Network coding – a guided tour

With an excursion into a proof.....

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<u>Collaborators:</u>
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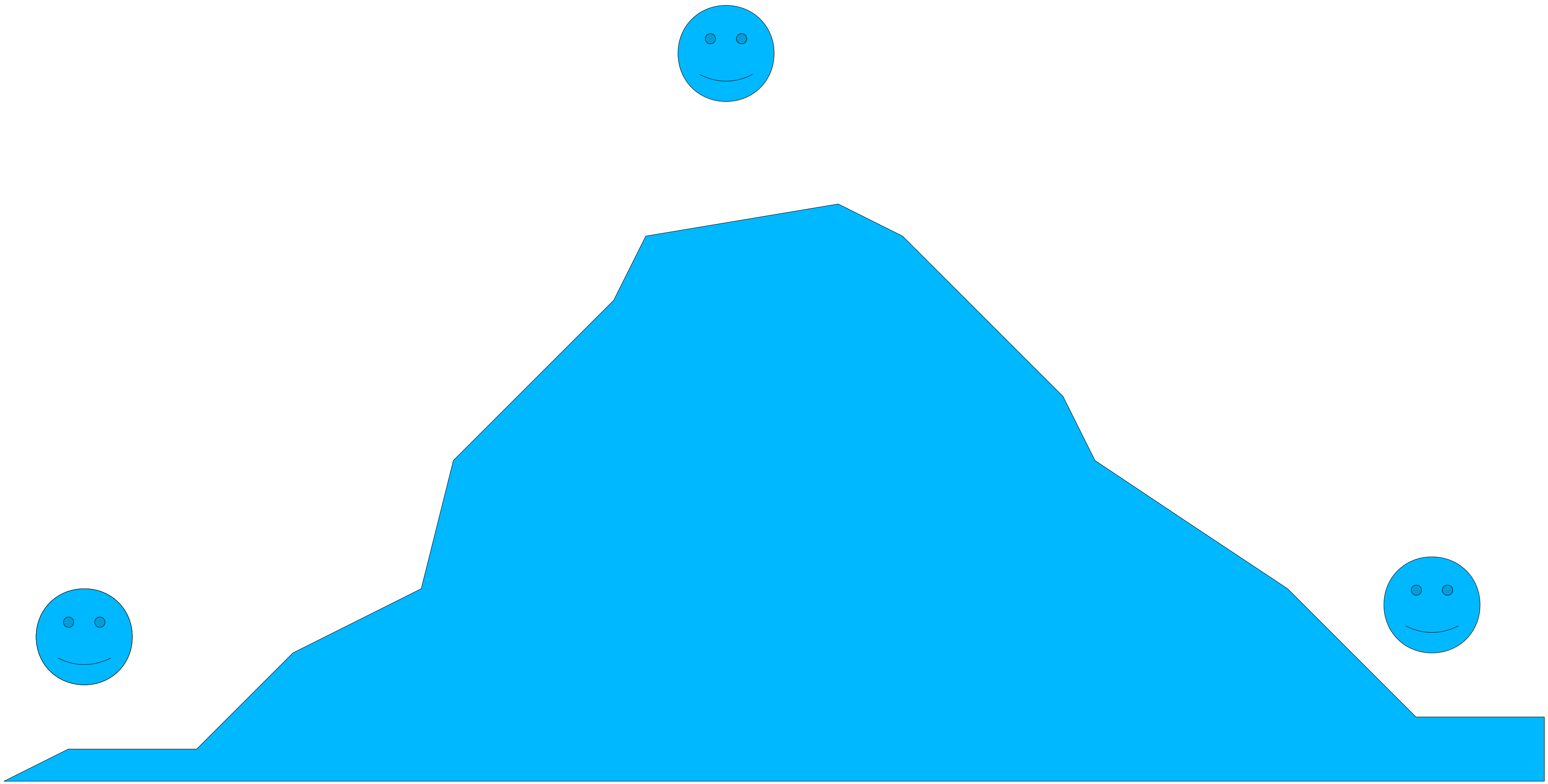
F. Kschischang, R. Yeung, T. Lutz, M. Thakur, G. Zeitler .....

Wireless Network Coding..



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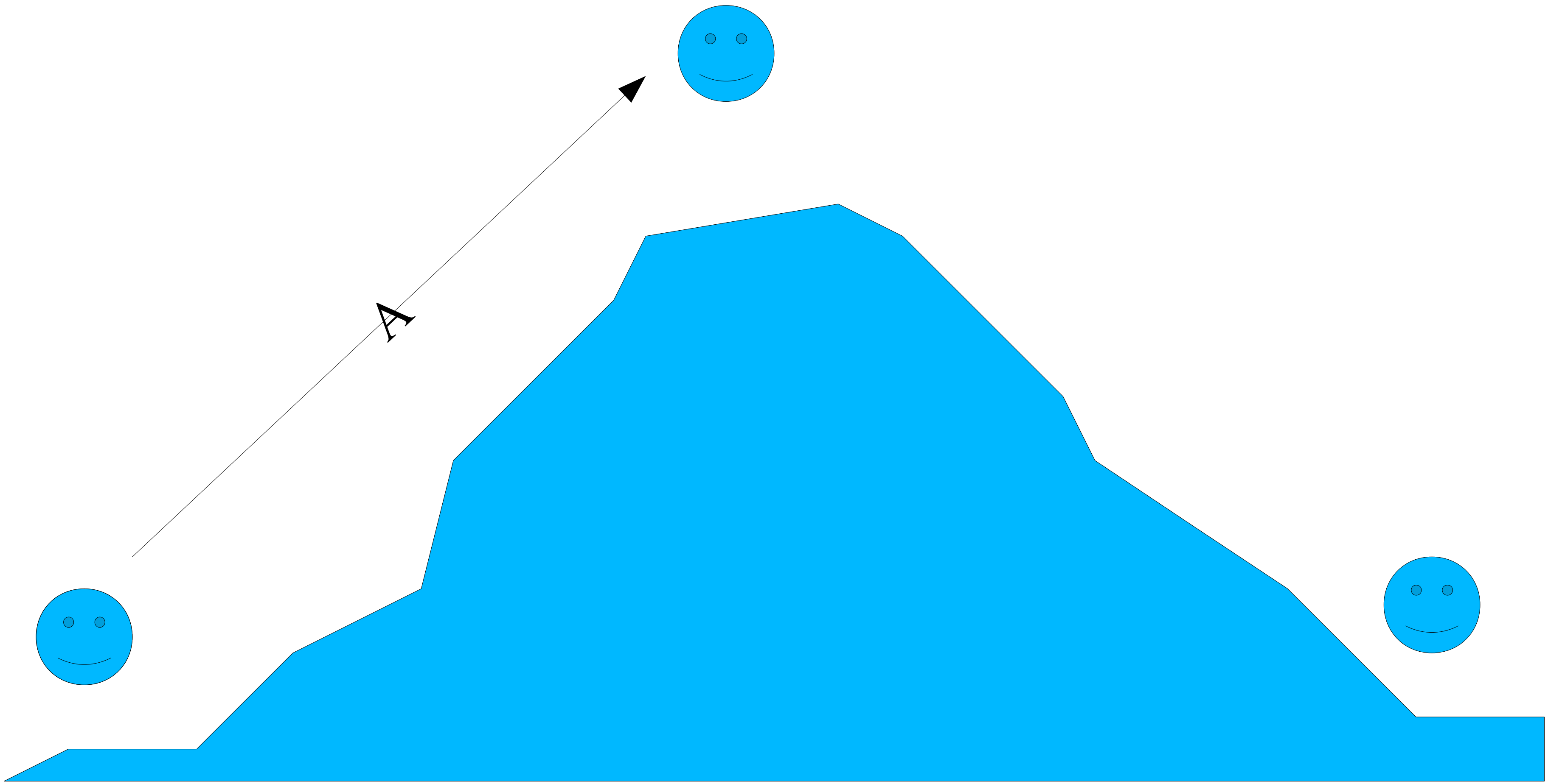

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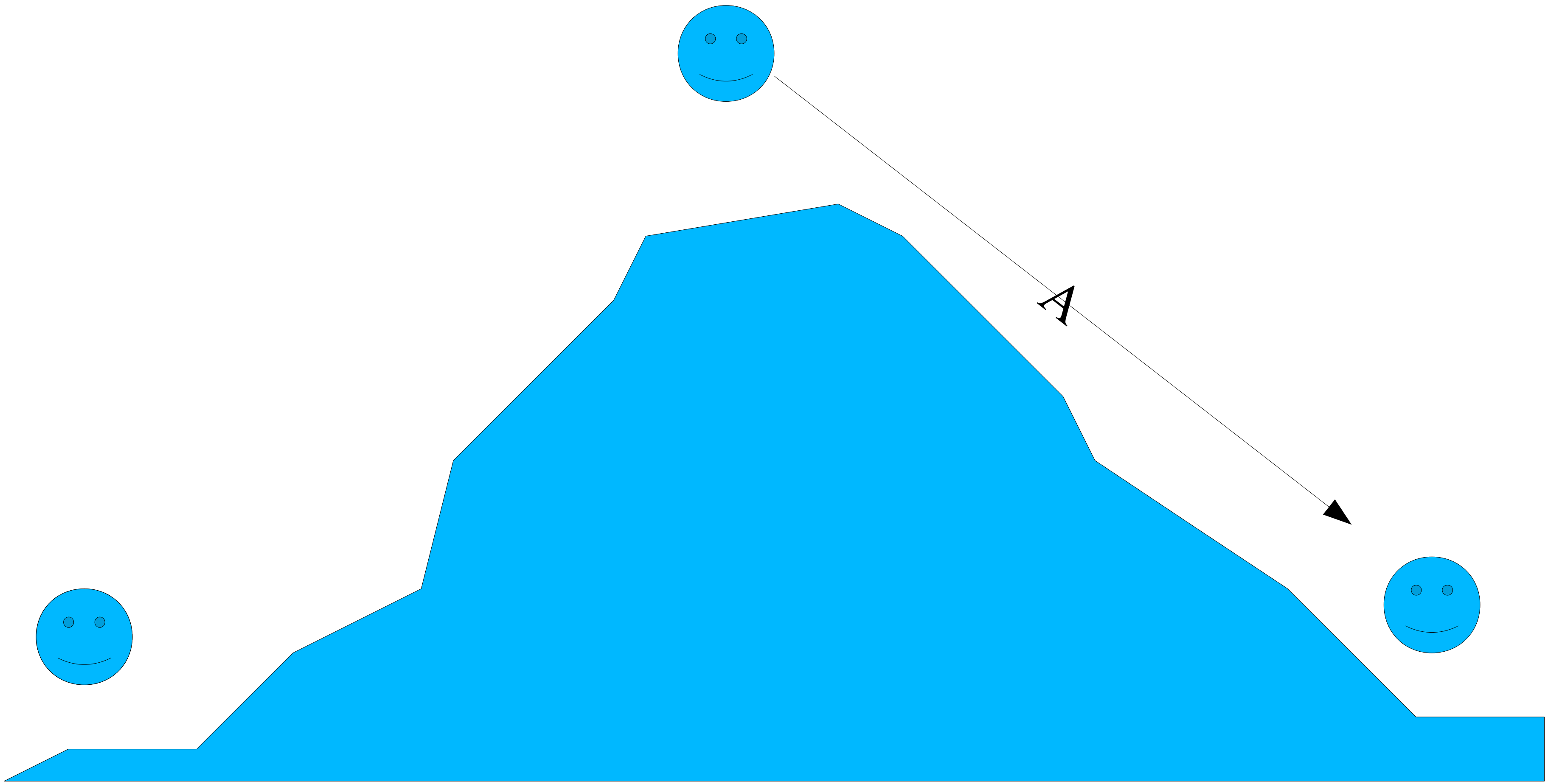

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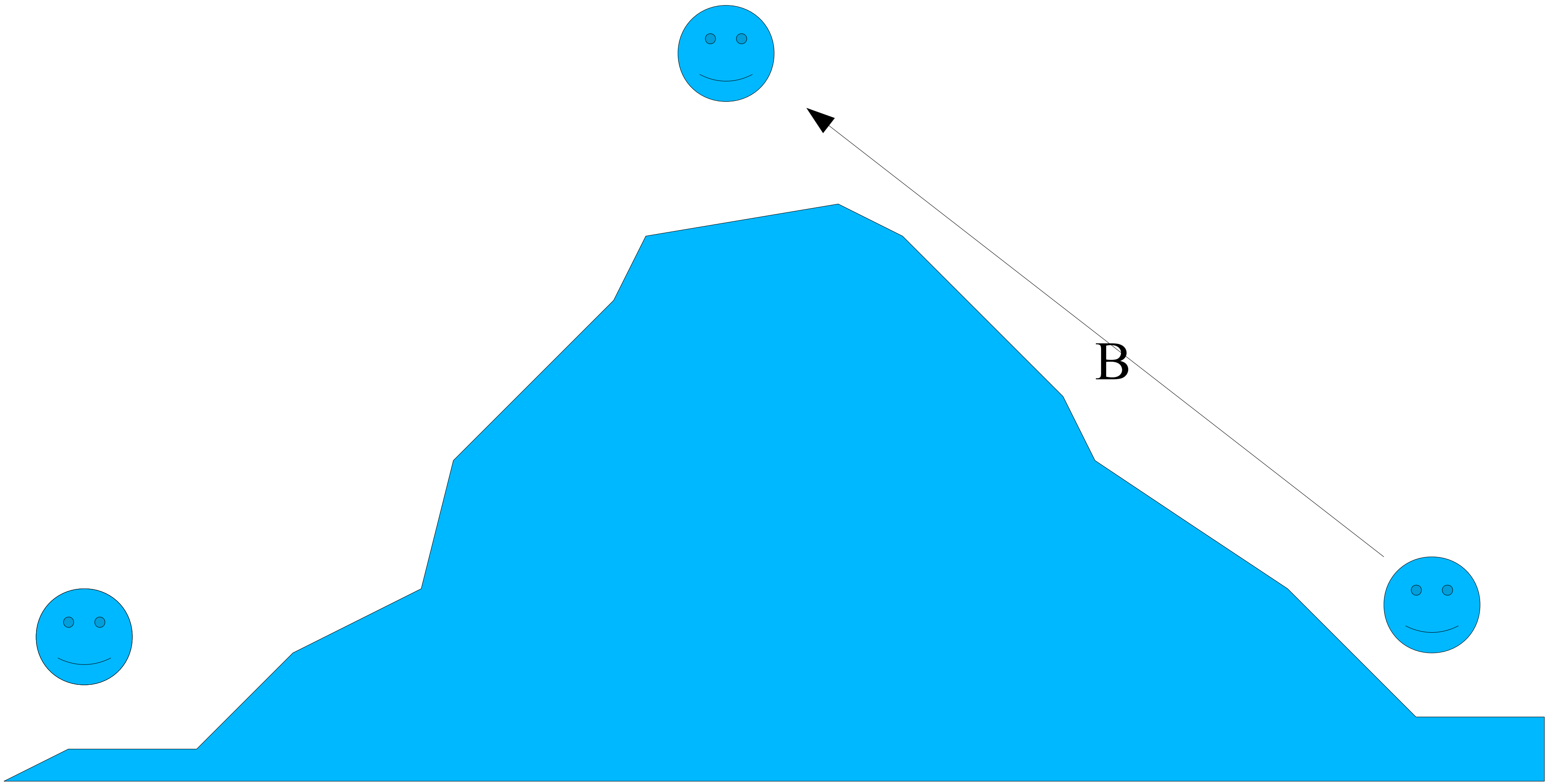

