Fundamentals and recent advances in vehicle platooning control

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Lecture content

1. Main Concepts in ACC
2. Control design in vehicle following mode
   1. requirements
   2. constant spacing
   3. constant time gap
3. Spacing error propagation in constant spacing policies
   1. Predecessor following control scheme
   2. Predecessor and leader following control scheme
4. Disturbance propagation
5. String stability in heterogenous platoons
   1. Predecessor following control scheme
   2. Predecessor and leader following control scheme

Note: The following notes have been extracted from

- "Vehicle Dynamics and Control" by R. Rajamani
- "Disturbance Propagation in Vehicle Strings", P. Seiler, A. Pant, K. Hendrick. *IEEE Transactions on Automatic Control, vol. 49, no. 10, October 2004*
- "Controller Design for String Stable Heterogeneous Vehicle Strings", E. Shaw, K. Hendrick. *46$^{\text{th}}$ IEEE Conference on Decision and Control, 2007*

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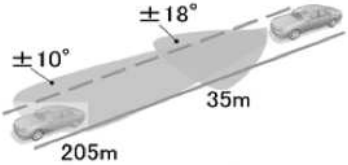
Main concepts and intro

ACC objectives

1. Maintaining a constant vehicle longitudinal speed in absence of preceding vehicles
2. Maintaing a “safe” distance from the preceding (slower) vehicle, if any

Actuators

1. Engine
2. Brakes



Sensors

1. Speed sensor (odometer)
2. Radar
    - Range through reflections
    - Range rate through doppler effect

Note. The ACC is an “autonomous” system. I.e., no wireless

Main concepts and intro

- First introduced in Japan in early nineties
- Originally thought as a "comfort and convenience" system
- According statistics (over 90% highways accident cause by human errors[1]) may impact safety as well
- Basis of many automated driving systems available on the market

[1] US Department of Transportation, 1992

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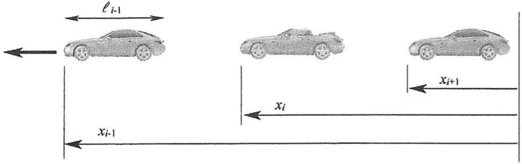
Vehicle following. Control requirements

Individual vehicle stability

String stability

The *string stability* property implies that, during velocity transients, the non-zero spacing errors do not amplify toward the tail of a string of ACC vehicles[a]

[a] Swaroop, 1995, Swaroop and Hedrick, 1996



Individual vehicle stability is trivial. ***We will focus on string stability***

Vehicle following. Vehicle model

Assumptions

- Two level hierarchical control
- Upper level calculates a desired acceleration to meet the control requirements
- Lower level calculates the engine and brake low level control inputs

Hence, model the i-th vehicle as either a double integrator

$$\ddot{x}_i = u_i,$$

or as

$$\ddot{x}_i = \frac{e^{-s\tau}}{a + sT} u_i,$$

where $u_i = \ddot{x}_{i_{des}}$. Typically,

$$-5m/s^2 \leq \ddot{x} \leq 2m/s^2$$

Vehicle following. String stability

Definition

Define

$$H(s) = \frac{\delta_i(s)}{\delta_{i-1}(s)}.$$

The chain of ACC vehicles is string stable if[a]

1. $\|H(s)\|_\infty \leq 1$
2. $h(t) > 0, \ \forall t \geq 0$

---

[a] Swaroop, 1995

Intuitively,

1. Condition 1 guarantees that $\|\delta_i\|_2 \leq \|\delta_{i-1}\|_2$
2. Condition 2 implies that the *steady state* spacing errors have the same sign

More rigorous explanation follows

...short detour to norms for signals and systems

Definitions (signals)

Consider a signal $u(t): \ t \in [-\infty, \infty] \rightarrow u \in \mathbb{R}$. Define the following norms

1. **1-Norm** $\|u\|_1 = \int_{-\infty}^{\infty} |u(t)| dt$
2. **2-Norm** $\|u\|_2 = \left(\int_{-\infty}^{\infty} u(t)^2 dt \right)^{1/2}$
3. **∞-Norm** $\|u\|_\infty = \sup_t |u(t)|$

Definitions (systems)

Consider a linear, time-invariant, causal system $y = g * u$, where g is the impulse response and $G = \mathscr{L}(g)$

1. **2-Norm** $\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega \right)^{1/2}$
2. **∞-Norm** $\|G\|_\infty = \sup_\omega |G(j\omega)|$

Useful results on gains[2]

2-norm/2-norm gain

Consider the system $y = g * u$, with $G = \mathscr{L}(g)$.

$$\|G\|_\infty = \sup \frac{\|y\|_2}{\|u\|_2}$$

Proof.

