On the Role of Hamiltonians for Dissipative Entanglement Engineering ⋆

Francesco Ticozzi * **and Lorenza Viola** **

* *Dipartimento di Ingegneria dell'Informazione, Università di Padova, via Gradenigo 6/B, 35131 Padova, Italy, and Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, 03755 Hanover, NH (USA) (e-mail: ticozzi@dei.unipd.it)*
** *Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, 03755 Hanover, NH (USA) (e-mail: lorenza.viola@dartmouth.edu)*

Abstract: Determining whether an entangled state of interest can be asymptotically prepared by realistic open-system dynamics has important applications across quantum engineering. This problem has been recently solved for *purely dissipative* quasi-local dynamics described by a continuous-time Markovian semigroup. Here, we extend our previous analysis by addressing the role of internal Hamiltonian dynamics as well as of Hamiltonian control resources for achieving the same task. We show how Hamiltonians that are *not* frustration-free can genuinely extend the class of stabilizable states. In particular, we present stabilizing Hamiltonians, along with necessary and sufficient conditions for their existence, for maximally entangled GHZ-states and translationally invariant W-states, none of which are generally stabilizable by dissipation alone.

Keywords: Quantum information and control; Global stability; Distributed models

1. INTRODUCTION

Entanglement is arguably one of the most intriguing features of quantum theory, allowing for a composite system to exist in states that exhibiting correlations beyond what is possible in a classical probabilistic framework. While entangled states are ubiquitous in matter, generating and manipulating entanglement in a controlled fashion is an outstanding challenge across quantum information processing and quantum engineering. In particular, entangled states are both an essential resource for quantum communication protocols and play an important role in boosting the efficiency of quantum algorithms over their classical counterpart (see e.g. Nielsen and Chuang (2000)). Remarkably, certain entangled states of (two- or three-dimensional) quantum registers, the so-called "cluster states", suffice for universal quantum computation within the one-way measurement-based model proposed by Raussendorf and Briegel (2001). Likewise, multi-particle entangled states of matter as well as light can enhance the sensitivity of estimation in quantum metrology applications, see e.g. Cappellaro et al. (2005); Boto et al. (2000). In order to practically exploit these advantages, these quantum states have to be generated and "kept alive" until ready for the intended use.

In addition to more traditional coherent-control strategies for generating entangled states based on the implementation of suitable sequences of unitary quantum gates, the use of *engineered dissipation* has recently attracted a growing attention as a tool for robust state preparation, by either relying on controlled interactions between the target interest with auxiliary ones, or by implementing measurements and feedback control loops (Poyatos et al. (1996); Beige et al. (2000); Carvalho et al. (2001); Ticozzi and Viola (2008)). With respect to unitary control protocols, open-system schemes based, in particular, on *Markovian semigroups* have the important feature of defining time-invariant dynamical models which, for stabilization purposes, are robust with respect to the initial condition.

A number of theoretical results on Markovian dynamics for entangled state preparation have been recently established (Kraus et al. (2008); Verstraete et al. (2009); Ticozzi and Viola (2009); Perez-Garcia et al. (2008); Ticozzi and Viola (2012)), including methods for performing universal quantum computation in a dissipative fashion (Verstraete et al. (2009)) and for dissipatively preparing novel phases of matter (Diehl et al. (2008)), along with experimental demonstrations of digital simulation of continuous-time Markov dynamics in trapped ions (Barreiro et al. (2011)) and macroscopic entangled-state engineering (Krauter et al. (2011)). In particular, in our earlier contribution (Ticozzi and Viola (2012)), we have provided necessary and sufficient conditions for an entangled state to be stabilizable by *purely-dissipative means* in the presence of realistic locality constraints. While these conditions translate into both an explicit stabilization test and an explicit form for the set of dissipative actions to be engineered, our derivation relied on two non-generic assumptions: first, we assumed that no uncontrolled ("drift") component of the dynamical generator was present; sec-

⋆ F.T. acknowledges support by the QUINTET and the QFuture projects of the University of Padova, Italy.

ond, that no Hamiltonian control resources were available, in addition to the dissipative ones.

In this work, we explore the general case, namely the possibility of achieving global asymptotic stabilization of a given entangled state for quasi-local, continuous-time Markov dynamics, by employing *both Hamiltonian and dissipators, possibly in the presence of an internal drift dynamics*. We develop necessary conditions on the Hamiltonian to make a state asymptotically stable under quasi-local constraints, in cases where dissipation alone is insufficient. In particular, we provide examples for Hamiltonians satisfying these properties and stabilizing physically relevant classes of multipartite entangled pure states. In the last Section, we discuss the role of underlying drift dynamics (both Hamiltonian and dissipative) for stabilization purposes, showing how different scenarios can be recast into the former drift-less problem, or can be otherwise tackled by adapting our previous results.

