U}^\dagger & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \tag{10}$$

$$L_k = \left(\begin{array}{c|c|c} L_{k,S} & 0 & 0 \\ \hline 0 & L_{k,C} & L_{k,W} \\ \hline 0 & L_{k,Y} & L_{k,Z} \end{array}\right).$$

In [22], a thorough analysis of the role of the Hamiltonian in the stabilization of the ME has been presented, including a constructive procedure to choose the controllable Hamiltonian parameters for the intended task. Using the same ideas, one can design an Hamiltonian that *destabilizes a certain subspace.* It can be explicitly shown that a time-invariant Hamiltonian H_c that "destabilizes" any state or subset with support in $\mathcal{H}_Z$ alone can be iteratively constructed. Actually, a *generic* Hamiltonian of the form

$$H_c = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & H_{c,C} & H_{c,W} \\ \hline 0 & H_{c,W}^\dagger & H_{c,Z} \end{array}\right)$$

will serve to our aim. Proving this fact rigorously goes beyond the scope of the present work, but the argument would follow that of Section III.B of [22].

B. Feedback law and proof of convergence

We are now ready to prove that the ideas developed in [19] for spin systems work also for the wider class of models we consider here, provided we allow for some freedom of choice in the open-loop Hamiltonian.

Theorem 4.1: Consider the SME (2) with $H_t = H + H_c + u(\rho_t)H_f$, with H_c chosen as in the previous Section, and L_0 normal with non-degenerate spectrum. Set the feedback control law $u(\rho_t)$ to be:

1) If $\mathrm{tr}(\rho_t\rho_d) \geq \gamma$:
$$u(\rho_t) = -\mathrm{tr}(i[H_f, \rho_t]\rho_d); \tag{11}$$
2) If $\mathrm{tr}(\rho_t\rho_d) \leq \gamma/2$, $u(\rho_t) = 1$;
3) If $\rho_t \in \mathcal{B} = \{\rho : \gamma/2 < \mathrm{tr}(\rho_t\rho_d) < \gamma\}$, then $u(\rho_t) = -\mathrm{tr}(i[H_f, \rho_t]\rho_d)$ if ρ_t last entered $\mathcal{B}$ through the boundary $\mathrm{tr}(\rho_d\rho) = \gamma$, and $u_t = 1$ otherwise.

Then there exists a $\gamma > 0$ such that $u(\rho_t)$ globally stabilizes (2) around ρ_d and $\mathbb{E}(\rho_t) \to \rho_d$ as $t \to \infty$.

To prove the Theorem, we begin with a Lemma that ensure local asymptotic stability in probability of ρ_d. Let us define $\mathcal{S}_{<\varepsilon} = \{\rho \in \mathfrak{D}(\mathcal{H}) | 1 - \mathrm{tr}(\rho\rho_d) < \varepsilon\}$. Recall that the feedback control (11) is chosen when $\rho \in \mathcal{S}_{<1-\gamma}$.

Lemma 4.1: The sample paths $\Phi(\rho, u)$ with $\rho \in \mathcal{S}_{<1-\gamma}$ that do not exit the set $\mathcal{S}_{<1-\gamma/2}$ converge in probability to ρ_d as $t \to \infty$.

Proof: Consider $V_2(\rho) = 1 - \mathrm{tr}(\rho_d\rho)^2$. It is straightforward to show that $V_2(\rho) \geq 0$, with equality if and only if $\rho = \rho_d$. Moreover, by using the fact that $[\rho_d, H] = [\rho_d, H_c] = [\rho_d, L_k] = 0$, we get:

$$\begin{aligned}\mathcal{A}V_2(\rho) \;&=\; 2u(\rho_t)\mathrm{tr}(i[H_f, \rho]\rho_d)\mathrm{tr}(\rho_d\rho) \\ &\quad -4\eta\left(2\Re(L_{0,S}) - \mathrm{tr}((L_0 + L_0^\dagger)\rho)\right)^2 \mathrm{tr}(\rho_d\rho)^2.\end{aligned}$$

Choosing $u(\rho_t) = -\mathrm{tr}(i[H_f, \rho_t]\rho_d)$ we have that $\mathcal{A}V_2(\rho) \leq 0$, and hence by the stochastic version of LaSalle Theorem [16], we have that the paths that do not exit the set $\mathcal{S}_{<1-\gamma/2}$ converge in probability to the largest invariant set in $\mathcal{Z} = \{\rho \in \mathfrak{D}(\mathcal{H}) | \mathcal{A}V_2(\rho) = 0\} \cap \mathcal{S}_{<1-\gamma/2}$. Since the choice (11) of control makes $\mathcal{A}V_2$ the sum of two negative-semidefinite parts, the points in $\mathcal{Z}$ must make zero both. Focusing on the second term, $\mathrm{tr}(\rho_d\rho_t) = 0$ only outside the set $\mathcal{S}_{<1-\gamma/2}$, so we must have

