TWO-DIMENSIONAL FILTERS: NEW ASPECTS OF THE REALIZATION THEORY

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Abstract

The input-output behaviour of a two-dimensional linear filter is defined by a formal power series in two variables. If the power series is rational, the dynamics of the filter is described by updating equations on finite dimensional local state spaces. The class of realizations considered in this paper is constituted by double indexed dynamical systems of reduced structure. The notions of local reachability and observability are defined in a natural way and an algorithm for obtaining a reachable and observable realization is given.

The minimality of the realizations is not guarantee by reachability and observability. In general the dimension of minimal realizations depends on the ground field and does not coincide with the rank of the Hankel matrix. Nevertheless the dimension of a minimal realization is the least rank in a family of Hankel matrices.

I. Introduction

Spatial filters [1-6] are extensively used in processing two-dimensional sampled data, such seismic data sections, digitized photographic data, and gravitational and magnetic maps.

The algebraic realization theory of spatial filters has been formulated by the authors in some published papers[7,8,11]. In this contribution we will derive additional results mainly with regard to a reduced structure of the updating equation for the local states. We shall make use of examples to evidentiate some interesting aspects of reachability, observability and minimality properties of filter realizations.

2. Realization of two-dimensional filters

We will consider two-dimensional digital filters with scalar inputs and outputs taken from an arbitrary field K. The input-output representation of such a filter is given by

$$\mathscr{S} \triangleq (T,\ U,\mathscr{U},\ Y,\mathscr{Y},\ F) \tag{1}$$

where $T = \mathbb{Z}\times\mathbb{Z}$ (partially ordered by the product of the orderings) is the discrete plane, U and Y are one-dimensional vector spaces over the field K, $\mathscr{U}$ and $\mathscr{Y}$ are the space of truncated formal Laurent series in two variables over K (whose precise description will be given below), and $F:\mathscr{U}\to\mathscr{Y}$ is the input-output map.

A typical element of $\mathscr{U}$ or $\mathscr{Y}$ will be written

$$r = \sum_{-k}^{\infty}{}_{i,j}(r,z_1^iz_2^j)z_1^iz_2^j, \qquad \text{for some integer } k$$

where $(r,z_1^iz_2^j)$ denotes the coefficient of $z_1^iz_2^j$.

The input-output map $F:\mathscr{U}\to\mathscr{Y}$ is assumed to satisfy the following axioms:

i) linearity

ii) two dimensional shift invariance:

$$F(z_1^iz_2^j) = z_1^iz_2^j\,F(r), \qquad i,j\in\mathbb{Z}$$

iii) two dimensional strict causality:

$$(u_1,z_1^iz_2^j) = (u_2,z_1^iz_2^j), \qquad i<t_1,\ j<t_2$$

implies:

$$(Fu_1,z_1^iz_2^j) = (Fu_2,z_1^iz_2^j),\ i\le t_1,\ j\le t_2,\ \forall u_1,\ u_2\in\mathscr{U}$$

Under assumption iii) it is easy to verify that the impulse response F(1) is a "strictly causal" power series, i.e..

$$F(1) = \sum_{1}^{\infty}{}_{ij}\,(F(1),\ z_1^iz_2^j)z_1^iz_2^j$$

More formally we can say that:

$$s \triangleq F(1)\in(z_1z_2)K[[z_1,z_2]] \triangleq K_c[[z_1,z_2]]$$

where $K[[z_1z_2]]$ denotes the ring of formal power series in two variables and $K_c[[z_1,z_2]]$ is the ideal of "strictly causal" power series.

From i) and ii) it follows that:

$$F(u) = su, \qquad \forall u\in\mathscr{U}$$

that is, two-dimensional filters (in their-output representation) are in one-to-one correspondence with formal power series $K_c[[z_1,z_2]]$.

Definition. A double indexed, linear, stationary, finite dimensional dynamical system Σ is defined by a pair of equations of the form:

$$x(h+1,\ k+1) = A_1x(h+1,k) + A_2x(h,k+1) + Bu(h,k) \tag{2}$$

$$y(h,k) = Cx(h,k)$$

where $A_i\in K^{n\times n}$, $i=1,2$, $C\in K^{1\times n}$, $B\in K^{n\times 1}$ and x belongs to some finite dimensional vector space $X = K^n$ (local state space).

The solution of equations (2) for $h\ge 0$, $k\ge 0$, is uniquely determined by u and by the values $x(h,0)$, $h = 1,2,\dots$, and $x(0,k)$, $k = 0,1,2,\dots$, (initial local states).