2. PRELIMINARIES

2.1 Quantum dynamical semigroups and locality notions

We consider a multipartite system $\mathcal{Q}$, composed of n (distinguishable) subsystems, labeled with index $a = 1, \ldots, n$, with associated d_a-dimensional Hilbert spaces $\mathcal{H}_a$. Thus, $\mathcal{H}_{\mathcal{Q}} = \bigotimes_{a=1}^{n} \mathcal{H}_a$. In the standard quantum-statistical framework, a *state* for $\mathcal{Q}$ is described by a density operator ρ, which is a trace-one, positive operator on the system Hilbert space $\mathcal{H}$. We shall denote with $\mathfrak{D}(\mathcal{H})$ the set of density operators on $\mathcal{H}$.

The dynamical models we are interested in are associated to Lindblad Master Equations (ME) ($\hbar = 1$):

$$\begin{aligned}\dot{\rho}(t) &= \mathcal{L}(\rho(t)) \\ &= -i[H,\, \rho(t)] + \sum_k \Big(L_k \rho(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k,\, \rho(t)\}\Big),\end{aligned} \tag{1}$$

that represent the general form of generators of completely-positive, trace-preserving continuous Markov semigroups $\{e^{\mathcal{L}t}, t \geq 0\}$ acting on $\mathfrak{D}(\mathcal{H})$. Here, H is an Hermitian operator associated with the Hamiltonian of the system, whilst the noise (or Lindblad) operators $\{L_k\}$ specify the non-Hamiltonian part of the generator, resulting in non-unitary irreversible dynamics.

The locality constraints on the semigroup dynamics are introduced by limiting the action of each operator L_k to affect only certain subsets of subsystems, which we call *neighborhoods*, with an equivalent condition being imposed on H. Following Ticozzi and Viola (2012), the neighborhoods can be specified in full generality as subsets of the set of indexes labeling the subsystems, that is,

$$\mathcal{N}_j \subseteq \{1, \ldots, n\}, \quad j = 1, \ldots, M.$$

We say that a *noise operator L is Quasi-Local (QL)* if there exists a neighborhood $\mathcal{N}_j$ such that:

$$L = L_{\mathcal{N}_j} \otimes I_{\bar{\mathcal{N}}_j},$$

where $L_{\mathcal{N}_j}$ accounts for the action of L on the subsystems included in $\mathcal{N}_j$, and $I_{\bar{\mathcal{N}}_j} := \bigotimes_{a \notin \mathcal{N}_j} I_a$ is the identity on the remaining subsystems. Similarly, a *Hamiltonian is QL* if it admits a decomposition into a *sum* of QL terms:

$$H = \sum_j H_j, \quad H_j = H_{\mathcal{N}_j} \otimes I_{\bar{\mathcal{N}}_j}.$$

A ME will be called QL if *both* its Hamiltonian and noise operators are QL [1]. The introduction of locality constraints based on neighborhoods allows us to encompass in a single definition different specific notions that have been used in the physical literature, where the locality notions are typically associated with sets of nearest-neighbor sites on a graph or lattice, or with Hamiltonian and noise generators being forced to act on a given maximum number t of subsystems (so-called "t-body" interactions), see also Kraus et al. (2008); Verstraete et al. (2009).

2.2 Standard form for QL stabilizing dynamics

A state ρ_d for a system driven by (1) is said to be *Globally Asymptotically Stable (GAS)* if for every initial condition ρ_0 we have

$$\lim_{t\to\infty} e^{\mathcal{L}t}[\rho_0] = \rho_d.$$

A necessary condition for a state $\rho_d = |\Psi\rangle\langle\Psi|$ to be GAS is that the state be invariant for the dynamics, namely $\mathcal{L}(\rho_d) = 0$: the following proposition, that has been proved in Ticozzi and Viola (2012), provides us with a simple way to check its invariance:

Proposition 1. Let the dynamics be driven by $\mathcal{L}$. Then $\rho_d = |\Psi\rangle\langle\Psi|$ is invariant for the dynamics if and only if

$$\begin{aligned} L_k|\Psi\rangle &= \ell_k|\Psi\rangle,\ \ell_k \in \mathbb{C},\ \forall k, \\ \tilde{H}|\Psi\rangle &= h|\Psi\rangle, h \in \mathbb{R}, \end{aligned} \tag{2}$$

with $\tilde{H} = H + \frac{i}{2}\sum_k(\ell_k^* L_k - \ell_k L_k^\dagger)$.

Building on the above result, we introduce below a convenient *standard form*, or more precisely, an equivalent representation of a stabilizing generator in terms of new Hamiltonian and noise operators with some useful properties.

Lemma 1. If a generator $\mathcal{L}$ associated to $\{H, L_k\}$ makes $\rho_d = |\Psi\rangle\langle\Psi|$ GAS, then the same generator can be associated to $\{\tilde{H}, D_k\}$, in such a way that $\tilde{H}|\Psi\rangle = h|\Psi\rangle$ and $D_k|\Psi\rangle = 0$, for all k.

