RESEARCH TOPICS IN LINEAR SYSTEMS ON PARTIALLY ORDERED TIME SETS

by

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I.- INTRODUCTION.-

Current research in systems on partially ordered time sets is strongly motivated by specific technical problems, such as image processing requirements, biological systems description, discretization of partial differential equations. Aside from the possible applications, another significant aspect of these studies relies in the extension of standard system theoretic results to more complex structures, the ultimate goal being the construction of satisfactory state space models on p.o. time sets.

This paper seeks to make a contribution to the structural analysis mentioned above by focusing on some recent results which show promise for the future.

The first part of the paper deals with the stability problem of 2-D systems. Although 2-D systems have received a considerable amount of contributions (an extensive list of references is included in [1]), "internal" stability criteria have been studied only very recently [2, 3]. The result we present is the generalization to 2-D systems of a classical theorem concerning asymptotic stability of finite dimensional linear systems.

In the second part of the paper a realization theory of linear systems over free group time sets is outlined. The tool of Nerode equivalence classes allows to derive recursive state equations and to characterize finite dimensional "realizable" input/output maps.

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II. - INTERNAL STABILITY OF 2-D SYSTEMS.

In the past few years several 2-D system models have been considered in the literature. We shall refer here to the model presented in [2], since it seems to be the most general one. Accordingly, a 2-D system is identified with the following pair of difference equations

$$
\begin{aligned}
x(h+1,k+1) &= A_1 x(h,k+1) + A_2 x(h+1,k) + B_1 u(h,k+1) + B_2 u(h+1,k) \\
y(h,k) &= Cx(h,k)
\end{aligned}
\tag{2.1}
$$

where (h,k) are elements of $\mathbb{Z} \times \mathbb{Z}$ ("time set") partially ordered by the product of the orderings, $x : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^n$ is a map whose value at time (h,k) is the "local state at time (h,k)", $u : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ and $y : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ are the "input" and the "output" maps respectively, and $u(h,k)$, $y(h,k)$ are the input and the output values at time (h,k). $A_1, A_2 \in \mathbb{R}^{n \times n}$, $B_1, B_2 \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ are matrices which completely characterize the 2-D system (2.1).

When an input u is given, solving (2.1) requires information about a set of local states we call an initial "global state". Let us define the global state $\mathcal{X}_i$ as the collection

$$\mathcal{X}_i = \{x(h,k), h+k=i\} .$$

Then the computation of $x(h,k)$ for any (h,k) in $\mathcal{T}_i = \{(h,k) : h+k \geq i\}$ can be performed from $\mathcal{X}_i$ and the values of u in $\mathcal{T}_i$.

The notion of internal stability of a 2-D system is related to the asymptotic behaviour of the states free evolution resulting from a bounded initial global state. Denote by $\|x\|$ the euclidean norm of x and introduce the shorthand notation $\|\mathcal{X}_i\| = \sup_{h+k=i} \|x(h,k)\|$.

Definition 2.1. - The 2-D system (2.1) is internally stable if, given $\varepsilon > 0$ and $\mathcal{X}_i$, $\|\mathcal{X}_i\| < \infty$ and assuming $u = 0$, there exists a positive integer M such that $\|x(h,k)\| < \varepsilon$ whenever $h+k > i+M$.

Remark. The straight line shaped global states $\mathcal{X}_i$ constitute a subclass of a wider set of possible initial conditions for (2.1). It turns out [3] that Definition 2.1 is easily extended to more general situations, and internal stability does not depend on any particular support of the global state. The search for internal stability criteria is a natural topic of investigation at this point. The first criterion we shall state provides the 2-D counterpart of the 1-D criterion which relates internal stability of the system $x(h+1) = Ax(h) + Bu(h)$ and the root location of the characteristic polynomial of A.

Theorem 2.2. [2] - The 2-D system (2.1) is internally stable iff the polynomial $\det(I - A_1 z_1 - A_2 z_2)$ is devoid of zeros in the closed polydisc

$$\mathcal{P}_1 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| \leq 1, |z_2| \leq 1\} .$$

Input/output stability tests for 2-D filters [4, 5] reduce to checking the (non) intersection of $\mathcal{P}_1$ with the complex variety of a polynomial $q \in \mathbb{R}[z_1, z_2]$. Theorem 2.2 makes those tests suitable for internal stability analysis. In particular, Huang's criterion [5] gives the following.