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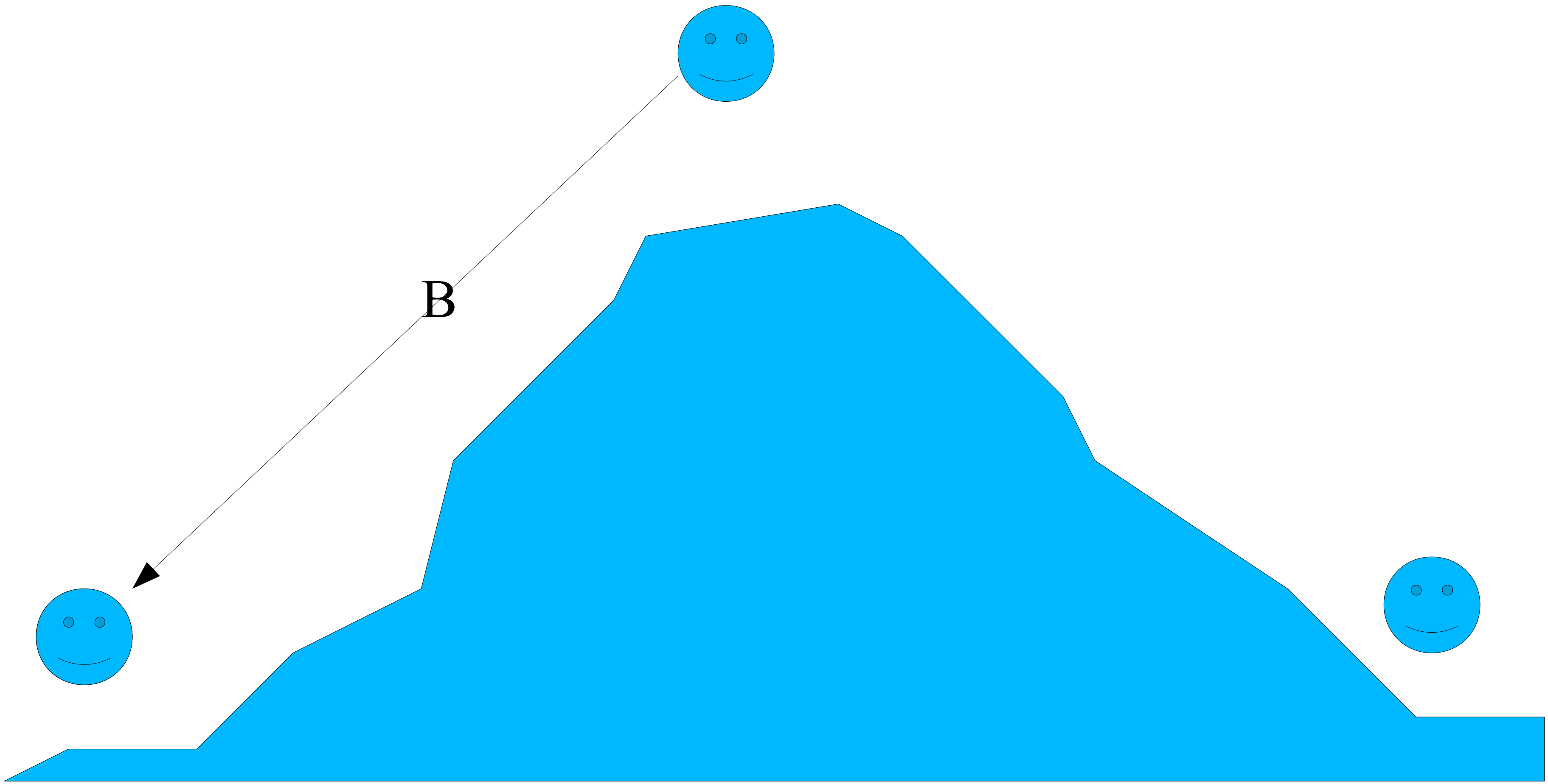
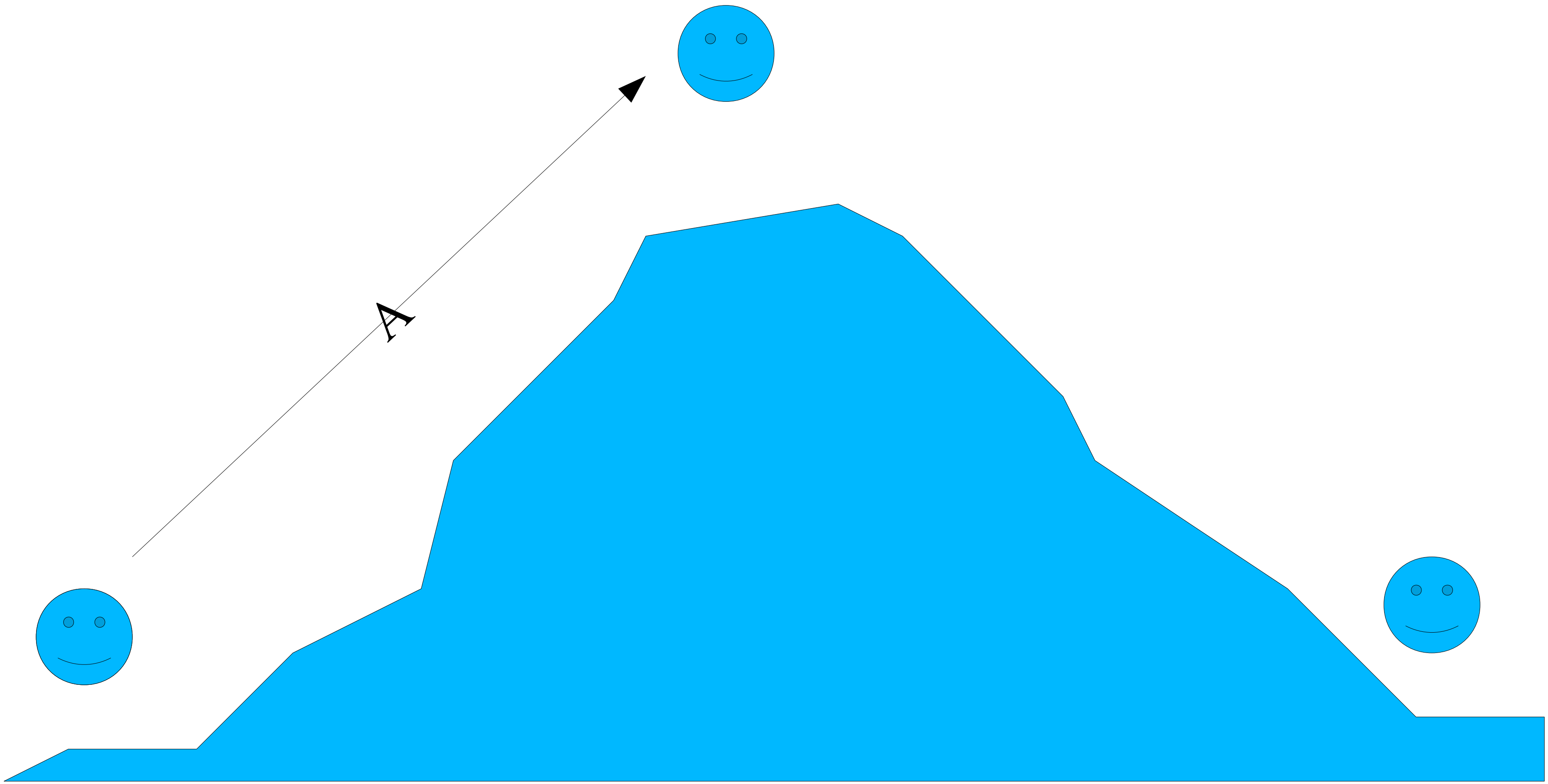
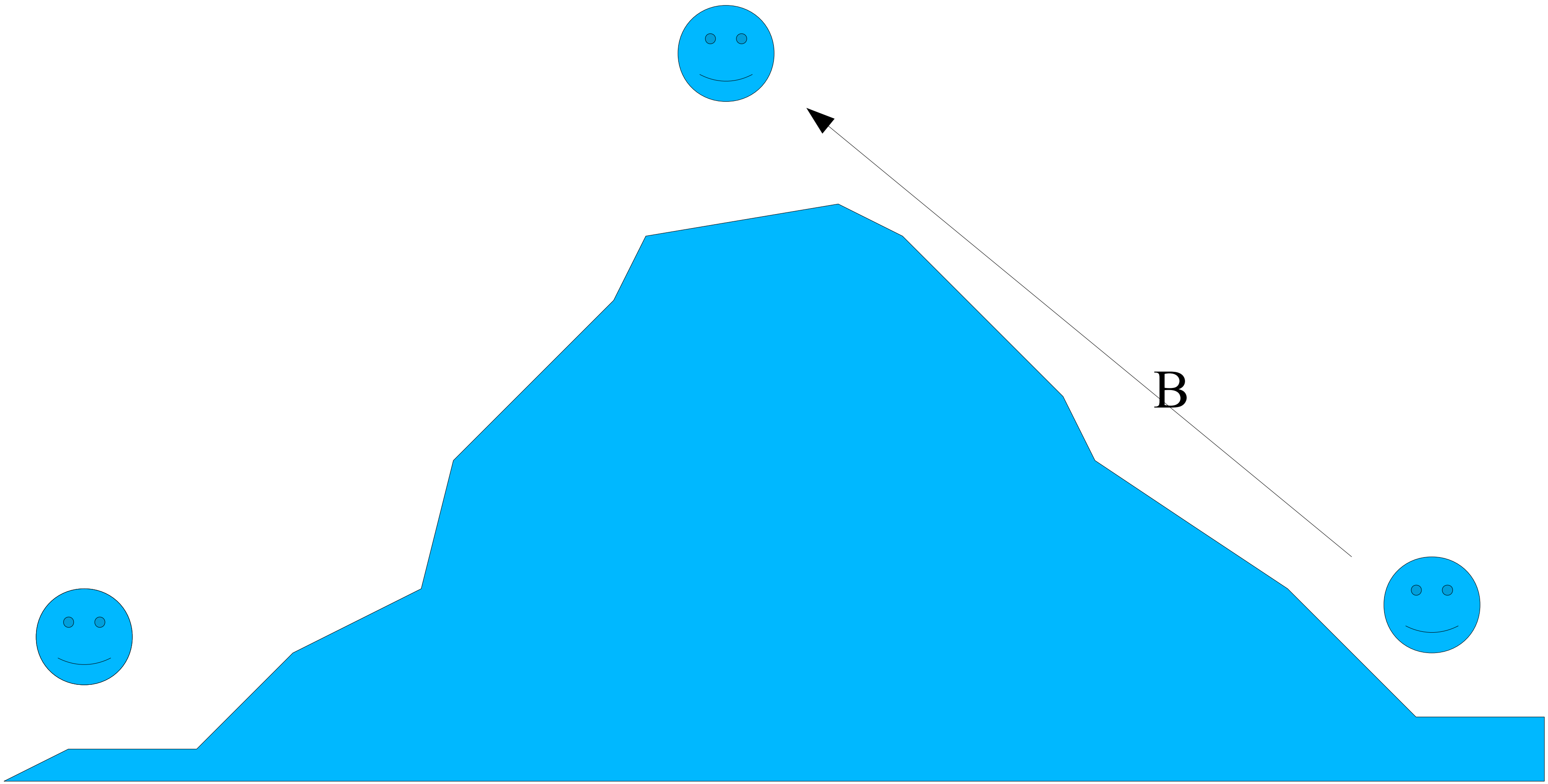


Wireless Network Coding.......



BETTER IS....

Wireless Network Coding........