By the Parseval's theorem

$$\begin{aligned}
\|y\|_2^2 &= \|Y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 |U(j\omega)|^2 d\omega \\
&\leq \|G\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(j\omega)|^2 d\omega \\
&= \|G\|_\infty^2 \|U\|_2^2 = \|G\|_\infty^2 \|u\|_2^2
\end{aligned}$$

Show now that $\|G\|_\infty$ is the least upper bound on the 2-norm/2-norm gain.
Choose u such that $\|u\|_2 = 1$ and show that $\|Y\|_2^2 = \|G\|_\infty^2$ $\square$

Useful results on gains

2-norm/∞-norm gain

Consider the system $y = g * u$, with $G = \mathscr{L}(g)$.

$$\|G\|_2 = \sup \frac{\|y\|_\infty}{\|u\|_2}$$

Proof.

Apply the Cauchy-Schwarz inequality

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau \right| \\ &\le \left(\int_{-\infty}^{\infty} g(t-\tau)^2 d\tau \right)^{1/2} \left(\int_{-\infty}^{\infty} u(\tau)^2 d\tau \right)^{1/2} \\ &= \|g\|_2 \|u\|_2 = \|G\|_2 \|u\|_2 \end{aligned}$$

Hence $\|y\|_\infty \le \|G\|_2 \|u\|_2$
Proof follows the same steps as before $\square$

Useful results on gains

∞-norm gain/2-norm

Consider the system $y = g * u$, with $G = \mathscr{L}(g)$.

$$\frac{\|y\|_2}{\|u\|_\infty} = \infty$$

Proof.

Choose a sinusoidal input signal with frequency ω, such that ω is not a zero of G. Hence $\|u\|_\infty = 1$ and $\|y\|_2^2$ is unbounded $\square$

Useful results on gains

∞-norm/∞-norm gain

Consider the system $y = g * u$, with $G = \mathscr{L}(g)$.

$$\|g\|_1 = \sup \frac{\|y\|_\infty}{\|u\|_\infty}$$

Proof.

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau \right| \leq \int_{-\infty}^{\infty} |g(t-\tau)u(\tau)|\, d\tau \\ &\leq \int_{-\infty}^{\infty} |g(t-\tau)|\, d\tau \|u\|_\infty = \|g\|_1 \|u\|_\infty \end{aligned}$$

Hence $\|y\|_\infty \leq \|g\|_1 \|u\|_\infty$

Proof follows the same steps as before □

Useful results on gains[3]

If $g(t) > 0 \; \forall t \geq 0$ then $\|g\|_1 = \|G\|_\infty$

Proof.

Be $\gamma_p = \sup \dfrac{\|y\|_p}{\|u\|_p}$ for a induced p-norm. Since $\dfrac{\|y\|_p}{\|u\|_p} \leq \|g\|_1$,

$$|G(0)| \leq \|G(j\omega)\|_\infty \leq \gamma_p \leq \|g\|_1.$$

If $g(t) > 0$ then

$$|G(0)| = \left| \int_0^\infty g(\tau) d\tau \right| \leq \int_0^\infty |g(\tau)| d\tau = \|g\|_1$$

□

[3] Swaroop, 1995

In summary

Table: System gains

	$\|u\|_2$	$\|u\|_\infty$
$\|y\|_2$	$\|G\|_\infty$	∞
$\|y\|_\infty$	$\|G\|_2$	$\|g\|_1$

Moreover, if $g(t) > 0 \; \forall t \geq 0$ then $\|g\|_1 = \|G\|_\infty$

Vehicle following. String stability

Definition

Define

$$H(s) = \frac{\delta_i(s)}{\delta_{i-1}(s)}.$$

The chain of ACC vehicles is string stable if[a]

1. $\|H(s)\|_\infty \leq 1$
2. $h(t) > 0, \ \forall t \geq 0$

[a] Swaroop, 1995

The main objective is to obtain

$$\|\delta_i\|_\infty \leq \|\delta_{i-1}\|_\infty,$$

i.e., $\|h\|_1 \leq 1$. This is equivalent to $\|H\|_\infty \leq 1$, with the additional condition $h(t) > 0, \ \forall t \geq 0$.

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Constant spacing control design

Define the *inter-vehicle spacing* as

$$\epsilon_i = x_i - x_{i-1} + \ell_{i-1},$$

where ℓ_{i-1} is the length of the $(i-1)$-th vehicle.

Define the *spacing error* as

$$\delta_i = x_i - x_{i-1} + L_{des},$$

where L_{des} is the desired distance and includes ℓ_{i-1}.

Consider a double integrator model for the vehicle and a *linear PD controller*

$$\ddot{x}_i = -k_p \delta_i - k_v \dot{\delta}_i$$

Constant spacing control design

Differentiate twice the spacing error

$$\ddot{\delta}_i = \ddot{x}_i - \ddot{x}_{i-1} = -k_p\delta_i - k_v\dot{\delta}_i + k_p\delta_{i-1} + k_v\dot{\delta}_{i-1}$$

Rearranging leads to the closed-loop error dynamics

$$\ddot{\delta}_i + k_v\dot{\delta}_i + k_p\delta_i = k_p\delta_{i-1} + k_v\dot{\delta}_{i-1},$$

corresponding to the transfer function

$$H(s) = \frac{\delta_i(s)}{\delta_{i-1}(s)} = \frac{k_p + k_v s}{s^2 + k_v s + k_p}$$

Problem. Find $k_p,\ k_v$ such that

$$\|H\|_\infty \leq 1$$

Constant spacing control design

Solution. For individual vehicle stability, it must be $k_v,\ k_p > 0$.

