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Computation of reachable and observable realizations of spatial filters

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The input–output behaviour of a two-dimensional linear filter is defined by a formal power series in two variables. If the power series is rational, the dynamics of the filter is described by updating equations on finite dimensional local state spaces. The notions of local reachability and observability are defined in a natural way and an algorithm for obtaining a reachable and observable realization is given.

In general reachability and observability do not imply the minimality of the realization. Nevertheless the dimension of a minimal realization is the least rank in a family of Hankel matrices.

1. Introduction

In the past few years there has been an increasing interest in two-dimensional filters (Anderson and Jury 1974, Shanks *et al.* 1972, Habibi 1972, Attasi 1973, Powell and Silvermann 1974, Roesser 1975). This type of filter is extensively used in processing two-dimensional sampled data, such as seismic data sections, digitized photographic data, and gravity and magnetic maps.

The aim of this paper is to afford the algebraic realization problem of spatial filters defined by their input–output map. This problem has been attacked by the authors (Fornasini and Marchesini 1975) from a system theoretic point of view defining the state via Nerode equivalence classes. Additional results in this direction will be developed here mainly with regard to the computation of reachable and observable realizations. The connection between minimality and observability and reachability will also be discussed.

2. Realization of two-dimensional filters

We will consider two-dimensional digital filters with scalar inputs and outputs taken from an arbitrary field K. The input–output representation of such a filter is given by

$$\mathscr{S} \triangleq (T, U, \mathscr{U}, Y, \mathscr{Y}, F) \tag{1}$$

where $T = Z \times Z$ (partially ordered by the product of the orderings) is the discrete plane, U and Y are one-dimensional vector spaces over the field K, $\mathscr{U}$ and $\mathscr{Y}$ are the space of truncated formal Laurent series in two variables over K (whose precise description will be given below), and $F : \mathscr{U} \to \mathscr{Y}$ is the input–output map.

A typical element of $\mathscr{U}$ or $\mathscr{Y}$ will be written

$$r = \sum_{-k}^{\infty}{}_{i,j}\, (r, z_1^i z_2^j) z_1^i z_2^j, \quad \text{for some integer } k$$

where $(r, z_1^i z_2^j)$ denotes the coefficient of $z_1^i z_2^j$.

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The input–output map $F : \mathscr{U} \to \mathscr{Y}$ is assumed to satisfy the following axioms :

(i) *linearity*

(ii) *two-dimensional shift invariance* :

$$F(z_1^i z_2^j r) = z_1^i z_2^j F(r), \quad i, j \in Z$$

(iii) *two-dimensional strict causality* :

$$(u_1, z_1^i z_2^j) = (u_2, z_1^i z_2^j), \quad i < t_1, \quad j < t_2$$

implies

$$(Fu_1, z_1^i z_2^j) = (Fu_2, z_1^i z_2^j), \quad i \leqslant t_1, \quad j \leqslant t_2, \quad \forall u_1, u_2 \in \mathscr{U}$$

Under assumption (iii) it is easy to verify that the impulse response $F(1)$ is a ' strictly causal ' power series, i.e.

$$F(1) = \sum_{1}^{\infty}{}_{ij} \; (F(1), z_1^i z_2^j) z_1^i z_2^j$$

More formally we can say that

$$s \triangleq F(1) \in (z_1 z_2) K[[z_1, z_2]] \triangleq K_c[[z_1, z_2]]$$

where $K[[z_1, z_2]]$ denotes the ring of formal power series in two variables and $K_c[[z_1, z_2]]$ is the ideal of ' *strictly causal* ' power series.

From (i) and (ii) it follows that

$$F(u) = su, \quad \forall u \in \mathscr{U}$$

that is, two-dimensional filters (in their input–output representation) are in one-to-one correspondence with formal power series $K_c[[z_1, z_2]]$.

A double indexed, linear, stationary, finite dimensional dynamical system Σ is defined by a pair of equations of the form (Fornasini and Marchesini 1975)

$$\left.\begin{aligned} x(h+1, k+1) &= A_0 x(h, k) + A_1 x(h+1, k) + A_2 x(h, k+1) + Bu(h, k) \\ y(h, k) &= Cx(h, k) \end{aligned}\right\} \qquad (2)$$

where $A_i \in K^{n \times n}$, $i = 0, 1, 2$, $C \in K^{1 \times n}$, $B \in K^{n \times 1}$ and x belongs to some finite dimensional vector space $X = K^n$ (local state space).

The solution of eqns. (2) for $h \geqslant 0$, $k \geqslant 0$ is uniquely determined by u and by the values $x(h, 0)$, $h = 1, 2, \ldots$, and $x(0, k)$, $k = 0, 1, 2, \ldots$ (initial local states).

Let now $x(h, 0) = x(0, k) = 0$, $h, k = 0, 1, \ldots$ and associate the monomial $x(h, k) z_1^h z_2^k \in K^{n \times 1}[[z_1, z_2]]$ with the local state $x(h, k)$.