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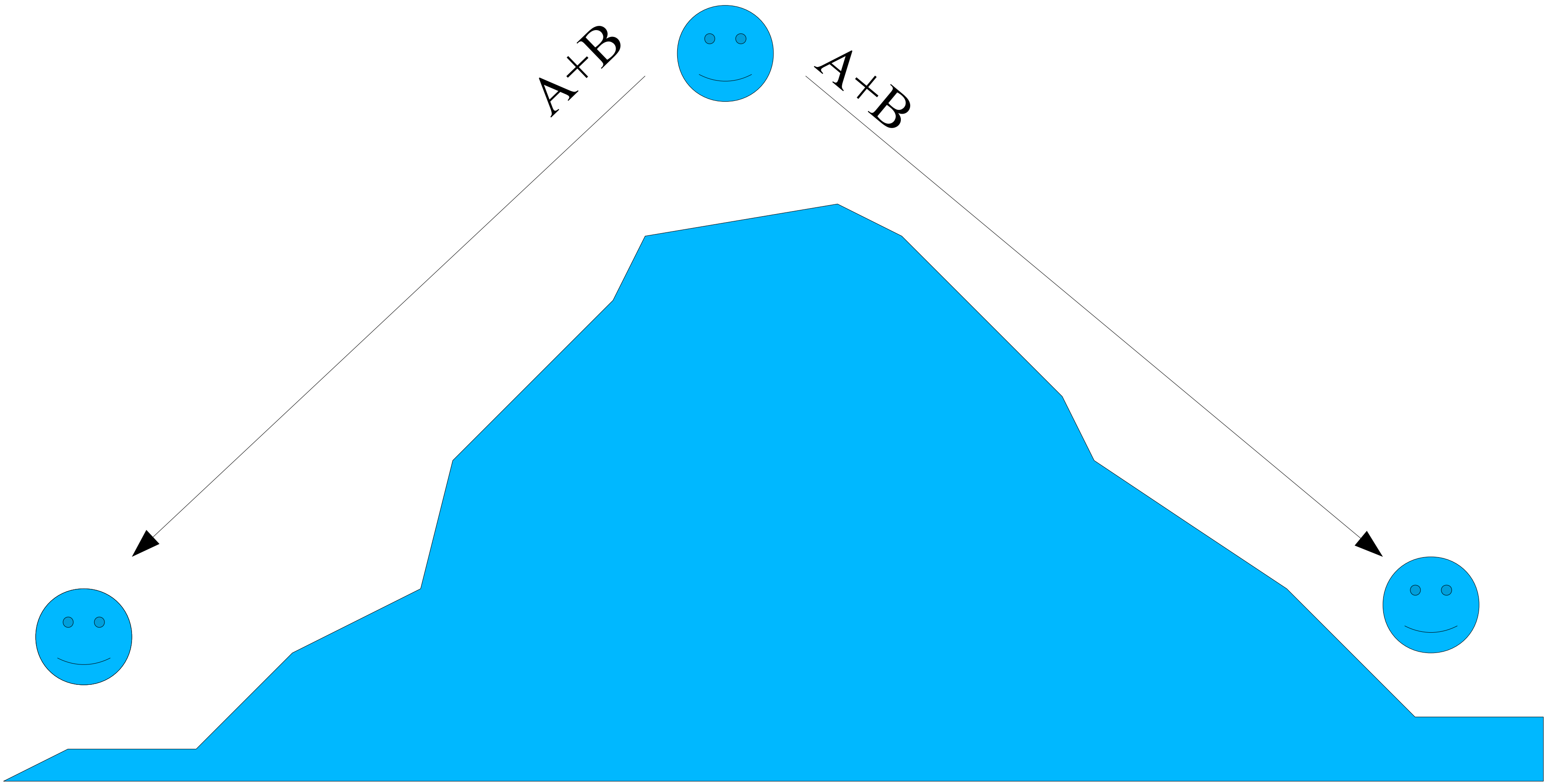
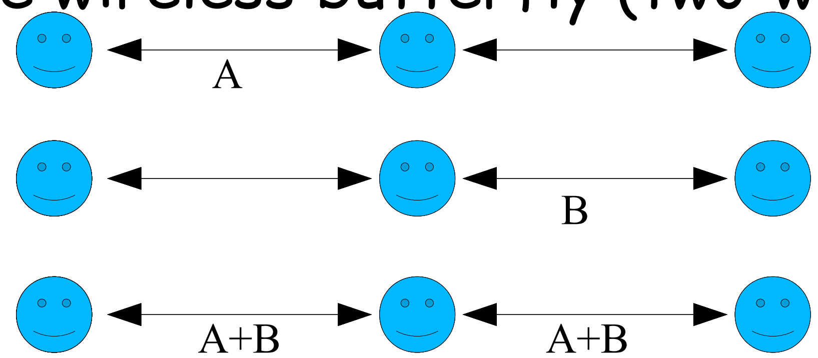
Wireless Network Coding.........

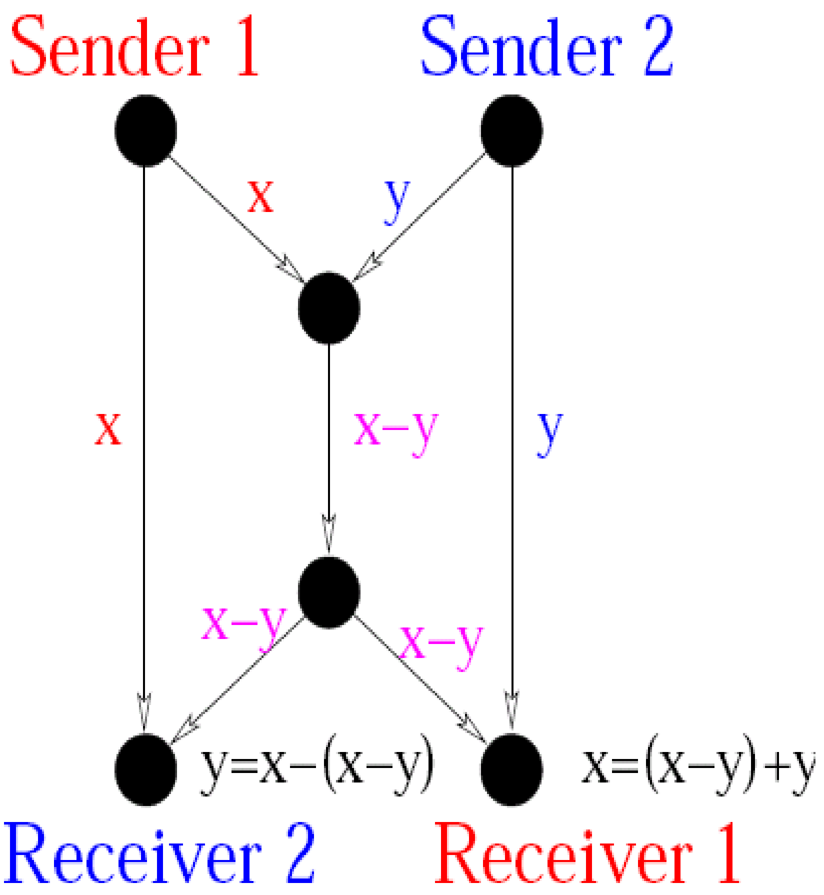


BETTER IS....this! Three channel uses only

Network coding.... the other example

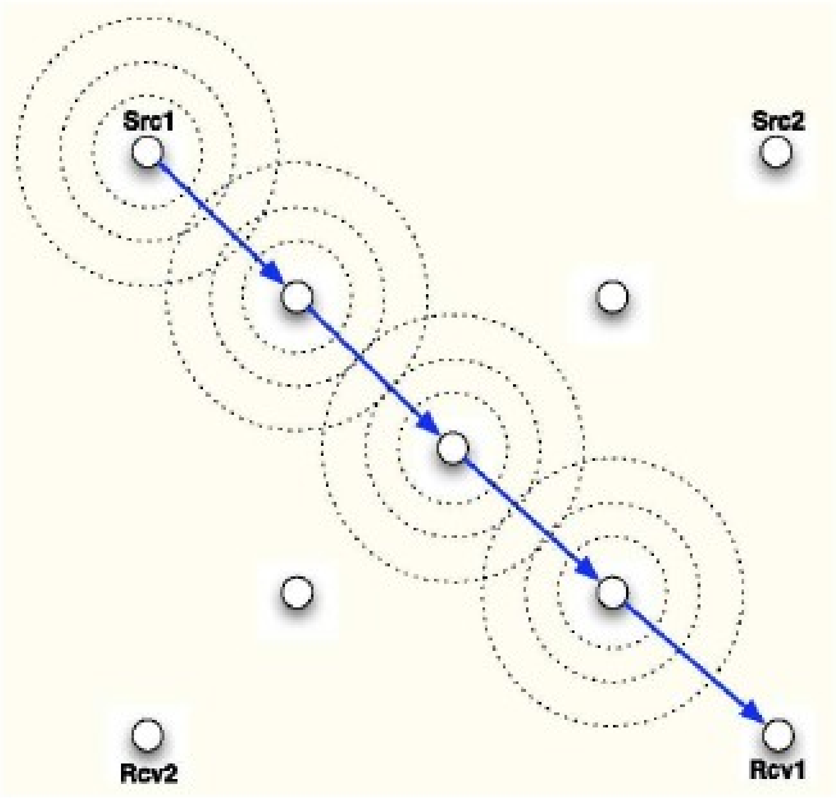
The wireless butterfly (two-way relay channel)



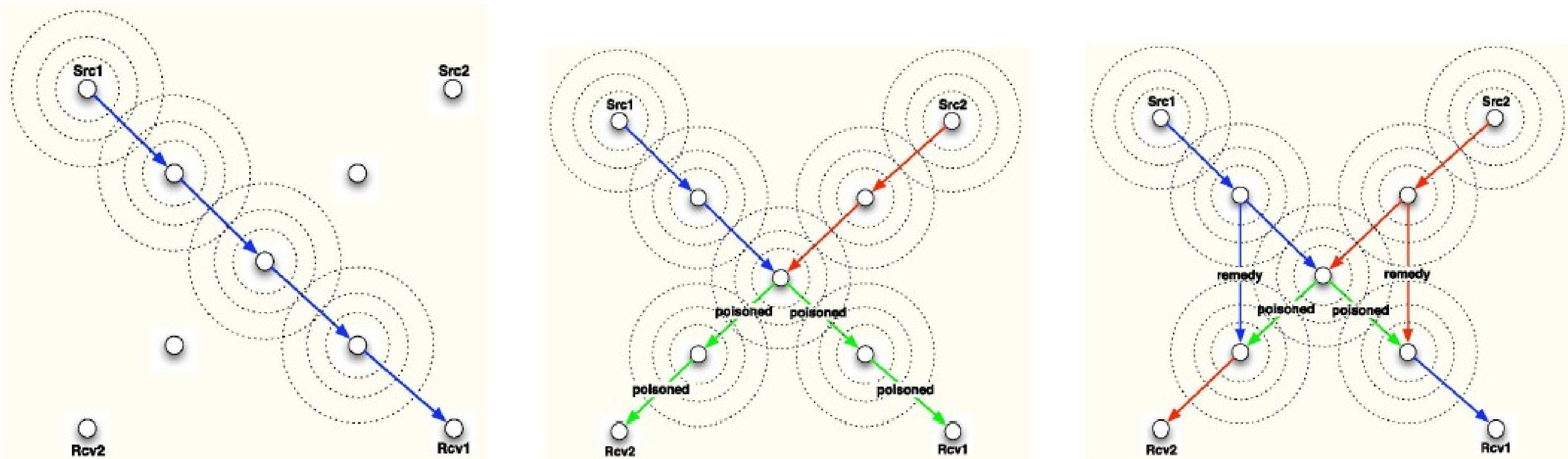


The "butterfly" example

R. Ahlswede, N. Cai, S.-Y. R. Li, R. Yeung, "Network Information Flow"
IEEE-IT, vol. 46, pp 1204-1216, 2000



The wireless situation



The wireless situation — congested areas in the network are better utilized: packets use "car pooling"

Network coding.... some thoughts and talk outline

1) Network coding is most often understood in the context of a rate region enlargement (the "main" theorem)
- combinatorial network coding
- random network coding
- flash multicast (COPE)

2)Network coding also provides the opportunity to utilize optimization theory to plan resource allocation in networks

3) A third benefit of network coding is the use as rateless codes in a packet erasure network.

The items above provide largely independent benefits. It is the combination of these three features that makes network coding a powerful tool

Network coding and the transportation model

The defining property of network coding is that nodes in a network are allowed to form outgoing symbols from incoming symbols in any way — not only time-multiplexing of data streams.

$\Rightarrow$

The goal of network coding is to provide a receiver R with enough evidence to solve an inference problem concerning the data that it wants to receive $\Rightarrow$ "$H(X_R|Y_R) = 0$" is the goal

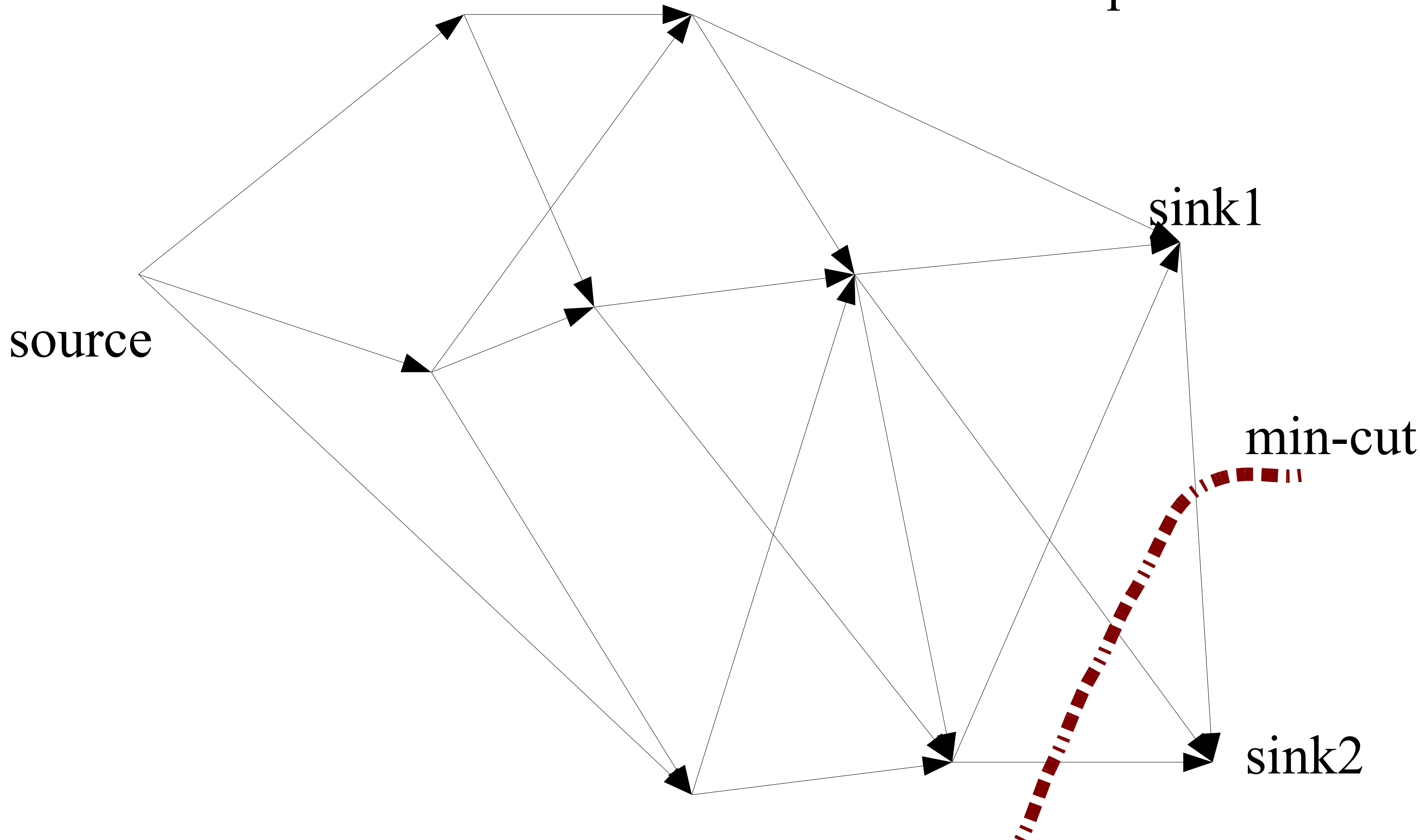
Network coding breaks with the transportation model:

A bit is not a car!