$$2\Re(L_{0,S}) - 2\mathrm{tr}(L_0^H\rho) = 0,$$

L_0^H being the Hermitian part of L_0 (Notice that L_0^H must have the same block-diagonal structure of L_0). Hence, $\mathrm{tr}(L_0^H\rho)$ should be a constant for each trajectory in $\mathcal{Z}$. Since

$$\begin{aligned}d\mathrm{tr}(L_0^H\rho) \;&=\; -iu_t\mathrm{tr}\left(L_0^H(\mathcal{F}(H_t, \rho) + \sum_k \mathcal{D}(L_k, \rho))\right)dt \\ &\quad +2\sqrt{\eta}\mathrm{tr}(L_0^H\mathcal{G}(L_0, \rho))dW_t,\end{aligned}$$

a necessary condition for the invariance is then

$$\begin{aligned}\mathrm{tr}(L_0^H\mathcal{G}(L_0, \rho)) \;&=\; \mathrm{tr}((L_0^HL_0 + L_0^\dagger L_0^H)\rho) - 2\mathrm{tr}(L_0^H\rho)^2 \\ &=\; 2\mathrm{Var}(L_0^H, \rho) - \mathrm{tr}(i[L_0^H, L_0^S]\rho) = 0,\end{aligned}$$

where $\mathrm{Var}(X, \rho) = \mathrm{tr}(X^2, \rho) - \mathrm{tr}(X\rho)^2$, and L_0^S denotes the skew-Hermitian part of L_0. Recall we assumed L_0 to be a normal matrix, so that $[L_0^H, L_0^S] = 0$. On the other hand $\mathrm{Var}(L_0^H, \rho) = 0$ if and only $\rho = \Pi_j$, with Π_j a (rank-one) spectral projector of L_0^H. Since the only such state in $\mathcal{S}_{<1-\gamma/2}$ is ρ_d, we get the conclusion. □

Consider again the nonnegative function $V_1(\rho) := \mathrm{tr}(\Pi_R\rho)$, which is linear in the state, and is zero if and only if the state is in $\mathfrak{I}_S(\mathcal{H})$. The following proposition can be used to prove that the $u(\rho_t) = 1$ destabilizes (in expectation) the $\mathcal{H}_R$ subspace.

Proposition 4.1: Assume $\mathcal{H}_R$ does not support invariant sets for the ME (4). Then for every fixed time T

$$\chi(\rho) = \min_{t\in[0,T]} \mathbb{E}V_1(\rho_t) < 1,\ \forall \rho_0 \in \mathfrak{I}_R(\mathcal{H}). \tag{12}$$

The details of the proof will be presented elsewhere: essentially it follows from the following basic property of linear systems in state-space representation:

Lemma 4.2: Consider a linear system $\dot{x}_t = Ax_t$, $x_t \in \mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$. Assume $x_0 \in \mathcal{H}_R$, $x_0 \neq 0$, and that $\mathcal{H}_R$ does not contain any A-invariant subspace. Then for any $T \geq 0$ there exists $t \leq T$ such that $x_t \notin \mathcal{H}_R$.

The proof of Theorem 4.1 now follows that of Theorem 4.2 of [19], with two main differences:

- *Lemma 4.3 and 4.4 in [19] are substituted by the following argument:*
  One can choose H_c such that there are no invariant subspaces for the (time-invariant) ME (4) when $H' = H + H_c + H_f$. Hence, Proposition 4.1 guarantees that when $u_t = 1$, for any finite T,
  $$\chi(\rho) = \min_{t \in [0,T]} \mathbb{E}V_1(\rho_t) < 1;$$
- *Lemma 4.8 in [19] is substituted by Lemma 4.1 above*;

The rest of the Lemmas and arguments of the proof do not depend on the specific form of the operators $H, \{L_k\}$ but on the properties of the solutions of (2), and carry over to our case directly.

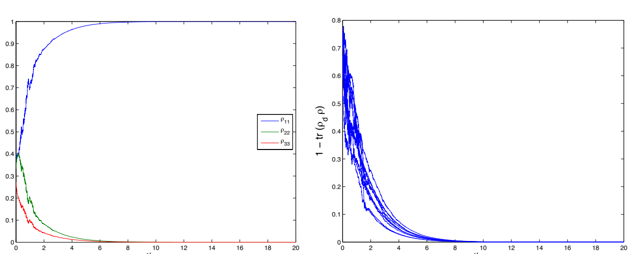
V. Examples

In this Section three examples are introduced in order to describe the various situation that may emerge in the stabilization of a pure state for a SME, following the strategy presented in this paper. In all examples $\dim(\mathcal{H}) = 3$ and $\dim(\mathcal{H}_S) = 1$ and we choose a basis such that the target state is represented by $\rho_d = \mathrm{diag}(1, 0, 0)$. A free Hamiltonian $H = \mathrm{diag}(0, 1, 2)$ is considered. In each simulation, the initial density matrix is chosen at random.