Corollary 2.3.- The 2-D system (2.1) is internally stable iff the matrix $A_1 + A_2 e^{j\omega}$ is stable (i.e. the magnitudes of its eigenvalues are less than 1) for every $\omega \in \mathbb{R}$.

Proof : Recall first Huang's criterion : a polynomial $q \in \mathbb{R}[z_1, z_2]$ is devoid of zeros in $\mathcal{P}_1$ iff i) $q(z_1, 0) \neq 0$ for $|z_1| \leq 1$ and ii) $q(z_1, z_2) \neq 0$ for $|z_1| = 1$ and $|z_2| \leq 1$. Assume now $A_1 + A_2 \exp(j\omega)$ to be stable, $\forall \omega \in \mathbb{R}$. The images of $\mathcal{P}_1$ given by the polynomial functions $q_1(z_1, \eta) = \det(I - A_1 z_1 - A_2 z_1 \eta)$ and $q_2(\eta, z_2) = \det(I - A_1 z_2 \eta - A_2 z_2)$ coincide respectively with the images of $\mathcal{P}_1 \cap \{(z_1, z_2) : |z_1| \geq z_2\}$ and $\mathcal{P}_1 \cap \{(z_1, z_2) : |z_2| \geq |z_1|\}$ given by the polynomial function $\det(I - A_1 z_1 - A_2 z_2)$. Since $q_1(0, \eta) \neq 0$ and $q_1(z_1, e^{j\omega}) \neq 0$ for $|z_1| \leq 1$ by the stability assumption on $A_1 + A_2 \exp(j\omega)$, $q_1(z_1, \eta)$ is devoid of zeros in $\mathcal{P}_1$. The same property holds for $q_2(\eta, z_2)$ proving $\det(I - A_1 z_1 - A_2 z_2) \neq 0$ in $\mathcal{P}_1$. Conversely, internal stability implies $\det(I - A_1 z_1 - A_2 z_1 \exp(j\omega)) \neq 0$ for $|z_1| \leq 1$ and $\omega \in \mathbb{R}$. Hence $A_1 + A_2 e^{j\omega}$ is stable. ∎

The second criterion extends to 2-D systems the following well known chain of equivalences : the system $x(h+1) = Ax(h) + Bu(h)$ is internally stable $\Leftrightarrow$ the series $\sum_{i=0}^{\infty} \|A^i\|$ converges $\Leftrightarrow$ $\|A^k\| < 1$ for some $k > 0$ $\Leftrightarrow$ the series

$\sum_{i=0}^{\infty} (A^i)^T A^i$ converges. In order to get corresponding statements for 2-D systems, let define inductively the matrices $A_1 \overset{r}{\sqcup\!\sqcup} \overset{s}{} A_2$, $r, s \in \mathbb{N}$:

$$A_1 \overset{r}{\sqcup\!\sqcup}\overset{0}{} A_2 = A_1^r, \quad A_1 \overset{0}{\sqcup\!\sqcup}\overset{s}{} A_2 = A_2^s$$

$$A_1 \overset{r}{\sqcup\!\sqcup}\overset{s}{} A_2 = A_1(A_1 \overset{r-1}{\sqcup\!\sqcup}\overset{s}{} A_2) + A_2(A_1 \overset{r}{\sqcup\!\sqcup}\overset{s-1}{} A_2) \text{ if } r, s > 0 .$$

Theorem 2.4. - The following are equivalent : i) (2.1) is internally stable, ii) the series $\sum_{r,s=0}^{\infty} \|A_1 \overset{r}{\sqcup\!\sqcup}\overset{s}{} A_2\|$ converges, iii) $\sum_{r+s=k} \|A_1 \overset{r}{\sqcup\!\sqcup}\overset{s}{} A_2\| < 1$ for some positive integer k, iv) the series $\sum_{r,s=0}^{\infty} (A_1 \overset{r}{\sqcup\!\sqcup}\overset{s}{} A_2)^T (A_1 \overset{r}{\sqcup\!\sqcup}\overset{s}{} A_2)$ converges.