The main theorem - this is not a proof

C:\Users\ralf\Talks\Summer School Brixen\sessionII.pdf

This is a proof!



The network can transport at most the "min-cut" amount of randomness towards any sink. By proper mixing of the data the max amount of randomness can be simultaneously achieved for all sinks.

The "main" network coding theorem

* Assume we want to transmit a source with entropy rate R.

* The maximal entropy rate that we can generate at any receiver is equal to the min-cut between the source and this receiver.

* Using network coding functions that mix data efficiently enough we can achieve the maximal entropy rate simultaneously at each receiver.

The multicast capacity is determined by the min-cut in the network and is achievable with random network coding

Some categories

Some directions in network coding:

- Combinatorial network coding

  R. Dougherty, C. Freiling, K. Zeger,
  Insufficiency of linear coding in network
  information flow, IEEE-IT

  field size, structure of networks, matroidal
  solutions, entropic vectors,

  "....diabolically constructed networks...."
  (M.Sudan)

- Opportunistic network coding for medium access

  S. Katti, H. Rahul, W.Hu, D. Katabi, M.
  Medard, J. Crowcroft
  XORs in The Air: Practical Wireless Network
  Coding

  exploiting dynamic and localized multicast opportunities,
  protocol issues, how proactive can we do this,.....

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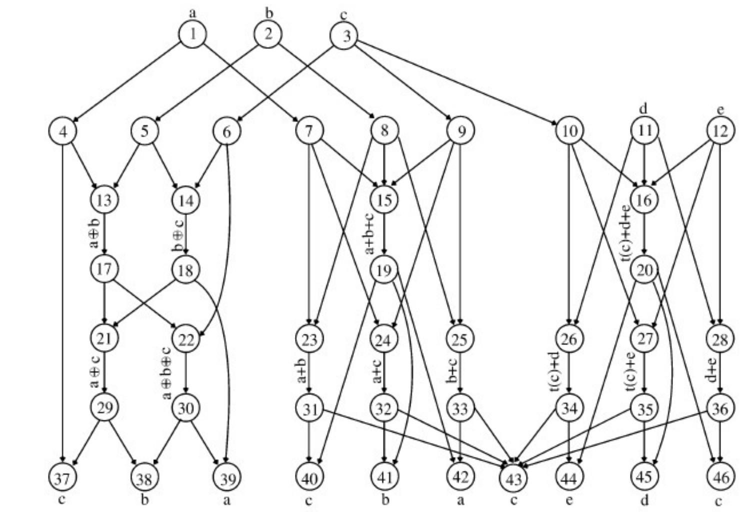

- Opportunistic network coding for medium access

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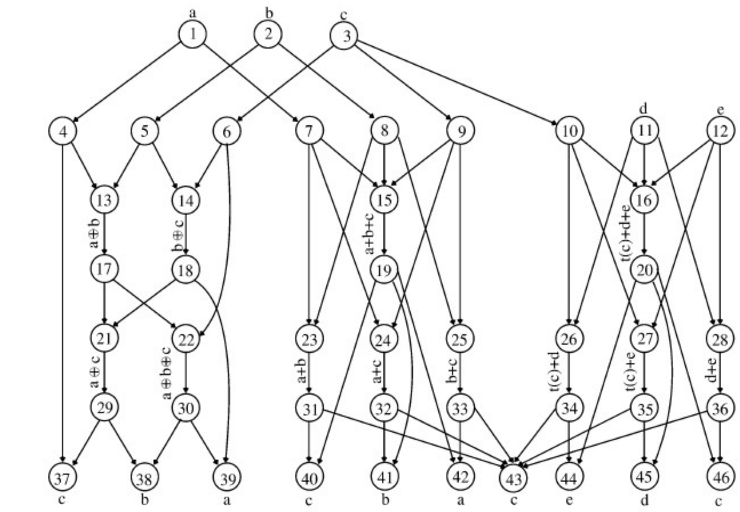

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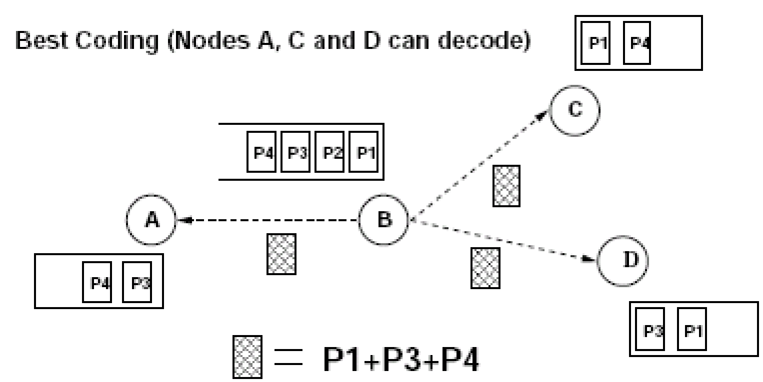
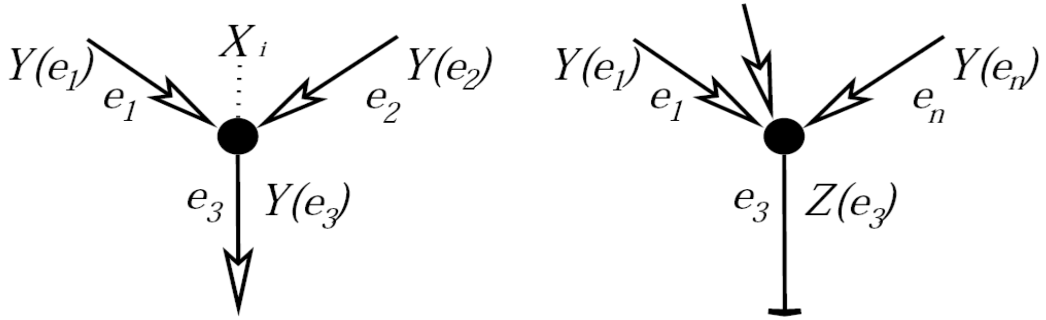


Random network coding

- <u>Random linear network coding</u>

In random network coding (multicast, gossip content distribution, distributed storage, broadcast) the information packets are interpreted as vectors over a finite field which are linearly combined with randomly chosen coefficients from that field. The approach will achieve the capacity region of the network with high probability.

All operations at network nodes are assumed linear!

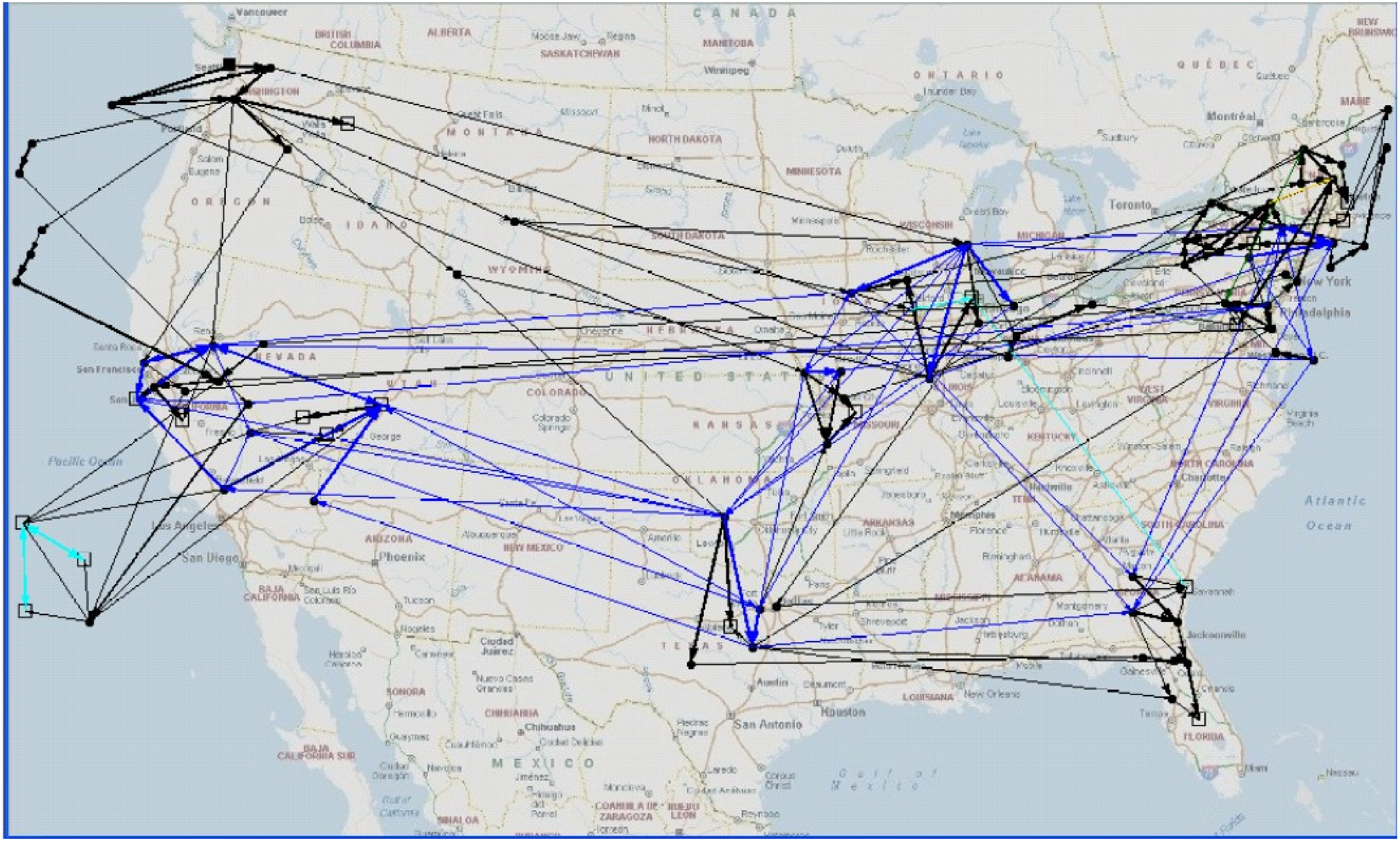


$$Y(e_3) = \sum_i \alpha_i X(v,i) + \sum_{j=1,2} \beta_j Y(e_j)$$

$$Z(v,j) = \sum_{j=1}^{n} \varepsilon_j Y(e_j).$$

Note that in random network coding the choice of code and operation of each node is chosen completely decentralized and does not require any centralized management!

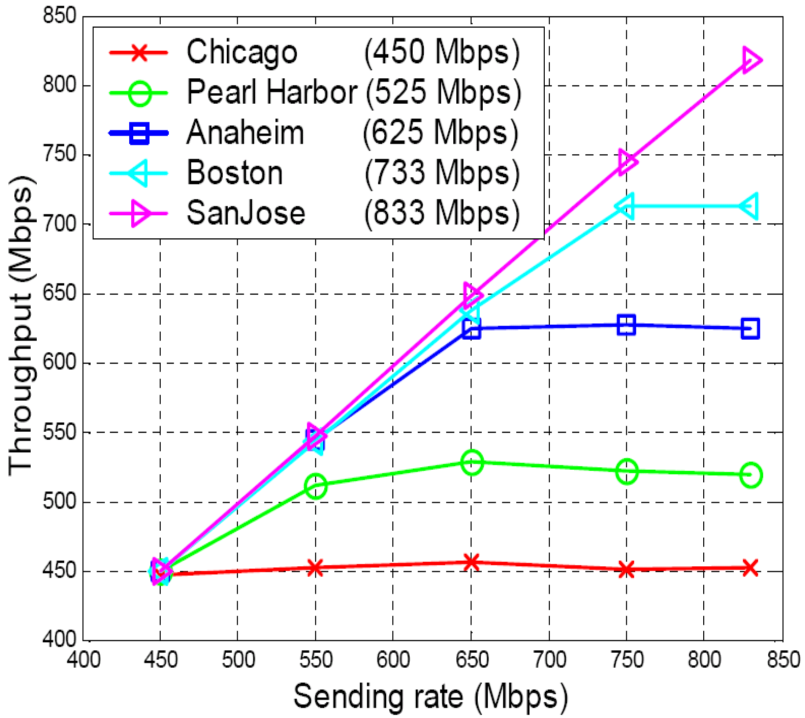
Chou, Wu, Jain: *Practical Network Coding*, Allerton 2003

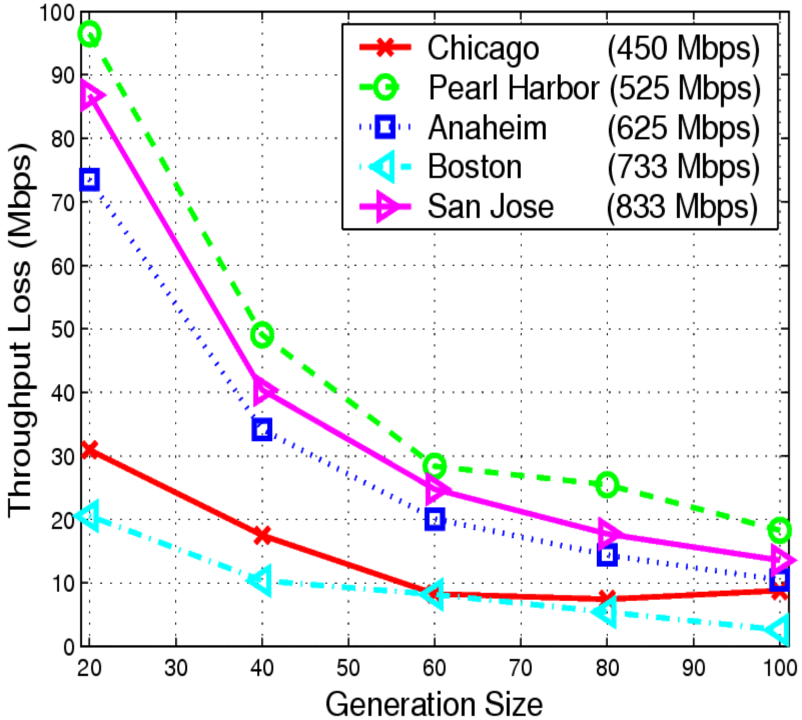


A Rocketfuel based test network

(89, 972); (89,207); G=100;q=2^16

http://www.cs.washington.edu/research/networking/rocketfuel/





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Random network coding

In random network coding for multicast, (content distribution, distributed storage, broadcast,...) the information packets are linearly combined with randomly chosen coefficients from a field. The approach will achieve the capacity region of the network with high probability.

- Note that in random network coding the choice of code and operation of each node is chosen completely decentralized and does not require any centralized management!
- The combined effect of the random choices is communicated by pilot tones at a controllable overhead.