Proof. Assume that $\rho_d = |\Psi\rangle\langle\Psi|$ is GAS for (3), then by Proposition 1 it follows that for all k,

$$L_k|\Psi\rangle = \ell_k|\Psi\rangle,\ \ell_k \in \mathbb{C},$$

and $\tilde{H}|\Psi\rangle = h|\Psi\rangle$, with $\tilde{H}$ given in 2. Then, if $\ell_k \neq 0$, use Lemma 2 in Ticozzi and Viola (2008) to substitute L_k with $D_k = L_k - \ell_k I$. □

By Lemma 1 we can always, without loss of generality, consider $\ell_k = 0$, $\forall k$, and $\tilde{H}$ an Hamiltonian that has $|\Psi\rangle$ as an eigenstate. In other words, we can always transform the semigroup operators to a *standard form*, so that $\ell_k = 0$ for all k. This is similar to what is done in the preliminary analysis of Kraus et al. (2008), where an equivalent process is built with the desired state as the unique dark state.

[1] While it is well known that the decomposition into Hamiltonian and dissipative part of Eq. (1) is not unique, it is easy to verify that the QL property remains well defined since the freedom in the representation does not affect the tensor structure of H and $\{L_k\}$.

Our main task is to find criteria to determine whether a target state is stabilizable with QL resources:

Definition 1. A pure state $\rho_d = |\Psi\rangle\langle\Psi| \in \mathfrak{D}(\mathcal{H}_\mathcal{Q})$, is Quasi-Locally Stabilizable (QLS) if there exist QL operators $\{\tilde{H}, D_k\}_{k=1,\dots,K}$ on $\mathcal{H}_\mathcal{Q}$, as in Lemma 1, such that ρ_d is GAS for the associated generator $\mathcal{L}$.

In Ticozzi and Viola (2012), we restricted our attention to the case in which $\tilde{H} = 0$, and the stabilization is enacted by the dissipators alone, that is:

Definition 2. A pure state $\rho_d = |\Psi\rangle\langle\Psi| \in \mathfrak{D}(\mathcal{H}_\mathcal{Q})$, is Dissipatively Quasi-Locally Stabilizable (DQLS) if there exist QL operators $\{D_k\}_{k=1,\dots,K}$ on $\mathcal{H}_\mathcal{Q}$, with $D_k|\Psi\rangle = 0$, such that ρ_d is GAS for

$$\dot{\rho} = \sum_k \Big(D_k \rho D_k^\dagger - \frac{1}{2}\{D_k^\dagger D_k, \rho\}\Big). \tag{3}$$

By virtue of Theorem 1 below, one can show that for DQLS states we can restrict to the case of a *single* noise operator per neighborhood ($K = M$) without loss of generality. A characterization of DQLS states, leading to a simple linear-algebraic algorithm to test whether a state is DQLS, may be obtained essentially based on the properties of the *reduced states* on the neighborhoods. Let us define:

$$\rho_{\mathcal{N}_k} = \text{trace}_{\bar{\mathcal{N}}_k}(\rho_d), \tag{4}$$

where $\text{trace}_{\bar{\mathcal{N}}_k}$ indicates the partial trace over the *tensor complement* of the neighborhood $\mathcal{N}_k$, namely $\mathcal{H}_{\bar{\mathcal{N}}_k} = \bigotimes_{a\notin\mathcal{N}_k} \mathcal{H}_a$. We thus have the following result (Ticozzi and Viola (2012)):

Theorem 1. A pure state $\rho_d = |\Psi\rangle\langle\Psi|$ is DQLS if and only if

$$\text{supp}(\rho_d) = \bigcap_k \text{supp}(\rho_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}) \equiv \bigcap_k \mathcal{H}_{\mathcal{N}_k}. \tag{5}$$

The proof of this Theorem leads to explicit connections with the physically-motivated concept of a *parent Hamiltonian.* In fact, consider a QL Hamiltonian $H = \sum_k H_k$, with $H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$. A pure state $\rho_d = |\Psi\rangle\langle\Psi|$ is called a *frustration-free ground state* if

$$\langle\Psi|H_k|\Psi\rangle = \min \lambda(H_k), \quad \forall k,$$

where $\lambda(\cdot)$ denotes the spectrum of a matrix. A QL Hamiltonian is called a *parent Hamiltonian* if it admits a *unique* frustration-free ground state (Perez-Garcia et al. (2007)). Suppose that a pure state admits a QL parent Hamiltonian H. Then the QL structure of H may be naturally used to derive a stabilizing semigroup, and it is easy to show that this condition is also necessary:

Corollary 2. A state $\rho_d = |\Psi\rangle\langle\Psi|$ is DQLS if and only if it is the ground state of a QL parent Hamiltonian.

We refer the reader to Ticozzi and Viola (2012) for a more detailed discussion of the relation of these results with those derived relying on the so-called matrix product state formalism, see also Perez-Garcia et al. (2007, 2008).

3. CAN A QL HAMILTONIAN STABILIZE NON-DQLS STATES?

A first natural question is the following: *Are all QLS states also DQLS states?* In other words, can the addition of a QL Hamiltonian to an engineered dissipative process of the form (3) make a non-DQLS state GAS for the new dynamics? In this section we provide a set of conditions that H and $\{D_k\}$ have to obey for this to happen.

