Minimal Resources for the Estimation of Trace-Preserving Quantum Channels

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***Abstract*— We determine the minimal experimental resources that ensure a unique solution in the estimation of trace-preserving quantum channels with both direct and convex optimization methods. A convenient parametrization of the constrained set is used to develop a globally converging Newton-type algorithm that ensures a physically admissible solution to the problem. Numerical simulation is provided to support the results, and indicate that the minimal experimental setting is sufficient to guarantee good estimates.**

I. Introduction

We consider an *identification problem arising in the reconstruction of quantum dynamical models from experimental data*. This is a key issue in many quantum information processing tasks [11], [5], [12], [10], [4]. For example, a precise knowledge of the behavior of a channel to be used for quantum computation or communications is needed in order to ensure that optimal encoding/decoding strategies are employed, and verify that the noise thresholds for hierarchical error-correction protocols, or for effectiveness of quantum key distribution protocols, are met [11], [5]. In many cases of interest, for example in free-space communication [15], channels are not stationary and to ensure good performances, repeated and fast estimation steps would be needed as a prerequisite for adaptive encodings. In addition to this, when the goal is to embed the system used for probing the channel in a moving vehicle or a satellite, one seeks the simplest implementation, or at least a compromise between estimation accuracy and the number of experimental resources needed.

Therefore, we here focus on: (i) characterizing the minimal experimental setting (in terms of available probe states and measured observables) needed for a consistent estimation of the channel; (ii) exploring how a minimal parametrization of the models can be exploited to reduce the complexity of the estimation algorithm; and (iii) testing (numerically) the *minimal experimental setting*, and compare it to "richer" experimental resources. In doing this, we present a general framework for the estimation of physically-admissible trace preserving quantum channels by minimizing a suitable class of (convex) loss functions which include commonly used maximum likelihood (ML) functionals. In the large body of literature regarding channel estimation, or *quantum process tomography* (see e.g. [12], [10] and references therein), the experimental resources are usually assumed to be given. Mohseni et al. [10] compare different strategies, but focus on the role of having entangled states as an additional resource, while we shall assume there is no additional quantum system to work with. The problem we study is closer in spirit to that taken in [14] while studying minimal *state* tomography.

Our analysis of the problem and the standard tomography methods (including "inversion" and ML methods) leads to a necessary and sufficient condition for *identifiability* of the channel and to the characterization of the minimal experimental resources (or *quorum*, in the language of [6]) for *Trace-Preserving (TP) channels estimation*. While the existing ML approaches introduce the TP constraint through a Lagrange multiplier [12], [10], the method we propose constrains the set of channels of the optimization problem to TP maps from the beginning. In a d level quantum system, this allow for an immediate reduction from d^4 to d^4-d^2 free parameters in the estimation problem. However, determining which conditions on probe states and output measurements must be satisfied to ensure identifiability has not been made explicit so far. Our analysis can also be considered as complementary to the one presented in [3], where the TP assumption is relaxed to include losses. We pursue a rigorous presentation of the results and we try to make contact with ideas and methods of (classical) system identification. The same explicit parametrization for TP channels is also used to develop a Newton-type algorithm with barriers, which ensures convergence in the set of physically-admissible maps. Numerical simulations show that experimental settings richer than the minimal one do not lead to better performances (fixed the total number of available "trials").

II. Preliminaries: Quantum channels and χ-representation

Consider a d-level quantum system with associated an Hilbert space $\mathcal{H}$ isomorphic to $\mathbb{C}^d$. The *state* of the system is described by a density operator, namely by a positive, unit-trace matrix

$$\rho \in \mathfrak{D}(\mathcal{H}) = \{\rho \in \mathbb{C}^{d\times d} | \rho = \rho^\dagger \geq 0,\, \mathrm{tr}(\rho) = 1\},$$

which plays the role of probability distribution in classical probability. A state is called *pure* if it is associated by an orthogonal projection matrix on a one-dimensional subspace. Measurable quantities or *observables* are associated with Hermitian matrices $X = \sum_k x_k \Pi_k$, with $\{\Pi_k\}$ the

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associated spectral family of orthogonal projections. Their spectrum $\{x_k\}$ represents the possible outcomes, and the probability of observing the kth outcome can be computed as $p_\rho(\Pi_k) = \text{tr}(\Pi_k\rho)$. A *quantum channel* is a map $\mathcal{E} : \mathfrak{D}(\mathcal{H}) \to \mathfrak{D}(\mathcal{H})$. It is well known [9], [11] that a physically admissible quantum channel must be linear and *Completely Positive* (CP), namely it admits an Operator-Sum Representation (OSR)

$$\mathcal{E}(\rho) = \sum_{j=1}^{d^2} K_j \rho K_j^\dagger \tag{1}$$

where $K_i \in \mathbb{C}^{d\times d}$ are called *Kraus operators*. In order to be *Trace Preserving* (TP), a necessary condition to map states to states, it must also hold that