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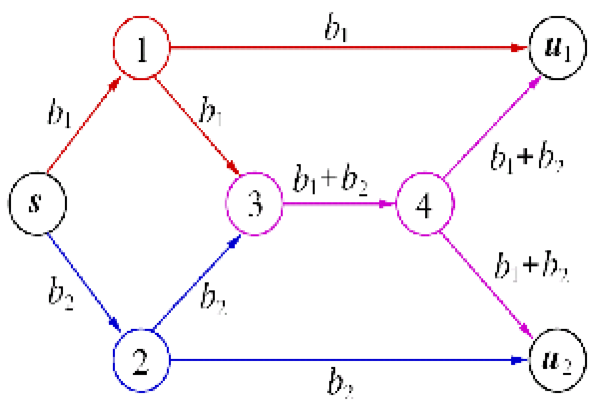
- The combined effect of the random choices is communicated by pilot tones at a controllable overhead.

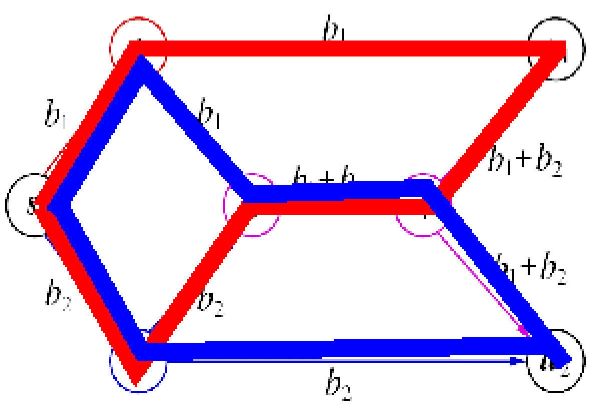
- If *every* node in a network forwards random combinations of packets, we will flood the network with information.....
- If we can combine network coding with an efficient resource allocation scheme that only forwards information on properly chosen links with proper rates we can avoid the flooding problem

→ Network optimization and the multicast........................

Multicast and packing/flow problems



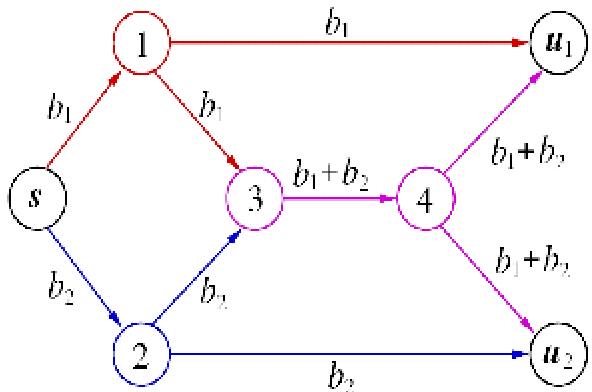
The original multicast problem has an interpretation as “random network coding”



Interpretation as packing of overlapping source destination **flows**

We hence have a chance to formulate the multicast problem as a flow problem.....

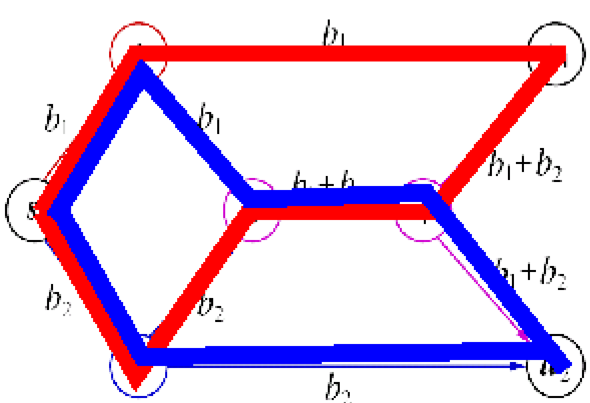
Multicast and packing/flow problems



Utilizing the flow formulation we can solve a linear program:

Minimize the sum of cost-max flow product for each edge

subject to: Observe the flow conservation at intermediate network nodes for each receiver individually



An LP problem for the multicast

If we assume a proportional cost assignment we obtain the following LP

$$\text{minimize} \sum_{(i,j)\in\mathcal{A}} a_{ij} z_{ij}$$

$$\text{subject to } c_{ij} \geq z_{ij}, \qquad \forall (i,j) \in \mathcal{A}$$

$$z_{ij} \geq x_{ij}^{(t)} \geq 0, \qquad \forall (i,j) \in \mathcal{A},\, t \in T$$

$$\sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij}^{(t)} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji}^{(t)} = \sigma_i^{(t)}, \quad \forall i \in \mathcal{N},\, t \in T$$

What we achieved so far.....

- A main benefit of network coding is the formulation of the multicast problem as a problem of "overlapping packing of flows"

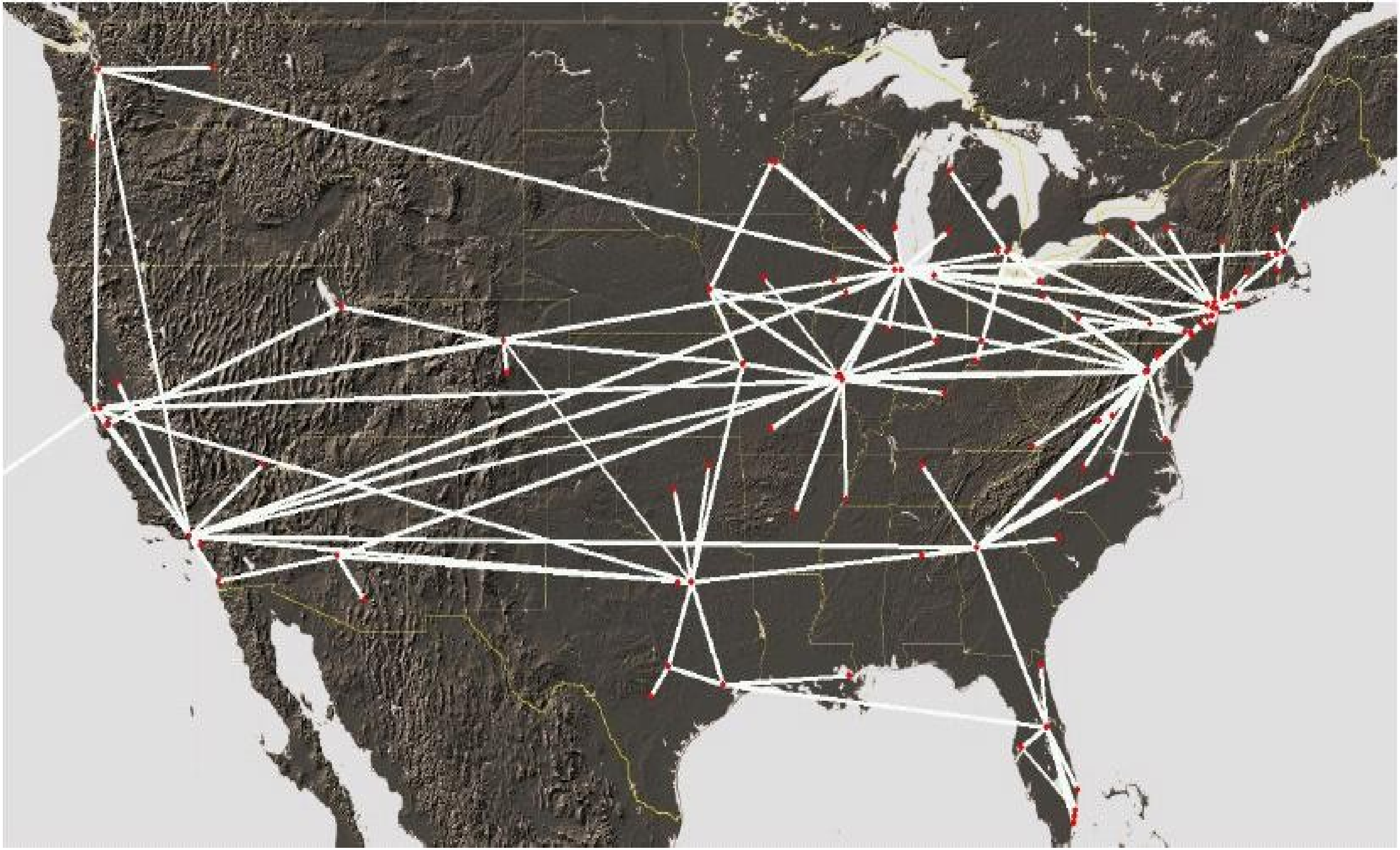
- This makes it possible to integrate network optimization and resource allocation techniques with multicast applications!

- The resulting linear (convex) programs can be efficiently solved by iterative distributed algorithms

- At the same time random network coding provides a way to construct the network code in a distributed way

.... the multicast problem enters the realm of network optimization....

Some plots...



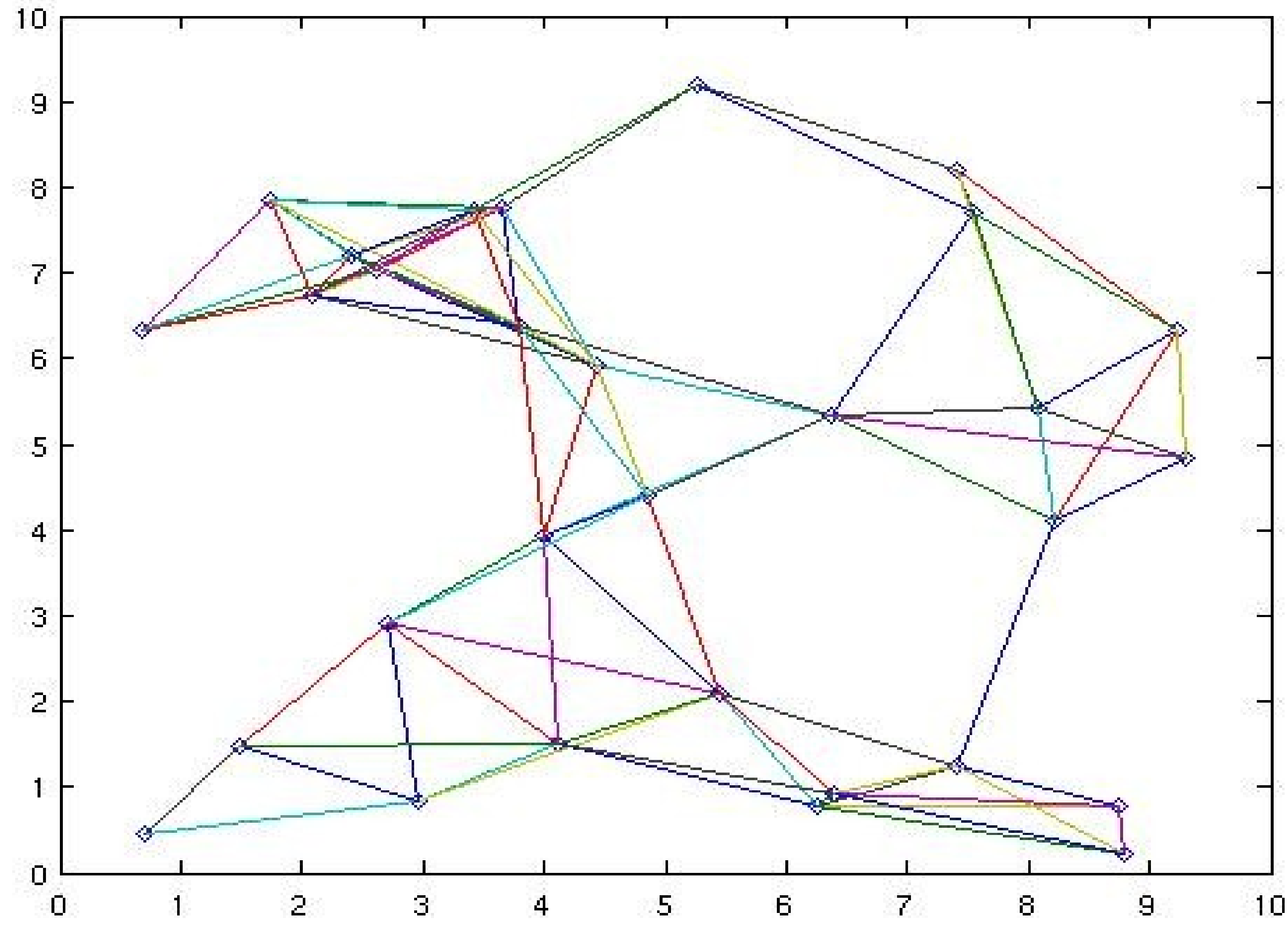
Rocketfuel's view of the AT&T backbone....

www.cs.washington.edu/research/networking/rocketfuel/

Some numbers...