The first result regards the $\{D_k\}$: this variation of Lemma 4 in Ticozzi and Viola (2012) shows that the support of a QLS state must be, as in the DQLS case, contained in the kernel of the noise operators written in standard form:

Lemma 2. Assume that the operators $\{\tilde{H}, D_k\}$ make $\rho_d = |\Psi\rangle\langle\Psi|$ QLS. Then, for each k, we have $\text{supp}(\rho_{\mathcal{N}_k}) \subseteq \ker(\tilde{\text{D}}_{\mathcal{N}_k})$.

Proof. In order to be QLS, the state must be invariant for $\mathcal{L}$. Hence, by Proposition 1, $|\Psi\rangle$ must be in the kernel of each D_k. Thus, with respect to the decomposition

$$\mathcal{H}_Q = \mathcal{H}_\Psi \oplus \mathcal{H}_\Psi^\perp,$$

with $\mathcal{H}_\Psi = \text{span}\{|\Psi\rangle\}$, every D_k must be of block form (Ticozzi and Viola (2009)):

$$D_k = \begin{bmatrix} 0 & D_{P,k} \\ 0 & D_{R,k} \end{bmatrix},$$

which immediately implies $D_k \rho_d D_k^\dagger = 0$. It then follows that $\text{trace}_{\bar{\mathcal{N}}_k}(D_k\rho_d D_k^\dagger) = 0 = \text{trace}_{\bar{\mathcal{N}}_k}(D_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k} \rho_d D_{\mathcal{N}_k}^\dagger \otimes I_{\bar{\mathcal{N}}_k})$. Therefore, it also follows that $D_{\mathcal{N}_k}\rho_{\mathcal{N}_k} D_{\mathcal{N}_k}^\dagger = 0$. If we consider the spectral decomposition $\rho_{\mathcal{N}_k} \equiv \sum_j q_j |\phi_j\rangle\langle\phi_j|$, with $q_j > 0$, the latter condition implies that, for each j, $\tilde{D}_{\mathcal{N}_k}|\phi_j\rangle\langle\phi_j|\tilde{D}_{\mathcal{N}_k}^\dagger = 0$. Thus, it must be $\text{supp}(\rho_{\mathcal{N}_k}) \subseteq \ker(\tilde{\text{D}}_{\mathcal{N}_k})$, as stated. □

Given Theorem 1 above, if the support of ρ_d is exactly the intersection of these kernels, then the state is DQLS. Let us then assume that

$$\mathcal{H}_0 \equiv \bigcap_k \mathcal{H}_{\mathcal{N}_k}, \quad \dim(\mathcal{H}_0) = n_0 \geq 2,$$

so that ρ_d is *not* DQLS. With this notation, we have the following:

Proposition 2. ρ_d is QLS but not DQLS only if $\mathcal{H}_\Psi \equiv \text{span}|\Psi\rangle$ is an invariant subspace for the Hamiltonian $\tilde{H}$, and no other invariant subspace is contained in (or equal to) $\mathcal{H}_0$. In particular, one can choose $\tilde{H}$ so that $\tilde{H}|\Psi\rangle = 0$.

Proof. If there were another invariant subspace for the Hamiltonian with support in $\mathcal{H}_0$, the latter would be, by definition of $\mathcal{H}_0$, also in the kernel of each D_k, and hence it would be invariant. The dynamics in this invariant subspace would be Hamiltonian, thus it would admit as many invariant states as the the dimension of the subspace. It then follows that ρ_d could not be GAS. If $\tilde{H}|\Psi\rangle = \lambda|\Psi\rangle$, $\lambda \neq 0$, we can always choose $\tilde{H}' = \tilde{H} - \lambda I$ instead, which is also QL if $\tilde{H}$ was. □

When such a $\tilde{H}$ exists, one must look for noise operators $\{D_k\}$ such that $\mathcal{H}_\Psi$ is the *only* invariant subspace for the whole generator, and hence makes ρ_d GAS. Lemma 2 suggests that the most effective choice of noise operators can stabilize $\mathcal{H}_0$, but cannot go further. In order to specify what the action of a stabilizing Hamiltonian would be, it is convenient to pick an orthonormal basis for $\mathcal{H}_0$, which includes the target state:

$$\mathcal{H}_0 = \text{span}\{|\Psi\rangle, |\phi_1\rangle, \dots, |\phi_r\rangle\}, \quad r = n_0 - 1.$$

One would hope that $\tilde{H}|\phi_j\rangle \notin \mathcal{H}_0$ for each j. However, fulfilling these conditions is clearly not necessary, and in fact it need not be possible given the QL constraint. In general, determining whether a QL $\tilde{H}$ *exists* for any non-DQLS state remains an open problem. Nonetheless, in the simple case where $n_0 = 2$, the above idea leads to a specialized formulation of Proposition 2:

Corollary 3. ρ_d is QLS but not DQLS, with $\mathcal{H}_0 = \text{span}\{|\Psi\rangle, |\phi_1\rangle\}$, only if there exists a QL Hamiltonian $\tilde{H}$ such that $\tilde{H}|\Psi\rangle = 0$ and

$$\tilde{H}|\phi_1\rangle \notin \mathcal{H}_0. \tag{6}$$

The necessary conditions summarized in the results above clearly show that a "good" effective Hamiltonian *cannot be a frustration-free Hamiltonian*, that is, it must destabilize $\mathcal{H}_0$. In the following section we will employ Corollary 3 to show that such an Hamiltonian *does exist* for two important classes of multipartite entangled states: maximally entangled GHZ states and translationally invariant W states for multi-qubit systems. Both these classes are *never* DQLS: for any number of qubits and non-trivial locality constraints the conditions of Theorem 1 cannot be satisfied (Ticozzi and Viola (2012)). However, we will explicitly show below that there exists a stabilizing $\tilde{H}$, proving they are QLS.