$$\sum_{j=1}^{d^2} K_j^\dagger K_j = I_d \tag{2}$$

where I_d is the $d \times d$ identity matrix. TPCP maps can be thought as the quantum equivalent of Markov transition matrices in the classical setting. An alternative way to describe a CPTP channel is offered by the χ-*representation*. Each Kraus operator $K_j \in \mathbb{C}^{d\times d}$ can be expressed as a linear combination (with complex coefficients) of $\{F_m\}_{m=1}^{d^2}$, F_m being the elementary matrix E_{jk}, with $m = (j-1)d + k$. Accordingly, the OSR (1) can be rewritten as

$$\mathcal{E}(\rho) = \sum_{m,n=1}^{d^2} \chi_{m,n} F_m \rho F_n^\dagger \tag{3}$$

where χ is the $d^2 \times d^2$ Hermitian matrix with element $\chi_{m,n}$ in position (m, n). It is easy to see that it must satisfy $\chi = \chi^\dagger \geq 0$ and (following from (2))

$$\sum_{m,n=1}^{d^2} \chi_{m,n} F_n^\dagger F_m = I_d. \tag{4}$$

The map $\mathcal{E}$ is completely determined by the matrix χ.

The χ matrix can be used directly to calculate the effect of the map on a given state, and the probability of measurements outcomes for the transformed state, as well as observable expectations. Before providing the explicit formulas, we need to recall the definition of *partial trace*. Consider two finite-dimensional vector spaces $\mathcal{V}_1\, \mathcal{V}_2$, with $\dim \mathcal{V}_1 = n_1^2$, $\dim \mathcal{V}_2 = n_2^2$. The partial trace can be defined as the only linear function such that for any pair $X \in \mathcal{M}_{n_1}$, $Y \in \mathcal{M}_{n_2}$:

$$\text{tr}_{\mathcal{V}_2}(X \otimes Y) = \text{tr}(Y)X.$$

An analogous definition can be given for the partial trace over $\mathcal{V}_1$. If the two vector spaces have the same dimension, $n_1 = n_2$, we will indicate with tr_1 and tr_2 the partial traces over $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively.

Let $\mathcal{E}_\chi$ be a CPTP map associated to a given χ: then it can be shown, by using the properties of the partial trace, that

$$\text{tr}_1(\chi) = I_d \tag{5}$$

and for any $\rho \in \mathfrak{D}(\mathcal{H})$, we have

$$\begin{aligned} \mathcal{E}_\chi(\rho) &= \text{tr}_2(\chi(I_d \otimes \rho^T)) \\ p_{\mathcal{E}(\rho)}(\Pi) &= \text{tr}(\mathcal{E}(\rho)\Pi) = \text{tr}(\chi(\Pi \otimes \rho^T)). \end{aligned}$$

III. Main Results: Identifiability Condition and Minimal Setting

A. The Channel Identification Problem

Consider the following setting: a quantum system prepared in a *known pure state* ρ is fed to an unknown channel $\mathcal{E}$. The system in the *output state* $\mathcal{E}(\rho)$ is then subjected to a projective measurement of an *observable*. By noting that an observable (being represented by an Hermitian matrix in our setting) admits a decomposition in orthogonal projections representing mutually incompatible quantum events, we can without loss of generality restrict ourselves to consider measurements associated to orthogonal projections $\Pi = \Pi^\dagger = \Pi^2$. For each one of these, the outcome x is in the set $\{0, 1\}$, and can be interpreted as a sample of the (classical) random variable X which has distribution

$$P_{\chi(x),\rho} = \begin{cases} p_{\chi,\rho}(\Pi), & \text{if } x = 1 \\ 1 - p_{\chi,\rho}(\Pi), & \text{if } x = 0 \end{cases} \tag{6}$$

where $p_{\chi,\rho}(\Pi) = \text{tr}(\mathcal{E}_\chi(\rho)\Pi)$ is the probability that the measurement of Π returns outcome 1 when the state is $\mathcal{E}_\chi(\rho)$.

Assume that the experiment is repeated with a series of known input (pure) states $\{\rho_k\}_{k=1}^L$, and to each trial the same orthogonal projections $\{\Pi_j\}_{j=1}^M$ are measured N times, obtaining a series of outcomes $\{x_l^{jk}\}$. We consider the sampled frequencies to be our *data*, namely

$$f_{jk} := \frac{1}{N}\sum_{l=1}^{N} x_l^{jk}. \tag{7}$$

The channel identification problem (or as it is referred to in the physics literature, the *quantum process tomography* problem [12], [11], [10]) we are concerned with consists in constructing a *Kraus map* $\mathcal{E}_{\hat{\chi}}$ that fits the experimental data (in some optimal way), in particular estimating a matrix $\hat{\chi}$ satisfying constraints (3),(4).