From (2) it follows that for each $u \in K[[z_1, z_2]]$

$$\begin{aligned} \sum_{0}^{\infty}{}_{h,k} x(h, k) z_1^h z_2^k = A_0 \left(\sum_{0}^{\infty}{}_{h,k} x(h, k) z_1^h z_2^k \right) z_1 z_2 \\ + A_1 \left(\sum_{0}^{\infty}{}_{h,k} x(h, k) z_1^h z_2^k \right) z_1 \\ + A_2 \left(\sum_{0}^{\infty}{}_{h,k} x(h, k) z_1^h z_2^k \right) z_2 + (z_1 z_2) Bu \end{aligned}$$

and then

$$(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)\Sigma_{h,\,k} x(h,k) z_1^h z_2^k = (z_1 z_2) B u$$

The polynomial $(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)$ belongs to $K^{n\times n}[z_1, z_2]$ and has an inverse in the ring of formal power series $K^{n\times n}[[z_1, z_2]]$. Its inverse is given by

$$(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} = \sum_{0}^{\infty}{}_i\; (A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^i$$

With the aid of this inverse we can relate the state to the input. In fact we have

$$\Sigma_{h,\,k} x(h,k) z_1^h z_2^k = (I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} (z_1 z_2) B u$$

This yields at once the input–output relation

$$y = C \sum_{h,\,k} x(h,k) z_1^h z_2^k = C(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} (z_1 z_2) B u$$

The series

$$C(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} (z_1 z_2) B$$

is called the *transfer function of* Σ.

We recall that a formal power series $s \in K[[z_1 z_2]]$ is *rational* if there exist polynomials $p, q \in K[z_1^{-1}, z_2^{-1}]$ with $\deg p \leqslant \deg q$, such that $qs = p$. The polynomial q is called a denominator of s.

Then transfer function $(z_1 z_2) C(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} B$ belongs to $(z_1 z_2) K[(z_1, z_2)] \triangleq K_c[(z_1, z_2)]$, where $K[(z_1, z_2)]$ denotes the ring of rational power series in two variables and $K_c[(z_1, z_2)]$ is the ideal of *causal rational power series*.

Definition

A doubly-indexed dynamical system $\Sigma = (A_0, A_1, A_2, B, C)$ *is a zero-state realization of a two-dimensional filter* $\mathscr{S}$ *represented by a series* $s \in K_c[[z_1, z_2]]$ *if*

$$s = (z_1 z_2) C (I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1} B \tag{4}$$

The dimension of a realization Σ *is the dimension of the local state space* X.

The minimality of the realization is naturally related to the dimension of X in the sense that a realization Σ is minimal when $\dim \Sigma' \leqslant \dim \Sigma'$ for any Σ' which realizes $\mathscr{S}$.

Proposition 1.1

Let $\mathscr{S}$ be represented by $s \in K_c[[z_1, z_2]]$. Then $\mathscr{S}$ is realizable by a double-indexed dynamical system if and only if $s \in K\ [(z_1 z_2)]$.

Proof

The necessity is a trivial consequence of (3). The sufficiency follows from the construction below.

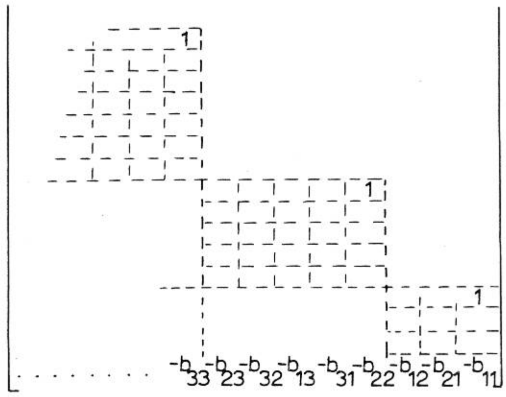
Let

$$s = \sum_{0}^{n-1}{}_{ij}\, a_{n-i,\,n-j} z_1^{-i} z_2^{-j} \Big/ \sum_{0}^{n}{}_{ij}\, b_{n-j} z_1^{-i} z_2^{-j} \quad b_{00} = 1$$

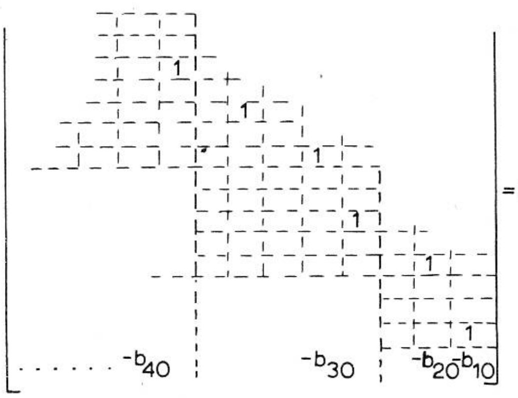
The matrices A_0, A_1, $A_2 \in K^{n^2 \times n^2}$, $B \in K^{n^2 \times 1}$, $C \in K^{1 \times n^2}$ defined by

$$[\ldots\ldots\ldots\ldots\ldots \quad 0 \quad 0 \quad 0 \quad 1\,] = B^{\mathrm{T}}$$

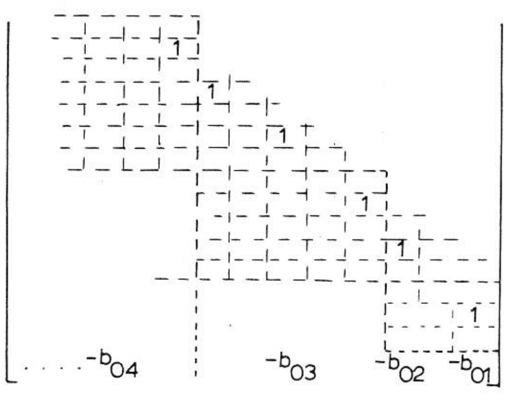
$$[\ldots\, a_{33} \quad a_{23} \quad a_{32} \quad a_{13} \quad a_{31} \quad a_{22} \quad a_{12} \quad a_{21} \quad a_{11}] = C$$



$= A_0$



$= A_1$



$= A_2$

satisfy the relation

$$s = (z_1 z_2) C (I - A_0 z_1 z_2 - A_1 z_1 - A_1 z_2)^{-1} B$$

Consequently the doubly-indexed dynamical system $\Sigma=(A_0, A_1, A_2, B, C)$ is a (not necessarily minimal) zero-state realization of $\mathscr{S}$.

