Abstract—The Dynamic-Frame-Aloha protocol has played an important role in random access systems and is nowadays used in the RFID environment, which has renewed investigations in this field. This paper, is aimed at providing a further contribution to the understanding of the issue, exploring the performance of this protocol under a population that is Poisson distributed, for different strategies in setting the frame length and backlog estimation mechanisms. Also provided are the performance of the absolute best strategy in the two cases in which either the frame is entirely explored or a new frame can be restarted at any slot.

I. INTRODUCTION

Random access protocols have played a fundamental role in communication systems starting with the appearance of the Aloha protocol [1], [2] back in 1970. Since then, a variety of such protocols have appeared and have influenced Satellite, Radio and Local Area Networks communications, being nowadays applied also in the RFID environment. Although many of them have been thoroughly investigated, many issues still remain open and are being addressed mainly by researchers in the RFID field. This paper, is aimed at providing a further contribution to the understanding of the issue.

In order to correctly operate, Aloha must be stabilized. This is possible only if some form of “channel feedback” is available, i.e., if the outcome of transmissions, EMPTY, SUCCESS, or COLLIDED, is known at each slot. Based on such information it is possible to adapt the transmission probability in S-Aloha [3][4][5] so that the theoretical throughput of $e^{-1} \approx 0.368$ pkt/slot can be reached.

An alternative proposal that has originated an entirely new family of random access protocols, namely tree-based protocols, has appeared in 1979 [6], [7]. Also these protocols require channel feedback and their operation is intrinsically stable. Colliding terminals are successively split into subgroups in such a way that, in the end, each subgroup is composed of a single terminal only, thus ensuring collision-free transmission. The simplest tree-based protocol yields a throughput of 0.347 pkt/slot, under Poisson arrivals. Improvements have been attained along two ways; first, by treating collided packets separately with respect to new arrivals until all collision have been solved [6]. This gives rise to the so called blocked access protocols (BA) in which the algorithm operates in cycles where each cycle solves collisions of packets arrived in the previous cycle. A further improvement is attained by treating at each cycle a number of packets with constant average $\gamma$, where $\gamma$ is optimized to provide maximum throughput. Along this way, the maximum attained throughput is reached in the Gallager-Tzibakov protocol, [8][7], and amounts to 0.487 pkt/slot.

The BA mechanism has also been adopted in S-Aloha giving rise to Frame Aloha (F-Aloha) [9], [10]. Similarly to S-Aloha, time is subdivided into time slots equal to a packet transmission time, and slots are grouped into frames. The difference with respect to S-ALOHA is that a terminal is permitted to transmit only one packet per frame, in a randomly chosen slot, and that only terminals that have collided in the preceding frame can re-schedule their transmissions in the current frame. Unfortunately, also F-Aloha is unstable and needs some control to be stabilized. A way to do this is to dynamically adapt the frame length according to an estimate of the packet backlog, hence the name Dynamic Frame Aloha (DF-Aloha). Along this way Schoute [10] has proposed an estimate that, under Poisson arrivals, DF-Aloha allows to reach a throughput as high as 0.427 pkt/slot.

In the past, in random-access systems, DF-Aloha has rarely been considered, owing to its lower efficiency with respect to the Tree protocol family. In fact, both families are based on the same channel feedback, i.e., the same overhead when a completely distributed system is considered. However, sometimes a centralized operation is preferred, that is an operation where the channel feedback is observed by a central station that also communicates to terminals the subset that is allowed to transmit either in a slot or in a frame. In this case, the communication overhead of the two families changes due to the differences in subsets enabled to transmit in a given frame or a given slot. Therefore, in centralized systems, once we take into account the cited overhead, DF-Aloha may become more efficient than Tree protocols.

DF-Aloha has been adapted, in more recent years, to Radio Frequency Identification (RFID), an automatic identification system where a reader interrogates a set of tags in order to identify each of them [11],[12]. Collisions occur among the responses of tags and DF-Aloha is used to arbitrate collisions so that all tags can be finally identified. Here the protocol is centralized and the reader acts as central station. The
performance figure is still the fraction of tags identified in a slot, the throughput, that in this environment is better denoted as "efficiency".

Usually, the number $N$ of tags to be identified is an unknown constant and, also in this case, an estimate $\hat{n}$ of the backlog $n$ is required to set the frame length in order to maximize efficiency. A novelty introduced in RFID is the capability of restarting a new frame each time the exploration of the current frame is no longer considered optimal.