B. Necessary and sufficient conditions for identifiability

It is well known [13], [12] that by imposing linear constraints associated to the TP condition (4), or equivalently (5), one reduces the d^4 real degrees of freedom of χ to $d^4 - d^2$. This will be made explicit in the following, by parameterizing χ in a "generalized" Pauli basis (also known as gell-mann matrices, Fano basis or coherence vector representation in the case of states [2], [12]). Usually the trace preserving constraint is not directly included in the standard tomography method [10], since in principle it should emerge from the physical properties of the channel, or it is imposed through a (nonlinear) Lagrange multiplier in the maximum likelihood approach [12]. Here, in order to investigate the minimum number of probe (input) states and measured projectors needed to uniquely determine χ, it is convenient to include this constraint from the very beginning. Doing so, we lose the possibility of exploiting a Cholesky factorization in order to impose positive semidefiniteness of χ: noentheless, semidefiniteness of the solution can be imposed directly in the algorithm, *e.g.* by using a barrier method. Before proceeding to the main results, a number of definitions are in order. Consider an orthonormal basis for $d^2 \times d^2$ Hermitian matrices of the form $\{\sigma_j \otimes \sigma_k\}_{j,k=0,1,\ldots,d^2-1}$, where $\sigma_0 = 1/\sqrt{d} I_d$, while $\{\sigma_j\}_{j=1,\ldots,d^2-1}$ is a basis for the traceless subspaces. We can now write

$$\chi = \sum_{jk} s_{jk} \sigma_j \otimes \sigma_k.$$

If we now substitute it into (5), we get:

$$I_d = \mathrm{tr}_1(\chi) = \sum_{jk} s_{jk} \mathrm{tr}(\sigma_j) \sigma_k = \sum_k \sqrt{d}\, s_{0k} \sigma_k,$$

and hence, since the σ_j are linearly independent, we can conclude that $s_{00} = 1$, $s_{0j} = 0$ for $j = 1, \ldots, d^2-1$. Hence, the free parameters for a TP map (at this point not necessarily CP, since we have not imposed the positivity of χ yet) are $d^4 - d^2$, and we can write any TP χ as $\chi = d^{-1} I_{d^2} + \sum_{j=1,k=0}^{d^2-1,d^2-1} s_{jk} \sigma_j \otimes \sigma_k$, or, in a more compact notation,

$$\chi(\underline{\theta}) = d^{-1} I_{d^2} + \sum_{\ell=1}^{d^4-d^2} \theta_\ell Q_\ell, \tag{8}$$

by rearranging the double indexes j, k in a single index ℓ, and defining the corresponding $Q_\ell = \sigma_j \otimes \sigma_k$. Thus, there exists a one to one correspondence among χ and the $d^4 - d^2$-dimensional real vector $\underline{\theta} = \left[\begin{array}{ccc} \theta_1 & \ldots & \theta_{d^4-d^2} \end{array} \right]^T$, and the χ matrices corresponding to TP maps form an affine space, its linear part being

$$\mathcal{S}_{TP} := \mathrm{span}\{Q_\ell\} = \mathrm{span}\{\sigma_j \otimes \sigma_k\}_{j=1,\ldots,d^2-1,k=0,\ldots,d^2-1}.$$

In order to find necessary and sufficient conditions for identifiability, it is convenient to define

$$B_{jk} = (\Pi_j - \frac{1}{d} I) \otimes \rho_k^T \tag{9}$$

and $\mathcal{B} = \mathrm{span}\{B_{jk}\}_{j=1,\ldots,M,k=1,\ldots,L}$. Intuitively, $\mathcal{B}$ represents the space of input/output combination that can be probed by the set of experimental resources $\{\rho_k\}, \{\Pi_j\}$ we choose. The definition of the B_{jk} is motivated by the fact that, since $Q_\ell = \sigma_{j\neq 0} \otimes \sigma_k$, it holds that

$$\mathrm{tr}(Q_\ell(\Pi_j \otimes \rho_k^T)) = \mathrm{tr}(Q_\ell B_{jk}). \tag{10}$$

By recalling that $\sigma_j, j = 1, \ldots d^2 - 1$ is a basis for the traceless subspace of Hermitian matrices it is immediate to show that $\mathcal{B} \subseteq \mathcal{S}_{TP}$. Finally, let us introduce the function g that maps the space of TP channels in the (theoretical) set of probabilities for the input states/measured projectors combinations:

$$\begin{aligned} g \quad : \quad & \mathbb{R}^{d^4-d^2} \to \mathbb{R}^{M\times L} \\ & \underline{\theta} \mapsto g(\underline{\theta}) \end{aligned}$$

where the component of $g(\underline{\theta})$ in position (j,k) is defined as

$$g_{jk}(\underline{\theta}) = p_{\chi(\underline{\theta}),\rho_k}(\Pi_j) = \mathrm{tr}(\chi(\underline{\theta})(\Pi_j \otimes \rho_k^T)). \tag{11}$$

The key result on identifiability is the following:

Proposition 3.1: g is injective if and only if $\mathcal{S}_{TP} = \mathcal{B}$.