Proof : i) $\Leftrightarrow$ ii) . By theorem 2.2, internal stability implies $\det(I-A_1z_1-A_2z_2) \neq 0$ in $\mathcal{P}_{1+\varepsilon} = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| \leq 1+\varepsilon, |z_2| \leq 1+\varepsilon\}$ for some real $\varepsilon > 0$. Hence $(I-A_1z_1-A_2z_2)^{-1}$ admits a normally convergent power series expansion in $\mathcal{P}_1$. Conversely, let $\sum_{r,s=0}^{\infty} \|A_1 \overset{r}{\sqcup\!\sqcup}\overset{s}{} A_2\| < \infty$, $\|\mathcal{X}_0\| = M < \infty$. If $(h, k) \in \mathcal{T}_t$, $t > 0$, letting $t \uparrow \infty$ gives

$$\|x(h, k)\| = \sum_{i+j=t} \|A_1 \overset{i}{\sqcup\!\sqcup}\overset{j}{} A_2\, x(h-i, k-j)\| \leq M \sum_{i+j=t} \|A_1 \overset{i}{\sqcup\!\sqcup}\overset{j}{} A_2\| \to 0$$

ii) $\Rightarrow$ iii) is obvious. The converse depends on the following lemma.

Lemma 2.5. - For any pair of non negative integers p and q

$$\sum_{\mu+\nu=p+q} \|A_1 \overset{\mu}{\sqcup\!\sqcup}\overset{\nu}{} A_2\| \leq \sum_{i+j=p} \|A_1 \overset{i}{\sqcup\!\sqcup}\overset{j}{} A_2\| \sum_{r+s=q} \|A_1 \overset{r}{\sqcup\!\sqcup}\overset{s}{} A_2\| .$$

Proof : Let $\mu, \nu, p \in \mathbb{N}$ and $\mu+\nu \geq p$. An easy inductive argument shows that

$$A_1 \overset{\mu}{\sqcup\!\sqcup}\overset{\nu}{} A_2 = \sum_{i+j=p,\, i+r=\mu,\, j+s=\nu} (A_1 \overset{i}{\sqcup\!\sqcup}\overset{j}{} A_2)(A_1 \overset{r}{\sqcup\!\sqcup}\overset{s}{} A_2) \tag{2.3}$$

(2.3) and standard norm inequalities imply

$$\sum_{\mu+\nu=p+q} \|A_1 \overset{\mu}{\sqcup\!\sqcup}\overset{\nu}{} A_2\| \leq \sum_{\mu+\nu=p+q} \quad \sum_{i+r=\mu,\, j+s=\nu,\, i+j=p} \|A_1 \overset{i}{\sqcup\!\sqcup}\overset{j}{} A_2\| \, \|A_1 \overset{r}{\sqcup\!\sqcup}\overset{s}{} A_2\|$$

$$\leqslant \sum_{i+j=p,\ r+s=q} \|A_1^i \sqcup\!\sqcup^j A_2\| \, \|A_1^r \sqcup\!\sqcup^s A_2\| \ . \quad \blacksquare$$

Now we continue with the proof of the implication iii) $\Rightarrow$ ii) in Theorem 2.4. Upon setting $N = \max\limits_{0 \leqslant \tau < k} \{ \sum\limits_{r+s=\tau} \|A_1^r \sqcup\!\sqcup^s A_2\| \}$ and $h = kq+\rho$, $0 \leqslant \rho < k$, $\forall h \in \mathbb{N}$, Lemma 2.5 gives $\sum\limits_{r+s=h} \|A_1^r \sqcup\!\sqcup^s A_2\| \leqslant N(\sum\limits_{i+j=k} \|A_1^i \sqcup\!\sqcup^j A_2\|)^q$. Convergence of $\sum\limits_{r,s=0}^{\infty} \|A_1^r \sqcup\!\sqcup^s A_2\|$ follows from the bound

$$\sum_{r,s=0}^{\infty} \|A_1^r \sqcup\!\sqcup^s A_2\| = \sum_{q=0}^{\infty} \sum_{\rho=0}^{k-1} \sum_{r+s=kq+\rho} \|A_1^r \sqcup\!\sqcup^s A_2\| \leqslant \sum_{q=0}^{\infty} Nk(\sum_{i+j=k} \|A_1^r \sqcup\!\sqcup^s A_2\|)^q$$

$$= Nk(1 - \sum_{i+j=k} \|A_1^i \sqcup\!\sqcup^j A_2\|)^{-1}$$

ii) $\Rightarrow$ iv) Convergence of $\sum\limits_{r,s=0}^{\infty} \|A_1^r \sqcup\!\sqcup^s A_2\|$ implies that of $\sum\limits_{r,s=0}^{\infty} \|A_1^r \sqcup\!\sqcup^s A_2\|^2$, hence of $\sum\limits_{r,s=0}^{\infty} (A_1^r \sqcup\!\sqcup^s A_2)^T (A_1^r \sqcup\!\sqcup^s A_2)$.

