SYSTEM THEORETIC APPROACH TO MULTIDIMENSIONAL DIGITAL DATA PROCESSING

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Summary

The introduction of state-space models makes possible to attack multidimensional data processing from a system theoretic point of view. The paper deals with the realization of two-dimensional digital filters by 2-D systems. The structural properties of these dynamical systems are reviewed and the stability criterion is used to obtain roundoff error bounds.

1. Introduction

The analysis of remotely sensed data requires a massive use of digital multidimensional data processing for images filtering. It is not our purpose here of getting into details concerning the various fields of applications of these techniques, but we will rather take into account some formal structure properties of multidimensional filtering, and primarily the recursiveness, with the aim of introducing state space models (2-D systems[1-8]).

The way a 2-D system operates, corresponds to the recursion performed by two-dimensional filters in that the one-step filter updating can be derived by a one-step state updating of a 2-D system. The amount of computation at each step strongly depends on the dimension of the state in the dynamical model and this makes worth while to look for state space realizations with minimal dimension.

Stability is a highly desirable property of the filtering process. When we operate with state space models the stability requirements are transferred to the dynamics of the state.

When two-dimensional filtering process is done by a digital machine, the dynamics of roundoff error propagation is described by a 2-D system.

The accuracy of the computation depends on the error accumulation and it is possible to estimate it by deriving error bounds. As we shall show in the last section of this paper these bounds are easily obtained when the digital machine is an asymptotically stable 2-D system.

2. Input-Output Recursiveness and State Equations

In this section we are concerned with a concise exposition of state space realization of two-dimensional filters[2,5,6,9]. This requires an axiomatic description of their behaviour in terms of input and output functions and input-output maps.

The sets of input and output functions of a two-dimensional digital filter are subclasses of $K^{\mathbb{Z}\times\mathbb{Z}}$, K being a generic field. These functions are represented as formal power series in two indeterminates z_1 and z_2. To characterize the set of these functions assume in $\mathbb{Z}\times\mathbb{Z}$ the product of orderings and introduce the notions of "past" and "future" of a point (h,k) $\mathbb{Z}\times\mathbb{Z}$.

We shall call "past" of (h,k) the set of points (i,j) such that $i\leqslant h$, $j\leqslant k$, $(i,j)\neq(h,k)$ and "future" of (h,k) the set of points (i,j) with $h\leqslant i$, $k\leqslant j$.

We say that a function $u\in K^{\mathbb{Z}\times\mathbb{Z}}$ is past-finite if the intersection of the support of u and the past of any point in $\mathbb{Z}\times\mathbb{Z}$ is a finite set. The set of functions $\mathcal{U}$ we shall assume as admissible inputs to the filter are the past-finite functions in $K^{\mathbb{Z}\times\mathbb{Z}}$.

With this in mind we define a linear, stationary, digital filter in input-output form, as a map $F:\mathcal{U}\to K^{\mathbb{Z}\times\mathbb{Z}}$ which satisfies the following axioms:

(i) linearity

(ii) stationary:

$$F(z_1^h z_2^k u) = z_1^h z_2^k F(u),\quad \forall h,k\in\mathbb{Z},\quad u\in\mathcal{U}$$

(iii) causality:

the support of $F(1)$ belongs to the future of $(0,0)$.

We can directly check that $\operatorname{Im}F\subset\mathcal{U}$ and that the impulse response $F(1)$ constitutes the transfer function of the filter, for we have (in formal power series notation):

$$F(u) = F(1)u \qquad \forall u\in\mathcal{U}$$

In this way the input-output representations of two-dimensional filters are in one-to-one correspondence with the formal power series in z_1 and z_2 with zero constant term, called "causal formal power series" and denoted by $K_c[[z_1,z_2]]$.

This result generalizes the well known connection existing between input-output representations of one-dimensional systems and formal power series in one indeterminate.

The realization of one-dimensional systems is done by introducing a time vector function $x(\cdot)$, called the state of the system, which has a separation property with respect to the past, in the sense that the knowledge of this vector at any instant t is sufficient to evaluate the output at $\tau\geq t$. It is a very important result of one-dimensional realization theory that the state vector is finite dimensional if and only if the power series characterizing the input-output map is rational. In this case one obtains a state updating equation of the following form:

$$x(k+1) = Ax(k) + B\,u(k)$$
$$y(k) = Cx(k)$$

with A, B, C matrices of suitable dimensions.

When one deals with two-dimensional filters it is no longer possible to attach a vector having a separation property to every point (h,k) in $\mathbb{Z}\times\mathbb{Z}$ in such a way that the knowledge of this vector and of the input in the future of (h,k) makes possible to compute the output in the future of (h,k). Clearly this fact is intrinsic to the structure of partial ordering of $\mathbb{Z}\times\mathbb{Z}$, since a separation property should interest an infinite set of points.