Remark. In a previous paper[7] we formulated the realization problem of spatial filters using Nerode equivalence and we obtained a local state space description in which the state $x(h+1,k+1)$ linearly depends on $x(h+1,k)$, $x(h,k+1)$ and $x(h,k)$. Then the associated updating equation has the following form:

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$$x(h+1,k+1) = A_o x(h,k) + A_1 x(h+1,k) + A_2 x(h,k+1) + Bu(h,k)$$

This kind of realization allowed us to analyze in a simple way the properties of filters characterized by recognizable power series.

Let now $x(h,0) = x(0,k) = 0$, $h,k = 0,1,\dots$ and associate the monomial $x(h,k)\, z_1^h z_2^k \in K^{n\times 1}[[z_1,z_2]]$ with the local state $x(h,k)$.

From (2) it follows that for each $u \in K[[z_1,z_2]]$

$$\Sigma_{o\,h,k}^{\infty} x(h,k) z_1^h z_2^k = A_1 (\Sigma_{o\,h,k}^{\infty} x(h,k) z_1^h z_2^k) z_1 + A_2 (\Sigma_{o\,h,k}^{\infty} x(h,k) z_1 z_2) z_2 + (z_1 z_2) Bu$$

and then

$$(I - A_1 z_1 - A_2 z_2)\Sigma_{h,k} x(h,k) z_1^h z_2^k = (z_1 z_2) Bu\ .$$

The po mial $(I - A_1 z_1 - A_2 z_2)$ belongs to $K^{n\times n}[z_1,z_2]$ and has an inverse in the ring of formal power series $K^{n\times n}$ z_1,z_2 Its inverse is given by:

$$(I - A_1 z_1 - A_2 z_2)^{-1} = \Sigma_{o\,i}^{\infty} (A_1 z_1 + A_2 z_2)^i$$

With the aid of this inverse we can relate the state to the input. In fact we have:

$$\Sigma_{h,k} x(h,k) z_1^h z_2^k = (I - A_1 z_1 - A_2 z_2)^{-1} (z_1 z_2) Bu$$

This yields at once the input-output relation:

$$y = C \sum_{h,k} x(h,k) z_1^h z_2^k = C(I - A_1 z_1 - A_2 z_2)^{-1} (z_1 z_2) Bu$$

The series

$$C(I - A_1 z_1 - A_2 z_2)^{-1} (z_1 z_2) B \qquad (3)$$

is called the transfer function of Σ.

We all that a formal power series $s \in K[[z_1 z_2]]$ is rational if there exist polynomials $p,q \in K[z_1^{-1}, z_2^{-1}]$ with $\deg p \leq \deg q$, such that $qs = p$. The polynomial q is called a denominator of s.

Then the transfer function $(z_1 z_2)\, C\, (I - A_1 z_1 - A_2 z_2)^{-1} B$ belongs to $(z_1 z_2) K[(z_1,z_2)] \triangleq K_c[(z_1,z_2)]$, where $K[(z_1,z_2)]$ denotes the ring of rational power series in two variables and $K_c[(z_1,z_2)]$ is the ideal of causal rational power series.

Definition. A double indexed dynamical system $\Sigma = (A_1, A_2, B, C)$ is a zero-state realization of a two-dimensional filter $\mathcal{S}$ represented by a series $s \in K_c[[z_1,z_2]]$ if

$$s = (z_1 z_2) C (I - A_1 z_1 - A_2 z_2)^{-1} B \qquad (4)$$

The dimension of a realization Σ is the dimension of the local state space X.

The minimality of the realization is naturally related to the dimension of X in the sense that a realization Σ is minimal when $\dim \Sigma \leq \dim \Sigma'$ for any Σ' which realizes $\mathcal{S}$.

Proposition 2.1. Let $\mathcal{S}$ be represented by $s \in K_c[[z_1,z_2]]$. Then $\mathcal{S}$ is realizable by a double indexed dynamical system if and only if $s \in K_c[(z_1 z_2)]$.

Proof. The necessity is a trivial consequence of (3).

The proof of the sufficiency part is based on the following observation due to B.F. Wyman [personal communication], i.e. if there exist $\bar{A}_o, \bar{A}_1, \bar{A}_2 \in K^{n\times 1}$, $\bar{B} \in K^{n\times 1}$, $\bar{C} \in K^{1\times n}$ such that:

$$s = (z_1 z_2)\bar{C}(I - \bar{A}_o z_1 z_2 - \bar{A}_1 z_1 - \bar{A}_2 z_2)^{-1} B$$

then the double indexed dynamical system $\Sigma = (A_1, A_2, B, C)$, where:

$$A_1 = \begin{bmatrix} \bar{A}_1 & \bar{A}_o \\ 0 & 0 \end{bmatrix},\quad A_2 = \begin{bmatrix} \bar{A}_2 & 0 \\ I & 0 \end{bmatrix},\quad B = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix},\quad C = \begin{bmatrix} \bar{C} & 0 \end{bmatrix}$$

is a zero-state realization of $\mathcal{S}$.