iv) $\Rightarrow$ iii) Assume by contradiction $\sum\limits_{r+s=k} \|A_1^r \sqcup\!\sqcup^s A_2\| \geqslant 1$ for every $k \geqslant 0$. Then $\sum\limits_{r+s=k} \|A_1^r \sqcup\!\sqcup^s A_2\|^2 \geqslant (k+1)^{-1}$, $k=0,1,2,\ldots,$ by an obvious constrained minimization argument. Divergence of harmonic series implies that of $\sum\limits_{r,s=0}^{\infty} \|A_1^r \sqcup\!\sqcup^s A_2\|^2$ and $\sum\limits_{r,s=0}^{\infty} (A_1^r \sqcup\!\sqcup^s A_2)^T (A_1^r \sqcup\!\sqcup^s A_2)$ would not converge. $\blacksquare$

III. - REALIZATION OF LINEAR SYSTEMS ON FREE GROUPS.

An easy detectable feature of (2.1) is the "pathwise" structure the local state updating rule induces on the input/output relation. The output value in (h, k) determined by the input value $u(i, j) = 1$, $(i, j) < (h, k)$ is

$$C[(A_1^{h-i-1} \sqcup\!\sqcup^{k-j} A_2) B_1 + (A_1^{h-i} \sqcup\!\sqcup^{k-j-1} A_2) B_2] \tag{3.1}$$

and repeated applications of (2.2) split up (3.1) in a bunch of addenda

$$C A_{i_1} A_{i_2} \ldots A_{i_{p-1}} B_{i_p} \qquad p = h+k-i-j \ , \ i = 1 \text{ or } 2 \tag{3.2}$$

biuniquely associated to the forward paths connecting (i,j) and (h,k) in $\mathbb{Z}\times\mathbb{Z}$. However, while (3.1) is immediately identified as a sample of an unit impulse response, the meaning of its components (3.2) is quite elusive. A single path contribution cannot be "separated" from other concomitant contributions, in the sense that i/o experiments on (2.1) are not sufficient for singling out values (3.2). This is related to the possibility of describing the same 2-D filter by means of infinitely many noncommutative power series, all of them having the same commutative image, i.e. the impulse response of the filter [6].

In this section we resort to a free group time structure for getting another interpretation of i/o maps defined by noncommutative power series and we derive a dynamical model which allows to reconstruct the maps from i/o experiments.

Given the alphabet $\Xi = \{\xi_1,\xi_2\}$, let consider the free group G generated by Ξ. G is determined up to isomorphisms [7], and we choose as a model of G the free monoid $\{\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}\}^*$ modulo the commutation rules $\xi_1\xi_1^{-1} = \xi_2\xi_2^{-1} = \xi_1^{-1}\xi_1 = \xi_2^{-1}\xi_2 = e$ (e identity of $\{\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}\}^*$. Then e is the identity of G, every element of G is uniquely represented by a word of $\{\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}\}^*$ in which ξ_1 (ξ_2) cannot be immediate predecessor nor immediate successor of ξ_1^{-1} (ξ_2^{-1}), and multiplication is induced by the concatenation in $\{\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}\}^*$. The free monoids $\Xi^* = \{\xi_1,\xi_2\}^*$ and $(\Xi^{-1})^*$ are imbedded in G in the natural way.

Given two elements a,b in G, we agree to write $a \leqslant b$ if $b = pa$ for some p in Ξ^*. G is partially ordered by the $\leqslant$ relation, and we shall refer to it as to the "time set". For any a in G the "past" and the "future" of a are the sets $P^a = \{b : a = pb,\ p \in \Xi^* \setminus \{e\},\ b \in G\}$ and $F_a = \{b : b = pa,\ p\in\Xi^*,\ b\in G\}$ respectively.