If the estimate were to provide the exact backlog value, then an asymptotical efficiency equal to 0.368 pkt/slot, the same as in Slotted Aloha, could be reached. In RFID the Shoute’s estimate is not so efficient as it is in random access. Actually, we have shown in [13] that asymptotically, for $N \rightarrow \infty$, it reaches an efficiency equal to 0.3112 pkt/slot. Therefore, several works on DF-Aloha RFID systems have investigated better backlog estimation procedures, such as maximum likelihood or Bayesian estimates. In [14] a way is shown to evaluate the “a posteriori” probability distribution of the backlog $n$ using all the past observation on the channel. The "a posteriori" probability is used to find the frame length that maximizes the throughput of the next slot. In practice, the procedure in [14] uses analytical expressions that are difficult to evaluate, although a closed-form expression of probability involved is given in [15].

Recently two papers [16], [17], have pointed out the non-optimality of the procedures that use the backlog estimate to determine the frame length. In fact, most of the procedures set the frame length equal to the backlog estimate according to the belief that this setting maximizes the efficiency of the procedure. In fact, it is well known that, considering a single frame, the maximum throughput is achieved when the frame length $r$ is equal to the terminal backlog size $n$. However, the entire backlog is solved in a period of several frames, and since the efficiency of single frames changes with the length of the frame, it is not obvious how to set that length of the first and subsequent frames. If the backlog size $n$ is known, it has been numerically shown that, in order to maximize the efficiency/throughput on the entire procedure, the frame length must be set equal to the backlog size $n$ at each frame. Recently in [18] we have proved this optimality rule analytically. However, this does not enable us to retain optimal the use of the estimate as if it were the real backlog value. In [16] the authors suggest a procedure to numerically find the best frame-length setting procedure when the initial population size $N$ is only known in distribution. They provide also some examples that show the non-optimality of the cited belief.

In [17] the authors refer to the case in which the frame can be restarted and show that, in this case, even with known backlog size $n$, setting the frame length equal to $n$ is non-optimal. The optimality procedures given in both these papers, however, are not simple to attain in practice, and also the goal of finding the best performance in specific cases is still open.

In this paper we evaluate the performance of DF-Aloha for random access with Poisson arrivals under different backlog estimation mechanisms, also considering the possibility of frame restart. We find that, when the entire frame is completely explored, the Schoute’s mechanism is very efficient, its performance being practically equal to the one provided by Bayesian estimation and very close to the absolute optimum.

When frame restart is allowed, we demonstrate a practical way to attain the Bayesian estimate in [14] and find that it provides an efficiency as high as 0.4687, a figure very close to the maximum ever observed in this environment, 0.487, that is reached using tree protocols [8][7]. We also propose an estimation mechanism by far simpler that the Bayesian one and that maximizes the throughput of the next slot. In practice, the procedure in [14] uses analytical expressions that are difficult to evaluate, although a closed-form expression of probability involved is given in [15].

The paper is organized as follows. In Section II, we analyze DF-Aloha when the entire frame is explored, and discuss Shoute’s, Bayesian and Optimum cases. In Section III we analyze DF-Aloha when the frame can be restarted, and discuss the Bayesian estimation. We also show a simpler but effective estimation procedure called Backlog Lower Bound, and finally show how to evaluate the absolute optimum performance. Our concluding remarks are reported in Section IV.

II. DYNAMIC FRAME-ALOHA

The average number of slots $L(N)$ needed to solve $N$ collisions with frames of constant length $r$ obeys the following recursive relation [10]:

$$L(N) = r + \sum_{s=0}^{m} p_N(s)L(N-s), \quad N \geq 2,$$

(1)

where $s$ is the number of successful slots in the frame, $m = \min(N - 2, r - 1)$, and $p_N(s)$ is the distribution of the number of successes $s$. Relation (1) can be solved with respect to $L(N)$ yielding

$$L(N) = r + \frac{\sum_{s=1}^{m} p_N(s)L(N-s)}{1-p_N(0)}, \quad N \geq 2,$$

(2)

This can be used recursively to derive all values $L(N)$ starting from the initial values

$$L(0) = L(1) = r.$$

As with S-Aloha, also in F-Aloha instability arises, in the sense that

$$\lim_{N \rightarrow \infty} \frac{N}{L(N)} = 0$$

for any finite $r$.