An explicit procedure for constructing $\bar{A}_o, \bar{A}_1, \bar{A}_2, \bar{B}, \bar{C}$ starting from the transfer function is available in the literature[7,8].

A different proof appealing to Hankel matrices of non commutative power series will appear shortly[10].

3. Reachability and Observability

From now on we shall assume that the formal power series s characterizing the input-output map of the filter $\mathcal{S}$ is rational. Hence there exists a realization given by a doubly indexed dynamical system $\Sigma = (A_1, A_2, B, C)$:

$$x(h+1,k+1) = A_1 x(h+1,k) + A_2 x(h,k+1) + Bu(h,k)$$
$$y(h,k) = C\, x(h,k)$$

such that:

$$s = (z_1 z_2) C (I - A_1 z_1 - A_2 z_2)^{-1} B.$$

We shall now extend the notions of reachability and observability for discrete-time systems to provide equivalent notions of local reachability and observability for two-dimensional filters. We say that a local state $\bar{x} \in X$ is "reachable" (from zero initial states) if there exist an input $u \in K[[z_1,z_2]]$ and integers $i > 0$, $j > 0$ such that $x(i,j) = \bar{x}$ when Σ starts from initial states $x(h,0) = x(0,k) = 0$, $h,k = 0,1,\dots$. Since doubly indexed dynamical systems are assumed to be stationary, we can introduce the following definitions:

Definition 1. A state $x \in X$ is reachable if $x = ((z_1 z_2)\, \Sigma_{o\,k}^{\infty} (A_1 z_1 + A_2 z_2)^k Bu, 1)$ for some $u \in \mathcal{U}$.

Definition 2. The reachable local state space is

$$X^R = \{x : x = ((z_1 z_2) \Sigma_{o\,k}^{\infty} (a_1 z_1 + A_2 z_2)^k Bu, 1),\ u \in \mathcal{U}\}$$

and the realization $\Sigma = (A_1, A_2, B, C)$ is L-reachable if $X = X^R$.

The reachable local state space X^R is spanned by the columns of the matrix

$$R_\infty = \begin{bmatrix} M_{oo}B & M_{10}B & M_{01}B & \dots \end{bmatrix}$$

where

$$M_{ij} = \left(\sum_{o}^{\infty}{}_k (A_1z_1 + A_2z_2)^k, \; z_1^iz_2^j\right)$$

Since dim X^R = rank R_∞, the realization $\Sigma = (A_1, A_2, B, C)$ is L-reachable if and only if R_∞ is full rank.

The notion of indistinguishable states is also extended in a very natural way.

Definition 3. A state $x \in X$ indistinguishable from the state 0 in X if

$$\sum_{o}^{\infty}{}_i C(A_1z_1 + A_2z_2)^i x = 0$$

Notice that the left hand term in the above relation represents the zero-input response of Σ determined by $x(0,0) = x$.

Definition 4. The indistinguishable local state space is

$$X^I \triangleq \{x:\; x \in X,\; \sum_{o}^{\infty}{}_i C(A_1z_1 + A_2z_2)^i x = 0\}$$

Since the space X^I is the null space of the infinite matrix:

$$O_\infty = \begin{bmatrix} C & M_{00} \\ C & M_{10} \\ C & M_{01} \\ & \vdots \end{bmatrix},$$

the realization $\Sigma = (A_1, A_2, B, C)$ is L-observable if $X^I = \{0\}$, that is if O_∞ full rank.

The matrices R_∞ and O_∞ contain an infinite number of elements.

Nevertheless the evaluation of their ranks, which is essential to reachability and observability analysis, can be confined to check the rank of two submatrices $R \in K^{n \times n^2}$ and $O \in K^{n^2 \times n}$ given by:

$$R = \left[M_{00}B \;\; M_{10}B \;\cdots\; M_{n-1,n-1}B\right]$$

and

$$O = \begin{bmatrix} C & M_{00} \\ C & M_{10} \\ & \vdots \\ C & M_{n-1,n-1} \end{bmatrix}$$

This statement will be proved in Lemma 3.1 and Proposition 3.1.

Lemma 3.1. Let A_1, A_2 belong to $K^{n \times n}$ and let

$$M_{ij} = \left(\sum_{o}^{\infty}{}_h (A_1z_1 + A_2z_2)^h, \; z_1z_2\right) \; i,j \in \mathbb{Z}$$

be the coefficients of $z_1^iz_2^j$ in the series $(I - A_1z_1 - A_2z_2)^{-1}$. Then there exist $b_{ij} \in K$, $i,j = 0,1,2,\ldots,n$, $b_{nn} \neq 0$ such that

$$\sum_{o}^{n}{}_{ij} M_{i-h,j-k} b_{ij} = 0$$

for all $(h,k) \notin \{(1,1),(1,2),\ldots,(n,n)\}$

The scalars b_{ij} can be assumed as the coefficients in the polynomial $\det(Iz_1^{-1}z_2^{-1} - A_1z_2^{-1} - A_2z_1^{-1})$.