Proof. Given (11), we have that

$$\begin{aligned} g_{jk}(\underline{\theta}_1) - g_{jk}(\underline{\theta}_2) &= \mathrm{tr}[(\chi(\underline{\theta}_1) - \chi(\underline{\theta}_2))(\Pi_j \otimes \rho_k^T)] \\ &= \mathrm{tr}[S(\underline{\theta}_1 - \underline{\theta}_2) B_{jk}] \\ &= \langle S(\underline{\theta}_1 - \underline{\theta}_2), B_{jk} \rangle \end{aligned}$$

where $S(\underline{\theta}_1 - \underline{\theta}_2) = \chi(\underline{\theta}_1) - \chi(\underline{\theta}_2) = \sum_{l=1}^{d^4-d^2} (\theta_{1,l} - \theta_{2,l}) Q_l \in \mathcal{S}_{TP}$. So, we have that

$$g(\underline{\theta}_1) = g(\underline{\theta}_2) \;\Leftrightarrow\; \langle S(\underline{\theta}_1 - \underline{\theta}_2), B_{jk} \rangle = 0 \;\; \forall\, j, k. \tag{12}$$

Assume $\mathcal{S}_{TP} = \mathcal{B}$: the only element of $\mathcal{S}_{TP}$ for which the r.h.s. of (12) could be true is zero. Since by definition $S(\underline{\theta}_1 - \underline{\theta}_2) = 0$ if and only if $\underline{\theta}_1 = \underline{\theta}_2$, g is injective. On the other hand, assume that $\mathcal{B} \subsetneq \mathcal{S}_{TP}$: therefore there exists $T \neq 0 \in \mathcal{S}_{TP} \bigcap \mathcal{B}^\perp$ such that

$$T = \sum_\ell \gamma_\ell Q_\ell, \quad \langle T, B_{jk} \rangle = 0 \;\forall j, k.$$

But this would mean that $\underline{\theta}$ and $\underline{\theta} + \underline{\gamma}$ have the same image $g(\underline{\theta})$, and hence g is not injective. ■

We anticipate here that g being injective is a necessary and sufficient condition for *a priori* identifiability of χ, and thus for having a unique solution of the problem for both inversion (standard process tomography) and convex optimization-based (e.g. maximum likelihood) methods, up to some basic assumptions on the cost functional, see Section III-D. As a consequence of these facts, we can determine the *minimal experimental resources*, in terms of input states and measured projectors, needed for faithfully reconstructing χ from noiseless data $\{f_{jk}^\circ\}$, where $f_{jk}^\circ = p_{\chi,\rho}(\Pi)$. In the light of Proposition 3.1, the minimal experimental setting is characterized by a choice of $\{\Pi_j, \rho_k\}$ such that $\mathcal{S}_{TP} = \mathcal{B}$. Recalling the definition of $\mathcal{B}$, through (9), it is immediate to see that $\mathcal{S}_{TP} = \mathcal{B}$ if and only if $\mathrm{span}\{\Pi_j - d^{-1} I_d\} = \mathrm{span}\{\sigma_j, j = 1, \ldots, d^2 - 1\}$ and $\mathrm{span}\{\rho_k\} = \mathbb{C}^{d\times d}$. We can summarize this fact as a corollary of Proposition 3.1.

Corollary 3.1: g is injective if and only if we have *at least* d^2 linearly independent input states $\{\rho_k\}$, and $d^2 - 1$ measured $\{\Pi_j\}$ such that

$$\mathrm{span}\{\Pi_j - d^{-1} I_d\} = \mathrm{span}\{\sigma_j, j = 1, \ldots, d^2 - 1\}.$$

We call such a set a *minimal experimental setting*. Notice that, using the terminology of [12], [6], the minimal *quorum* of observables consists of $d^2 - 1$ properly chosen elements. While in most of the literature at least d^2 observables are considered [7], [10], we showed it is in principle possible to spare a measurement channel at the output. A physically-inspired interpretation for this fact is that, since we *a priori* know, or assume, that the channel is TP, measuring the component of the observables along the identity does not provide useful information. This is clearly not true if one relaxes the TP condition, as it has been done in [3]: in that case, by the same line of reasoning, d^2 linearly independent observables are the necessary and sufficient for g to be injective.

As an example relevant to many experimental situation, consider the qubit case, i.e. $d = 2$. A minimal set of projector has to span the traceless subspace of $\mathbb{C}^{2\times 2}$: one can choose e.g.:

$$\Pi_j = \frac{1}{2}I_2 + \sigma_j,\ j = x, y, z.$$

$$\rho_{x,y} = \frac{1}{2}I_2 + \sigma_{x,y}, \quad \rho_{\pm} = \frac{1}{2}I_2 \pm \sigma_z. \tag{13}$$

It is clear that there is an asymmetry between the role of output and inputs: in fact, exchanging the number of $\{\Pi_j\}$ and $\{\rho_k\}$ can not lead to an injective g.