F-Aloha can be stabilized if the frame length can be adapted dynamically to the backlog existing at the beginning of the frame. This version is referred to as Dynamic Frame Aloha (DF-Aloha) [10]. In fact, it is well known that the maximum throughput in a frame is achieved when the frame length $r$ is equal to the terminal backlog size $n$. Equation (2) can be used to show numerically that, if $N$ is known, in order to minimize $L(n)$, i.e., to maximize efficiency, the frame length must be set
equal to the backlog size \( n \) at each frame. Recently in [18] we have proved this optimality rule analytically. Efficiency results for this case are shown in the first row of Table I, where we can appreciate the fact that the performance decreases as \( N \) increases, asymptotically reaching the well known value of \( e^{-1} \).

In our case, \( N \) is known in distribution. A procedure commonly used in this case is to derive an estimation \( \hat{n} \) of the backlog size at the beginning of each frame and set the frame length to this value, i.e., \( r = \hat{n} \). Many backlog estimation mechanisms have been proposed. In [10], Schoute aims at maximizing the throughput of a satellite channel running DF-Aloha, where traffic is assumed to be represented by a Poisson process with intensity \( \gamma \) packet/slot. The throughput to be maximized is given by:

\[
\eta_t = \frac{\gamma}{L_\gamma} \tag{3}
\]

where

\[
L_\gamma = \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} e^{-\gamma} L(n). \tag{4}
\]

Schoute proposes to estimate the backlog \( \hat{n} \) at the end of each frame by counting the number of collided slots \( c \) at the end of the previous frame. Assuming that the procedure is able to keep a frame size equal to the backlog \( n \), the number of terminals transmitting in a slot can be approximated by a Poisson distribution of average 1, such that the average number of terminals in a collided slot is [10],

\[
H = \frac{1 - e^{-1}}{1 - 2e^{-1}} \approx 2.39 \tag{5}
\]

and the estimate is \( \hat{n} \) = round\((Hc)\). He also shows that the average number of slots \( L(N, r) \) to solve \( N \) collisions starting with a frame of length \( r \) is given by:

\[
L(N, r) = r + \sum_{s=0}^{m} \sum_{c=0}^{[N/2]} p_{s,c}(N, r) L(N - s, \tilde{r}) \quad N \geq 2, \tag{6}
\]

where \( p_{s,c}(N, r) \) is the joint distribution of the number of successes \( s \) and collisions \( c \), \( m = \min\{N - 2, r - 1\} \), and \( \tilde{r} = N - s \) is function of \( N, r, s, c \). Equations (6) can be solved starting from \( L(0, r) = L(1, r) = r \).

In Table I we have re-derived the throughput for different values of \( N \) with initial frame length \( r_0 = N, r_0 = 1, r_0 = 10 \) and \( r_0 = 100 \). Values up to \( N = 100 \) have been evaluated using (6), whereas values for \( N = 500 \) and \( N = 1000 \) have been obtained by simulating the algorithm. These values represent also the performance of an RFID system based on DF-Aloha with the Schoute estimation mechanism, when the population size is \( N \). As we have already mentioned, this mechanism is tailored for a multiple access system with an average arrivals per slot close to one and, therefore, we do not expect it to be optimal for the RFID environment. As a matter of fact, in [13] we have demonstrated that the asymptotical efficiency of the Schoute’s mechanism is as low as 0.311.

Table II shows the optimum \( \gamma \) and throughput with Poisson distribution as evaluated by (3) with \( r_0 = 1, 2, 3 \). We see that the absolute optimum is for \( r_0 = 1, \gamma = 1.11 \) and the corresponding throughput is 0.4271 [10].

\[
\begin{array}{cccc}
\gamma_{\text{opt}} & r_0 = 1 & r_0 = 2 & r_0 = 3 \\
\hline
S_{\text{opt}} & 1.11 & 2.15 & 3.1 \\
\end{array}
\tag{7}
\]

A. Bayesian estimation

In the work of Floerkemeier [14] a way is suggested to evaluate the probability

\[
P(E, S, C|j, r, n) \tag{7}
\]

of observing exactly \( E \) empty, \( S \) successful, and \( C \) collided slots in \( j \) observed slots of a frame composed of \( r \) slots, being \( n \) the number of transmitting terminals. In our case we have \( j = r \) and (7) can be used to evaluate the probability that \( n \) terminals are replying in the current frame, given all evidence \( z_{1:t} \) from previous frames up to the current one \( t \):

\[
P(n|z_{1:t}) = \alpha P(z_t|n, z_{1:t-1})P(n|z_{1:t-1}) \tag{8}
\]

where \( \alpha \) denotes a normalizing constant. Since outcomes in frame \( t \), given the number of terminals transmitting, is independent of previous frames (8) reduces to.

\[
P(n|z_{1:t}) = \alpha P(z_t|n)P(n|z_{1:t-1}) \tag{9}
\]

Equation (9), provides a recurrent expression by which the "a posteriori" probability distribution of the original population size \( n \), at frame \( t \) can be derived by the same expression at frame \( t - 1 \). It allows to evaluate the "a posteriori" probability distribution of the original population size \( n \), conditioned to all the past history, starting from the "a priori" distribution of the number of transmitting terminals.
In Table III we show the optimum $\gamma$ and throughput with Poisson distribution (3) for the Bayesian estimate $n$ of the backlog. The Bayesian strategy, that takes into account all the history, does not improve the performance with respect to the Schoute’s estimate, a simple estimate based only the observation of the current frame.