3. Reachability and observability

From now on we shall assume that the formal power series s characterizing the input–output map of the filter $\mathscr{S}$ is rational. Hence there exists a realization given by a doubly-indexed dynamical system $\Sigma=(A_0, A_1, A_2, B, C)$:

$$x(h+1, k+1)=A_0x(h, k)+A_1x(h+1, k)+A_2x(h, k+1)+Bu(h, k)$$
$$y(h, k)=Cx(h, k)$$

such that

$$s=(z_1z_2)C(I-A_0z_1z_2-A_1z_1-A_2z_2)^{-1}B$$

We shall now extend the notations of reachability and observability for discrete-time systems to provide equivalent notions of local reachability and observability for two-dimensional filters. We say that a local state $\bar{x}\in X$ is 'reachable' (from zero initial states) if there exist an input $u\in K[[z_1, z_2]]$ and integers $i>0, j>0$ such that $x(i, j)=\bar{x}$ when Σ starts from initial states $x(h, 0)=x(0, k)=0, \; h, k=0, 1, \ldots$. Since doubly-indexed dynamical systems are assumed to be stationary, we can introduce the following definitions :

Definition 1

A state $x\in X$ *is reachable if* $x=\left((z_1z_2)\sum_{0}^{\infty}{}_k\,(A_0z_1z_2+A_1z_1+A_2z_2)^kBu, 1\right)$ *for some* $u\in\mathscr{U}$.

Definition 2

The reachable local state space is

$$X^{\mathrm{R}}=\left\{x:\; x=\left((z_1z_2)\sum_{0}^{\infty}{}_k\,(A_0z_1z_2+A_1z_1+A_2z_2)^kBu, 1\right),\quad u\in\mathscr{U}\right\}$$

and the realization $\Sigma=(A_0, A_1, A_2, B, C)$ *is* 1*-reachable if* $X=X^{\mathrm{R}}$.

The reachable local state space X^{R} is spanned by the columns of the matrix

$$R_\infty=[M_{00}B\;\; M_{10}B\;\; M_{01}B\;\ldots]$$

where

$$M_{ij}=\left(\sum_{0}^{\infty}{}_k\,(A_0z_1z_2+A_1z_1+A_2z_2)^k, z_1{}^iz_2{}^j\right)$$

Since $\dim X^{\mathrm{R}}=\operatorname{rank} R_\infty$, *the realization* $\Sigma=(A_0, A_1, A_2, B, C)$ *is* 1*-reachable* if and only if R_∞ is full rank.

The notion of indistinguishable states is also extended in a very natural way.

Definition 3

A state $x\in X$ *is indistinguishable from the state* 0 *in* X *if*

$$\sum_{0}^{\infty}{}_i\, C(A_0z_1z_2+A_1z_1+A_2z_2)^ix=0$$

Notice that the left-hand term in the above relation represents the zero-input response of Σ determined by $x(0, 0)=x$.

Definition 4

The indistinguishable local state space is

$$X^{\mathrm{I}} \triangleq \left\{ x : \; x \in X, \; \sum_{0}^{\infty}{}_{i} \; C(A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^i x = 0 \right\}$$

Since the space X^{I} is the null space of the infinite matrix :

$$O_\infty \triangleq \begin{bmatrix} C \; M_{00} \\ C \; M_{10} \\ C \; M_{01} \\ \vdots \end{bmatrix}$$

the realization $\Sigma=(A_0, A_1, A_2, B, C)$ *is* 1-*observable* if $X^{\mathrm{I}}=\{0\}$, that is if O_∞ is full rank.

The matrices R_∞ and O_∞ contain an infinite number of elements. Nevertheless the evaluation of their ranks, which is essential to reachability and observability analysis, can be confined to check the rank of two submatrices $R \in K^{n \times n^2}$ and $O \in K^{n^2 \times n}$ given by

$$R=[M_{00}B \;\; M_{10}B \; \ldots \; M_{n-1,\, n-1}B]$$

and

$$O=\begin{bmatrix} C \; M_{00} \\ C \; M_{10} \\ \vdots \\ C \; M_{n-1,\, n-1} \end{bmatrix}$$

This statement will be proved in Lemma 3.1 and Proposition 3.1.

Lemma 3.1

Let A_0, A_1, A_2 belong to $K^{n \times n}$ and let

$$M_{ij}=\left(\sum_{0}^{\infty}{}_{h} \; (A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^h, \; z_1^i z_2^j \right), \quad i, j \in Z$$

be the coefficients of $z_1^i z_2^j$ in the series $(I - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2)^{-1}$. Then there exist $b_{ij} \in K$, $i, j=0, 1, 2, \ldots, n$, $b_{nn} \neq 0$ such that

$$\sum_{0}^{n}{}_{ij} \; M_{i-h,\, j-k} b_{ij} = 0$$

for all $(h, k) \notin \{(1, 1), (1, 2), \ldots, (n, n)\}$.