Rewrite $H(s)$ as

$$H(s) = \underbrace{\frac{k_p}{s^2 + k_v s + k_p}}_{H_1(s)} \underbrace{\left(\frac{k_v}{k_p} s + 1\right)}_{H_2(s)}$$

In order to have $\|H_1\|_\infty < 1$, the damping must be larger than 0.707, i.e.,

$$\frac{k_v}{2\sqrt{k_p}} \geq 0.707 \ \Rightarrow\ k_v \geq 1.4141\sqrt{k_p}$$

H_2 has to be below one up to the resonant frequency $\sqrt{k_p}$. Hence,

$$\frac{k_p}{k_v} \geq \sqrt{k_p} \ \Rightarrow\ \sqrt{k_p} > k_v$$

Constant spacing control design

Solution. In conclusion, the following conditions have to be satisfied

$$k_v \geq 1.4141\sqrt{k_p},\ \sqrt{k_p} > k_v,\ k_p, k_v > 0$$

String stability can't be achieved with a PD controller based on constant spacing policy

Question. Can string stability be achieved with any other ***linear controller***?

Answer. No, unless. . . .

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Constant time gap control design

In Constant Time Gap (CTG) control policy, the desired inter-vehicle distance varies with the speed

Define the *spacing error* as

$$\delta_i = x_i - x_{i-1} + L_{des},$$

where $L_{des} = \ell_{i-1} + h\dot{x}_i$ and h is the time gap

Consider a double integrator model for the vehicle and the control law

$$u_i = -\frac{1}{h}\left(\dot{\epsilon}_i + \lambda\delta_i\right) [4]$$

The error dynamics become

$$\dot{\delta}_i = -\lambda\delta_i$$

[4] Chien, 1993

Constant time gap control design

Analyze the string stability property of the CTG policy

Combine the first order vehicle model and the control law $u_i = -\frac{1}{h}\left(\dot{\epsilon}_i + \lambda\delta_i\right)$. Obtain

$$\tau \dddot{x}_i + \ddot{x}_i = -\frac{1}{h}\left(\dot{\epsilon}_i + \lambda\delta_i\right)$$

Differentiate twice the spacing error $\delta_i = \epsilon_i + h\dot{x}_i$ and replace $\dddot{x}_i$ to obtain

$$\ddot{\epsilon}_i = \ddot{\delta}_i + \frac{1}{\tau}\left(\dot{\delta}_i + \lambda\delta_i\right)$$

Solve for ϵ_i and replace in $\delta_i - \delta_{i-1} = \epsilon_i - \epsilon_{i-1} + h\dot{\epsilon}_i$ to obtain

$$H(s) = \frac{\delta_i}{\delta_{i-1}} = \frac{s + \lambda}{h\tau s^3 + hs^2 + (1 + \lambda h) + \lambda}$$

Problem. Find condition on τ and h such that $\|H\|_\infty \leq 1$

Constant time gap control design

Theorem

Proof

Consider the transfer function

$$H(s) = \frac{\delta_i}{\delta_{i-1}} = \frac{s+\lambda}{h\tau s^3 + hs^2 + (1+\lambda h) + \lambda}$$

Substitute $s = j\omega$

$$H(s)\,|_{s=j\omega} = \frac{j\omega+\lambda}{(\lambda - h\omega^2) + j\omega\,(1+\lambda h - \tau h\omega^2)}$$

Calculate

$$|H(s)|^2 = \frac{\omega^2+\lambda^2}{\left(\lambda - h\omega^2\right)^2 + \omega^2\left(1+\lambda h - \tau h\omega^2\right)^2}$$

Constant time gap control design

Proof (Cont.)

Imposing $|H(j\omega)| \leq 1$ leads to

$$\omega^2 + \lambda^2 \leq \left(\lambda - h\omega^2\right)^2 + \omega^2\left(1 + \lambda h - \tau h \omega^2\right)^2$$

Squaring the terms in parentheses and rearranging

$$\tau^2 h^2 \omega^4 + \left(h^2 - 2\tau h - 2\tau\lambda h^2\right)\omega^2 + \lambda^2 h^2 \geq 0$$

Study positiveness of $a\omega^4 + b\omega^2 + c$. Rewrite

$$\begin{aligned} a\omega^4 + b\omega^2 + c &= a\left(\omega^4 + 2\frac{b}{2a}\omega^2 + \frac{c}{a}\right) \\ &= a\left[\left(\omega^2 + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \end{aligned}$$

Constant time gap control design

Proof (Cont.)