B. Optimum strategy

In [16] it has been shown that setting $r = \hat{r}$ is not necessarily an optimal strategy. Authors suggest an approach based on Markov chains to find the optimal strategy. Unfortunately the numerical approach is rather involved, so we have resorted to an almost exhaustive investigation by exploring the tree of the different strategies.

A strategy is made of the frame lengths chosen at the end of a frame, and it depends on the past history. The set of the strategies is

$$\mathcal{R} = \{\text{NewFrame}(r)\}_{r=2}^{r_{\text{max}}},$$

where "NewFrame"$(r)$ means that a new frame of length $r$ must be started, and $r_{\text{max}}$ is the maximum length for a frame. The number of participating terminals is generally known through a probability mass function, also indicated with beliefs, $\{n\} = \{P(N = n)\}_{n=0}^{N_{\text{max}}}$.

The expected number of slots needed for solving $\{n\}$, starting with a frame of length $r$ is

$$L(n,r) = r + \sum_{s=0}^{m} \sum_{c=0}^{\lfloor N_{\text{max}}/2 \rfloor} p_{s,c}(n,r) \min_{t \in \mathcal{R}} L(\{n\}_{s,c}, t),$$

(10)

where $m = \min\{N_{\text{max}} - 2, r - 1\}$, the joint probability $p_{s,c}(n,r)$ given the beliefs and $r$ is

$$p_{s,c}(n,r) = \sum_{n=0}^{N_{\text{max}}} P(n)p_{s,c}(n,r),$$

being $p_{s,c}(n,r)$ the same appearing in (6), and

$$\{n\}_{s,c} \sim \begin{cases} P(n+s)p_{s,c}(n+s,r) & 0 \leq n \leq N_{\text{max}} - s \\ 0 & N_{\text{max}} - s < n \leq N_{\text{max}} \end{cases}$$

are the updated beliefs.

The recursive evaluation of (10) induces the tree of the strategies. Each node of the tree has at most $|\mathcal{R}| = r_{\text{max}} - 1$ branches, and the leaves of the tree are those nodes in which $n$ is known, in fact the best strategy for known $n$ is $r = n$.

Since the visit of all the branches of the call tree may become infeasible, we resort to a method that yields a lower bound $L_{LB}$ and an upper bound $L_{UB}$ to $L$. We assume a pruning procedure that drop some of the branches of the tree. Considering the set of pruned branches that leave the same node with a backlog $n$, the best pruned strategy is the one that presents a length $l_o$ such that $l_o = \min(l_i)$. The pruning procedure prevents from knowing $l_o$; nevertheless, we can always find a lower bound $l_{lb}$, and an upper bound $l_{ub}$ to $l_o$, that can be used with the pruned tree to evaluate $L_{LB}$ and $L_{UB}$.

In our case we have assumed $l_{lb}$ as the performance of the ideal protocol in which $n$ is known, and whose efficiencies are reported in Table I, whereas $l_{ub}$ is assumed as the performance of Schoute’s protocol. The pruning strategy takes into account the probability of reaching that particular node and stops the exploration of its branches if this probability is under a given threshold. Diminishing this threshold increases the number of nodes visited and narrows the difference between bounds.

We have evaluated the lower bound and the upper bound of the best strategy, when starting with a frame of length 1 and with an initial backlog distribution represented by a Poisson distribution, truncated to $N_{\text{max}} = 8$. The threshold probability used for pruning the strategy tree has been set to $10^{-4}$. With this setting the two bounds coincide up to the forth decimal digit. Then we have chosen $\gamma$ so that the coincident bounds are minimal. The results are shown in Table III. Again, we observe an almost zero improvement with respect to the Bayesian/Schoute’s estimates, a surprising results. In the same Table we have reported also the performance of the Ideal case, that refers to a perfect estimate of the backlog. This result has been derived from the data that appears in row one in Table I. Its performance, 0.4377 pkt/slot, constitutes the ultimate upper bound to the performance of any estimation procedure.