Remark: Even if we succeed in finding a Hamiltonian $\tilde{H}$ satisfying Proposition 2, we still do not have in general a constructive procedure for determining the stabilizing D_k. What can be shown, as we will illustrate with an example in Section 4.1, is that *not all choices of $\{D_k\}$ satisfying Lemma 2 are effective.* In fact, one needs to be careful in ensuring that the interplay between $\tilde{H}$ and the noise operators introduces enough "mixing" and does not allow for the existence of other invariant sets. We claim that a *generic* choice of noise operators that satisfy Lemma 2 and stabilize $\mathcal{H}_0$ will suffice: while a formal proof will provided elsewhere, the basic idea is to follow the argument of Section III.B of Ticozzi et al. (2011).

4. DQLS $\neq$ QLS: EXAMPLES

4.1 GHZ states

Let the system be a n-qubit quantum register, with total dimension $N = 2^n$. A representative of the Greenberger-Horne-Zeilinger (GHZ) class [2] is the state $\rho_{\text{GHZ}} = |\Psi\rangle\langle\Psi|$, where

$$|\Psi\rangle \equiv |\Psi_{\text{GHZ}}\rangle = (|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}.$$

As shown in Ticozzi and Viola (2012), GHZ states are not DQLS, except in cases where the QL constraint becomes trivial. In fact, any reduced state on any (nontrivial) neighborhood is an equiprobable mixture of $|000\ldots0\rangle$ and $|111\ldots1\rangle$. It is then immediate to see that

$$\mathcal{H}_0 = \text{span}\{|000\ldots0\rangle, |111\ldots1\rangle\} = \bigcap_k \text{supp}(\rho_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}),$$

and thus ρ_{GHZ} is not DQLS.

[2] Observe that the QLS property is invariant under arbitrary local unitary transformations, similar to the DQLS case examined in Ticozzi and Viola (2012).

We now show how adding a QL Hamiltonian can render ρ_{GHZ} GAS. Given Corollary 3, we need $\tilde{H}$ such that:

$$\tilde{H}(|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2} = 0,$$
$$\tilde{H}(|000\ldots0\rangle - |111\ldots1\rangle)/\sqrt{2} \notin \mathcal{H}_0.$$

In order for this to happen, we need to find $\tilde{H}$ such that

$$\tilde{H}|000\ldots0\rangle = -\tilde{H}|111\ldots1\rangle.$$

We now take into account the QL structure of $\tilde{H} = \sum_k H_k$, and the fact that each QL component acts on *at most* a finite number n_k of "symbols", that is, 0 or 1 in the factorized components of the states $|0\rangle^{\otimes n}, |1\rangle^{\otimes n}$, respectively. If we assume that $n_k < n/2$, it follows that

$$\tilde{H}|000\ldots0\rangle \in \text{span}\Big\{|x_1,\ldots,x_n\rangle, x_j \in \{0,1\}, \sum_j x_j < n/2\Big\},$$

$$\tilde{H}|111\ldots1\rangle \in \text{span}\Big\{|x_1,\ldots,x_n\rangle, x_j \in \{0,1\}, \sum_j x_j > n/2\Big\}.$$

Hence, the two vectors must be orthogonal, since they belong to subspaces spanned by two orthogonal sets of vectors. This means that a Hamiltonian satisfying the requirements of Corollary 3 does not exists. In other words, we need $\tilde{H}$ to be able to "flip" *at least $n/2$ qubits* in the completely factorized basis states $|0\rangle^{\otimes n}, |1\rangle^{\otimes n}$.

If we take the neighborhood to include $n_k = n/2$ qubits, we can always construct a QL Hamiltonian such that

$$\tilde{H}(|000\ldots0\rangle = (|1\ldots10\ldots0\rangle - |0\ldots01\ldots1\rangle)/\sqrt{2},$$
$$\tilde{H}(|111\ldots1\rangle = (-|1\ldots10\ldots0\rangle + |0\ldots01\ldots1\rangle)/\sqrt{2},$$

with the vectors in the r.h.s. containing exactly $n/2$ zeroes and $n/2$ ones, which clearly satisfies the requirement. This can be obtained by considering two disjoint sets $\mathcal{S}_{\ell=1,2}$ each including one half of the qubits if n is even (or $(n+1)/2, (n-1)/2$ in the odd-n case), and for $\ell = 1, 2$, choose

$$\tilde{H}_\ell = (-1)^{\ell-1}\Big(\bigotimes_{a\in\mathcal{S}_\ell} \sigma_x^{(a)}\Big) \otimes I_{\bar{\mathcal{S}}_\ell}.$$