Proof. Since

$$\sum_{o}^{\infty}{}_h (A_1z_1 + A_2z_2)^h = \frac{(z_1z_2)^{-1}\operatorname{adj}(Iz_1^{-1}z_2^{-1} - A_1z_2^{-1} - A_2z_1^{-1})}{\det(Iz_1^{-1}z_2^{-1} - A_1z_2^{-1} - A_2z_1^{-1})}$$

$$= \frac{1}{\sum_{o}^{n}{}_{ij} b_{ij} z_1^{-i}z_2^{-j}} \sum_{1}^{n}{}_{r,s} N_{rs} z_1^{-r}z_2^{-s}, \quad b_{nn} = 1,\; N_{rs} \in K^{n \times n}$$

is follows that

$$\sum_{ij} M_{ij} z_1^iz_2^j \sum_{o}^{n}{}_{hk} b_{hk} z_1^{-h}z_2^{-k} = \sum_{1}^{n}{}_{rs} N_{rs} z_1^{-r}z_2^{-s}$$

Equating the coefficients of the same powers in both sides one gets the proof.

Proposition 3.1. Let M_{ij} as in Lemma 3.1. Then

$$\operatorname{span}(M_{ij}, i,j \in \mathbb{Z}) = \operatorname{span}(M_{ij}, i,j = 0,1,\ldots,n-1) \triangleq \mathcal{M}.$$

Proof. It is sufficient to prove that if r,s are non negative integers and either $r \geq n$ or $s \geq n$, then

$$M_{ij} \in \mathcal{M},\; i \leq r,\; j \leq s,\; (i,j) \neq (r,s)$$

implies $M_{rs} \in \mathcal{M}$. In fact by Lemma 3.1

$$\sum_{o}^{n}{}_{ij} M_{i-n+r,j-n+s} b_{ij} = 0$$

so that

$$M_{rs} = -\sum_{\substack{o \\ (i,j)\neq(n,n)}}{}_{i,j} M_{i-n+r,j-n+s} b_{ij}$$

Remark 1. The above result can be refined when $r \geq n$ and $s < n$. In fact

$$M_{rs} \in \operatorname{span}(M_{ij},\; i = 0,\ldots, n-1,\; j = 0,\ldots s)$$

Remark 2. The Cayley-Hamilton theorem is a particular case of Proposition 3.1 when $A_1 = 0$.

Applying Proposition 3.1 we can write rank R_∞ = rank R, rank O_∞ = rank O, which proves the following

Proposition 3.2. A realization Σ is L-reachable (L-observable) if and only if R (O) is full rank.

The matrices R and O are called reachability matrix and observability matrix associated with the realization $\Sigma = (A_1, A_2, B, C)$.

4. Computation of reachable and observable realizations

So far we have seen that reachability and observability of a realization are strictly connected with the ranks of the reachability matrix R and the observability matrix O. We are now concerned with the following problem: suppose that we have a realization $\Sigma = (A_1, A_2, B, C)$ of dimension n and we would like to construct a L-reachable and L-observable realization, starting from Σ. To solve this problem we introduce two algorithms which act independently to give a L-reachable or a L-observable realization. Of course the alternate application of these provides realizations which are eventually both L-reachable and L-observable.

Let us assume that $\Sigma = (A_1, A_2, B, C)$ is a realization of dimension n of a given filter Σ, and let R be the reachability matrix of Σ. Assume rank $R = r < n$. The algorithm for constructing a L-reachable realization of dimension r is based on the following two steps.

Step 1. Construct a matrix $T \in Gl(n,K)$ having the last n-r rows orthogonal to the space spanned by the columns of R. Consider then the realization $(\hat{A}_1, \hat{A}_2, \hat{B}, \hat{C})$ characterized by

$$\hat{A}_1 = T A_1 T^{-1}$$
$$\hat{A}_2 = T A_2 T^{-1}$$
$$\hat{B} = T B$$
$$\hat{C} = C T^{-1}$$

The matrix T induces a change of basis in X. The first r elements of this new basis are a basis for X^R.