The scalars b_{ij} can be assumed as the coefficients in the polynomial

$$\det\,(I z_1^{-1} z_2^{-1} - A_0 - A_1 z_2^{-1} - A_2 z_1^{-1})$$

Proof

Since

$$\sum_{0}^{\infty}{}_{h}\ (A_0 z_1 z_2 + A_1 z_1 + A_2 z_2)^h = \frac{(z_1 z_2)^{-1}\ \mathrm{adj}\ (I z_1^{-1} z_2^{-1} - A_0 - A_1 z_2^{-1} - A_2 z_1^{-1})}{\det\ (I z_1^{-1} z_2^{-1} - A_0 - A_1 z_2^{-1} - A_2 z_1^{-1})}$$

$$= \frac{1}{\sum_{0}^{n}{}_{ij}\ b_{ij} z_1^{-i} z_2^{-j}} \sum_{1}^{n}{}_{r,\, s}\ N_{rs} z_1^{-r} z_2^{-s},$$

$$b_{nn} = 1, \quad N_{rs} \in K^{n \times n}$$

it follows that

$$\Sigma_{ij} M_{ij} z_1^{i} z_2^{j} \sum_{0}^{n}{}_{hk}\ b_{hk} z_1^{-h} z_2^{-k} = \sum_{1}^{n}{}_{rs}\ N_{rs} z_1^{-r} z_2^{-s}$$

Equating the coefficients of the same powers in both sides one gets the proof.

Proposition 3.1

Let M_{ij} *as in Lemma* 3.1. *Then*

$$\mathrm{span}\ (M_{ij},\ i,\ j \in Z) = \mathrm{span}\ (M_{ij},\ i,\ j = 0,\ 1,\ \ldots,\ n-1) \triangleq \mathscr{M}$$

Proof

It is sufficient to prove that if r, s are non-negative integers and either $r \geqslant n$ or $s \geqslant n$, then

$$M_{ij} \in \mathscr{M}, \quad i \leqslant r, \quad j \leqslant s, \quad (i,\ j) \neq (r,\ s)$$

implies

$$M_{rs} \in \mathscr{M}$$

In fact by Lemma 2.1

$$\sum_{0}^{n}{}_{ij}\ M_{i-n+r,\ j-n+s} b_{ij} = 0$$

so that

$$M_{rs} = -\frac{1}{b_{nn}} \sum_{\substack{0 \\ (i,\ j) \neq (n,\ n)}}^{n}{}_{i,\ j}\ M_{i-n+r,\ j-n+s} b_{ij}$$

Remark 1

The above result can be refined when $r \geqslant n$ and $s < n$. In fact

$$M_{rs} \in \mathrm{span}\ (M_{ij},\ i = 0,\ \ldots,\ n-1,\ j = 0,\ \ldots,\ s)$$

Remark 2

The Cayley–Hamilton theorem is a particular case of Proposition 3.1 when $A_0 = A_1 = 0$.

Applying Proposition 3.1 we can write rank R_∞ = rank R, rank O_∞ = rank O, which proves the following :

Proposition 3.2

A realization Σ *is* 1-*reachable* (1-*observable*) *if and only if* R (O) *is full rank.*

The matrices R and O are called *reachability matrix* and *observability matrix* associated with the realization $\Sigma=(A_0, A_1, A_2, B, C)$.

4. Computation of reachable and observable realizations

So far we have seen that reachability and observability of a realization are strictly connected with the ranks of the reachability matrix R and the observability matrix O. We are now concerned with the following problem : suppose that we obtained by some algorithm (see § 2) a realization $\Sigma=(A_0, A_1, A_2, B, C)$ of dimension n and we would like to construct a 1-reachable and 1-observable realization, starting from Σ. To solve this problem we introduce two algorithms which act independently to give a 1-reachable or a 1-observable realization. Of course the alternate application of these provides realizations which are eventually both 1-reachable and 1-observable.

Let us assume that $\Sigma=(A_0, A_1, A_2, B, C)$ is a realization of dimension n of a given filter Σ, and let R be the reachability matrix of Σ. Assume rank $R=r<n$. The algorithm for constructing a 1-reachable realization of dimension r is based on the following two steps.

Step 1

Construct a matrix $T \in Gl(K, n)$ having the last $n-r$ rows orthogonal to the space spanned by the columns of R. Consider then the realization $(\hat{A}_0, \hat{A}_1, \hat{A}_2, \hat{B}, \hat{C})$ characterized by

$$\begin{aligned}\hat{A}_0&=TA_0T^{-1}\\ \hat{A}_1&=TA_1T^{-1}\\ \hat{A}_2&=TA_2T^{-1}\\ \hat{B}&=TB\\ \hat{C}&=CT^{-1}\end{aligned}$$

The matrix T induces a change of basis in X. The first r elements of this new basis are a basis for X^{R}.