Hence $a\omega^4 + b\omega^2 + c > 0$ if

1. $a,\ b,\ c > 0$
2. $b < 0,\ a > 0,\ c > 0$ and $4ac - b^2 > 0$, i.e., $b^2 - 4ac < 0$

Distinguish the following two cases

1. $b > 0$ corresponds to $h^2 - 2\tau h - 2\lambda\tau h^2 > 0$. Hence

   $$h > \frac{2\tau}{1 - 2\lambda\tau}.$$

   For small λ, this is possible if $h > 2\tau$.
2. $b < 0,\ a > 0,\ c > 0$ and $b^2 - 4ac < 0$ corresponds to

   $$\left(h^2 - 2\tau - 2\lambda\tau h^2\right)^2 - 4\tau^2 h^4 \lambda^2$$

Constant time gap control design

Proof (Cont.)

2. Simplify to obtain

$$
\begin{aligned}
\lambda &< \frac{4\tau h - h^2 - 4\tau^2}{8\tau^2 h - 4\tau h^2}, \\
\lambda &< \frac{-\left(2\tau - h\right)^2}{4\tau h\left(2\tau - h\right)}.
\end{aligned}
$$

   Since $\lambda > 0$, it must be $h > 2\tau$.

By relaxing the inequality in $a\omega^4 + b\omega^2 + c > 0$, $h \geq 2\tau$ follows.

By 1) and 2) also follows that if $h \geq 2\tau$ a λ can be found such that $|H(j\omega)| < 1$. □

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Motivation and intro

Recall that for a string of ACC vehicles we have found out that

String stability can't be achieved with a PD controller based on constant spacing policy

...moreover

Question. Can string stability be achieved with any other ***linear controller***?

Answer. No, unless. . . .

The objective of this lecture is to analyze the string stability of vehicle convoys for any linear controller based on ***constant spacing policy***

Recall that string stability can be achieved with other spacing policy like, e.g., ***constant time gap***

Spacing error propagation in constant spacing policies

The results presented next are based on the following assumptions

Assumptions

1. Identical vehicles modeled by $G(s)$
2. $G(s)$ is linear, strictly proper, single-input-single-output and with two integrators
3. Identical control laws
4. Constant spacing policy

Hereafter, $G(s)$ can be assumed as follows

$$G(s) = \frac{X(s)}{U(s)} = \frac{1}{ms^2(1+s\tau)},$$

with $X(s) = \mathcal{L}(x(t))$ and $U(s) = \mathcal{L}(u(t))$

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Predecessor following control scheme

Each vehicle longitudinal motion can be modeled as

$$X_i(s) = H(s)\left(U_i(s) + D_i(s)\right) + \frac{x_i(0)}{s}, \quad 1 \leq i \leq N,$$

where $D_i(s)$ is an input disturbance and $x_i(0) = -iL_{des}$.

The *spacing error* is given by

$$\Delta_i(s) = X_{i-1}(s) - X_i(s) - \frac{L_{des}}{s}$$

Assume momentarily $D_i(s) = 0$ and a local feedback control law based on the spacing error w.r.t. the predecessor. I.e., $U_i(s) = K(s)\Delta_i(s)$.

Calculate $\Delta_1(s)$ (drop the s argument)

$$\Delta_1 \quad = \quad X_0 - X_1 - \frac{L_{des}}{s}$$

Predecessor following control scheme

calculation of Δ_1 *(cont.)*

$$\begin{aligned}
\Delta_1 &= X_0 - HU_1 + \frac{L_{des}}{s} - \frac{L_{des}}{s} \\
&= X_0 - HK\Delta_1 \\
&\Rightarrow \quad \Delta_1 = \underbrace{\frac{1}{1+HK}}_{S} X_0
\end{aligned}$$

Calculate Δ_i

$$\begin{aligned}
\Delta_i &= X_{i-1} - X_i - \frac{L_{des}}{s} \\
&= HK\Delta_{i-1} - (i-1)\frac{L_{des}}{s} - HK\Delta_i + i\frac{L_{des}}{s} - \frac{L_{des}}{s} \\
&= HK\Delta_{i-1} - HK\Delta_i \\
&\Rightarrow \quad \Delta_i = \underbrace{\frac{HK}{1+HK}}_{T} \Delta_{i-1}, \quad i = 1, \ldots, N
\end{aligned}$$

Predecessor following control scheme

Remarks

1. The transfer function from X_0 to Δ_1 is the *sensitivity function*
2. The transfer function from Δ_{i-1} to Δ_i is the *complementary sensitivity function*
3. We would like to have
   1. $|S(j\omega)|$ small for all frequencies (limiting the first spacing error)
   2. $|T(j\omega)|$ small for all frequencies (limiting the error propagation)
4. Classical trade-off between sensitivity and complementary sensitivity functions
5. Given a $K(s)$ stabilizing the closed-loop system, $H(s)K(s)$ has two poles in the origin.