Step 2. Write $\hat{A}_1$, $\hat{A}_2$ in partitioned form:

$$\hat{A}_k = \begin{bmatrix} \hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)} \\ \hat{A}_{21}^{(k)} & \hat{A}_{22}^{(k)} \end{bmatrix}, \ \hat{A}_{11}^{(k)} \in K^{r\times r}, \ k = 1,2$$

and partition $\hat{B}$ and $\hat{C}$ conformably:

$$\hat{B} = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}, \quad \hat{B}_1, \hat{C}_1^T \in K^{r\times 1}$$

Then (A_1,A_2,B,C) and $(\hat{A}_{11}^{(1)}, \hat{A}_{11}^{(2)}, \hat{B}_1, \hat{C}_1)$ realize the same filter, and $(\hat{A}_{11}^{(1)}, \hat{A}_{11}^{(2)}, \hat{B}_1, \hat{C}_1)$ is L-reachable. In fact, let $x(h,k)$ be the state reached by the effect of an input $u \in \mathcal{U}$, and assume as a basis in X the basis corresponding to $(\hat{A}_1, \hat{A}_2, \hat{B}, C)$. With respect to such a basis the last n-r components of $x(h,k)$ are zero and the system

$$\begin{bmatrix} x_1(h+1,k+1) \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11}^{(1)} & \hat{A}_{12}^{(1)} \\ \hat{A}_{21}^{(1)} & \hat{A}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} x_1(h+1,k) \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{A}_{11}^{(2)} & \hat{A}_{12}^{(2)} \\ \hat{A}_{21}^{(2)} & \hat{A}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} x_1(h,k+1) \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u(h,k)$$

$$y(h,k) = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \begin{bmatrix} x_1(h,k) \\ 0 \end{bmatrix}$$

realizes the same input-output map as

$$x_1(h+1,k+1) = \hat{A}_{11}^{(1)} x_1(h+1,k) + \hat{A}_{11}^{(2)} x_1(h,k+1) + \hat{B}_1 u(h,k)$$
$$y(h,k) = \hat{C}_1 x_1(h,k)$$

Assume now rank $O' = r' < n$. The algorithm for obtaining a L-observable realization is also based on two steps and is substantially the same as the above reachability algorithm, although the proof of the second step is based on somehow different reasonings.

Step I. Construct a matrix $Q^{-1} \in Gl(n,K)$ having the last n-r' columns orthogonal to the space spanned by the rows of O. The realization $(\tilde{A}_1, \tilde{A}_2, \tilde{B}, \tilde{C})$ defined by

$$\tilde{A}_1 = Q A_1 Q^{-1}$$
$$\tilde{A}_2 = Q A_2 Q^{-1}$$
$$\tilde{B} = Q B$$
$$\tilde{C} = C Q^{-1}$$

satisfies

$$C(I - A_1 z_2 - A_2 z_2)^{-1} B =$$
$$= \tilde{C}(I - \tilde{A}_1 z_1 - \tilde{A}_2 z_2)^{-1} \tilde{B} =$$
$$= \Sigma_{ij} \tilde{C} \tilde{M}_{ij} \tilde{B} z_1^i z_2^j$$

In the associated observability matrix

$$\tilde{O} = \begin{bmatrix} \tilde{C} \tilde{M}_{00} \\ \tilde{C} \tilde{M}_{10} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{C} \tilde{M}_{n-1,n-1} \end{bmatrix}$$

the elements in the last n-r' columns are zeros.

Step II. Write $\tilde{A}_1, \tilde{A}_2, \tilde{B}, \tilde{C}$ in partitioned form as in step 2 above, and notice that $\tilde{C} = [\tilde{C}_1 \ 0]$, $\tilde{C}_1 \in K^{1\times r'}$. To show that $(\tilde{A}_{11}^{(1)}, \tilde{A}_{11}^{(2)}, \tilde{B}_1, \tilde{C}_1)$ is a realization we have to prove that

$$\Sigma_{ij} \tilde{C} \tilde{M}_{ij} \tilde{B} z_1^i z_2^j = \tilde{C}(I - \tilde{A}_1 z_1 - \tilde{A}_2 z_2) \tilde{B} =$$
$$= \tilde{C}_1 (I - \tilde{A}_{11}^{(1)} z_1 - \tilde{A}_{11}^{(2)} z_2)^{-1} \tilde{B}_1 \triangleq$$
$$\triangleq \Sigma_{ij} \tilde{C}_1 M_{ij}^{*} \tilde{B}_1 z_1^i z_2^j$$

namely that

$$\tilde{C} \tilde{M}_{ij} \tilde{B} = \tilde{C}_1 M_{ij}^{*} \tilde{B}_1 \qquad i,j = 0,1,\ldots$$