Example 1: Consider a SME having the following measurement and (unmonitored) noise operators

$$L_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix}.$$

From Proposition 3.1 and Theorem 3.1 the SME is GAS. The typical behavior of the diagonal elements of the density matrix, also known as *energy level populations* as they correspond to the probability of measuring the state in one of the Hamiltonian eigenvalues, is depicted in Fig. 1(a), and the evolution of $V_1(\rho_t) = 1 - \mathrm{tr}(\rho_d \rho_t) = 1 - \rho_{11}$ for ten sample trajectories is shown in Fig. 1(b). This provides us with a good illustration of the convergence features in this case: after a short transient, the trajectories are exponentially attracted towards ρ_d.



(a) Typical energy population dynamics (b) $V_1(\rho_t)$ trajectories for ten random initial states.

Fig. 1. Example 1: Environment-assisted stabilization without Hamiltonian control.

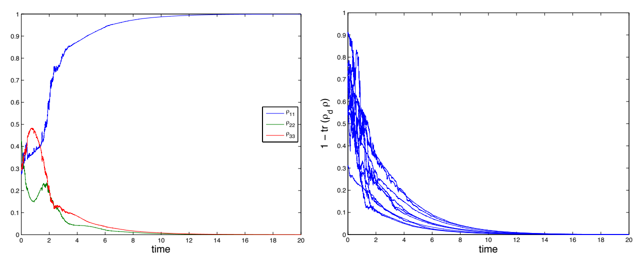
Example 2: If

$$L_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

then Theorem 3.1 holds and the system is stabilized by means of e.g. the following open-loop Hamiltonian

$$H_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

See Fig.2(a) for a representative evolution of the energy-level populations, and Fig. 2(b) for an illustration of the convergence of ten trajectories. In this case a longer transient before monotone convergence emerges, in which oscillations induced by the Hamiltonian terms are evident in the populations of the energy levels.



(a) Typical energy population dynamics (b) $V_1(\rho_t)$ trajectories for ten random initial states.

Fig. 2. Example 2: Environment-assisted stabilization with time-invariant Hamiltonian control.

Example 3: Consider the case of

$$L_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } L_1 = 0.$$

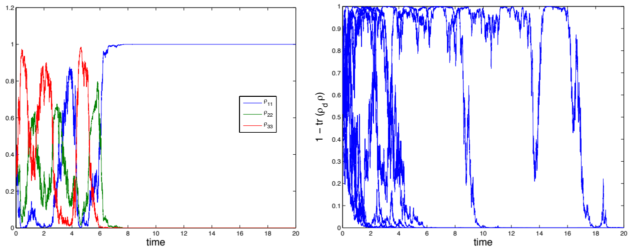
Since $L_{0,P} = 0$, the measurement is not destabilizing $\mathcal{H}_R$. Hence open-loop control alone is not effective and we have to use Theorem 4.1. Choosing $\gamma = 0.6$ and

$$H_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad H_f = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we obtain Fig. 3(a), Fig. 3(b). Notice the qualitative difference in the convergence with respect to the previous cases: the equilibrium ρ_d is not globally attractive under the feedback law (11) and some trajectories do exit the set $\mathcal{S}_{<1-\gamma}$ a few times before converging.

VI. Conclusions

In this paper we showed how open-loop Hamiltonian control can be sufficient for pure state (or subspace) stabilization when suitable measurement processes can be enacted, or dissipative environments are present. The key is to have non-hermitian noise operators, with an upper block-triangular matrix representation, that can effectively "pump" the state towards the desired one. In this case, stabilization of a



(a) Typical energy population dynamics (b) $V_1(\rho_t)$ trajectories for ten random initial states.

Fig. 3. Example 3: Feedback-assisted stabilization.

pure state or a subspace for a time-invariant ME implies stabilization in probability for the underlying SME, linking the result to the analysis developed in [24], [25], [22]. It is worth remarking that, in practical applications, the use of real-time feedback is an extremely challenging task since it entails the computation-intensive, real-time integration of the SME: open-loop, time-independent solutions appear to be a more practical option in many real world settings [22]. In addition to this, numerical simulations show how the use of environment-assisted stabilization leads to more regular behaviors and, generally, faster convergence. When open-loop stabilization is not viable, we showed how adding a switching feedback controller on top of a suitably designed Hamiltonian perturbation, pure state stabilization can be achieved in expectation. We assumed to have some freedom in constructing open-loop, time invariant Hamiltonians, and considered additional noise operators in addition to those induced by the measurement: it has been shown that, when the action of the noise does not prevent invariance of the target set, it can actually be useful for the stabilization task: for example, the presence of extra noise, "mixing" the degrees of freedom in the $\mathcal{H}_R$ subspace could simplify the structure of the open-loop Hamiltonian, or even make it completely superfluous (see Example 1).

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