If, compatibly with our QL constraints, there exists two neighborhoods $\mathcal{N}_{k_1,k_2}$ such that

$$\mathcal{S}_{\ell=1,2} \subset \mathcal{N}_{k_{\ell=1,2}},$$

then, with a proper choice of the noise operators, a GHZ state can then be rendered GAS. For example, for 3 qubits on a line, we can choose $\mathcal{S}_1 = \{1\}, \mathcal{S}_2 = \{2,3\}$, so that under 2-body QL constraints $\mathcal{N}_1 = \{1,2\}, \mathcal{N}_2 = \{2,3\}$, we can implement:

$$\tilde{H} = \sigma_x^{(1)} - \sigma_x^{(2)} \otimes \sigma_x^{(3)},$$
$$D_1 = I_2 \otimes (|00\rangle\langle 01| + |11\rangle\langle 10|),$$
$$D_2 = I_2 \otimes (|00\rangle\langle 01| + i|11\rangle\langle 10|).$$

Remark: The phase factor in D_2 is not coincidental: in fact, the more symmetric choice $D_2' = I_2 \otimes (|00\rangle\langle 01| + |11\rangle\langle 10|)$ would leave the -1-eigenspace of $\bigotimes_{a=1,2,3} \sigma_x^{(a)}$ invariant for the generator associated to $\tilde{H}, D_1, D_2'$, while ρ_{GHZ} is the unique invariant state for $\tilde{H}, D_1, D_2$.

This simple example is enough to prove that DQLS $\subsetneq$ QLS, namely that *there exist pure entangled states that are not stabilizable by dissipation alone but can be made GAS by the addition of a suitable QL Hamiltonian.*

Furthermore, this way of making GHZ states GAS is general, and, in the light of the observation above, it requires neighborhoods of the minimum possible size. Notice that the neighborhood size scales linearly ($\approx n/2$) with the overall system size, thus as $n \to +\infty$ the needed range of interaction size goes to infinity.

4.2 W states

Consider again an n-qubit system with the same notation as in the previous example. A representative of the W class is the state $\rho_{\rm W} = |\Psi\rangle\langle\Psi|$, with
$|\Psi\rangle \equiv |\Psi_{\rm W}\rangle = (|100\ldots0\rangle + |010\ldots0\rangle + \ldots + |000\ldots1\rangle)/\sqrt{n}$.
This state is a permutation-invariant superposition of all computational basis states with a single 1, and has reduced states on any non-trivial neighborhood that are statistical mixtures of $|000\ldots0\rangle$ and a smaller W state $|\Psi_{\rm W'}\rangle$, of the dimension of the neighborhood. Accordingly,

$$\mathcal{H}_0 = \mathrm{span}\{|000\ldots0\rangle, |\Psi_{\rm W}\rangle\} = \bigcap_k \mathrm{supp}(\rho_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}),$$

and thus $\rho_{\rm W}$ is *not* DQLS. We are still in a case where Corollary 3 can be applied.

It is easy to show that, for example, the following 2-body Hamiltonian satisfies the requirements of Corollary 3:

$$\tilde{H} = \sigma_x^{(1)} \otimes P_0 - P_0 \otimes \sigma_x^{(n)} \tag{7}$$

$$= \sigma_x^{(1)} \otimes \Big(\sum_{a=2}^{n-1} (\sigma_z^{(a)} - (n-3)I^{(a)})\Big) + \Big(\sum_{a=2}^{n-1} (\sigma_z^{(a)} - (n-3)I^{(a)})\Big) \otimes \sigma_x^{(n)}, \tag{8}$$

where P_0 is an operator on $\bigotimes_{a=2}^{n-1} \mathcal{H}_a$ such that:

$$P_0|\Psi_{W_{n-2}}\rangle = 0, \quad P_0|0\rangle^{\otimes(n-2)} = |0\rangle^{\otimes(n-2)}.$$

While we derived this particular Hamiltonian relying on symmetry considerations, to our scope it suffices to verify by direct computation that

$$\tilde{H}|\Psi_W\rangle = 0, \quad \tilde{H}|0\rangle^{\otimes n} = |10\ldots0\rangle - |0\ldots01\rangle,$$

as needed. This shows how W-states on an arbitrary number of qubits can in principle be QLS by allowing for *arbitrary* 2-body interactions.

5. QL STABILIZATION WITH DRIFT DYNAMICS

We will now address the possibility of a pre-existing QL drift dynamics. It will be convenient to discuss the case of having a given Hamiltonian component first, and to introduce the more general case next.

5.1 Drift Hamiltonian

Imagine that a drift QL Hamiltonian H_0 describes the free dynamics of the system of interest, and that as before the target state is $\rho_d = |\Psi\rangle\langle\Psi|$.