   Hence, $T(0) = 1$ and $\|T\|_\infty \geq 1$
6. Actually $\|T\|_\infty > 1$ as shown next

Predecessor following control scheme

Theorem (analogous to Bode's integral formula)

Assume that the loop transfer function H(s)K(s) of a feedback system goes to zero faster than $1/s$ as $s \to \infty$ and let $T(s)$ be the complementary sensitivity function. The complementary sensitivity function satisfies the following integral

$$\int_0^\infty \frac{\ln|T(j\omega)|}{\omega^2} d\omega = \pi \sum \frac{1}{z_i},$$

where z_i are right half-plane zeros.

Observe that,

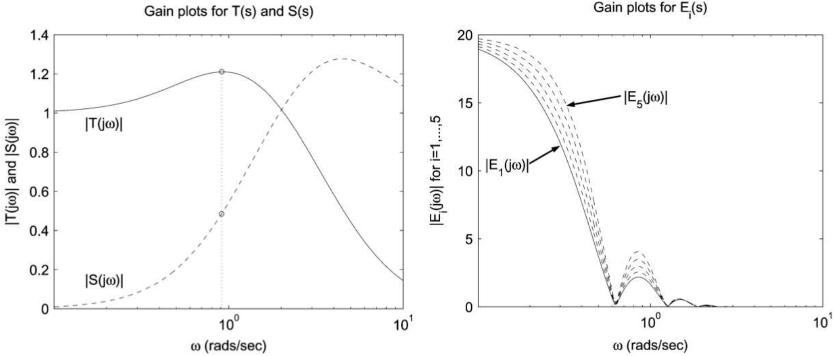
1. Since $H(s)$ is strictly proper, $|T(j\omega)| \to 0$ as $\omega \to \infty$. Hence, $\ln|T(j\omega)| < 0$ at high frequencies.
2. The theorem implies that $\ln|T(j\omega)| > 0$ for some frequency. Hence, $|T(j\omega)| > 1$

Example

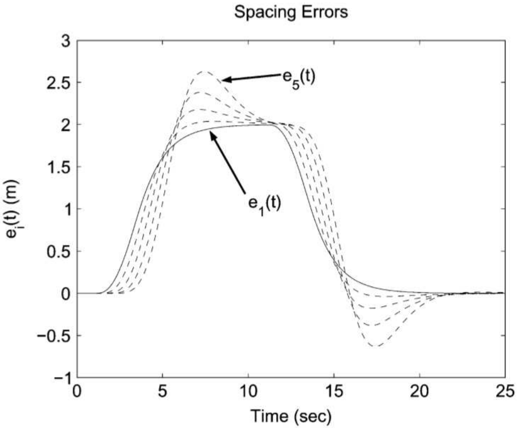
Consider the following vehicle model and controller

$$H(s) = \frac{1}{s^2(1+0.1s)}, \quad K(s) = \frac{1+2s}{1+0.05s}$$

Assume the lead vehicle accelerates from rest to 20 m/s over 12 s using the control input $U_0(s) = \frac{1}{s^2}\left(e^{-s} - e^{-3s} - e^{-11s} + e^{-13s}\right)$, corresponding to a trapezoidal input



Example (Cont.)



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Predecessor and leader following control scheme

Modify the local control strategy in order to add info from the leader

$$U_i(s) = K_P(s)\Delta_i(s) + K_l(s)\left(X_0(s) - X_i(s) - i\frac{L_{des}}{s}\right)$$

Calculate Δ_1

$$\begin{aligned}
\Delta_1 &= X_0 - X_1 - \frac{L_{des}}{s} \\
&= HU_1 - \frac{L_{des}}{s} + \frac{L_{des}}{s} - \frac{L_{des}}{s} \\
&= X_0 - HK_P\Delta_1 - HK_l\underbrace{\left(X_0 - X_1 - \frac{L_{des}}{s}\right)}_{\Delta_1} \\
&\Rightarrow \quad \Delta_1 = \underbrace{\frac{1}{1 + H\left(K_P + K_l\right)}}_{S_{lp}(s)} X_0
\end{aligned}$$

Predecessor and leader following control scheme

Calculate Δ_i

$$
\begin{aligned}
\Delta_i &= X_{i-1} - X_i - \frac{L_{des}}{s} \\
&= HU_{i-1} - (i-1)\frac{L_{des}}{s} - HU_i + i\frac{L_{des}}{s} - \frac{L_{des}}{s} \\
&= HK_P\Delta_{i-1} + HK_l\left[X_0 - X_{i-1} - (i-1)\frac{L_{des}}{s}\right] \\
&- HK_P\Delta_i - HK_l\left[X_0 - X_i - i\frac{L_{des}}{s}\right] \\
&= HK_P\Delta_{i-1} - HK_P\Delta_i - HK_l\underbrace{\left[X_{i-1} + (i-1)\frac{L_{des}}{s} - X_i - i\frac{L_{des}}{s}\right]}_{\Delta_i}
\end{aligned}
$$

$$
\Rightarrow \quad \Delta_i = \underbrace{\frac{HK_P}{1 + H\left(K_P + K_l\right)}}_{T_{lp}(s)}\Delta_{i-1}
$$

Predecessor and leader following control scheme

Clearly, K_P and K_l can now be easily designed in order to guarantee $\|T_{lp}\|_\infty < 1$.