Observe that for $i,j > 0$

$$\tilde{C} \tilde{M}_{ij} = \tilde{C} \tilde{M}_{i-1,j} \tilde{A}_1 + \tilde{C} \tilde{M}_{i,j-1} \tilde{A}_2$$
$$\tilde{C}_1 M_{ij}^{*} = \tilde{C}_1 M_{i-1,j}^{*} \tilde{A}_{11}^{(1)} + \tilde{C}_1 M_{i,j-1}^{*} \tilde{A}_{11}^{(2)}$$

and assume by induction that the first r' elements of $\tilde{C} \tilde{M}_{i-1,j}$, $\tilde{C} \tilde{M}_{i,j-1}$, coincide with $\tilde{C}_1 M_{i-1,j}^{*}$, $\tilde{C}_1 M_{i,j-1}^{*}$, respectively. Then

$$\tilde{C} \tilde{M}_{i-1,j} \tilde{A}_1 = \begin{bmatrix} \tilde{C}_1 M_{i-1,j}^{*} & 0 \end{bmatrix} \begin{bmatrix} \tilde{A}_{11}^{(1)} & \tilde{A}_{12}^{(1)} \\ \tilde{A}_{21}^{(1)} & \tilde{A}_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} \tilde{C}_1 M_{i-1,j}^{*} \tilde{A}_{11}^{(1)} & * \end{bmatrix}$$

and similarly

$$\tilde{C} \tilde{M}_{i,j-1} \tilde{A}_2 = \begin{bmatrix} \tilde{C}_1 M_{i,j-1}^{*} \tilde{A}_{11}^{(2)} & * \end{bmatrix}$$

from which it is clear that the first r' elements in $\tilde{C} \tilde{M}_{ij}$ are the same as in $\tilde{C}_1 M_{ij}^{*}$. Of course the last n-r' elements are zero because of the structure of $\tilde{O}$.

Evidently the result that we have proved can also be stated in the following form:

Proposition 4.1. Let $\Sigma = (A_1,A_2,B,C)$ be any realization of $\mathcal{S}$. Then a L-reachable and L-observable realization can be constructed in a finite number of steps from Σ following the procedure above.

Corollary. Every minimal realization is completely L-reachable and L-observable.

Remark. As we shall see in section 5, in general the converse of the above corollary does not hold.

5. Examples of realizations

As it was remarked at the end of section 4, L-reachability and L-observability properties of a realization do not imply the minimality of the realization.

The following examples prove this fact and furthermore they give some insights into the general problem of the dependence of the realization on the field K.

Example 1. Let K be a field of characteristic 0 and consider the filter:

$$s = (z_1 z_2)\frac{1-z_1}{1-z_1^2-z_2^2}$$

The following double indexed dynamical systems

$$\Sigma_1 = (A_1^{(1)}, A_2^{(1)}, B^{(1)}, C^{(1)}):$$

$$A_1^{(1)} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ A_2^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C^{(1)} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and $\Sigma_2 = (A_1^{(2)}, A_2^{(2)}, B^{(2)}, C^{(2)})$:

$$A_1^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ A_2^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ B^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C^{(2)} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$$

are L-reachable and L-observable realizations of s over the field K. However Σ_2 is not a minimal realization.

Example 2. Consider the filter

$$s = (z_1 z_2)\frac{1}{1-z_1^2-z_2^2}$$

and assume $K = \mathbb{C}$. It is easy to check that s has a minimal realization of dimension 2 given by $\Sigma_1 = (A_1^{(1)}, A_2^{(1)}, B^{(1)}, C^{(1)})$ with

$$A_1^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ A_2^{(1)} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},\ B^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C^{(1)} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

We now show that if we assume $K = \mathbb{R}$, the filter s is no longer realizable in dimension 2.

First make the following observation. If $\Sigma = (A_1, A_2, B, C)$ is a realization of a filter characterized by a strictly causal power series $s = \sum_{o\,i,j}^{\infty}(s, z_1^i z_2^j)z_1^i z_2^j$, the dynamical systems $\Sigma_1 = (A_1, B, C)$ and $\Sigma_2 = (A_2, B, C)$ realize the formal power series in one indeterminate

$$s_1 = \sum_{o\,i}^{\infty}(s, z_1^i z_2)z_1^i \text{ and } s_2 = \sum_{o\,j}^{\infty}(s, z_1 z_2^j)z_2^j$$

respectively. In this example

$$s_1 = z_1 + z_1^3 + z_1^5 + \dots$$

$$s_2 = z_2 + z_2^3 + z_2^5 + \dots$$

A minimal realization of s_1 and s_2 is given by

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Hence the class of minimal realizations of s_1 and s_2 over R is

$$\mathscr{P} = \{(P^{-1}\bar{A}P,\ P^{-1}\bar{B},\ \bar{C}P),\ P \in Gl(2,\mathbb{R})\}.$$

Suppose now that $\Sigma = (A_1, A_2, B, C)$ realizes s in dimension 2 over $\mathbb{R}$. Hence (A_1, B, C) and (A_2, B, C) are minimal realizations of s_1 and s_2 and consequently they belong to $\mathscr{P}$. Then we can find $S \in Gl(2,\mathbb{R})$ such that

$$S^{-1}A_1 S = \bar{A},\ S^{-1}B = \bar{B},\ CS = \bar{C}$$

and check that $\bar{\Sigma} = (S^{-1}A_1S, S^{-1}A_2S, S^{-1}B, CS) = (\bar{A}, S^{-1}A_2S, \bar{B}, \bar{C})$ still realizes the filter s.