In order to establish whether ρ_d can be made GAS by *purely dissipative dynamics* together with the action of H_0, we can first check whether $H_0|\Psi\rangle = \lambda|\Psi\rangle$, in which case the state is invariant. If not, consider a QL decomposition of $H_0 = \sum_k H_k$, and decompose H_k in matrix blocks according to $\mathcal{H} = \mathcal{H}_\Psi \oplus \mathcal{H}_R$, that is:

$$H_k = \begin{bmatrix} H_{S,k} & H_{P,k} \\ H_{P,k}^\dagger & H_{R,k} \end{bmatrix}.$$

For each k, define:

$$D'_k = \begin{bmatrix} 1 & D'_{P,k} \\ 0 & 0 \end{bmatrix}, \quad D'_{P,k} = 2iH_{P,k}.$$

By Corollary 1 in Ticozzi and Viola (2008), it follows that adding D'_k as a noise operator for each neighborhood makes ρ_d invariant for the global dynamics. Hence, we can find an alternative (equivalent) representation of the generator with the added noise operators $\{D''_k\}$ in standard form and H'_0 obeying $H'_0|\Psi\rangle = 0$. Now we can proceed to:

(1) Determine whether ρ_d would be DQLS in the absence of H'_0, by applying Theorem 1.
(2) If ρ_d is DQLS, then a *generic* choice of stabilizing $\{D_k\}$ will make ρ_d GAS.
(3) If ρ_d is not DQLS, we can check whether H'_0 satisfies the necessary conditions of Proposition 2. If it does, we know that *under some suitable locality notions*, there can exist stabilizing dissipators. If not, ρ_d cannot be made GAS.

Note that if complete QL Hamiltonian control is available as an additional resource, the situation is obviously simpler: since the QL H_0 is known, a control Hamiltonian $H_1 = -H_0$ can be added to cancel the action of the drift. Then the problem becomes completely equivalent to the drift-less case, for which we have Theorem 1 in the DQLS case or, in the QLS case, the set of necessary conditions given in Lemma 2, Proposition 2 and/or Corollary 3.

5.2 Drift Hamiltonian and dissipation

Assume that a QL Markovian drift dynamics is driving the system of interest, say, $\dot{\rho} = -i[H_0, \rho] + \mathcal{L}_D[\rho]$, where $\mathcal{L}_D[\rho]$ denotes some noise generator in Lindblad form, associated to QL noise operators $\{L_k\}$.

The first thing to check is whether, for each neighborhood $\mathcal{N}_k$, $L_k|\Psi\rangle = \lambda_k|\Psi\rangle$. In fact, this is a necessary condition for the invariance of ρ_d that *must be satisfied by all noise operators irrespective of them being controlled or not* (Corollary 1 in Ticozzi and Viola (2008)). Hence, if $L_k|\Psi\rangle \neq \lambda_k|\Psi\rangle$ for some k, then ρ_d cannot be made GAS.

If $L_k|\Psi\rangle = \lambda_k|\Psi\rangle$ for all k, then using Lemma 1 we can write an equivalent generator in standard form, with operators $\{\tilde{H}_0, \tilde{L}_k\}$ such that $\tilde{L}_k|\Psi\rangle = 0$. These noise operators must satisfy Lemma 2. In this way, we have mapped the problem back the one considered in Section 5.1, where we had only Hamiltonian drift.

(1) In case we have only dissipative control capabilities, we need to determine if the Hamiltonian $\tilde{H}_0$ destabilizes the desired state. A destabilizing Hamiltonian can then be compensated in a way similar to the one given in the previous section, but now choosing

$$D'_{P,k} = 2iH_{P,k} - \tilde{L}_{S,k}^\dagger \tilde{L}_{P,k}.$$

The free dynamics plus the QL $\{D'_k\}$ can then be associated to operators in standard form $\{H'_0, D'_k\}$.

(2) If the state is DQLS in the absence of H_0', then again a generic choice of $\{D_k\}$ will suffice. If the state is not DQLS, we can check whether H_0' satisfies the conditions of Proposition 2 and/or Corollary 3. If so, we are left with the problem of finding effective QL noise operators, otherwise the state is not QLS.

(3) In case we also have complete QL Hamiltonian control, we can cancel H_0' as above, and the problem is reduced to determining whether ρ_d is DQLS or QLS.

6. QL STABILITY AND STABILIZATION: WHERE DO WE STAND?

Building on our previous analysis and results (Ticozzi and Viola (2012)), we have provided evidence that for certain quasi-locality notions *the set of DQLS states is strictly smaller than the set of QLS states*, as the latter contains also the celebrated GHZ and W states. This points to QL control Hamiltonians as key resources for dissipative entanglement engineering in multipartite quantum systems.

While a characterization of QLS states is still missing, we have provided in Proposition 2 and/or Corollary 3 *necessary conditions* for a QL Hamiltonian to be an effective aid in stabilizing the desired entangled pure state. If such a Hamiltonian exists, we have so far been able to easily find effective QL noise operators that make the desired state GAS (and hence QLS) on a case-by-case basis. Our claim is that a *generic choice of the noise operators* $\{D_k\}$ *is effective*, provided that they satisfy Lemma 2 and that the *only* invariant subspace for the dissipative part of the dynamics is:

$$\mathcal{H}_0 = \bigcap_k \text{supp}(\rho_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}) = \bigcap_k \mathcal{H}_{\mathcal{N}_k}.$$

A detailed proof of this claim will be worked out elsewhere, relying on the fact that the sub-manifold of states that can be made invariant by adding an Hamiltonian to a dissipative generator has a lower dimensionality with respect to the whole set, and exploiting representation tools for Markovian dynamics such as the "dissipation-induced decomposition" introduced in Ticozzi et al. (2011).