III. DF-ALOHA WITH FRAME RESTART

In this section we consider the capability of Frame Aloha of restarting a new frame when the further exploration of the current one is no longer useful.

A. Bayesian estimation

We follow the work of Floerkemeier [14] already cited in Section II-A. Here equations (8) and (9) are enriched with the evidence $a_{1:j}$ from the first $j$ slots in the current frame, in addition to the evidence $z_{1:t}$ from previous frames, and get:

$$P(n|z_{1:t}, a_{1:j}) = \alpha P(n|z_{1:t}, a_{1:j-1})P(a_j|n, z_{1:t}, a_{1:j-1})$$

(11)

where $\alpha$ denotes a normalizing constant. Since consecutive frames are independent given the number of terminals transmitting, the following holds

$$P(a_j|n, z_{1:t}, a_{1:j-1}) = P(a_j|n, a_{1:j-1}).$$

(12)

Again, equation (12), provides a recurrent expression by which the "a posteriori" probability distribution of the original population size $n$, at slot $j$ frame $t$ can be derived by the same expression at slot $j-1$ of frame $t$, starting from the "a priori" distribution of the number of transmitting terminals. The "a posteriori" probability $p_n$ is then used to find the frame length $r$ that maximizes the throughput of the next slot

$$E[S(r)] = \sum_n S(n,r)p_n = \sum_n \frac{n}{r} \left(1 - \frac{1}{r}\right)^{n-1} p_n.$$

(13)
This maximization is carried out slot by slot and a new frame is restarted whenever the frame length so determined changes with respect to the old one.

Unfortunately in [14] the expression for (12) is provided only by way of transforms, a method hardly suited to evaluate distributions when $n$ is large. It is based on probability (7) that is hard to evaluate when $j \neq r$. A close-form expression of probability (7) is given in [15], but even this formula presents evaluation complexity. Therefore, to derive results about this procedure, we have resorted to a new evaluation method that evaluates (7) iteratively in $r$, the number of slots in the frame. This procedure is illustrated in Appendix A.

Throughput (3) of this procedure is reported in Table IV. In the case of the first column a new frame is restarted if $\hat{n}$ does not match the number of remaining slots in the frame. We also report the performance of two other cases that behave slightly better. In the first modification we do not allow to interrupt the report the performance of two other cases that behave slightly better. In the first modification we do not allow to interrupt the second frame, which is composed of two slots. This provides a better estimate even with a slight penalty in the instantaneous throughput, and results in higher efficiency. In the case denoted as Floerkemeier we maintain the first modification and restart a new frame according to the maximum throughput measure (13) as suggested in [14].

### B. Backlog Lower Bound

The estimation procedure exposed in the previous section may be too complex to be implemented. However, we have observed that most of the gain of the Bayesian technique lies in avoiding the exploration of the last slot of the frame when a collision is certain. A simple mechanism capable to provide a lower bound on the backlog size can retain the same advantage. Thus, we adopt the following mechanism, named Backlog Lower Bound (BLB), that represents both a lower bound and good estimate on the backlog size $n$:

$$\hat{n}_1 = 2$$
$$\hat{n}_k = \max\{\hat{n}_{k-1} - s_k, 2c_k\}$$

being $s_k$ and $c_k$ respectively the number of successes and observed collisions in frame $k$ up to that point. Here the max operation presents the advantage of including a relevant part of the memory of the past frames. The number of transmissions estimated in a collided slot is $2c_k$. This choice is adequate since we have verified that if it is used in place of the Schoute’s estimate 2.39$c_k$, owing to the small $\gamma$, the performance of the Schoute’s protocol is practically not changed.

The BLB protocol operates as follows: the last slot of the frame $k$ gives rise to a collision whenever we have $\hat{n}_k - s_k \geq 2$ and $c_k = 0$, and, to avoid a waste of a slot, a new frame is started; otherwise, the last slot is observed, $\hat{n}_k$ is updated and a new frame is started. In both cases the length of the new frame is $r_{k+1} = \hat{n}_{k+1}$.

