Capture Analysis in Wireless Radio Systems with Multi-Packet Reception Capabilities

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Motivations

- Uncoordinated transmissions by multiple nodes cause “packets collisions”
  - Examples: IEEE 802.11 cell, WSN, MANET

- “Capture” occurs when one or more of the overlapping signals are successfully decoded by the receiver

- Multipacket reception capabilities have been shown to be a key enabling factor for high-capacity wireless networks
  - [Wang&Garcia-Luna-Aceves, INFOCOM08]

- Understanding the capture behavior of the receiver for different statistics of the received signal powers is of great interest!
Previous work: Physical Model

- $P_j$: power of the $j$-th signal at the receiver
- $N_0$: noise power
- $\gamma_j$: SINR of the $j$-th signal
- $b$: capture threshold

- $\gamma_j > b \Rightarrow j$-th signal is captured
- $\gamma_j < b \Rightarrow j$-th signal is missed

- $b > 1 \Rightarrow$ can capture at most one signal at a time
  - [Zorzi&Rao, JSAC1994, TVT1997] derive the probability that one signal is above the capture threshold

- $b < 1 \Rightarrow$ can capture multiple signals at a time
  - [Nguyen&Ephremides&Wieselthier, ISIT06, ISIT07] derive the probability that there is at least one signal above the capture threshold

$$\gamma_j = \frac{P_j}{\sum_{h \neq j} P_h + N_0}$$
Contribution of this work

**Complete capture probability distribution**

- Derive the probability $C_r(n)$ that $r$ out of $n$ signals are above the capture threshold $b<1$
- Provide a **scalable method for the numerical evaluation** of this probability distribution
- Propose a **simple approximate expression** for the capture probabilities

**System Throughput**

- Investigate the system throughput when varying the multi-packet reception capability
System Model

Notation
- \( n \): collision of size
- \( P_j \): power of the \( j \)th signal
- \( \Lambda \): Aggregate received power
- \( \gamma_j \): SINR of the \( j \)-th signal
- \( b \): capture threshold

Assumptions
- \( P_j \) are i.i.d. random variables with common
  - PDF: \( f_P(x) \), \( x \) in \((P_m, P_M)\)
  - CDF: \( F_P(x) \), \( x \) in \((P_m, P_M)\)
- Statistics depend on the distance to RX and the propagation model
- The noise term is negligible
Problem statement

- The capture condition can be expressed as

\[ \gamma_j = \frac{P_j}{\Lambda - P_j} > b \Rightarrow P_j = \frac{b}{b + 1} \Lambda = b' \Lambda \]

- \( b' \) is termed *modified capture threshold*

- \( b' \Lambda \) is named *absolute capture threshold*

- We aim at determining the expression of

\[ C_n(r) = \Pr[r \text{ signals out of } n \text{ are captured}] \]
Because of the problem symmetry we have

\[ C_n(r) = \binom{n}{r} \Pr(P_{1:r} > \Lambda b', P_{r+1:n} \leq \Lambda b') \]

Conditioning on \( \Lambda = x \) we get...

\[ c_n(r) = \int_0^\infty \Pr(P_{1:r} > xb', P_{r+1:n} \leq xb' | \Lambda = x) f_\Lambda(x) dx \]

and applying Bayes' rule...

\[ c_n(r) = \int_0^\infty \Pr(\Lambda \equiv x | P_{1:r} > xb', P_{r+1:n} \leq xb') \Pr(P_{1:r} > xb', P_{r+1:n} \leq xb') dx \]

\[ (1 - F_p(xb'))^r F_p(xb')^{n-r} \]

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Conditioned aggregate received power $\Lambda_r$

- The issue is now to compute the PDF

$$f_{\Lambda_r}(x) = \Pr(\Lambda \equiv x \mid P_{1:r} > xb', P_{r+1:n} \leq xb')$$

- $\Lambda_r$ is the aggregate power given that $r$ signals are above $\Lambda_r b'$, and $n-r$ are below $\Lambda_r b'$

- We introduce the auxiliary r.v.

$$\tilde{\Lambda}_r(u) = \sum_{h=1}^{r} \alpha_h(u) + \sum_{k=1}^{n-r} \beta_k(u)$$

$iid$ rvs with PDF $f_{\alpha(u)}(x) = f_P(x \mid P > u)$

$iid$ rvs with PDF $f_{\beta(u)}(x) = f_P(x \mid P \leq u)$

- $\tilde{\Lambda}_r(u)$ gives the aggregate power given that $r$ signals are above $u$, and $n-r$ are below $u \Rightarrow$ setting $u = xb'$ we get

$$f_{\Lambda_r}(x) = f_{\tilde{\Lambda}_r}(xb') (x)$$
Since $\alpha_h(u)$ and $\beta_h(u)$ are independent, we get

\[
f_{\tilde{\Lambda},r}(u)(x) = \left( f_{\alpha(u)} \otimes f_{\alpha(u)} \otimes \cdots \otimes f_{\alpha(u)} \otimes f_{\beta(u)} \otimes f_{\beta(u)} \otimes \cdots \otimes f_{\beta(u)} \right)(x)
\]