C. Process Tomography by inversion

Assume that $\mathcal{S}_{TP} = \mathcal{B}$, and that the data $\{f_{jk}\}$ have been collected. Since f_{jk} is an estimate of $p_{\chi(\underline{\theta}),\rho_k}(\Pi_j)$, consider the following least mean square problem

$$\min_{\underline{\theta}\in\mathbb{R}^{d^4-d^2}} \|\underline{g}(\underline{\theta}) - \underline{f}\| \tag{14}$$

where $\underline{g}(\underline{\theta})$ and $\underline{f}$ are the vectors obtained by stacking the $g_{jk}(\underline{\theta})$ and f_{jk} $j = 1\ldots M$, $k = 1\ldots L$, respectively. In view of (8) and (11) we have that $\underline{g}(\underline{\theta}) = T\underline{\theta} + d^{-1}\underline{1}$ where

$$T = \begin{bmatrix} \ddots & \vdots & \\ & \mathrm{tr}(B_{jk}Q_\ell) & \\ & \vdots & \ddots \end{bmatrix} \tag{15}$$

and $\underline{1}$ is a vector of ones. Notice that the ℓth column of T is formed with the inner products of Q_ℓ with each B_{jk}. Since $\mathcal{S}_{TP} = \mathcal{B}$, the Q_ℓ are linearly independent and the B_{jk} are the generators of $\mathcal{B}$, then T is *full column rank*, namely has rank $d^4 - d^2$. Hence, in principle, one can reconstruct $\hat{\theta}$ as

$$\hat{\underline{\theta}} = T^{\#}(\underline{f} - \frac{1}{d}\underline{1}), \tag{16}$$

$T^{\#}$ being the Moore-Penrose pseudo inverse of T [8]. If the experimental setting is minimal, the usual inverse suffices. However, as it is well known, when computing χ from real (noisy) data, the positivity character is typically lost [12], [1].

D. Convex methods: general framework

More robust approaches for the estimation of physically-acceptable χ (or equivalent parametrizations) have been developed, most notably by resorting to Maximum Likelihood methods [7], [13], [12], [16]. The optimal channel estimation problem can be stated, by using the parametrization for $\chi(\underline{\theta}) = d^{-1}I_{d^2} + \sum_\ell \theta_\ell Q_\ell$ presented in the previous section, as it follows: consider a set of data $\{f_{jk}\}$ as above, and a cost functional $J(\underline{\theta}) := h \circ g(\underline{\theta})$ where $h : \mathbb{R}^{M\times L} \to \mathbb{R}$ is a suitable function which depends on the data $\{f_{jk}\}$. We aim to find

$$\hat{\theta} = \arg\min_{\underline{\theta}} J(\underline{\theta}) \tag{17}$$

subject to $\underline{\theta}$ belonging to some constrained set $\mathcal{C} \subset \mathbb{R}^{d^4-d^2}$. In our case $\mathcal{C} = \mathcal{A}_+$ or $\mathcal{C} = \mathcal{A}_+ \cap \mathcal{I}$, with $\mathcal{A}_+ = \{\underline{\theta} \mid \chi(\theta) \geq 0\}$, while $\mathcal{I} = \{\underline{\theta} \mid 0 < \mathrm{tr}(\chi(\underline{\theta})(\Pi_j \otimes \rho_k^T)) < 1,\ \forall j, k\}$. The second constraint may be used when a cost functional which is not well-defined for extremal probabilities, or in order to ensure that the estimated channel exhibits some noise in each of the measured directions, as it is expected in realistic experimental settings. Since the analysis does not change significantly in the two settings, we will not distinguish between them where it is not strictly necessary. Finally, it can be proven that $\mathcal{C}$ is a bounded set.

Here we focus on the following issue: under which conditions on the experimental setting (or, mathematically, on the set $\mathcal{B}$ defined above) do the optimization approach have a unique solution? In either of the cases above, $\mathcal{C}$ is the intersection of convex nonempty sets: in fact, $\mathcal{S}_{TP}$ and $\chi \geq 0$ are convex and so must be the corresponding sets of $\underline{\theta}$, and it is immediate to verify that $\mathcal{I}$ is convex as well; all of these contain $\underline{\theta} = 0$, corresponding to $\frac{1}{d}I_{d^2}$, and hence they are non empty. In the light of this, it is possible to derive sufficient conditions on J for existence and uniqueness of the minimum in the presence of arbitrary constraint set $\mathcal{C}$. Define $\partial\mathcal{C}_0 := \partial\mathcal{C} \setminus (\partial\mathcal{C} \cap \mathcal{C})$.

Proposition 3.2: Assume h is continuous and strictly convex on $g(\mathcal{C})$, and

$$\lim_{\underline{\theta}\to\partial\mathcal{C}_0} J(\underline{\theta}) = \lim_{\underline{\theta}\to\partial\mathcal{C}_0} h \circ g(\underline{\theta}) = +\infty. \tag{18}$$

If $\mathcal{S}_{TP} = \mathcal{B}$, then the functional J has a unique minimum point in $\mathcal{C}$.