Network	Approach	Average multicast cost			
		2 sinks	4 sinks	8 sinks	16 sinks
Telstra (au)	DST approximation	17.0	28.9	41.7	62.8
	Network coding	13.5	21.5	32.8	48.0
Sprint (us)	DST approximation	30.2	46.5	71.6	127.4
	Network coding	22.3	35.5	56.4	103.6
Ebone (eu)	DST approximation	28.2	43.0	69.7	115.3
	Network coding	20.7	32.4	50.4	77.8
Tiscali (eu)	DST approximation	32.6	49.9	78.4	121.7
	Network coding	24.5	37.7	57.7	81.7
Exodus (us)	DST approximation	43.8	62.7	91.2	116.0
	Network coding	33.4	49.1	68.0	92.9
Abovenet (us)	DST approximation	27.2	42.8	67.3	75.0
	Network coding	21.8	33.8	60.0	67.3

A wireless network



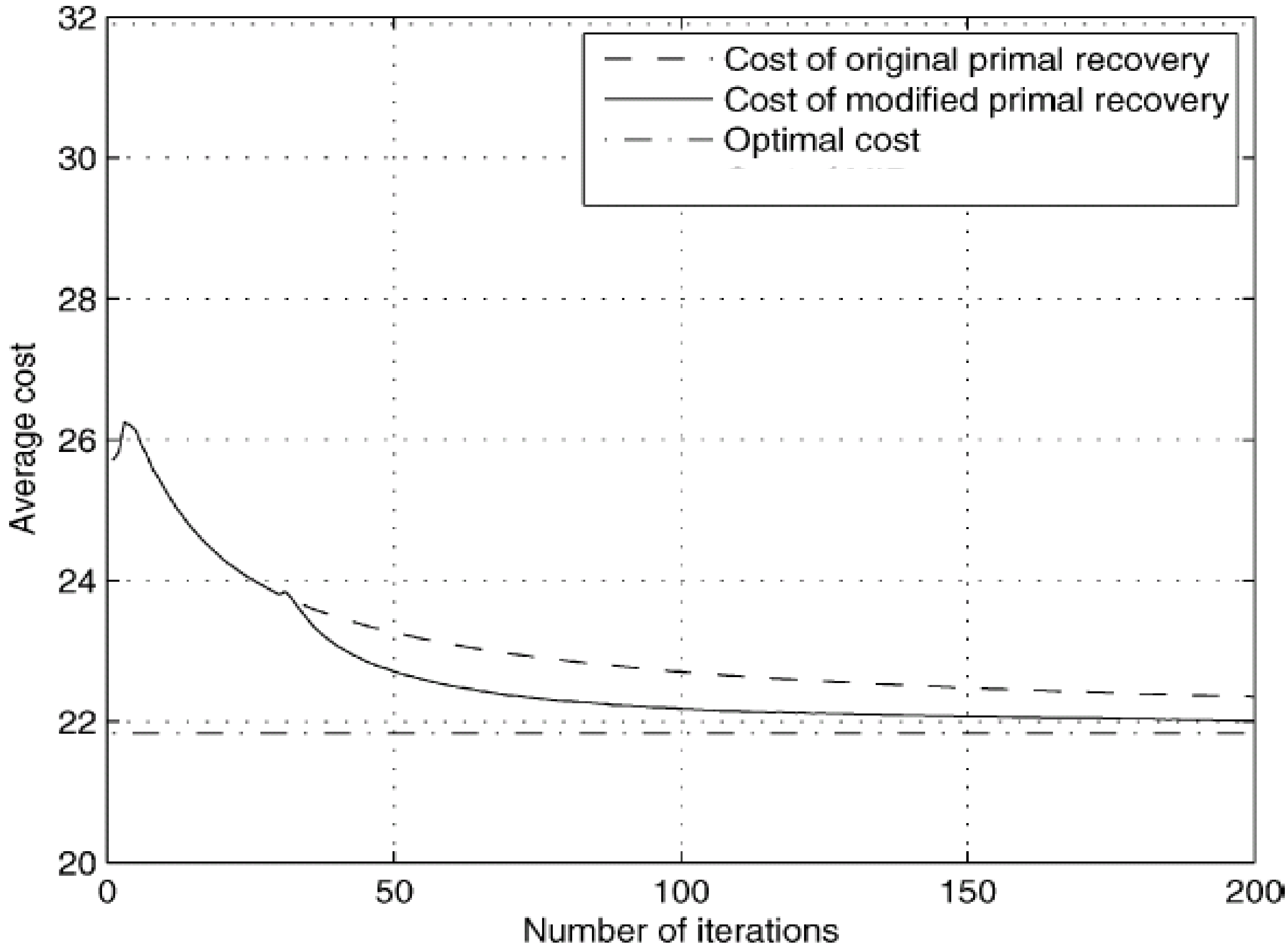
More (wireless) numbers

TECHNISCHE UNIVERSITÄT MÜNCHEN

Network size	Approach	Average multicast energy			
		2 sinks	4 sinks	8 sinks	16 sinks
20 nodes	MIP algorithm	30.6	33.8	41.6	47.4
	Network coding	15.5	23.3	29.9	38.1
30 nodes	MIP algorithm	26.8	31.9	37.7	43.3
	Network coding	15.4	21.7	28.3	37.8
40 nodes	MIP algorithm	24.4	29.3	35.1	42.3
	Network coding	14.5	20.6	25.6	30.5
50 nodes	MIP algorithm	22.6	27.3	32.8	37.3
	Network coding	12.8	17.7	25.3	30.3

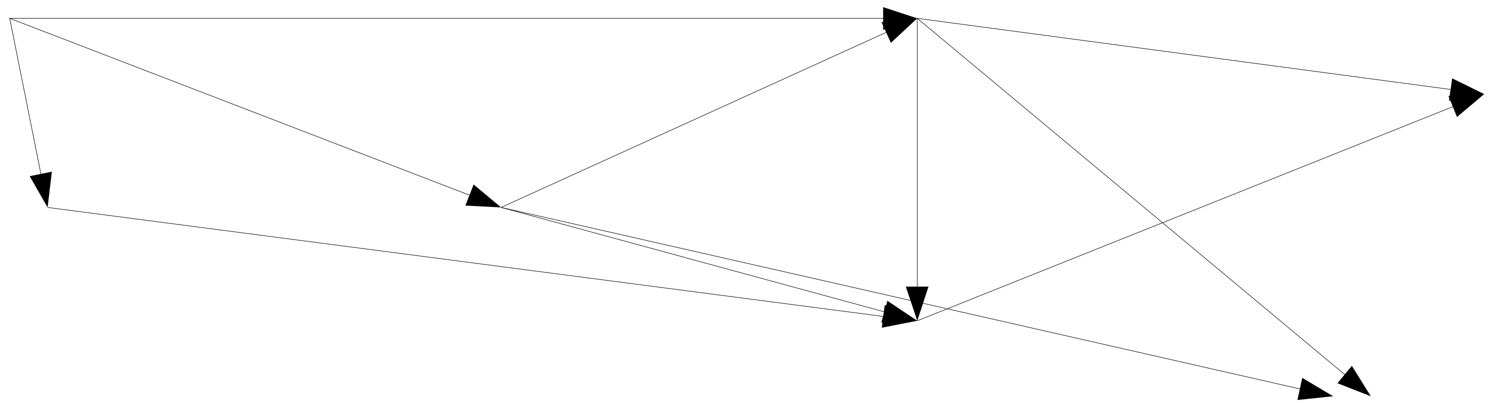
MIP: J. E. Wieselthier, G. D. Nguyen, and A. Ephremides, "Energy-efficient broadcast and multicast trees in wireless networks," *Mobile Netw. Applic.*, vol. 7, pp. 481–492, 2002.

Convergence speed

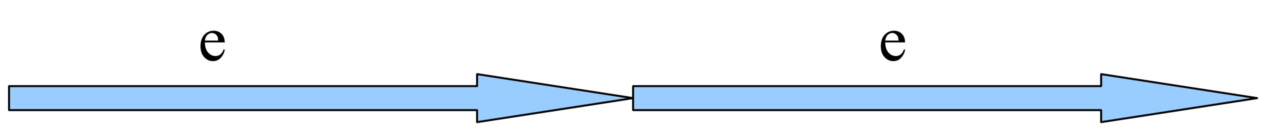


Network codes as packet erasure codes

We consider coding on packet erasure networks



A prototype toy example:



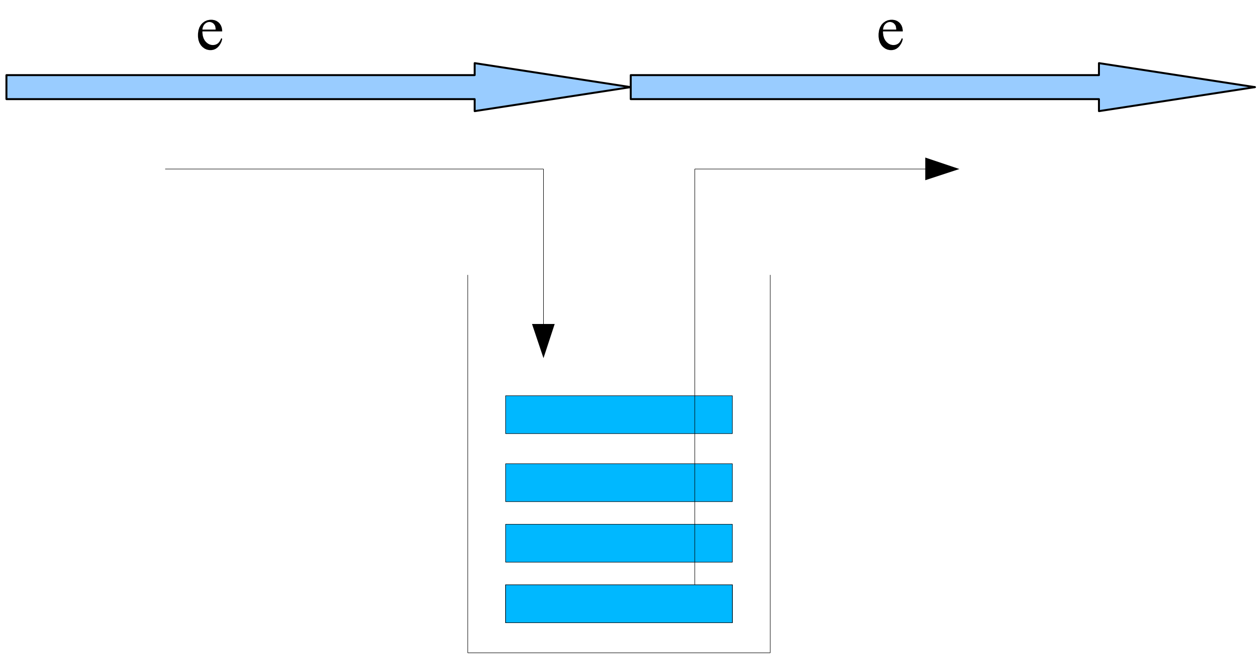
End-to-end coding (LT codes) achieves a "capacity" of $(1-e)^2$

Network codes as packet erasure codes

"Classically" one would decode and re-encode at the center node.

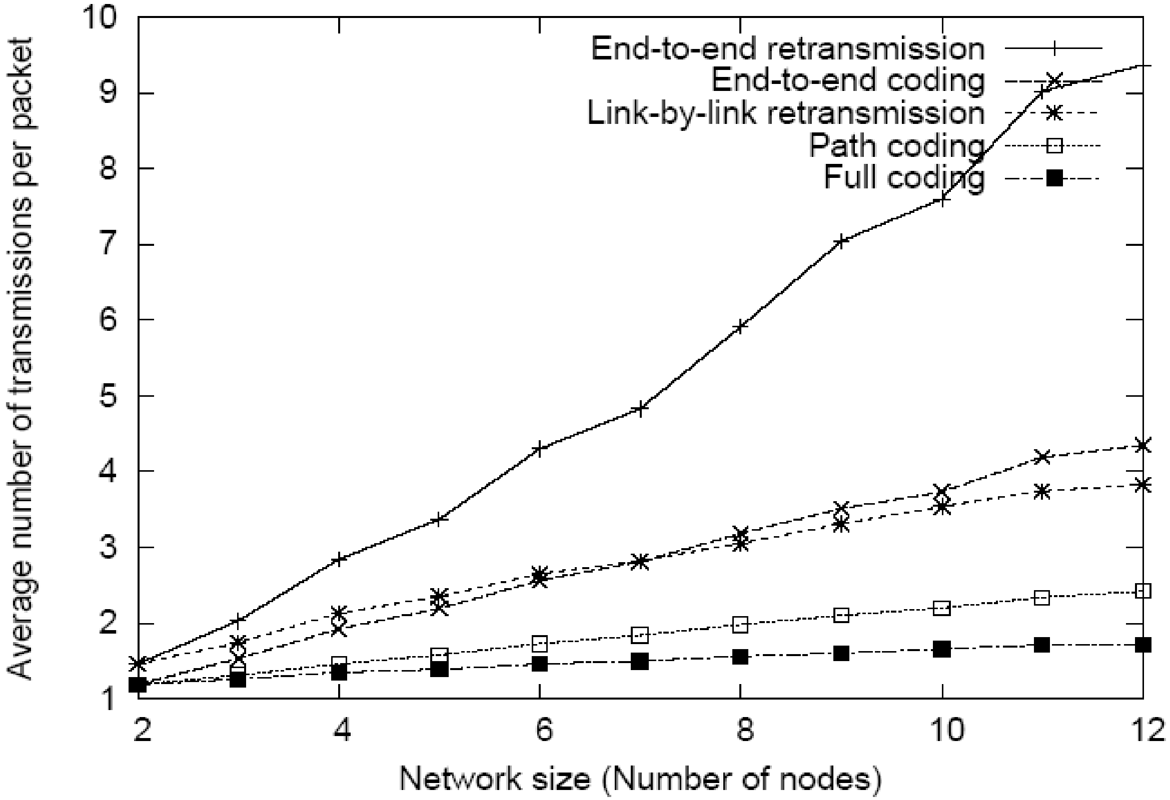
- long latency
- potentially significant complexity

Random network coding solution:



Random network coding also achieves the capacity of 1-e without significant complexity or delay!

Network codes as packet erasure codes



Average number of transmissions required per packet in random networks of varying size. Sources and sinks were chosen randomly according to a uniform distribution. Paths or subgraphs were chosen in each random instance to minimize the total number of transmissions required, except in the cases of end-to-end retransmission and end-to-end coding, where they were chosen to minimize the number of transmissions required by the source node.

Summary

- Network coding is a multifaceted phenomenon with several significant contributions:

- The rate region enlargement -> the butterfly example

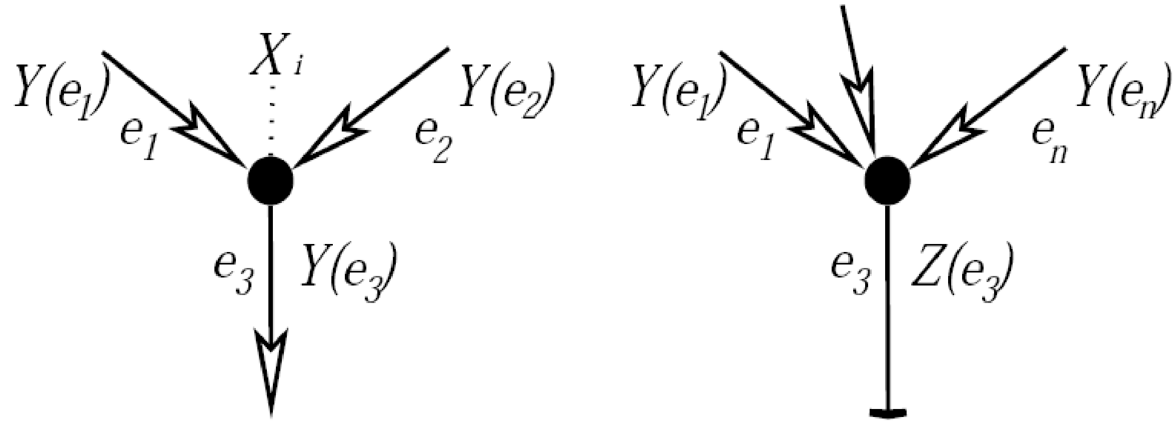
- The possibility to cast classically hard problems, i.e. the multicast, as standard optimization problems

- Network codes as the "better" packet erasure codes ...

- Opportunities: The joint optimization of the channel access and the network coded multicast, COPE, CODECAST, network coding for delay, security against wiretapping, overlay networks...