Step 2

Write $\hat{A}_0, \hat{A}_1, \hat{A}_2$ in partitioned form :

$$\hat{A}_k=\begin{bmatrix}\hat{A}_{11}{}^{(k)} & \hat{A}_{12}{}^{(k)}\\ \hat{A}_{21}{}^{(k)} & \hat{A}_{22}{}^{(k)}\end{bmatrix}, \quad \hat{A}_{11}{}^{(k)} \in K^{r\times r}, \quad k=0, 1, 2$$

and partition $\hat{B}$ and $\hat{C}$ conformably :

$$\hat{B}=\begin{bmatrix}\hat{B}_1\\ 0\end{bmatrix}, \quad \hat{C}=[\hat{C}_1 \ \hat{C}_2], \quad \hat{B}_1, \hat{C}_1{}^{\mathrm{T}} \in K^{r\times 1}$$

Then (A_0, A_1, A_2, B, C) and $(\hat{A}_{11}{}^{(0)}, \hat{A}_{11}{}^{(1)}, \hat{A}_{11}{}^{(2)}, \hat{B}_1, \hat{C}_1)$ realize the same filter, and $(\hat{A}_{11}{}^{(0)}, \hat{A}_{11}{}^{(1)}, \hat{A}_{11}{}^{(2)}, \hat{B}_1, \hat{C}_1)$ is 1-reachable. In fact, let $x(h, k)$ be the state reached by the effect of an input $u \in \mathscr{U}$, and assume as a basis in X

the basis corresponding to (A_0, A_1, A_2, B, C). With respect to such a basis the last $n-r$ components of $x(h, k)$ are zero and the system

$$\begin{bmatrix} \hat A_{11}{}^{(0)} & \hat A_{12}{}^{(0)} \\ \hat A_{21}{}^{(0)} & \hat A_{22}{}^{(0)} \end{bmatrix}\begin{bmatrix} x_1(h, k) \\ 0 \end{bmatrix} + \begin{bmatrix} \hat A_{11}{}^{(1)} & \hat A_{12}{}^{(1)} \\ \hat A_{21}{}^{(1)} & \hat A_{22}{}^{(1)} \end{bmatrix}\begin{bmatrix} x_1(h+1, k) \\ 0 \end{bmatrix}$$

$$+\begin{bmatrix} \hat A_{11}{}^{(2)} & \hat A_{12}{}^{(2)} \\ \hat A_{21}{}^{(2)} & \hat A_{22}{}^{(2)} \end{bmatrix}\begin{bmatrix} x_1(h, k+1) \\ 0 \end{bmatrix} + \begin{bmatrix} \hat B_1 \\ 0 \end{bmatrix} u(h, k) = \begin{bmatrix} x_1(h+1, k+1) \\ 0 \end{bmatrix}$$

$$y(h, k) = [\hat C_1 \ \ \hat C_2]\begin{bmatrix} x_1(h, k) \\ 0 \end{bmatrix}$$

realizes the same input–output map as :

$$\hat A_{11}{}^{(0)} x_1(h, k) + \hat A_{11}{}^{(1)} x_1(h+1, k) + \hat A_{11}{}^{(2)} x_1(h, k+1) + \hat B_1 u(h, k) = x_1(h+1, k+1)$$
$$y(h, k) = \hat C_1 x_1(h, k)$$

Assume now rank $O' = r' < n$. The algorithm for obtaining a 1-observable realization is also based on two steps and is substantially the same as the above reachability algorithm, although the proof of the second step is based on somehow different reasonings.

Step I

Construct a matrix $Q^{-1} \in Gl(K, n)$ having the last $n-r'$ columns orthogonal to the space spanned by the rows of O. The realization $(\tilde A_0, \tilde A_1, \tilde A_2, \tilde B, \tilde C)$ defined by

$$\begin{aligned} \tilde A_0 &= Q A_0 Q^{-1} \\ \tilde A_1 &= Q A_1 Q^{-1} \\ \tilde A_2 &= Q A_2 Q^{-1} \\ \tilde B &= QB \\ \tilde C &= CQ^{-1} \end{aligned}$$

satisfies

$$\begin{aligned} & C(I - A_0 z_1 z_2 - A_1 z_2 - A_2 z_2)^{-1} B \\ & \quad = \tilde C(I - \tilde A_0 z_1 z_2 - \tilde A_1 z_1 - \tilde A_2 z_2)^{-1} \tilde B \\ & \quad = \Sigma_{ij} \tilde C \tilde M_{ij} \tilde B z_1{}^i z_2{}^j \end{aligned}$$

In the associated observability matrix

$$\tilde O = \begin{bmatrix} \tilde C \ M_{00} \\ \tilde C \ M_{10} \\ \vdots \\ \tilde C \ \tilde M_{n-1,\, n-1} \end{bmatrix}$$

the elements in the last $n-r'$ columns are zeros.

Step II

Write $\tilde{A}_0$, $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$, $\tilde{C}$ in partitioned form as in step 2 above, and notice that $\tilde{C}=[\tilde{C}_1\ 0]$, $\tilde{C}_1 \in K^{1\times r'}$. To show that $(\tilde{A}_{11}{}^{(0)}, \tilde{A}_{11}{}^{(1)}, \tilde{A}_{11}{}^{(2)}, \tilde{B}_1, \tilde{C}_1)$ is a realization we have to prove that

$$\begin{aligned}\Sigma_{ij}\tilde{C}\tilde{M}_{ij}\tilde{B}z_1{}^iz_2{}^j &= \tilde{C}(I-\tilde{A}_0z_1z_2-\tilde{A}_1z_1-\tilde{A}_2z_2)\tilde{B}\\ &= \tilde{C}_1(I-\tilde{A}_{11}{}^{(0)}z_1z_2-\tilde{A}_{11}{}^{(1)}z_1-\tilde{A}_{11}{}^{(2)}z_2)^{-1}\tilde{B}_1\\ &\triangleq \Sigma_{ij}\tilde{C}_1M_{ij}\tilde{B}_1z_1{}^iz_2{}^j\end{aligned}$$

namely that

$$\tilde{C}\tilde{M}_{ij}\tilde{B}=\tilde{C}_1M_{ij}{}^*\tilde{B}_1, \quad i, j=0, 1, \ldots$$