The key remaining open problems are then to provide an *explicit characterization* of QLS states, possibly suggesting a way to classify states at least for commonly used QL constraints, and to find a *systematic* way to synthesize, whenever possible, QL Hamiltonian satisfying Proposition 2. Addressing the *scalability* of both DQLS and QLS stabilization procedure is another challenging yet practically important area for further investigation. The possibility of considering switched dynamics, feedback control and time-dependent generators offer promising research directions in this sense, in the effort of surpassing the stringent limitations imposed by QL continuous-time dynamics.

ACKNOWLEDGEMENTS

L.V. gratefully acknowledges hospitality by the Department of Information Engineering of the University of Padova, where part of this work was performed.

REFERENCES

Barreiro, J.T., Müller, M., Schindler, P., Nigg, D., Monz, T., Chwalla, M., Hennrich, M., Roos, C.F., Zoller, P., and Blatt, R. (2011). An open-system quantum simulator with trapped ions. *Nature*, 470, 486–491.

Beige, A., Braun, D., Tregenna, B., and Knight, P.L. (2000). Quantum computing using dissipation to remain in a decoherence-free subspace. *Phys. Rev. Lett.*, 85(8), 1762–1765.

Boto, A.N., Kok, P., Abrams, D.S., Braunstein, S.L., Williams, C.P., and Dowling, J.P. (2000). Quantum interferometric optical lithography: Exploiting entanglement to beat the diffraction limit. *Phys. Rev. Lett.*, 85(13), 2733.

Cappellaro, P., Emerson, J., Boulant, N., Ramanathan, C., Lloyd, S., and Cory, D.G. (2005). Entanglement-assisted quantum metrology. *Phys. Rev. Lett.*, 94(2), 020502.

Carvalho, A.R.R., Milman, P., de Matos Filho, R.L., and Davidovich, L. (2001). Decoherence, pointer engineering, and quantum state protection. *Phys. Rev. Lett.*, 86(22), 4988–4991.

Diehl, S., Micheli, A., Kantian, A., Kraus, B., Büchler, H.P., and Zoller, P. (2008). Quantum states and phases in driven open quantum systems with cold atoms. *Nature Phys.*, 4, 878–883.

Kraus, B., Diehl, S., Micheli, A., Kantian, A., Büchler, H.P., and Zoller, P. (2008). Preparation of entangled states by dissipative quantum Markov processes. *Phys. Rev. A*, 78(4), 042307.

Krauter, H., Muschik, C.A., Wasilewski, W., Petersen, J.M., Cirac, J.I., and Polzik, E.S. (2011). Entanglement generated by dissipation and steady state entanglement of two macroscopic objects. *Phys. Rev. Lett.*, 107, 080503:1–4.

Nielsen, M.A. and Chuang, I.L. (2000). *Quantum Computation and Quantum Information*. Cambridge Univ. Press.

Perez-Garcia, D., Verstraete, F., Wolf, M.M., and Cirac, J.I. (2007). Matrix product state representations. *Quantum Inf. Comput.*, 7, 401–430.

Perez-Garcia, D., Verstraete, F., Wolf, M.M., and Cirac, J.I. (2008). PEPS as unique ground states of local hamiltonians. *Quant. Inf. Comp.*, 8, 0650–0663.

Poyatos, J.F., Cirac, J.I., and Zoller, P. (1996). Quantum reservoir engineering with laser cooled trapped ions. *Phys. Rev. Lett.*, 77(23), 4728–4731.

Raussendorf, R. and Briegel, H.J. (2001). A one-way quantum computer. *Phys. Rev. Lett.*, 86(22), 5188.

Ticozzi, F., Lucchese, R., Cappellaro, P., and Viola, L. (2011). Hamiltonian control of quantum dynamical semigroups: Stabilization and convergence speed. *submitted, e-print arXiv:1101.2452.*

Ticozzi, F. and Viola, L. (2008). Quantum Markovian subsystems: Invariance, attractivity and control. *IEEE Trans. Aut. Contr.*, 53(9), 2048–2063.

Ticozzi, F. and Viola, L. (2009). Analysis and synthesis of attractive quantum Markovian dynamics. *Automatica*, 45, 2002–2009.

Ticozzi, F. and Viola, L. (2012). Stabilizing entangled states with quasi-local quantum dynamical semigroups. *Phil. Trans. R. Soc. London, Ser. A, in press, e-print arXiv:1112.4860.*

Verstraete, F., Wolf, M.M., and Cirac, J.I. (2009). Quantum computation and quantum-state engineering driven by dissipation. *Nature Phys.*, 5, 633–636.