Since $(S^{-1}A_2S, \bar{B}, \bar{C})$ must realize s_2, there exists $T \in Gl(2,\mathbb{R})$ such that

$$T^{-1}(S^{-1}A_2S)T = \bar{A} \tag{5}$$

$$T^{-1}\bar{B} = \bar{B} \tag{6}$$

$$\bar{C}\,T = \bar{C} \tag{7}$$

From (6) and (7) we have

$$T = \begin{bmatrix} 1 & 0 \\ 0 & t_{22} \end{bmatrix},\quad t_{22} \in \mathbb{R} \tag{8}$$

Then the structure of $S^{-1}A_2S$ follows from (5) and (8):

$$S^{-1}A_2S = \begin{bmatrix} 0 & t_{22} \\ t_{22}^{-1} & 0 \end{bmatrix},\quad t_{22} \in \mathbb{R}.$$

Since $\bar{\Sigma}$ realizes s, we have also

$$0 = (s, z_1^2 z_2^2) = \bar{C}(\bar{A}\,S^{-1}A_2S + S^{-1}A_2S\,\bar{A})\bar{B} =$$
$$= t_{22} + t_{22}^{-1}.$$

Hence

$$t_{22}^2 = -1$$

which contradicts the assumption $T \in Gl(2,\mathbb{R})$.

6. Minimality of realizations

From the examples given in section 5 it appears very clearly that the minimality problem for double indexed dynamical systems shows some peculiar aspects which do not allow us to derive the realization theory of two dimensional filters by simply generalizing linear systems theory.

One of the first key points failing in this context is the rule played by the Hankel matrix. In linear system theory the Hankel matrix associated with a rational series can be obtained multiplying the (infinite) observability matrix by the (infinite) reachability matrix and its rank gives the dimension of minimal realizations. These properties do not hold for two-dimensional filters as we shall show below.

Let $s \in K_c[[z_1,z_2]]$, and let $s' = (z_1z_2)^{-1}s$. Then the Hankel matrix $\mathscr{H}(s)$ associated with s is given by[9]

$$\mathscr{H}(s) = \begin{bmatrix} (s',1) & (s',z_1) & (s',z_2) & (s',z_1^2) & (s',z_1z_2) & (s',z_2^2)\dots \\ (s',z_1) & (s',z_1^2) & (s',z_1z_2) & (s',z_1^3) & (s',z_1^2z_2) & (s',z_1z_2^2)\dots \\ (s',z_2) & (s',z_1z_2) & (s',z_2^2) & (s',z_1^2z_1) & (s',z_1z_2^2) & (s',z_2^3)\dots \\ (s',z_1^2) & (s',z_1^3) & (s',z_1^2z_2) & (s',z_1^4) & (s',z_1^3z_2) & (s',z_1^2z_2^2)\dots \\ \dots\dots & & & & & \end{bmatrix}$$

. has been proved[9] that the rank of $\mathscr{H}(s)$ is finite if and only if s is rational and a denominator q of s can be factorized as $q = q_1q_2$ with $q_1 \in K[z_1^{-1}]$, $q_2 \in K[z_2^{-1}]$. The series satisfying this property are elements of the ring $K[(z_1)] \otimes K[(z_2)] \triangleq K^{rec}[(z_1,z_2)]$ called the ring of "recognizable series".

The following remarks are a consequence of the above mentioned characteristic property of recognizable series and give an account of what is the situation with Hankel matrix for two-dimensional filters:

i) if s is rational but $s \notin K_c^{rec}[(z_1,z_2)]$, then rank $\mathscr{H}(s) = \infty$. However there exist finite dimensional realizations of s (see Proposition 2.1);

ii) if $s \in K^{rec}[(z_1,z_2)]$, then rank $\mathscr{H}(s)$ is finite. However the dimension of minimal realizations of s does not coincide with rank $\mathscr{H}(s)$ as the following example shows. Assume $s \in K^{rec}[(z_1,z_2)]$ be given by:

$$s = (z_1z_2)\,\frac{1+z_1+z_2}{1+z_1+z_2+z_1z_2}$$

.ierefore the formal power series expansion of $(z_1z_2)^{-1}s$ is expressed by:

$$s = (1+z_1+z_2)\sum_{0}^{\infty}{}_k(-1)^k(z_1+z_2+z_1z_2)^k =$$

$$= 1 - z_1z_2 + z_1z_2^2 + z_1^2z_2 + \dots$$

and the Hankel matrix is then:

$$\mathscr{H}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & \dots\dots \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & \dots\dots \\ 0 & -1 & 0 & 1 & \dots\dots \\ 0 & 0 & 1 & \dots\dots \\ -1 & 1 & 1 & \dots\dots \\ 0 & 1 & \dots\dots \\ \dots\dots \end{bmatrix}$$