Let us provide the main ides of the proof, leaving the details for an extended version of this paper: since h is strictly convex on $g(\mathcal{C})$ and, in view of Proposition 3.1, the linear function g is injective on $\mathcal{C}$, J is strictly convex on $\mathcal{C}$. So, we only need to show that J takes a minimum value on $\mathcal{C}$. In order to do so, it is sufficient to show that the image of $(-\infty, r]$ under the map J^{-1} is a compact set.

E. ML Binomial functional

Assume a certain set of data $\{f_{jk}\}$ have been obtained, by repeating N times the measurement of each pair (ρ_k, Π_j). For technical reasons (strict convexity of the ML functional on the optimization set) and experimental considerations (noise typically irreversibly affects any state), it is typically assumed that $0 < f_{jk} < 1$. The probability of obtaining a series of outcomes with $c_{jk} = f_{jk}N$ ones for the pair (j, k) is then

$$P_\chi(c_{jk}) = \binom{N}{c_{jk}} \mathrm{tr}(\chi\Pi_j \otimes \rho_k^T)^{c_{jk}}[1 - \mathrm{tr}(\chi\Pi_j \otimes \rho_k^T)]^{N-c_{jk}} \tag{19}$$

so that the overall probability of $\{c_{jk}\}$, may be expressed as: $P_\chi(\{c_{jk}\}) = \prod_{j=1}^{M}\prod_{k=1}^{L} P_\chi(c_{jk})$. By adopting the Maximum Likelihood (ML) criterion, once fixed the $\{c_{jk}\}$ describing the recorded data, the optimal estimate $\hat{\chi}$ of χ is given by maximizing $P_\chi(\{c_{jk}\})$ with respect to χ over a suitable set $\mathcal{C}$. Let us consider our parametrization of the TP $\chi(\underline{\theta})$ as in (8) . If we assume $0 < \mathrm{tr}(\chi(\underline{\theta})(\Pi_j \otimes \rho_k^T)) < 1$, since the logarithm function is monotone, it is equivalent (up to a constant emerging from the binomial coefficients) to minimize over $\mathcal{C} = \mathcal{A}_+ \cap \mathcal{I}$ [1] the function

$$\begin{aligned} J(\underline{\theta}) &= -\frac{1}{N}\log P_{\chi(\underline{\theta})}(\{c_{jk}\}) + \sum_{j,k}\log\begin{pmatrix} N \\ c_{jk}\end{pmatrix} \\ &= -\sum_{j,k} f_{jk}\log[\mathrm{tr}(\chi(\underline{\theta})(\Pi_j\otimes\rho_k^T)] \\ &\quad +(1-f_{jk})\log[1-\mathrm{tr}(\chi(\underline{\theta})(\Pi_j\otimes\rho_k^T))]. \quad (20)\end{aligned}$$

Here, $h(X) = -\sum_{j,k} f_{jk}\log(x_{jk})+(1-f_{jk})\log(1-x_{jk})$ with $x_{jk} = [X]_{jk}$ and $X \in \mathbb{R}^{M\times L}$ is strictly convex on $\mathbb{R}^{M\times L}$ because $0 < f_{jk} < 1$ by assumption. Notice that $\partial\mathcal{C}_0$ is the set of $\underline{\theta} \in \mathcal{A}_+$ for which there exists at least one pair $(\tilde{i},\tilde{k})$ such that $\mathrm{tr}(\chi(\underline{\theta})(\Pi_{\tilde{j}}\otimes\rho_{\tilde{k}}^T)) = 0, 1$. Suppose that $\mathrm{tr}(\chi(\underline{\theta})(\Pi_{\tilde{j}}\otimes\rho_{\tilde{k}}^T)) \to 0$ as $\underline{\theta}\to\partial\mathcal{C}_0$. Therefore, $\log[\mathrm{tr}(\chi(\underline{\theta})(\Pi_j\otimes\rho_k^T))] \to -\infty$. Since $c_{\tilde{j},\tilde{k}} > 0$ by assumption, we have that

$$\begin{aligned} \lim_{\underline{\theta}\to\partial\mathcal{C}_0} J(\underline{\theta}) &= -\lim_{\underline{\theta}\to\partial\mathcal{C}_0}\sum_{j,k} f_{jk}\log[\mathrm{tr}(\chi(\underline{\theta})(\Pi_j\otimes\rho_k^T))] \\ &\quad +(1-f_{jk})\log[1-\mathrm{tr}(\chi(\underline{\theta})(\Pi_j\otimes\rho_k^T))] \\ &= -f_{\tilde{j},\tilde{k}}\lim_{\underline{\theta}\to\partial\mathcal{C}_0}\log[\mathrm{tr}(\chi(\underline{\theta})(\Pi_{\tilde{j}}\otimes\rho_{\tilde{k}}^T))] \\ &= +\infty. \end{aligned}$$

In similar way, we obtain the same result from the other case, and the conditions for existence and uniqueness of the minimum of Proposition 3.2 are satisfied.