THANK YOU

Random network coding



$$Y(e_3) = \sum_i \alpha_i X(v,i) + \sum_{j=1,2} \beta_j Y(e_j)$$

$$Z(v,j) = \sum_{j=1}^{n} \varepsilon_j Y(e_j).$$

Operations are over a finite field $\mathbb{F}_{2^m}$ with <u>randomly chosen</u> coefficients. The compound effect of all the choices can be efficiently communicated in a packet header.

Network coding as coding

Summary

Network coding is a multifaceted phenomenon with three significant contributions:

- The rate region enlargement -> the butterfly example
- The possibility to cast classically hard problems, i.e. the multicast, as standard optimization problems
- Network codes as the “better” fountain code ...
- The joint optimization of the channel access and the network coded multicast is a particular intriguing area of research

THANK YOU

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Network coding is a multifaceted phenomenon with three significant contributions:

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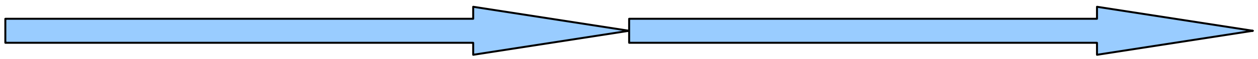
THANK YOU

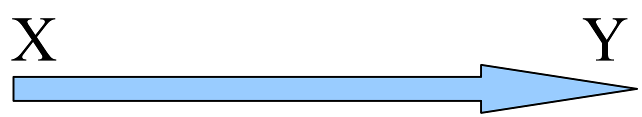
A simple (?) question

What is the capacity of a noiseless two hop network

simple links transmitting one bit per unit time.

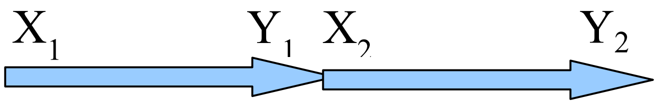


Each individual link is an error free when it is "on"
The joint distribution of Input X and Output Y is:



$P(X,Y);\ X,Y$	0	1
0	$1/2$	0
1	0	$1/2$

A simple (?) question



X1 and X2 cannot be nonzero simultaneously, X1,X2 assume values in {1,-1,NS}

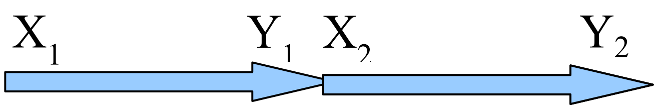
The conditional probability P(Y1|X1,X2):

$Y_1, \vert X_1, X_2$	N, N	$N, 0$	$N, 1$	$0, N$	$0, 0$	$0, 1$	$1, N$	$1, 0$	$1, 1$
E	1	0	0	0	0	0	0	0	0
0	0	1	0	1	1	0	0	1	0
1	0	0	1	0	0	1	1	0	1

assuming the half-duplex constraint on binary bit pipes

If X_2 is "on" we have $Y_1=X_2$ otherwise $Y_1=X_1$

A simple (?) question



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The conditional probability P(Y1|X1,X2):

$Y_1, \| X_1, X_2$	N,N	$N,0$	$N,1$	$0,N$	$0,0$	$0,1$	$1,N$	$1,0$	$1,1$
E	1	0	0	0	0	0	0	0	0
0	0	1	0	1	1	0	0	1	0
1	0	0	1	0	0	1	1	0	1

assuming the half-duplex constraint on binary bit pipes

This conditional pdf satiesfies the conditions for a physically degraded relay channel (Cover & Thomas; G. Kramer)

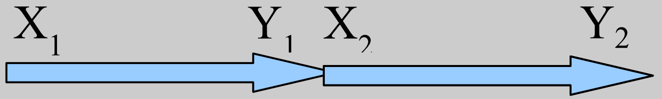
$$P(Y_2, Y_1 | X_1, X_2) = P(Y_1 | X_1, X_2) P(Y_2 | Y_1, X_2)$$

with capacity:

$$\sup_{P(X_1,X_2)} \min\{I(X_1, X_2; Y), I(X; Y_1 | X_1)\}$$

The capacity of the binary input half duplex chain of two links is 1.1388 bits/channel use

A simple (?) question

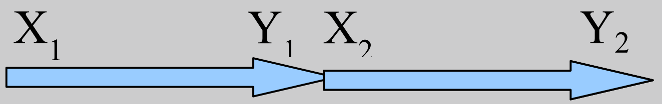


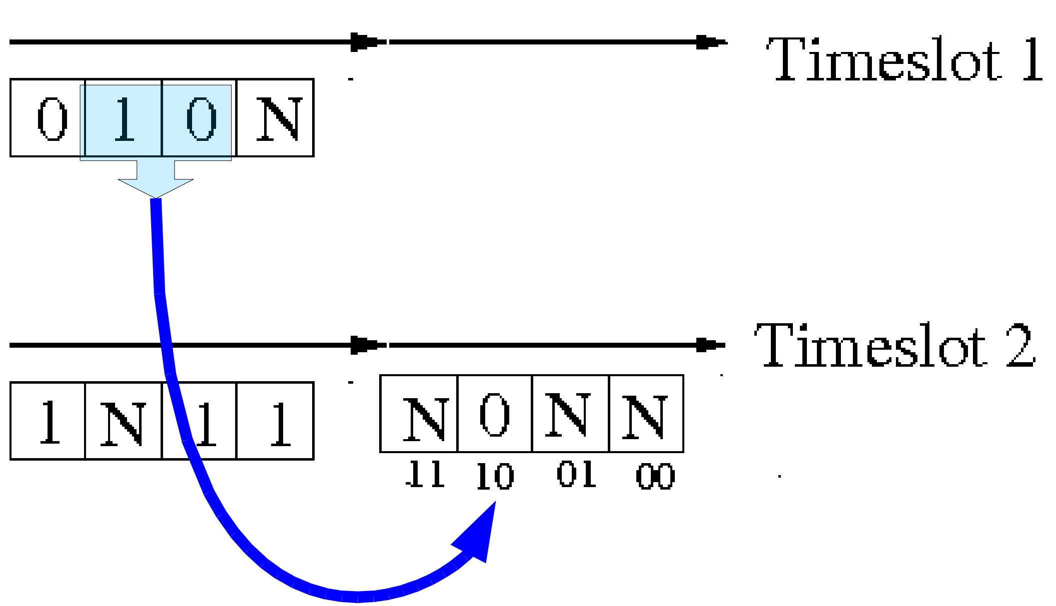
Timeslot 1

0	1	0	N

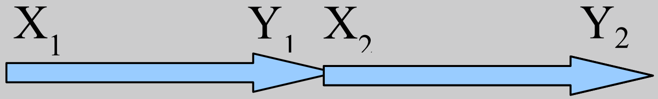
three information bits grouped as 0 and 10