Let K be a field. A K-valued past-compact support function on G is a map $u : G \to K$ which satisfies $\# \{\text{spt } u \cap P^a\} < \infty$, for any a in G ("spt" means "support"). We shall adopt the formal series notation $u = \sum_{a\in G} (u,a)a$, where (u,a) denotes the value of u at time a. The set $K \lessdot G \gtrdot$ of K valued p.c.s. functions on G inherits from K^G the K-vector space structure and its subspaces $K<G>$ (space of compact support functions on G) $K<\xi_1,\xi_2>$, $K<\xi_1^{-1},\xi_2^{-1}>$, $K\langle\langle\xi_1,\xi_2\rangle\rangle$ are K-algebras, the ring multiplication being defined as a convolution product

$$uv = \sum_a (u,a)a \sum_b (v,b)b = \sum_c \sum_{ab=c} (u,a)(v,b)c \quad . \tag{3.3}$$

Past compactness allows to introduce on $K\langle\langle G\rangle\rangle$ a right or left module structure over anyone of the algebras above just by extending the convolution product (3.3).

Definition 3.1. - Let $\mathcal{U}$ (input space) = $\mathcal{Y}$ (output space) = $K\langle\langle G\rangle\rangle$. A function $F : \mathcal{U} \to \mathcal{Y}$ is an i/o map on G if it is i) linear : $F(u\alpha + v\beta) = F(u)\alpha + F(v)\beta$, ii) time invariant : $F(u\xi_i^{-1}) = F(u)\xi_i^{-1}$, $i = 1, 2$ iii) causal :$(F(u), a) = (F(v), a)$ if $u \equiv v$ on P^a , $\forall u, v \in \mathcal{U}$, $\alpha, \beta \in K$, $a \in G$.

Let $\underline{1} \in \mathcal{U}$ denote the unit impulse function at time e : $(\underline{1}, e) = 1$, $(\underline{1}, a) = 0$ if $a \neq e$. Then $F(\underline{1}) \in K\langle\langle \xi_1, \xi_2\rangle\rangle$ and $(F(\underline{1}), e) = 0$ follow from linearity and causality assumptions. Time invariance and linearity in turn imply that F is a $K\langle G\rangle$ (in particular a $K\langle \xi_1^{-1}, \xi_2^{-1}\rangle$) right module homomorphism. Consequently, for any $u \in \mathcal{U}$, $F(u)$ is obtained from $F(\underline{1}) : F(u) = F(\underline{1})u$.

Denote by $u|^a$ and $u|_a$ the truncations of $u \in K\langle\langle G\rangle\rangle$ to the sets P^a and F_a respectively. Then $(F(u), a)$ is completely determined by $u|^a$ and, if $a \leq b$, $(F(u), b)$ depends on $u|^a$ because $\mathrm{spt}(u|^b) \supseteq \mathrm{spt}(u|^a)$. The question arises as to whether it is possible to "store" in some finite dimensional state vector at time a the amount of information about $u|^a$ which is relevant for computing $F(u)$ in F_a , and to built up a linear dynamical model describing the time evolution of state vectors. The way we undertake follows closely the canonical one in System Theory [8].

Definition 3.2. - u_1 and u_2 in $\mathcal{U}$ are Nerode-equivalent at time $a \in G$ ($u_1 \underset{a}{\sim} u_2$) if $(F(u_1|^a + v))|_a = (F(u_2|^a + v))|_a$ for any v in $\mathcal{U}$ such that $\mathrm{spt}\, v \cap P^a = \emptyset$.

Let introduce the map $f : K\langle \xi_1^{-1}, \xi_2^{-1}\rangle \to K\langle\langle \xi_1, \xi_2\rangle\rangle : u \mapsto (F(u))|_e$. f conveys as much information about the i/o relation as F does, since

$(F(u), a) = (f(ua^{-1}|^e), e)$ for any a in G and any u in $\mathcal{U}$. Upon setting $y.w = (yw)|^e$, y in $K\langle\langle \xi_1, \xi_2 \rangle\rangle$, w in $K\langle \xi_1^{-1}, \xi_2^{-1} \rangle$, $K\langle\langle \xi_1, \xi_2 \rangle\rangle$ can be viewed as a $K\langle \xi_1^{-1}, \xi_2^{-1} \rangle$ right module : this makes f a $K\langle \xi_1^{-1}, \xi_2^{-1} \rangle$ right module homomorphism and the quotient space $X = K\langle \xi_1^{-1}, \xi_2^{-1} \rangle / \ker f$ a $K\langle \xi_1^{-1}, \xi_2^{-1} \rangle$ right module. Proposition 3.3 allows to identify (as K-spaces) X and the set of Nerode classes at any time a.

<u>Proposition 3.3</u>.- <u>Let</u> $[u]$ <u>denote the coset</u> $u + \ker f$. <u>Then</u> $u_1 \underset{a}{\sim} u_2$ <u>iff</u> $[u_1 a^{-1}|^e] = [u_2 a^{-1}|^e]$.