We now discuss *consistency* of this method. Let $\underline{\theta}^\circ$ be the "true" parameter and $\chi = \chi(\underline{\theta}^\circ)$ be the corresponding χ-matrix of the "true" channel. First observe that, once fixed the sample frequencies f_{jk} (or, equivalently, c_{jk}),

$$J(\underline{\theta}) \geq -\sum_{j,k} f_{jk}\log[f_{jk}] + (1-f_{jk})\log[1-f_{jk}],$$

so that if there exists $\hat{\underline{\theta}} \in \mathcal{C}$ such that $\mathrm{tr}[\chi(\hat{\underline{\theta}})(\Pi_{\tilde{j}}\otimes\rho_{\tilde{k}}^T)] = f_{jk}$, then such a $\hat{\underline{\theta}}$ is optimal. Hence, in particular, the (unique) optimal solution corresponding to the f_{jk} equal to the "true" probabilities $\mathrm{tr}[\chi(\Pi_j\otimes\rho_k^T)]$ is exactly $\underline{\theta}^\circ$. On the other hand, as the number of experiments N increases, the sample frequencies f_{jk} tend to the "true" probabilities $\mathrm{tr}[\chi(\Pi_j\otimes\rho_k^T)]$. Therefore, in view of convexity of J and of the continuity of J and its first two derivatives, the corresponding optimal solution tends to the "true" parameter $\underline{\theta}^\circ$. This proves consistency of the estimation.

IV. Simulation results

A. Performance comparison

We use the following notation:
• *IN method* to denote the process tomography by inversion of Section III-C.
• *ML method* to denote the ML method, using the functional (20) of Section III-E. In order to find $\hat{\underline{\theta}}$, we used a Newton-type algorithm with logarithmic barriers (the details will be presented in a forthcoming publication) converges: After a finite number of steps, it converges in a quadratic way to the minimum point. Here, we want to compare the performance of IN and ML method for the qubit case $d = 2$. Consider a set of CPTP map $\{\chi_l\}_{l=1}^{100}$ randomly generated and the minimal setting (13). Once the number of measurements N for each couple (ρ_k, Π_j) is fixed, we consider the following comparison procedure:
• At the l-th experiment, let $\{c_{jk}^l\}$ be the data corresponding to the map χ_l. Then, compute the corresponding frequencies $f_{jk}^l = c_{jk}^l/N$.
• From $\{f_{jk}^l\}$ compute the estimates $\hat{\chi}_l^{IN}$ and $\hat{\chi}_l^{ML}$ using IN and ML method respectively.
• Compute the relative errors

$$e_{IN}(l) = \frac{\|\hat{\chi}_l^{IN} - \chi_l\|}{\|\chi_l\|},\ e_{ML}(l) = \frac{\|\hat{\chi}_l^{ML} - \chi_l\|}{\|\chi_l\|}. \quad (21)$$

• When the experiments are completed, compute the mean of the relative error

$$\mu_{IN} = \frac{1}{100}\sum_{l=1}^{100} e_{IN}(l),\ \mu_{ML} = \frac{1}{100}\sum_{l=1}^{100} e_{ML}(l). \quad (22)$$

In Figure 1 the results obtained for different lengths N of measurements related to $\{c_{jk}^l\}$ are depicted. The mean error norm of ML method is smaller than the one corresponding to the IN method, in particular when N is small (typical situation in the practice). In addition, more than half of the estimates obtained by the IN method are not positive semidefinite, i.e not physically acceptable, even when N is large. Finally, we observe that for both methods the mean error decrease as N grows, providing evidence for their consistency.

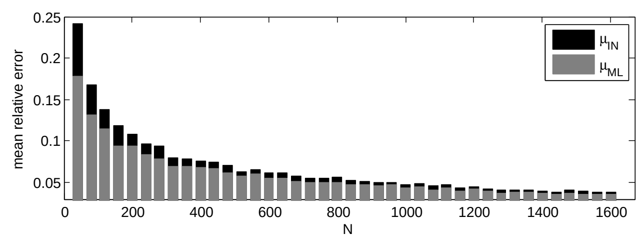

Fig. 1. Comparison performance IN vs ML method. N is the total number of measurements for each (ρ_k, Π_j), μ is the mean relative error as introduced in (22).