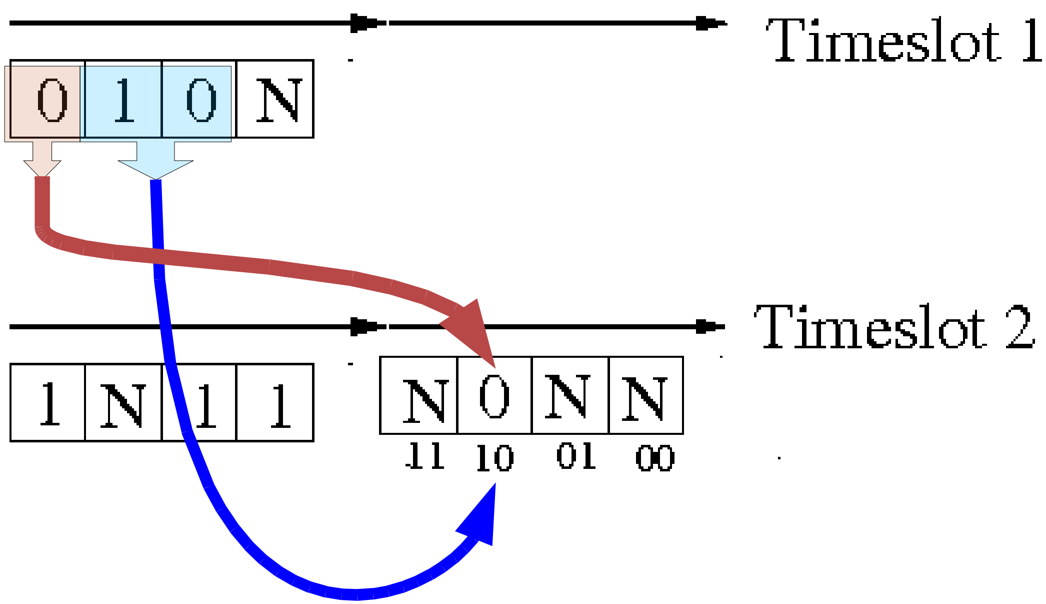
A simple (?) question



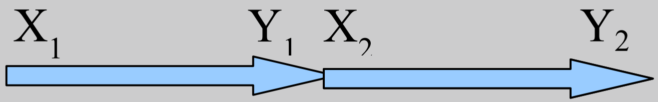


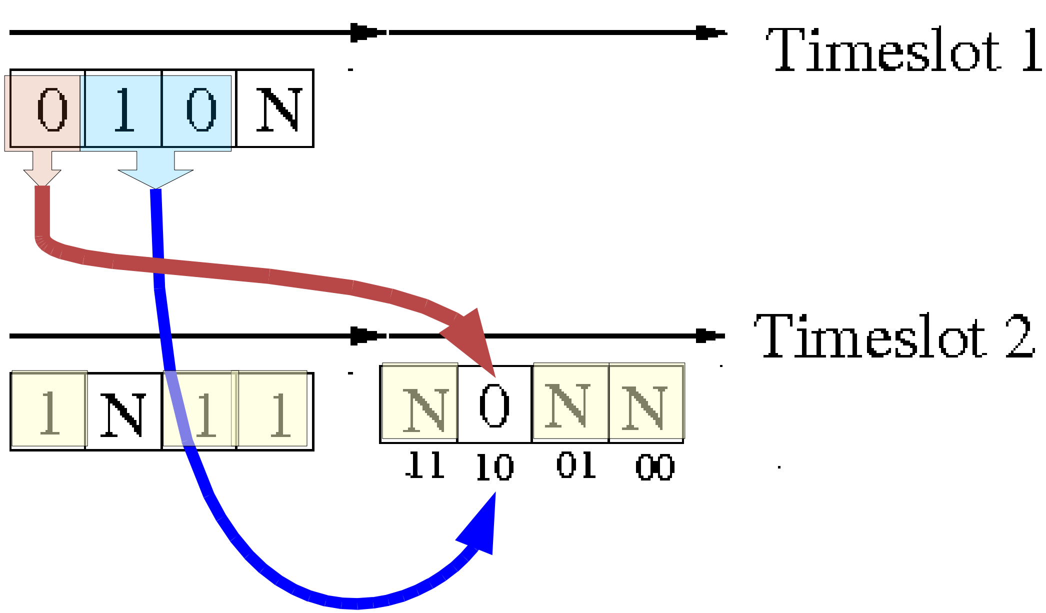
A simple (?) question



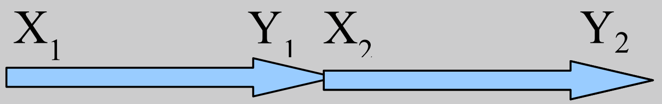


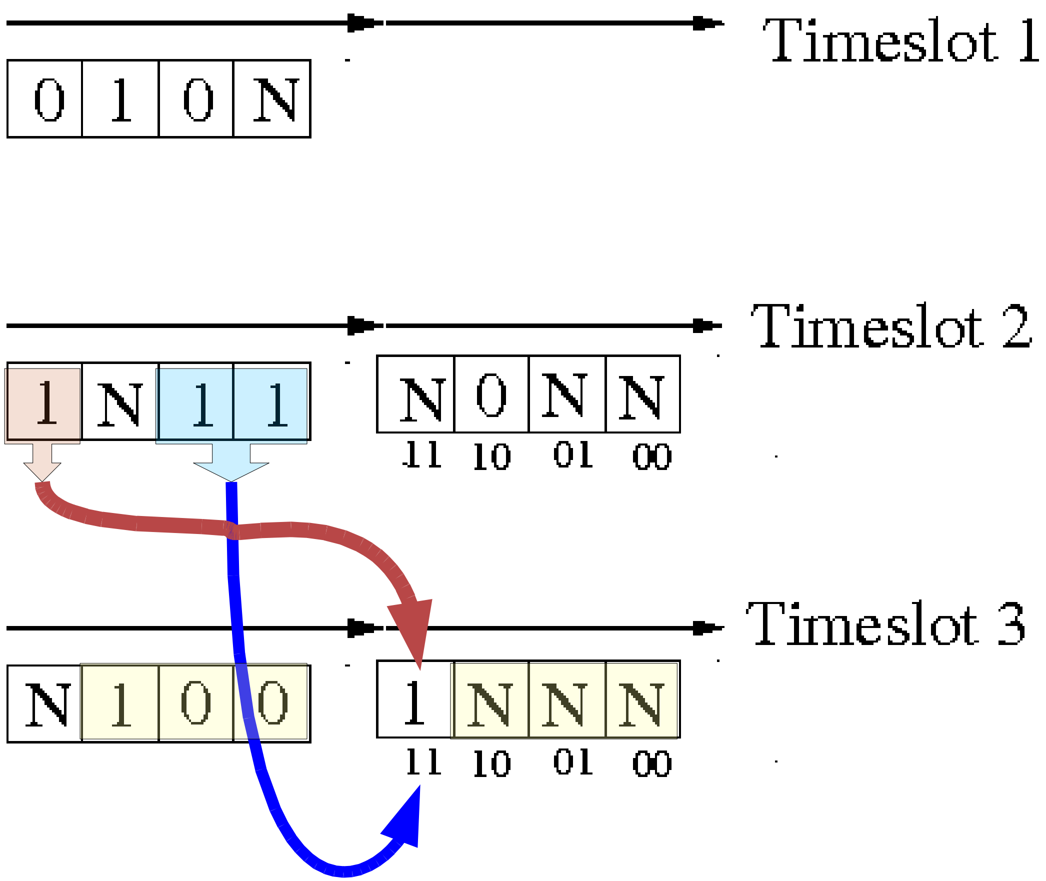
A simple (?) question



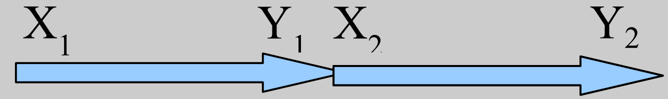


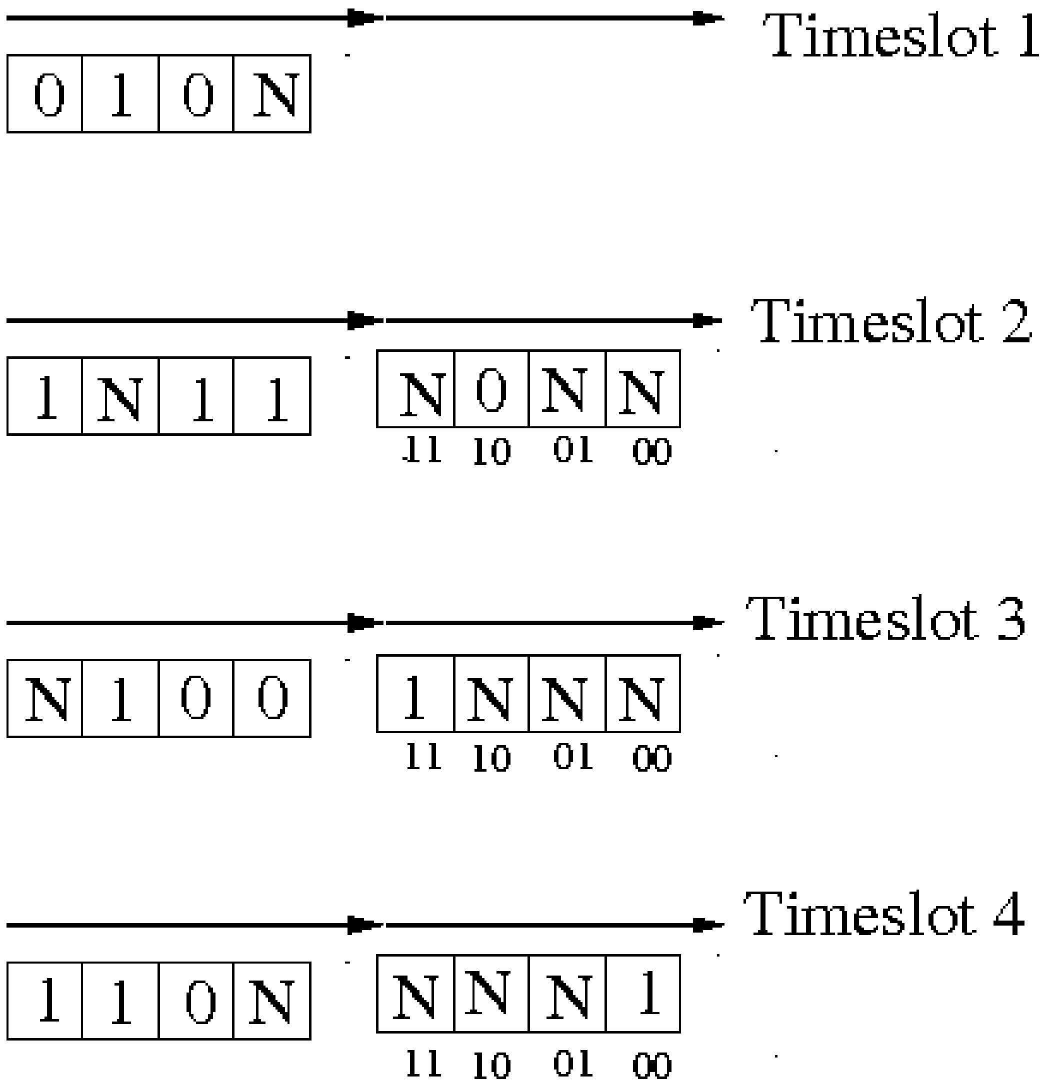
A simple (?) question





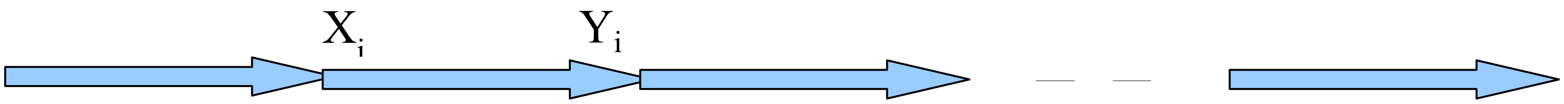
A simple (?) question





This strategy achieves 0.75 bit per channel use while respecting the half-duplex constraint. The crucial step is to allow memory in X1 and X2. Allowing for more memory eventually hits a limit of 1.138 bit per channel use.

More results........

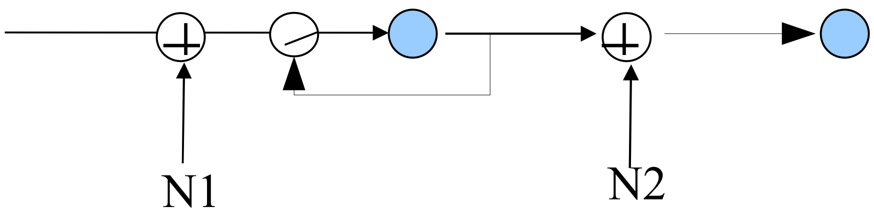


For a chain of links with half-duplex constraints the capacity converges to 1 bit per channel use.

The dependencies of all link inputs have to becarefully balanced in order to achieve this.

More results........

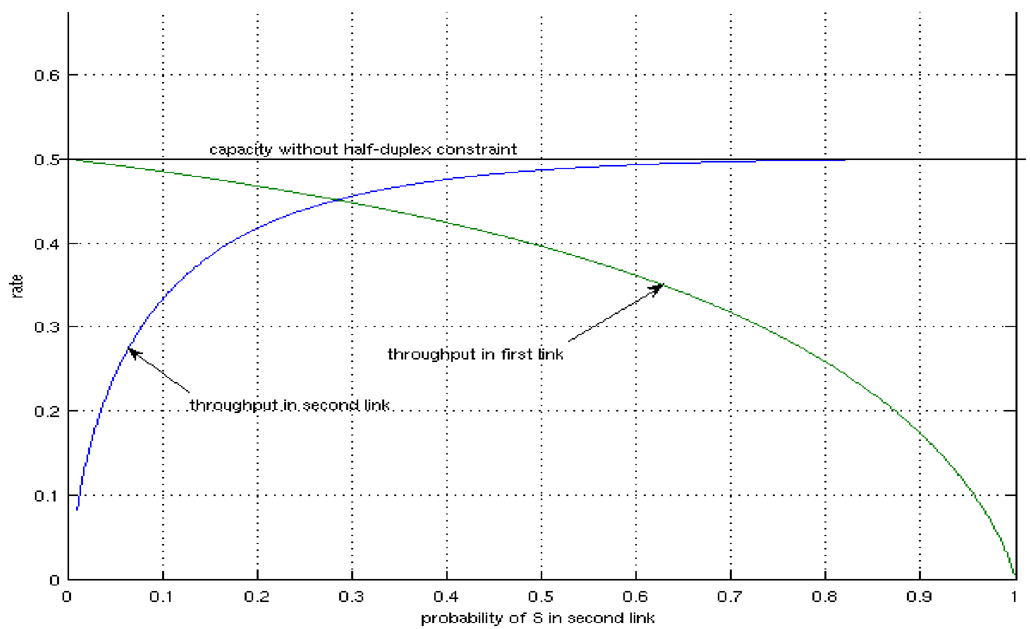
For a gaussian half-duplex channel (N0<<P2):



$$Y_1 = \begin{cases} X_1 + N_1 & X_2 = NS \\ X_2 + N_0 & \text{otherwise} \end{cases}$$

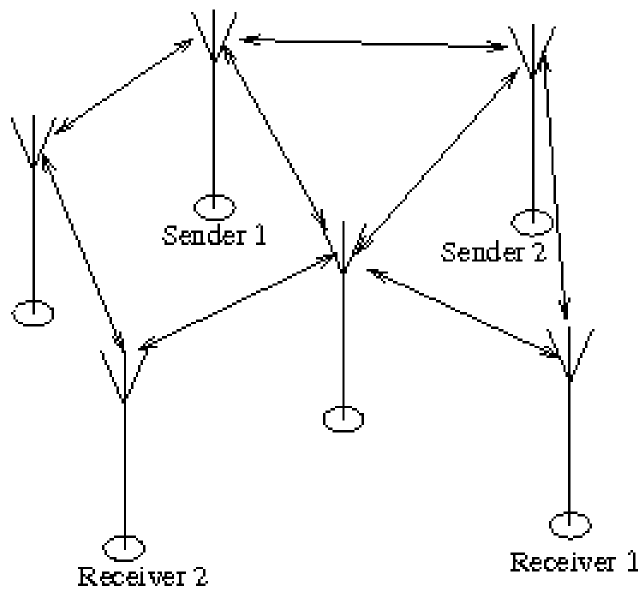
Capacity: ½ $E[1-T](\log(1+SNR_1/E[1-T]))$ where T is a Bernoulli r.v. satisfying:

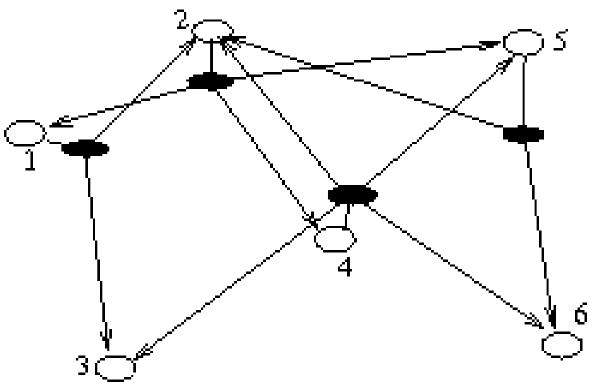
$$\frac{1}{2}(1-E[T])(log(1+SNR_1/(1-E[T]))) = \max_{f_X : E[X^2]/E[T]=P_2} h(N_2 + EX)$$



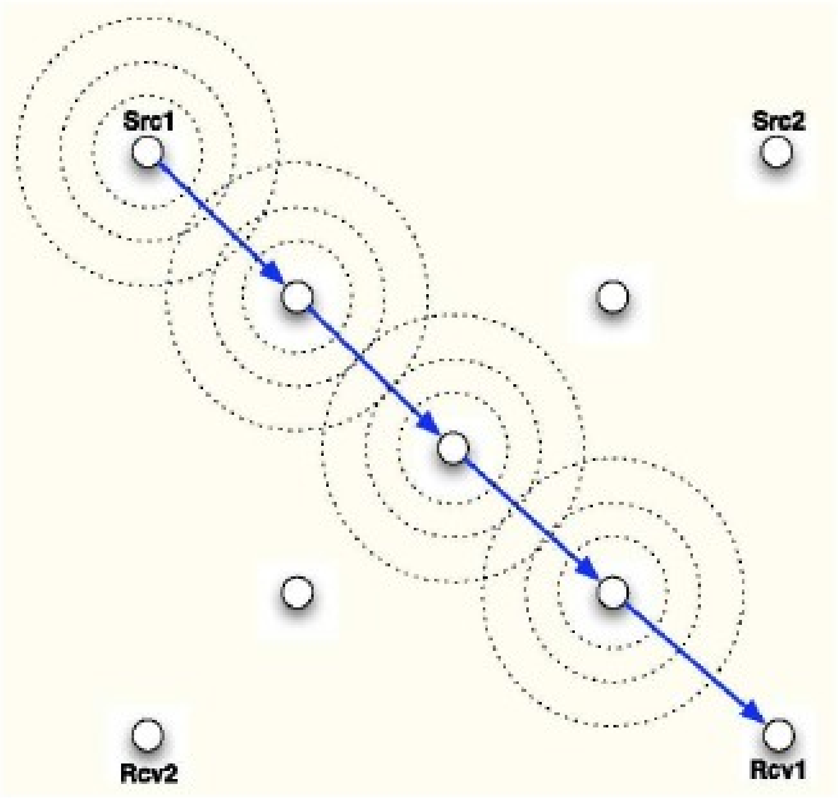
The wireless butterfly (incarnation II)

Half-duplex and interference constraint

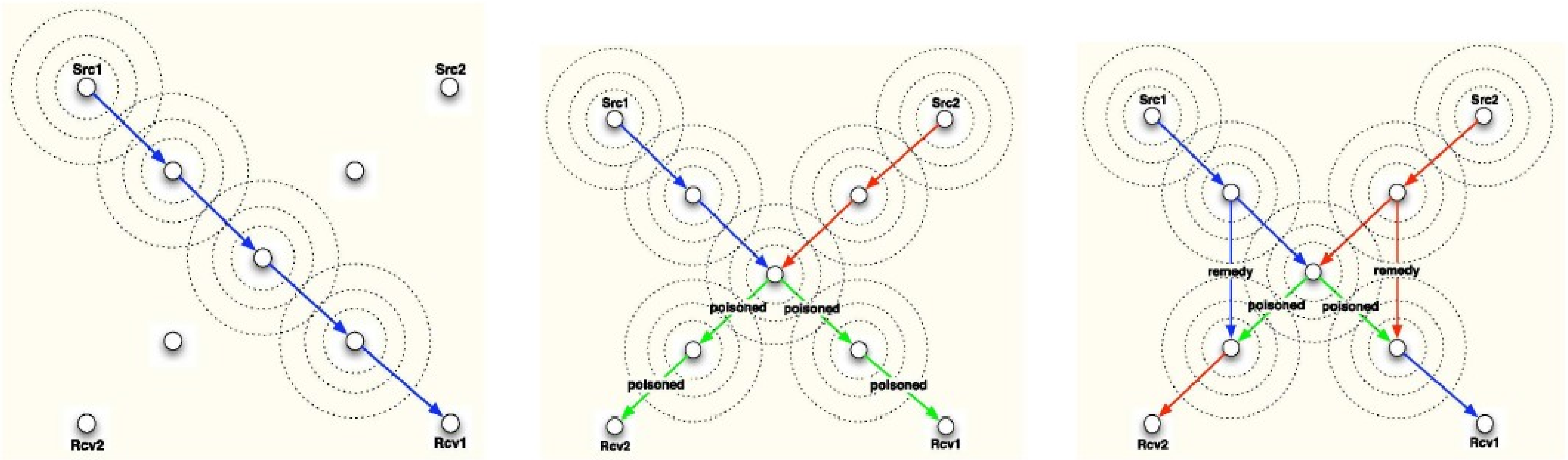




Without network coding and time-sharing: 0.25 bits/(unit time x connections)
With network coding and time sharing: 0.33 bits/(unit time x connections)
With network coding and half-duplex constraint: $\geq$ 0.57 bits/(unit time x connections)



The wireless situation



The wireless situation — congested areas in the network are better utilized: packets use "car pooling"