<u>Proof</u> : Notice that, by the linearity assumption, we can set $v = 0$ in definition 3.2. So $u_1 \underset{a}{\sim} u_2 \Leftrightarrow (F(u_1 a^{-1}|^e))|_e = (F(u_2 a^{-1}|^e))|_e \Leftrightarrow u_1 a^{-1} \underset{e}{\sim} u_2 a^{-1} \Leftrightarrow f(u_1 a^{-1}|^e) = f(u_2 a^{-1}|^e) \Leftrightarrow [u_1 a^{-1}|^e] = [u_2 a^{-1}|^e]$. ∎

The updating of Nerode classes is a consequence of the module structure in X :

$$(3.4)\qquad \begin{aligned}[ua^{-1}|^e] &= [ua^{-1}|^{\xi_1^{-1}} + ua^{-1}|^{\xi_2^{-1}} + (ua^{-1}, \xi_1^{-1})\xi_1^{-1} + (ua^{-1}, \xi_2^{-1})\xi_2^{-1}] = \\ &= [u(\xi_1^{-1}a)^{-1}|^e]\,\xi_1^{-1} + [u(\xi_2^{-1}a)^{-1}|^e]\,\xi_2^{-1} + (u, \xi_1^{-1}a)[\xi_1^{-1}] + (u, \xi_2^{-1}a)[\xi_2^{-1}]\end{aligned}$$

The Nerode class at time a linearly depends on the Nerode classes and input values at times $\xi_1^{-1}a$ and $\xi_2^{-1}a$: in fact ξ_1^{-1} and ξ_2^{-1} act on X as linear transformations. Let $\bar{f} : X \to K\langle\langle \xi_1, \xi_2 \rangle\rangle$ be the map f induces on its quotient space X. Then $(F(u), a) = (f(ua^{-1}|^e), e) = (\bar{f}([ua^{-1}|^e]), e)$ shows that the output value $(F(u), a)$ is a linear functional on X.

If $\dim X = n < \infty$ and a basis in X has been chosen, the vectors $[ua^{-1}|^e]$ are represented by K-valued row n-tuples $x(a)$, the linear transformations ξ_1^{-1} and ξ_2^{-1} by matrices A_1, A_2 in $K^{n\times n}$, and $[\xi_1^{-1}]$, $[\xi_2^{-1}]$ by n-tuples B_1 and B_2 in $K^{1\times n}$:

$$(3.5)\qquad x(a) = x(\xi_1^{-1}a)A_1 + x(\xi_2^{-1}a)A_2 + (u, \xi_1^{-1}a)B_1 + (u, \xi_2^{-1}a)B_2 \ .$$

Moreover there exists C in $K^{n\times 1}$ such that

$$(3.6)\qquad y(a) = (F(u), a) = x(a)\,C \ .$$

A pair of equations like (3.5, 6) is called an FG system. If an input u in $K\langle\langle G\rangle\rangle$ is applied to an FG system "at rest" (i.e. $x(a)=0$ if $\operatorname{spt} u \cap p^a = 0$), the i/o relation the FG system generates is an i/o map on G whose impulse response is the noncommutative power series $(B_1\xi_1 + B_2\xi_2)(I-A_1\xi_1-A_2\xi_2)^{-1}C$.

Definition 3.4.- The FG system (3.5, 6) is a realization of an i/o map F on G if $F(\underline{1}) = (B_1\xi_1 + B_2\xi_2)(I-A_1\xi_1-A_2\xi_2)^{-1}C$.

Notice that the Nerode equivalence construction we described above provides a realization of F whenever $X = K\langle\xi_1^{-1}, \xi_2^{-1}\rangle/\ker f$ is finite dimensional. The following proposition is an easy consequence of the properties of noncommutative power series and Hankel matrices [9].

Proposition 3.5.- An i/o map F on G is realizable iff $F(\underline{1})$ is a recognizable power series. Then the Nerode equivalence realization exists and is the minimum dimensional one.

Remark.- When X is finite dimensional, the Schreier Lewin Theorem [10] relates the dimension of X to the number g of free generators of $\ker f$: $n = \dim X = g-1$. An intrinsic characterization of realizable i/o maps in terms of recurrent relations on the impulse response obtained from the free generators of X constitutes a topic of current investigation.

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