**Inverse Fourier Transform**

\[
f_{\tilde{\Lambda},r}(u)(x) = \int_{-\infty}^{+\infty} \left[ \Psi_{\alpha(u)}(\xi) \right]^{r} \left[ \Psi_{\beta(u)}(\xi) \right]^{n-r} e^{j2\pi x \xi} d\xi
\]

**Fourier Transform**

Putting all the pieces together we get the final expression

\[
C_n(r) = \binom{n}{r} \int_{0}^{\infty} \left( 1 - F_P(xb') \right)^{r} F_P(xb')^{n-r} \times \\
\left( \int_{-\infty}^{+\infty} \left[ \Psi_{a(xb')}^{(\xi)} \right]^{r} \left[ \Psi_{\beta(xb')}^{(\xi)} \right]^{n-r} e^{j2\pi x \xi} d\xi \right) dx
\]
Capture distribution:
approximate expression

- If $n \gg r=0$ or $r=n$ or $r \approx n/2$ the central limit theorem applies

\[
\tilde{\Lambda}_r(u) = \sum_{h=1}^{r} \alpha_h(u) + \sum_{k=1}^{n-r} \beta_k(u)
\]

\[
N\left(r m_{\alpha(u)}, r \sigma^2_{\alpha(u)}\right) \quad N\left((n-r) m_{\beta(u)}, (n-r) \sigma^2_{\beta(u)}\right)
\]

- From which we get the following approximate expression of $C_n(r)$:

\[
\tilde{C}_n(r) = \binom{n}{r} \int_{nP_m}^{nP_m} \frac{\exp \left( - \frac{(x-r m_{\alpha(xb')})-(n-r) m_{\beta(xb')})^2}{2(r \sigma^2_{\alpha(xb')}+(n-r) \sigma^2_{\beta(xb')})} \right)}{\sqrt{2\pi \left(r \sigma^2_{\alpha(xb')} + (n-r) \sigma^2_{\beta(xb')} \right)}}
\times (1 - F_P(b'x)^r) \times F_P(b'x)^{n-r} \, dx
\]
Throughput

- $k$: reception capability
  - max number of signals that can be simultaneously decoded
- $S_n(k)$: average number of signals captured by a system with reception capacity $k$ and a collision size $n$

\[ S_n(k) = \sum_{r=1}^{k-1} r \binom{n}{r} + k \sum_{r=k}^{n} \binom{n}{r} \]

- Note: previous literature focused on $S_n(1)=1-\binom{n}{0}$ only!
Case study

Path Loss (PL)
- TXs uniformly distributed in a circle of radius $R$ centered in RX
  \[ f_P(x) = \frac{2}{R^2 \eta} \left( x^{-\frac{2}{\eta}} - 1 - x^{-\frac{1}{\eta}} - 1 \right) \]

Rayleigh Fading (RF)
- TXs at equal distance from RX but signals affected by multi-path fading
  \[ f_P(a) = e^{-a} \]

Path Loss & Rayleigh Fading (PLRF)
- TXs uniformly scattered around RX, within a disk of radius $R$ with signals affected by independent Rayleigh fading
  \[ f_P(a) = \frac{1 - e^{-a R^2} (1 + a R^2)}{a^2 R^2} \]
When $n \ll 1/b'$, all signals are captured with high probability.

When $n > 1/b'$, the distribution of the number of signals captured is bell-shaped.

When $n \gg 1/b'$, fewer and fewer signals are captured.

$C_n(r)$ in PL scenario

Path Loss $b = 0.02$, $\eta = 2$, $R = 10$

$r \leq 1/b' = 51$
Max performance are closely approached even with partial reception capability.

Rayleigh fading augments diversity of received signal strength & increase capture probabilities for large values of n

$S(6) \approx S(11)$
Approximate vs exact expressions

- $S_n(1) = 1 - C_n(0) = \text{Pr}[\text{capturing at least one signal}]
  - metric considered in most of the previous literature
- The accuracy of the approx. of $C_n(0)$ is very good in all the considered cases
- The approximation of $C_n(r)$ is not very good when either $r$ or $n-r$ are positive but small (not shown here)
We proposed a novel approach for computing the probability $C_n(r)$ of capturing $r$ out of $n$ signals

- Nicely scalable with the number of signals

We provided an approximate expression that is excellent in estimating the probabilities of $r=0^*$ and $r=n$

- *usually the only metric considered in the literature

We applied the method to study the system throughput when varying the multi-packet reception capabilities

- We observed that optimal performance can be closely approached even with partial reception capability

We are now investigating whether the method can be extended to systems with more complex PHY

- Different rates, power control, iterative interference cancellation,…
Spare slides
Bluestein FFT algorithm

- FFT samples are equally spaced over the entire signal bandwidth
- When raising the FTs to a power >1 most of such samples reduce to zero!
- Bluestein's FFT algorithm (BFFT) ```squeezes``` the samples into a fraction of the original bandwidth, so that samples are still significant after power raising