Observe that for $i, j>0$

$$\tilde{C}\tilde{M}_{ij}=\tilde{C}\tilde{M}_{i-1,\,j}\tilde{A}_1+\tilde{C}\tilde{M}_{i,\,j-1}\tilde{A}_2+\tilde{C}\tilde{M}_{i-1,\,j-1}\tilde{A}_0$$

$$\tilde{C}_1M^*_{ij}=\tilde{C}_1M^*_{i-1,\,j}\tilde{A}_{11}{}^{(1)}+\tilde{C}_1M^*_{i,\,j-1}\tilde{A}_{11}{}^{(2)}+\tilde{C}_1M^*_{i-1,\,j-1}\tilde{A}_{11}{}^{(0)}$$

and assume by induction that the first r' elements of $\tilde{C}\tilde{M}_{i-1,\,j}$, $\tilde{C}\tilde{M}_{i,\,j-1}$, $\tilde{C}\tilde{M}_{i-1,\,j-1}$ coincide with $\tilde{C}_1M^*_{i-1,\,j}$, $\tilde{C}_1M^*_{i,\,j-1}$, $\tilde{C}_1M^*_{i-1,\,j-1}$ respectively. Then

$$\tilde{C}\tilde{M}_{i-1,\,j}\tilde{A}_1=[\tilde{C}_1M^*_{i-1,\,j}\ \ 0]\begin{bmatrix}\tilde{A}_{11}{}^{(1)} & \tilde{A}_{12}{}^{(1)}\\ \tilde{A}_{21}{}^{(1)} & \tilde{A}_{22}{}^{(1)}\end{bmatrix}=[\tilde{C}_1M^*_{i-1,\,j}\ \ \tilde{A}_{11}{}^{(1)}]$$

and similarly

$$\tilde{C}\tilde{M}_{i,\,j-1}\tilde{A}_2=[\tilde{C}_1M^*_{i,\,j-1}\ \ \tilde{A}_{11}{}^{(2)}\ \ *]$$

$$\tilde{C}\tilde{M}_{i-1,\,j-1}\tilde{A}_0=[\tilde{C}_1M^*_{i-1,\,j-1}\ \ \tilde{A}_{11}{}^{(0)}\ \ *]$$

from which it is clear that the first r' elements in $\tilde{C}\tilde{M}_{ij}$ are the same as in $\tilde{C}_1M^*_{ij}$. Of course, the last $n-r'$ elements are zero because of the structure of $\tilde{O}$.

Evidently the result that we have proved can also be stated in the following form :

Proposition 4.1

Let $\Sigma=(A_0, A_1, A_2, B, C)$ be any realization of $\mathscr{S}$. Then a 1*-reachable and* 1*-observable realization can be constructed in a finite number of steps from Σ following the procedure above.*

Corollary

Every minimal realization is completely 1-reachable and 1-observable.

Remark. As we shall see in § 6, in general the converse of the above corollary does not hold.

5. Filters characterized by recognizable series

The Hankel matrix associated with a series $s \in K[[z_1, z_2]]$ is given by

$$\mathscr{H}(s) = \begin{bmatrix} (s, 1) & (s, z_1) & (s, z_2) & (s, z_1^2) & (s, z_1 z_2) & (s, z_2^2) & \cdots\cdots \\ (s, z_1) & (s, z_1^2) & (s, z_1 z_2) & (s, z_1^3) & \cdots\cdots & & \\ (s, z_2) & (s, z_1 z_2) & \cdots\cdots & & & & \\ \cdots\cdots & & & & & & \end{bmatrix}$$

In Fliess (1970) it is proved that the rank of $\mathscr{H}(s)$ is finite if and only if s is rational and a denominator q of s can be factorized as $q = q_1 q_2$ with $q_1 \in K[z_1^{-1}]$, $q_2 \in K[z_2^{-1}]$. The series satisfying this property are the elements of the ring $K[(z_1)] \otimes K[(z_2)] \triangleq K^{\mathrm{rec}}[(z_1, z_2)]$ called the ring of '*recognizable series*'.

A series s is recognizable (Fliess 1970) iff there exist an integer $m \geqslant 1$ and four matrices $B \in K^{m \times 1}$, $C \in K^{1 \times m}$, A_1, $A_2 \in K^{m \times m}$, with $A_1 A_2 = A_2 A_1$, such that

$$s = \sum_{ij\,0}^{\infty} C A_1{}^i A_2{}^j B z_1{}^i z_2{}^j$$

This fact can be stated in a more formal way saying that s is recognizable iff there exist an integer $m \geqslant 1$, a representation μ on $K^{m \times m}$ of the commutative monoid generated by z_1 and z_2, two matrices $C \in K^{1 \times m}$ and $B \in K^{m \times 1}$ such that

$$s = \Sigma_{ij} (C\mu(z_1{}^i z_2{}^j)B, z_1{}^i z_2{}^j) z_1{}^i z_2{}^j$$

The integer m is the dimension of the representation and one proves that rank of $\mathscr{H}(s)$ gives the dimension of the minimal representation.