The performance of this algorithm can be evaluated by suitably changing relation (6). Denoting by $L(n,r,b)$ the expected number of slots needed to resolve $n$ terminals starting with a frame of length $r$ and lower bound $b$ for the backlog, the following holds:

$$L(n,r,b) =$$
$$= r + \sum_{s=0}^{m} \sum_{c=2}^{\lfloor n/2 \rfloor} p_{s,c}(n,r)L(n-s,\hat{r},\max\{b-s,2c\})$$
$$+ \sum_{s=0}^{\hat{r}-1} p_{s,1}(n,r) \left( -\frac{1}{r} + L(n-s,\hat{r},\max\{b-s,2\}) \right)$$
$$+ \sum_{s=b-1}^{m} p_{s,1}(n,r)L(n-s,\hat{r},\max\{b-s,2\}) \quad n \geq 2,$$

where $\hat{r} = \max\{b-s,2c\}$ is the estimate for the new frame length, $m = \min\{n-2,r-1\}$. The double summation of (6) has been rearranged to isolate the case $c = 1$: when $c = 1$ and $0 \leq s \leq b-2$ the last slot can be saved with probability $r^{-1}$, hence the term $-1/r$ is added in the second summation.

In Table VI we have reported the throughput under Poisson statistics, and the value of $\gamma$ that maximizes such throughput. As we can see this suboptimal mechanism provides a Poisson performance only 2% below the Bayes mechanism.

### C. Optimum strategy

The strategy that leads to the best throughput performance with frame restart can be investigated much in the same way as we did in Section II-B.

If the current frame can be interrupted, then at each slot there is available a set of strategies $S$ made as follows:

$$S = \{\text{Continue}, \{\text{NewFrame}(r)\}_{r=2}^{\text{max}}\},$$

where the strategy "Continue" means that the current frame must be further explored, and "NewFrame($r$)" means that a new frame of length $r$ must be set. The total number of choices that can be taken at each slot is $|S| = r_{\text{max}}$, but this number can be significantly lowered by neglecting the strategies that are surely suboptimal at a certain slot.

It is evident that the optimal decision to be taken after having observed a slot must depend on the entire past history, being the history made of the slots’ outcomes and past decisions. Specifically, the past history serves to update the beliefs about the backlog, $\{n\}$, and about the number of terminals that will transmit in the rest of the current frame, $\{R\}$, that are the only statistics needed to take the optimal decision. These beliefs are inferred from the slots’ outcomes and are recursively updated as described by (11), (12) and in Appendices A and B.

Being in the $j$-th slot of a frame with length $r$, and having beliefs for the backlog $\{n\}$ and for the remaining terminals $\{R\}$, the average number of slots needed to conclude the

<table>
<thead>
<tr>
<th>$S_{\text{opt}}$</th>
<th>$S_{\text{mod1}}$</th>
<th>$S_{\text{mod2}}$</th>
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</thead>
<tbody>
<tr>
<td>1.11</td>
<td>1.16</td>
<td>1.18</td>
</tr>
<tr>
<td>0.4624</td>
<td>0.4680</td>
<td>0.4687</td>
</tr>
</tbody>
</table>
distribution represented by a Poisson distribution truncated to starting with a frame of length $r$. Lower bound procedure to obtain $l$.

$L(\{n\}, \{R\}, r, j) = 1 + \sum_{o \in \{E,S,C\}} p(O_j = o|\{R\}, r) \min_{s \in S} L(\{n\}_{s,o}, \{R\}_{s,o}, r, s, j)$,

$$p(O_j = E|\{R\}, r) = \sum_{n=0}^{N_{\max}} \left(1 - \frac{1}{r - j + 1}\right)^n R_n$$

$$p(O_j = S|\{R\}, r) = \sum_{n=0}^{N_{\max} - 1} \frac{(n + 1)(r - j)^n}{(r - j + 1)^{n+1}} R_{n+1}$$

$$p(O_j = C|\{R\}, r) = 1 - p(O_j = E|\{R\}, r) - p(O_j = S|\{R\}, r)$$

Note that (15) can be evaluated in a recursive manner, where each function call makes at most $3|S| = 3r_{\max}$ calls. Since the visit of all the branches of the call tree may become infeasible, we adopt a pruning procedure as the one exposed in Section II-B, that allows to determine a lower bound $L_{LB}$ and an upper bound $L_{UB}$ to the actual value of $L$.

We have first evaluated the Ideal case where $N$ is known. In this case we have assumed zero as $l_{LB}$ and the Backlog Lower Bound procedure to obtain $l_{LB}$. Results are shown in Table V, where we can verify that, for $2 < n < 8$, the optimal frame length is $r_o = n - 1$, a fact already cited in [17], that differentiates this case with frame restart with respect to the complete exploration of the frame, whose optimal strategy is always setting $r_o = n$ [18].