For example, if we chose $K_P(s) = K_l(s)$ then $T_{lp}(0) = 0.5$.

Consider the case of the previous example, if we chose $K_P(s) = K_l(s) = 1/2K(s)$ then $T_{lp}(s) = 1/2T(s)$. In the predecessor following scheme, $\|T\|_\infty = 1.21$ while $\|T_{pl}\|_\infty = 0.605$.

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Disturbance propagation

Recall we have modeled each vehicle as

$$X_i(s) = H(s)\left(U_i(s) + D_i(s)\right) + \frac{x_i(0)}{s}, \quad 1 \le i \le N,$$

Now we analyze the effects of the disturbance D_i on the stability of the string

Calculate Δ_1 and Δ_i

$$\begin{aligned}
\Delta_1 &= X_0 - X_1 - \frac{L_{des}}{s} \\
&= X_0 - HD_1 - HU_1 + \frac{L_{des}}{s} - \frac{L_{des}}{s} \\
&\Rightarrow \underline{\Delta_1 = X_0 - HD_1 - HU_1}
\end{aligned}$$

$$\begin{aligned}
\Delta_i &= X_{i-1} - X_i - \frac{L_{des}}{s} \\
&= HD_{i-1} + HU_{i-1} - HD_i - HU_i - (i-1)\frac{L_{des}}{s} + i\frac{L_{des}}{s} - \frac{L_{des}}{s} \\
&\Rightarrow \underline{\Delta_i = HD_{i-1} - HD_i + HU_{i-1} - HU_i}
\end{aligned}$$

Disturbance propagation

Introduce the vectors

$$\bar{\Delta} = [\Delta_1, \ldots, \Delta_N]\,,\ \bar{D} = [D_1, \ldots, D_N]\,,\ \bar{U} = [U_1, \ldots, U_N]$$

The spacing error dynamics can be compactly rewritten as

$$\bar{\Delta} = P_{11} \begin{bmatrix} X_0 \\ \bar{D} \end{bmatrix} + P_{12}\bar{U},$$

with

$$P_{11} = \left[\begin{array}{c|ccccc} 1 & -H & & & & \\ 0 & H & -H & & & \\ \vdots & & \ddots & \ddots & & \\ 0 & & & H & -H & \end{array}\right],\ P_{12} = \begin{bmatrix} -H & & & \\ H & -H & & \\ & & \ddots & \ddots \\ & & H & -H \end{bmatrix}$$

Disturbance propagation

The control law $U_i(s) = K_P(s)\Delta_i(s) + K_l(s)\left(X_0(s) - X_i(s) - i\frac{L_{des}}{s}\right)$ can be rewritten as

$$\begin{aligned} U_i(s) &= K_P(s)\Delta_i(s) + K_l(s)\left(X_0(s) - X_1(s) - \frac{L_{des}}{s}\right. \\ &+ \left. X_1(s) - X_2(s) - i\frac{L_{des}}{s} \ldots + X_{i-1}(s) - X_i(s) - i\frac{L_{des}}{s}\right) \\ &= K_P(s)\Delta_i(s) + K_l(s)\sum_{k=1}^{i}\Delta_k \end{aligned}$$

Hence, $\bar{U} = \bar{K}\bar{\Delta}$ with

$$\bar{K} = \begin{bmatrix} K_l + K_P & & & \\ K_l & K_l + K_P & & \\ \vdots & \ddots & \ddots & \\ K_l & \ldots & K_l & K_l + K_P \end{bmatrix}$$

Disturbance propagation

By substituting $\bar{U}$ in $\bar{\Delta} = P_{11} \begin{bmatrix} X_0 \\ \bar{D} \end{bmatrix} + P_{12}\bar{U}$, write the error vector as

$$\bar{\Delta} = \begin{bmatrix} 1 \\ T_{lp} \\ \vdots \\ T_{lp}^{N-1} \end{bmatrix} S_{lp} X_0 - S_{lp} H \begin{bmatrix} 1 & & & \\ T_{lp} - 1 & 1 & & \\ \vdots & \ddots & \ddots & \\ (T_{lp} - 1)\, T_{lp}^{N-2} & \dots & T_{lp} - 1 & 1 \end{bmatrix} \bar{D}$$

Contribution from the first term falls in the analysis done for the *leader and predecessor following scheme.* Focus on the second term

Disturbance propagation

Consider the matrix

$$\begin{bmatrix} 1 & & & \\ T_{lp}-1 & 1 & & \\ \vdots & \ddots & \ddots & \\ (T_{lp}-1)\,T_{lp}^{N-2} & \dots & T_{lp}-1 & 1 \end{bmatrix}$$

Recall that $T_{lp} = \dfrac{HK_P}{1+H\,(K_P+K_l)}$

Distinguish the following two cases

1. $K_P = 0$. I.e., only the information from the leader is used. In this case $T_{lp} = 0$. There is not disturbance propagation and D_i affects Δ_{i+1} through $S_{lp}H$
2. $K_l = 0$. I.e., only the information from the preceding vehicle is used. We already know that $|T_{lp}(j\omega)| > 1$ for some ω. Moreover, disturbances propagate and D_i affects Δ_{i+1} through $(T_{lp}-1)\,S_{lp}H$. On the other hand $|\,(T_{lp}-1)\,| \ll 0$ at low frequencies and the disturbances are attenuated.