B. Minimal setting

Let $\mathcal{T}_{M,L}$ denote the set of the experimental settings with L input states and M observables satisfying Proposition 3.1. Accordingly the set of the minimal experimental settings is $\mathcal{T}_{d^2-1,d^2}$. Here, we consider the case $d = 2$. We want to compare the performance of the minimal settings in $\mathcal{T}_{3,4}$ with those settings that employ more input states and observables. We shall do so by picking a test channel, finding a minimal setting that performs well, and comparing its performance with a non minimal setting in $\mathcal{T}_{M,L}$, $M > 3, L \geq 4$ that performs well in this set while the total number N_T of trials

[1] If the optimization is constrained to $\mathcal{A}_+\cap\mathcal{I}$, we are guaranteed that f_{jk} will tend to be positive for a sufficiently large numbers of trials.

is fixed. Consider the Kraus map (1) representing a perturbed amplitude damping operation ($\gamma = 0.5$) with

$$K_1 = \sqrt{0.9}\begin{bmatrix} \sqrt{0.5} & 0 \\ 0 & 0 \end{bmatrix}, K_2 = \sqrt{0.9}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{0.5} \end{bmatrix},$$

$K_3 = \sqrt{0.1}/2 I_2$, $K_j = \sqrt{0.1}/2\sigma_{l(j)}$, $j = 4, 5, 6$, $l(j) = x, y, z$ corresponding to the χ-representation

$$\chi = \begin{bmatrix} 0.95 & 0 & 0 & 0.6364 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0.6364 & 0 & 0 & 0.5 \end{bmatrix}.$$

We set the total number of trials $N_T = 3600$. Fixed the set $\mathcal{T}_{M,L}$ $M \geq 3$ $L \geq 4$, we take into account the following procedure:

- Set $N = N_T \setminus (LM)$ and choose a randomly generated collection $\{\mathrm{T}_m\}_{m=1}^{100}$, $\mathrm{T}_m \in \mathcal{T}_{M,L}$.
- Perform 50 experiments for each T_m. At the l-th experiment we have a sample data $\{f_{jk}^m(l)\}$ corresponding to χ and T_m. From $\{f_{jk}^m(l)\}$ compute the estimate $\hat{\chi}_m(l)$ using the ML method and the corresponding error norm $e_m(l) = \|\hat{\chi}_m(l) - \chi\|/\|\chi\|$.
- When the experiments corresponding to T_m are completed, compute the mean error norm $\mu_m = \frac{1}{50}\sum_{l=1}^{50} e_m(l)$.
- When we have μ_m for $m = 1 \ldots 100$, compute

$$\bar{\mu}_{L,M} = \min_{m\in\{1,\ldots,100\}} \mu_m.$$

In Figure 2, $\bar{\mu}_{L,M}$ is depicted for different values of M

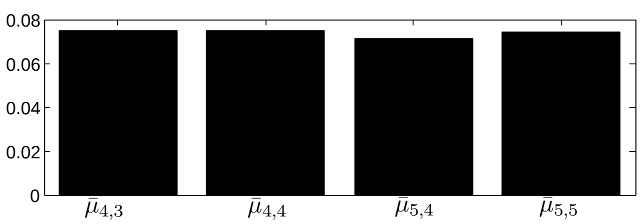

Fig. 2. $\bar{\mu}_{L,M}$ for different values of L and M.

and L. As we can see, incrementing the number of input states/observables does not lead to an improvement in the performance index. Analogous results have been observed with other choices of test maps and N_T. Finally, in Figure 3 the true χ and the averaged estimation $\bar{\chi}_{ML} = \frac{1}{50}\sum_{l=1}^{50} \chi_m(l)$ are depicted, with $m = \arg\min_{m\in\{1,\ldots,100\}} \mu_m$ for $M = 3$ and $L = 4$.

V. Conclusions

In this paper we determined the minimal conditions on the experimental setting that guarantee a unique and consistent estimation of a CPTP channel, for both inversion and convex approaches. Then, we employed a maximum likelihood approach to compare the estimation performances using the best minimal experimental setting *versus* the richer ones. Numerical simulations evidence that a minimal setting provides an estimation accuracy comparable to "richer" ones.

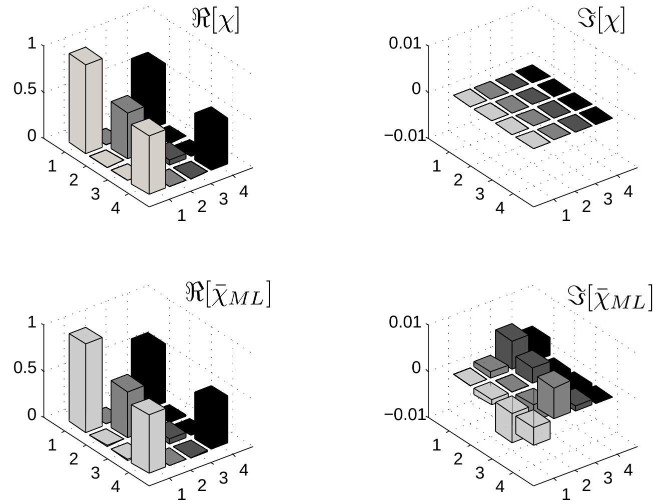

Fig. 3. Real and imaginary part of χ (top) and the averaged estimation $\bar{\chi}_{ML}$ (bottom). The vertical scale of the imaginary part has been magnified in order to show that the errors are below 0.01.

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