If we assume that the series s, characterizing the two-dimensional filter $\mathscr{S}$, is recognizable, we can exploit the above-mentioned properties of this class of series for getting an interesting subclass of realizations of $\mathscr{S}$.

Proposition 5.1

Let s belong to $K_c{}^{\mathrm{rec}}[(z_1, z_2)]$ and let s be represented by (5). *Then* $(-A_1 A_2, A_1, A_2, B, C)$ *is a realization of the filter $\mathscr{S}$. Vice versa, let (A_0, A_1, A_2, B, C) be a realization of $\mathscr{S}$ satisfying $-A_0 = A_1 A_2 = A_2 A_1$. Then the filter $\mathscr{S}$ is characterized by a series s belonging to $K_c{}^{\mathrm{rec}}[(z_1, z_2)]$.*

Proof

By (5) we have

$$s = (z_1 z_2) C \left(\sum_{ij\,0}^{\infty} A_1{}^i A_2{}^j z_1{}^i z_2{}^j \right) B = (z_1 z_2) C \sum_{i\,0}^{\infty} (A_1 z_1 + A_2 z_2 - A_1 A_2 z_1 z_2)^i B$$

$$= (z_1 z_2) C (I - A_1 z_1 - A_2 z_2 + A_1 z_1 z_2)^{-1} B$$

This proves that $(-A_1 A_2, A_1, A_2, B, C)$ is a realization of $\mathscr{S}$. The converse is immediate.

The realizations satisfying the condition $A_0 = -A_1 A_2 = -A_2 A_1$ are called '*representations*' of $\mathscr{S}$. As will evident, the study of this subclass of realizations is particularly simple, the Hankel matrix associated with the series characterizing the filter has the same structural properties as in the linear system theory, and reachability, observability and minimality are also related in the same way. In fact we have :

(i) When $\mathscr{S}$ is characterized by a recognizable series s, the matrices R_∞ and O_∞, associated with a representation $(-A_1A_2, A_1, A_2, B, C)$ assume the form

$$R=[B \;\; A_1B \;\; A_2B \;\; A_1{}^2B \;\; A_1A_2B \;\; A_2{}^2B \ldots]$$

$$O=\begin{bmatrix} C \\ C\;A_1 \\ C\;A_2 \\ C\;A_1{}^2 \\ C\;A_1A_2 \\ C\;A_2{}^2 \\ \vdots \end{bmatrix}$$

so that :

$$\mathscr{H}((z_1z_2)^{-1}s)=O_\infty R_\infty$$

(ii) The rank of $\mathscr{H}(s)$, $s\in K_c{}^{\mathrm{rec}}[(z_1z_2)]$, provides the dimension of a minimal realization in the class of representation. Moreover, if $(-A_1A_2, A_1, A_2, B, C)$ and $(-\tilde{A}_1\tilde{A}_2, \tilde{A}_1, \tilde{A}_2, \hat{B}, \tilde{C})$ are two minimal representations of dimension $m=\operatorname{rank}\mathscr{H}(s)$, then there exists a non-singular matrix $T\in K^{\overline{m}\times\overline{m}}$ such that

$$T\tilde{A}_iT^{-1}=A_i, \quad i=1,2$$
$$T\tilde{B}=B$$
$$\tilde{C}T^{-1}=C$$

Property (i) was pointed out in Attasi (1973) and Fornasini and Marchesini (1975). The following proposition is a direct consequence of (i) and (ii).

Proposition 5.2

Let $s\in K_c{}^{\mathrm{rec}}[(z_1, z_2)]$. *Then a representation* $(-A_1A_2, A_1, A_2, B, C)$ *is minimal iff the reachability and observability matrices are full rank.*

Proof

Assume that O_∞ and R_∞ are full rank. Then $\operatorname{rank} O_\infty=\operatorname{rank} R_\infty=\operatorname{rank}\mathscr{H}(s)$ which is the dimension of the minimal representation.

Conversely, if $(-A_1A_2, A_1, A_2, B, C)$ is a minimal representation of dimension n, then $n=\operatorname{rank}\mathscr{H}(s)=\operatorname{rank} O_\infty R_\infty$.

Consequently $\operatorname{rank} O_\infty=\operatorname{rank} R_\infty=n$.

6. Minimality of realizations

As it was remarked at the end of § 4, 1-reachability and 1-observability properties of a realization do not imply the minimality of the realization.

This can be proved by means of the following example.

Example

Assume $s \in R^{\mathrm{rec}}[(z_1, z_2)]$ be given by

$$s = (z_1 z_2) \frac{1 + z_1 + z_2}{1 + z_1 + z_2 + z_1 z_2}$$

Therefore the formal power series expansion of $(z_1 z_2)^{-1} s$ is expressed by

$$(1 + z_1 + z_2) \sum_{0}^{\infty} {}_k \, (-1)^k (z_1 + z_2 + z_1 z_2)^k = 1 - z_1 z_2 + z_1 z_2^2 + z_1^2 z_2 + \ldots$$

Since rank $\mathscr{H}((z_1 z_2)^{-1} s) > 2$, the dimension of minimal representations is greater than two. Hence the filter characterized by s admits 1-reachable and 1-observable realizations (i.e. the minimal representations) with dimension greater than two.