Disturbance propagation

In conclusion

1. With leader information only there is not disturbance propagation.
2. Information from the predecessor reduces the effect of the disturbance on the spacing error.

More rigorously,

Theorem

Assume H has two poles in the origin and the closed loop is stable. Be $\bar{G}_{d\delta}(s)$ the transfer matrix from the disturbance to the spacing errors.

1. If $\|T_{lp}\|_\infty > 1$, then given any $M > 0$, there exists a N (platoon length) such that $\|\bar{G}_{d\delta}\|_\infty > M$
2. If $\|T_{lp}\|_\infty < 1$, then there exists a $M > 0$, such that $\|\bar{G}_{d\delta}\|_\infty \leq M$, $\forall N$

Disturbance propagation

Remarks on the Theorem

1. If $\|T_{lp}\|_\infty > 1$ (e.g., in a predecessor following scheme) the gain from the disturbance to the spacing errors grows with the platoon length
2. If $\|T_{lp}\|_\infty < 1$ (e.g., in a predecessor and leader following scheme) the gain from the disturbance to the spacing errors is bounded as the platoon length increases

The predecessor and leader following scheme is scalable. Nevertheless *communication is needed.*

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String stability in heterogenous platoons

The results presented so far are based on the following assumptions

Assumptions

1. **Identical vehicles modeled by $G(s)$**
2. $G(s)$ is linear, strictly proper, single-input-single-output and with two integrators
3. **Identical control laws**
4. Constant spacing policy

We now remove Assumptions 1 and 3

For simplicity, we assume the platoon is made of three types of vehicles, with *slow*, *medium* and *fast* dynamics

Objective: analyze the string stability of the predecessor and the predecessor-leader following schemes

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Heterogenous platoons. Predecessor following scheme

Calculate Δ_1 and Δ_i

$$
\begin{aligned}
\Delta_1 &= X_0 - X_1 - \frac{L_{des}}{s} \\
&= X_0 - H_1 U_1 + \frac{L_{des}}{s} - \frac{L_{des}}{s} = X_0 - H_1 K_1 \Delta_1 \\
&\Rightarrow \Delta_1 = \underbrace{\frac{1}{1 + H_1 K_1}}_{S_1} X_0
\end{aligned}
$$

$$
\begin{aligned}
\Delta_i &= X_{i-1} - X_i - \frac{L_{des}}{s} \\
&= H_{i-1} K_{i-1} \Delta_{i-1} - H_i K_i \Delta_i - (i-1)\frac{L_{des}}{s} + i\frac{L_{des}}{s} - \frac{L_{des}}{s} \\
&\Rightarrow \Delta_i = \underbrace{\frac{H_{i-1} K_{i-1}}{1 + H_i K_i}}_{\tilde{T}_{i-1}} \Delta_{i-1},\ i = 1, \ldots, N
\end{aligned}
$$

Heterogenous platoons. Predecessor following scheme

From

$$\Delta_1 = \underbrace{\frac{1}{1+H_1K_1}}_{S_1} X_0,\ \ \Delta_i = \underbrace{\frac{H_{i-1}K_{i-1}}{1+H_iK_i}}_{\tilde{T}_{i-1}} \Delta_{i-1},\ \ i = 1,\ldots,N,$$

rewrite the error dynamics as

$$\begin{aligned}
\Delta_i &= \frac{H_{i-1}K_{i-1}}{1+H_iK_i}\Delta_{i-1} \\
&= \frac{H_{i-1}K_{i-1}}{1+H_iK_i}\cdot\frac{H_{i-2}K_{i-2}}{1+H_{i-1}K_{i-1}}\Delta_{i-2} \\
&= \frac{H_{i-1}K_{i-1}}{1+H_iK_i}\cdot\frac{H_{i-2}K_{i-2}}{1+H_{i-1}K_{i-1}}\cdot\ldots\cdot\frac{1}{1+H_1K_1} \\
&= \frac{1}{1+H_iK_i}\cdot\frac{H_{i-1}K_{i-1}}{1+H_{i-1}K_{i-1}}\cdot\ldots\cdot\frac{H_1K_1}{1+H_1K_1}X_0 \\
&= S_i\left(\prod_{k=1}^{i-1} T_k\right) X_0
\end{aligned}$$

Heterogenous platoons. Predecessor following scheme

In

$$\Delta_i = S_i \left(\prod_{k=1}^{i-1} T_k \right) X_0,$$

S_i and T_i are the sensitivity and complementary sensitivity functions of the i-th vehicle.