Note that rank $\mathscr{H}(s) > 2$. Nevertheless a minimal realization of dimension 2 does exist, i.e. $\Sigma = (A_1, A_2, B, C)$:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},\; A_2 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix},\; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\; C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

iii) Since the rank of the matrix $O_\infty R_\infty$ is finite, in general $\mathscr{H}(s) \neq O_\infty R_\infty$. Links between $\mathscr{H}(s)$ and $O_\infty R_\infty$ have been clarified in a different context[4,7,8].

In the sequel we will associate with the commutative series s a family of non commutative power series in two variables whose Hankel matrices have finite rank. We will prove that the minimal rank of these matrices is the dimension of minimal realizations of s.

A non commutative power series r in two indeterminates x_1,x_2 over a a field belongs to the ring of recognizable series $r \in K<(x_1,x_2)>$ if there exist an integer m, a representation $\mu: X^* \to K^{m\times m}$, two matrices $B \in K^{m\times 1}$ and $C \in K^{1\times m}$ such that

$$r = \sum_{w \in X^*} C\mu(w)Bw$$

where X^* is the free monoid generated by x_1,x_2.

This is equivalent to say that any non commutative recognizable power series r can be expressed as:

$$r = \sum_{1}^{\infty}{}_i C(A_1x_1 + A_2x_2)^i B = C(I - A_1x_1 - A_2x_2)^{-1}B$$

where $A_1 = \mu(x_1)$, $A_2 = \mu(x_2)$, (A_1, A_2 do not necessarily commute).

The dimension of the minimal representation of r is given by rank $\mathscr{H}(r)$[9].

Define the algebra morphism $\phi: K<(x_1,x_2)> \to K[[z_1,z_2]]$ by $\phi(k) = k$, $\forall k \in K$, $\phi(x_1) = z_1$, $\phi(x_2) = z_2$. Since

$$\phi:\; C(I - Ax_1 - A_2x_2)^{-1}B \to C(I - A_1z_1 - A_2z_2)^{-1}B\;, \qquad (9)$$

all rational series in the commutative variables z_1 and z_2 are obtained by varying A_1,A_2,B,C, and the map ϕ is onto $K[(z_1,z_2)]$. Then we can associate a commutative series $\phi(r) \in K[(z_1,z_2)]$ with each non commutative series $r \in K<(x_1,x_2)>$. By (9) each representation of r induces a realization of $\phi(r)$.

The following diagram

$$\begin{tikzcd} K<(x_1,x_2)> \arrow[r, "\phi"] \arrow[d] & K[(z_1,z_2)] \\ \dfrac{K<(x_1,x_2)>}{\ker\phi} \arrow[ur] & \end{tikzcd}$$

commutes and $\bar\phi$ is an isomorphism. Consequently given $s \in K_c[(z_1,z_2)]$, minimal realizations of s correspond to minimal representations in the class of representations of $\bar\phi^{-1}(s)$.

One has thus arrived at the following Proposition:

Proposition 6.1. Let s belong to $K_c[(z_1,z_2)]$. Then the dimension of minimal realization of s is given by:

$$\min_{r \in \bar\phi^{-1}(s)} \operatorname{rank}\mathscr{H}(r)\,.$$

7. Conclusions

In this paper by pursuing the idea of introducing a state space model of two-dimensional filters, the realization problem has been further investigated along the directions outlined in previous works[7,8].

The class of realizations introduced in this paper is characterized by a local state updating equation of the following form:

$$x(h+1,k+1) = A_1 x(h+1,k) + A_2 x(h,k+1) + Bu(h,k)$$

After defining the concepts of reachability and observability, we have presented an algorithm for obtaining a reachable and observable realization starting from a generic one.

The minimality of the realizations is not guarantee by reachability and observability. In general the dimension of minimal realizations depends on the field K and does not coincide with the rank of $\mathcal{H}(s)$. Nevertheless we can associate the commutative series s with non commutative recognizable power series whose representations provide all the realizations of s. Hence the problem of determining minimal realizations of s can be solved looking for minimal representations of non commutative power series.

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