However, these minimal representations do not constitute minimal realizations. In fact the doubly-indexed dynamical system $\Sigma = (A_0, A_1, A_2, B, C)$ with

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \;\; 1]$$

is a realization with dimension two.

From the above remarks it appears that the Hankel matrix $\mathscr{H}(s)$ is not relevant for evaluating the dimension of minimal realizations of s. In fact, when s is rational, but not recognizable, rank $\mathscr{H}(s)$ is infinite, and for s recognizable rank $\mathscr{H}(s)$ furnishes solely the dimension of minimal representations, which are not necessarily minimal as realizations.

In the sequel we will associate with the commutative series s a family of non-commutative power series in three variables whose Hankel matrices have finite rank. We will prove that the minimal rank of these matrices is the dimension of minimal realizations of s.

Since rational series in non-commutative variables are recognizable (Fliess 1970), for any non-commutative rational series $r \in K\langle(x_1, x_2, x_3)\rangle$ there exist an integer m, a representation $\mu : X \to K^{m \times m}$, two matrices $B \in K^{m \times 1}$ and $C \in K^{1 \times m}$ such that

$$r = \sum_{w \in X^*} C\mu(w) B w$$

where X^* is the free monoid generated by x_1, x_2, x_3.

This is equivalent to saying that any non-commutative rational power series r can be expressed as

$$r = \sum_{1}^{\infty} {}_i \, C(A_1 x_1 + A_2 x_2 + A_0 x_3)^i B = C(I - A_1 x_1 - A_2 x_2 - A_0 x_3)^{-1} B$$

where $A_1=\mu(x_1)$, $A_2=\mu(x_2)$, $A_0=\mu(x_3)$ and A_1, A_2, A_3 do not necessarily commute.

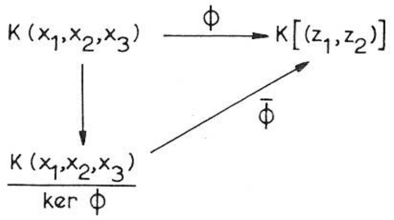
The dimension of the minimal representation of r is given by rank $\mathscr{H}(r)$.

Define the algebra morphism $\phi : K\langle(x_1, x_2, x_3)\rangle \to K[[z_1, z_2]]$ by $\phi(k)=k$, $\forall k \in K$, $\phi(x_1)=z_1$, $\phi(x_2)=z_2$, $\phi(x_3)=z_1z_2$. Since

$$\phi : C(I-A_1x_1-A_2x_2-A_0x_3)^{-1}B \to C(I-A_1z_1-A_2z_2-A_0z_1z_2)^{-1}B \qquad (5)$$

all rational series in the commutative variables z_1 and z_2 are obtained by varying A_1, A_2, A_0, B, C and the map ϕ is onto $K[(z_1, z_2)]$. Then we can associate a commutative series $\phi(r) \in K[(z_1, z_2)]$ with each non-commutative series $r \in K\langle(x_1, x_2, x_3)\rangle$. By (5) each representation of r induces a realization of $\phi(r)$.

The following diagram



commutes and $\bar{\phi}$ is an isomorphism. Consequently, given $s \in K_c[(z_1, z_2)]$, a minimal realization of s is a minimal representation in the class of representations of $\bar{\phi}^{-1}(s)$.

One has thus arrived at the following proposition :

Proposition 6.1

Let s belong to $K_c[(z_1, z_2)]$. *Then the dimension of minimal realization of s is given by*

$$\min_{r \in \bar{\phi}^{-1}(s)} \operatorname{rank} \mathscr{H}(r)$$

7. Conclusions

In this paper we introduced the realization of two-dimensional filters and we discussed the related concepts of reachability and observability. We have presented an algorithm for obtaining a reachable and observable realization starting from a generic one.

In general, the reachability and observability properties do not guarantee that we are dealing with a minimal realization, as we proved by means of an example. If we restrict ourselves to the subclass of realizations, called ‘ representations ’, essentially characterized by commutative matrices, the Hankel matrix associated with the recognizable series plays the same rule as in linear system theory. In this case 1-reachable and 1-observable representations are also minimal and their dimension is given by the rank of the Hankel matrix.

If we consider the whole class of realizations of a rational power series s, the dimension of minimal realizations is not related to the rank of $\mathscr{H}(s)$. Nevertheless we can associate the commutative series s with a family of non-commutative recognizable power series whose representations provide all the realizations of s. Hence the problem of determining minimal realizations of s can be solved looking for minimal representations of non-commutative power series.

References

Anderson, B. D. O., and Jury, E. I., 1974, *I.E.E.E. Trans. Circuits Syst.*, **21**, 300.
Attasi, A., 1973, IRIA Rapport No. 31, September.
Fliess, M., 1970, *Bull. Soc. Math.*, **94**, 231.
Fornasini, E., and Marchesini, G., 1975, *Variable Structure with Application to Economics and Biology*, edited by A. Ruberti and R. Mohler, Springer Lecture Notes in Economics and Math. Syst., No. 111.
Habibi, A., 1972, *Proc. Instn elect. electron. Engrs*, **60**, 878.
Powell, S. R., and Silverman, L. M., 1974, *I.E.E.E. Trans. autom. Control*, **19**, 8.
Roesser, R. P., 1975, *I.E.E.E. Trans. autom. Control*, **20**, 1.
Shanks, J. L., Treitel, S., and Justice, J. H., 1972, *I.E.E.E. Trans. audio Electroacoust.*, **20**, 115.