Recall that $\|T_i\|_\infty > 1$. Hence $\| \prod_{k=1}^{i-1} T_k \|_\infty$ is unbounded as N grows

More in details, from

$$\Delta_i = \frac{H_{i-1}K_{i-1}}{1 + H_i K_i} \Delta_{i-1},$$

it follows that

1. if the preceding vehicle has faster dynamics $\|\tilde{T}_i\|_\infty > 1$ and the error will grow
2. if the preceding vehicle has slower dynamics $\|\tilde{T}_i\|_\infty < 1$ and the error will shrink

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Heterogenous platoons. Predecessor-leader following

Consider the control law

$$U_i(s) = K_{P,i}(s)\Delta_i(s) + K_{l,i}(s)\left(X_0(s) - X_i(s) - i\frac{L_{des}}{s}\right),$$

where $K_{P,i} = p_i K_i$ and $K_{l,i} = l_i K_i$, K_i is the low level controller of the i-th vehicle, $p_i,\ l_i$ are positive scalar smaller than 1 and such that $p_i + l_i = 1$

Calculate Δ_1 and Δ_i

$$\Delta_1 = \underbrace{\frac{1}{1 + H_1 K_1}}_{S_1} X_0$$

$$\Delta_i = \hat{T}_{i-1}\Delta_{i-1} + A_{i-1}\sum_{k=1}^{i-1} E_k,\ i = 2, \ldots, N$$

Heterogenous platoons. Predecessor-leader following

$\hat{T}_i$ and A_i are defined as

$$\hat{T}_i = \frac{H_i K_{P,i}}{1 + H_{i+1}\left(K_{P,i+1} + K_{l,i+1}\right)}$$

$$A_i = \frac{H_i K_{l,i} - H_{i+1} K_{l,i+1}}{1 + H_{i+1}\left(K_{P,i+1} + K_{l,i+1}\right)}$$

We have seen that heterogenous string stable (unstable) platoons do not attenuate (amplify) the spacing errors uniformly (i.e., $\|\delta_i\|_\infty < \|\delta_{i-1}\|_\infty$)

Hence, we have to define a stable string of heterogenous vehicles

Definition

A heterogenous vehicle string is string stable if the propagating errors stay uniformly bounded for all string lengths and vehicle type orderings

Heterogenous platoons. Predecessor-leader following

Define the transfer functions

$$\begin{aligned} T_{p,x} &= \frac{H_x K_{P,x}}{1 + H_x\left(K_{P,x} + K_{l,x}\right)}, \\ T_{l,x} &= \frac{H_x K_{l,x}}{1 + H_x\left(K_{P,x} + K_{l,x}\right)}, \\ & \qquad\qquad\qquad x \in \{s, m, f\} \end{aligned}$$

Rewrite the error dynamics as

$$\begin{aligned} \Delta_i &= F_i X_0 \\ F_1 &= 1 - T_{p,x_1} - T_{l,x_1}, \\ F_i &= \prod_{j=1}^{i-1} T_{p,x_j} - \prod_{j=1}^{i} T_{p,x_j} \\ &+ \left(1 - T_{p,x_i}\right)\left(\left(\sum_{k=1}^{i-2}\left(\prod_{j=k+1}^{i-1} T_{p,x_j}\right) T_{l,x_k}\right) + T_{l,x_{i-1}}\right) - T_{l,x_i} \end{aligned}$$

Heterogenous platoons. Predecessor-leader following

Remark

Clearly, if there exists a finite $M > 0$ such that $\|F_i\|_\infty \leq M \; \forall \; i$ and vehicle type orderings, then $\|\Delta_i\|_\infty$ stays uniformly bounded for all i and vehicle type orderings. I.e., the heterogenous platoon is string stable according to the definition

The following result holds

Theorem

Assume $H(s)$ has two poles in the origin ad the closed-loop is stable. There exists a finite $M > 0$ such that $\|F_i\|_\infty \leq M \; \forall \; i$ and vehicle type orderings *if and only if* $\|T_{p,x}\|_\infty < 1$ for all vehicle types.

Heterogenous platoons. Predecessor-leader following

Guidelines for controller design

1. Design the local controller for each vehicle such that $\|T_{p,x}\|_\infty < 1$
2. In order to bound the errors in such a way controllers do not have to be redesigned when vehicle ordering and/or platoon length change a conservative design is needed
3. In case of homogenous platoon, the largest error is the first $\Delta_1 = S_1 X_0$. The largest possible Δ_1 is when the first follower is a *slow* vehicle. In this case $|S_1|$ is the largest at low frequencies.
4. The local controllers can then be designed in order to achieve $\|e_i\|_\infty \leq \|e_1\|_\infty, \ \forall \ i$