In the second row of Table VI we have reported the lower bound and the upper bound of the optimum strategy, when starting with a frame of length 1 and with an initial backlog distribution represented by a Poisson distribution truncated to $N_{\max} = 8$, with mean value $\gamma$ (first row). For the evaluation of the lower bound we have used the value of $L$ obtained for the case of Ideal shown in Table V. For the upper bound we have used the value of $L$ obtained by Backlog Lower Bound procedure.

For sake of comparison in Table VI we have also reported the throughput of the Bayesian estimate already given in section III-A. We have also reported the Poisson throughput of the ideal procedure using the values reported in Table V. The results shows that there is small room for an improvement with respect to the Bayesian estimate.

Finally, Figure 1 shows the initial part of the optimal strategy tree corresponding to the bounds reported. The nodes of the tree represent different stages of the exploration of the frames. In each node a string represents the outcomes of slots (S,E,C), spaces representing the frame boundaries, the number in parentheses the length of the next slot, and the dash indicates the further exploration of the frame. We see that the second frame is always composed by two slots that are always explored except when a collision is certain.

### IV. Conclusions

In this paper we have investigated the performance of the Dynamic Frame Aloha protocol under a population that is Poisson distributed, for different strategies in setting the frame length and backlog estimation mechanisms, in the two cases in which either the frame is entirely explored or a new frame can be restarted at any slot. We have re-derived the performance with estimation mechanism already proposed in literature, such as the Shoute’s and Bayesian estimates. We also have proposed a new estimate, for the Frame restart case, that is simpler than the Bayesian one but almost as efficient. We have also devised a new method to determine the performance of absolute best strategies. We have found that, if the frame is entirely explored, the Shoute’s estimate performs very well, exactly as Bayesian does, reaching 0.4271 pkt/slot, practically the same as the absolute optimum 0.4275. When the frame can be restarted, the Bayesian estimate can provide up to 0.4687, while our proposal, the Backlog Lower Bound, much simpler but far, can reach 0.4583. In this case, the optimum performance lies somewhere between 0.4693 and 0.4739. The results attained appear to show that, contrary to common belief, Frame Aloha can reach efficiency very close to the best observed in random access protocol, 0.487, reached by the Gallager-Tzybakov version of the Tree Protocols.
APPENDIX A

Referring to (12), we want to compute the expression
\[ P(o_j|n_0, o_{1:j-1}) \]
that can be written as
\[ P(o_j|n, r, o_{1:j}) = \frac{P(o_j|n, r)}{P(o_{1:j-1}|n, r)} \]
having indicated also the frame length \( r \), \( j \leq r \). If we denote by \( n_j \) the number of terminals transmitting in the first \( j \) slots of the frame we have
\[ P(o_{1:j}|n, r) = \sum_{n_j=0}^{n} P(o_{1:j}|n_j, n, r) \cdot P(n_j|n, r) = \]
\[ = \sum_{n_j=0}^{n} P(o_{1:j}|n_j, j) \cdot \binom{n}{n_j} \left( \frac{j}{n} \right)^{n_j} \left( 1 - \frac{j}{n} \right)^{n-n_j}. \]

Now we show that term \( P(o_{1:j}|n_j, j) \) can be easily derived when \( P(o_{1:j-1}|n_j, j - 1) \) is known, thus allowing a recursive evaluation in \( j \) for each \( n_j \), starting from \( P(o_1|n_1, 1) \).

We write
\[ P(o_{1:j}|n_j, j) = P(o_{1:j-1}|o_j, n_j, j) P(o_j|n_j, j), \]
Terms \( P(o_j|n_j, j) \) are given by the binomial expressions
\[ P(E|n_j, j) = \left( 1 - \frac{j}{n} \right)^{n_j}, \]
\[ P(S|n_j, j) = \frac{n_j}{j} \left( 1 - \frac{j}{n} \right)^{n_j-1}, \]
\[ P(C|n_j, j) = 1 - P(E|n_j, j) - P(S|n_j, j). \]
Terms \( P(o_{1:j-1}|y_j, n_j, j) \) are evaluated as
\[ P(o_{1:j-1}|O_j = E, n_j, j) = P(o_{1:j-1}|n_j, j - 1), \]
\[ P(o_{1:j-1}|O_j = S, n_j, j) = P(o_{1:j-1}|n_j, j - 1, j - 1), \]
\[ P(o_{1:j-1}|O_j = C, n_j, j) = \frac{1}{P(O_j = C|n_j, j)} \times \]
\[ \sum_{x=0}^{n_j-2} P(o_{1:j-1}|n_j-1 = x, j - 1) \binom{n_j}{x} \left( 1 - \frac{j}{n} \right)^{x} \left( \frac{1}{j} \right)^{n_j-x}. \]

The initialization of the iterative method is
\[ P(E|n_1, r = 1) = 1 \text{ for } n_1 = 0 \text{ and } 0 \text{ elsewhere,} \]
\[ P(S|n_1, r = 1) = 1 \text{ for } n_1 = 1 \text{ and } 0 \text{ elsewhere,} \]
\[ P(C|n_1, r = 1) = 1 \text{ for } n_1 > 1 \text{ and } 0 \text{ elsewhere.} \]

APPENDIX B

Let us denote with
- \( Y_i \) the random variable that counts the terminals that transmit from the \((i+1)\)-th slot to the last slot of the frame, and by \( y_i \) its current value;
- \( X_i \), the number of terminals that transmit in the \( i \)-th slot of the frame, and by \( x_i \) its current value;
- \( O_i \), the outcome of the \( i \)-th slot of the frame, and by \( o_i \) its current value, that can assume one of the values \( \{E, S, C\} \).

Given the a priori distribution of the number of remaining terminals at the beginning of the frame
\[ P(Y_0 = n) \quad 0 \leq n \leq N_{\text{max}}, \]
the distribution of \( Y_1 \) given the number of terminals \( X_1 \) that have transmitted in the first slot is
\[ P(Y_1 = n|X_1 = x) = P(Y_0 = n + x|X_1 = x) \]
\[ \propto P(X_1 = x|Y_0 = n + x) \cdot P(Y_0 = n + x) \]
\[ = \left( \binom{n + x}{x} \frac{1}{x} \left( \frac{1}{x} \right)^n \right) P(Y_0 = n + x) \]
for \( n = 0, \ldots, N_{\text{max}} - x \). Note that typically the reader is not able to identify the number of colliding terminals in a slot, hence we have to find the distribution of \( Y_1 \) conditioned to the available observation \( O_1 \), that is easily found by observing that
\[ P(Y_1 = n|O_1 = E) = P(Y_1 = n|X_1 = 0), \]
\[ P(Y_1 = n|O_1 = S) = P(Y_1 = n|X_1 = 1), \]

and

\[ P(Y_1 = n|O_1 = C) = P(Y_1 = n|X_1 < 1) \]

\[ = \sum_{x=0}^{N_{\text{max}}-n} P(X_1 = x)P(Y_1 = n|X_1 = x) \]

\[ = \sum_{x=0}^{N_{\text{max}}-n} \left( \frac{n+x}{x} \right) \left( \frac{1}{r} \right)^x \left( 1 - \frac{1}{r} \right)^n P(Y_0 = n + x), \]

Generalizing, given the distribution

\[ P(y_{k-1}|o_1, o_2, \ldots, o_{k-1}) = P(y_{k-1}|o_{k-1}) \]

the distribution of \( Y_k \), having observed the slots up to the \( k \)-th, results into

\[ P(Y_k = n|o_{k-1}, Y_{k-1} = x) \]

\[ = P(Y_{k-1} = n + x|o_{k-1}, X_k = x) \]

\[ \propto P(X_k = x|o_{k-1}, Y_{k-1} = n + x)P(Y_{k-1} = n + x|o_{k-1}) \]

\[ = \left( \frac{n+x}{x} \right) \left( \frac{r-k}{r-k+1} \right)^{n+x} P(Y_{k-1} = n + x|o_{k-1}). \]

Since the observations are drawn from the set \( \{E, S, C\} \), the updating reads as follows

\[ P(Y_k = n|o_{k-1}, o_k = E) \propto \left( \frac{r-k}{r-k+1} \right)^n P(Y_{k-1} = n|o_{k-1}), \]

\[ P(Y_k = n|o_{k-1}, o_k = S) \propto \left( \frac{n+1}{r-k+1} \right) \left( \frac{r-k}{r-k+1} \right)^{n+1} P(Y_{k-1} = n + 1|o_{k-1}), \]

\[ P(Y_k = n|o_{k-1}, o_k = C) = P(Y_k = n|o_{k-1}, X_k > 1) \]

\[ = \sum_{x=0}^{N_{\text{max}}-n} P(X_k = x|o_{k-1})P(Y_{k-1} = n + x|o_{k-1}, X_k = x) \]

\[ = \sum_{x=0}^{N_{\text{max}}-n} \left( \frac{n+x}{x} \right) \left( \frac{r-k}{r-k+1} \right)^{n+x} P(Y_{k-1} = n + x|o_{k-1}). \]

REFERENCES