However every point in the plane $\mathbb{Z}\times\mathbb{Z}$ can be associated with a "local state" vector which is uniquely determined by its past and whose evolution in its future is governed by a difference-equation of the following type[8]:

$$x(h+1,k+1) = A_1x(h+1,k) + A_2x(h,k+1) + B_1u(h+1,k) + B_2u(h,k+1) \tag{1}$$
$$y(h,k) = Cx(h,k)$$

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where A_1, A_2, B_1, B_2, C are matrices of suitable dimensions.

Since we deal with difference equations, we need a set of initial data for solving (1).

Let consider in $\mathbb{Z} \times \mathbb{Z}$ the sets $\mathcal{C}$, called "separation sets", which are built in such a way that the knowledge of all local states in $\mathcal{C}$ and of the input in the future of any point belonging to $\mathcal{C}$ (future of $\mathcal{C}$) is necessary and sufficient to compute the output in the future of $\mathcal{C}$. Some simple examples of separation sets are given in fig. 1.

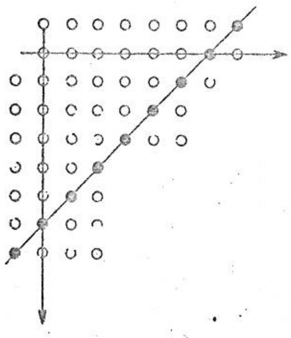

fig. 1a

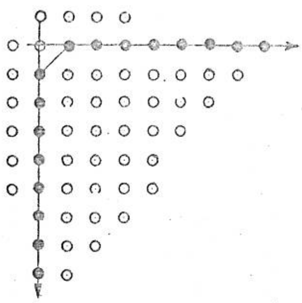

fig. 1b

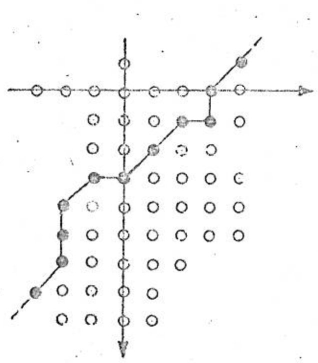

fig. 1c

If we start with zero local states on a separation set $\mathcal{C}$, the input-output map we obtain from (1) satisfies axioms (i) - (iii). So, we say that the "2-D system" $\Sigma = (A_1, A_2, B_1, B_2, C)$ given by equations (1) can be interpreted as a realization of a two-dimensional filter. The transfer function of the 2-D system (1) is given in terms of A_1, A_2, B_1, B_2, C by $C(I - A_1 z_1 - A_2 z_2)^{-1} (B_1 z_1 + B_2 z_2)$ which is clearly a rational function.

Hence a two-dimensional filter is realized by a 2-D system only if it is characterized by a rational transfer function. Actually the converse holds too as stated by the following proposition[2,5]

Proposition. Let $s \in K_c[[z_1, z_2]]$ characterize a two-dimensional filter. Then the filter is realizable by a 2-D system if and only if s is rational.

3. Zero State Equivalent Realizations

In multidimensional digital data processing the problems related to the amount of data to be simultaneously handled are much more severe than in one-dimensional case. The introduction of a local state structure allows the problem to be set up recursively so that the amount of data processing at each step considerably reduces. In any case since this amount of computation strictly depends on the dimension of local state space, we are led to look for realizations having minimal dimension. As usually by dimension of a realization we mean the dimension of its local state space.

Any two-dimensional filter characterized by a rational transfer function is realized by infinitely many 2-D systems. Two 2-D systems Σ_1 and Σ_2 are zero-state equivalent if they realize the same filter. In this way the set of 2-D systems is partitioned in equivalence classes (with respect to zero-state equivalence) in one-to-one correspondence with the set of filters with rational transfer functions.

It is interesting to mention some relevant properties shared by these classes[5,6,8]:

(a) the equivalence class containing the n-dimensional 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C)$ includes the set of "similar" 2-D systems:

$$\{(T^{-1}A_1 T, T^{-1}A_2 T, T^{-1}B_1, T^{-1}B_2, CT), T \in G\ell(n,K)\}$$

b) each equivalence class includes locally reachable, locally observable and locally reachable and observable realizations. Local reachability and local observability tests for a n-dimensional 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C)$ consist in checking the full rank of the following matrices:

$$R = \left[B_1 \vdots B_2 \vdots A_1 B_1 \vdots A_1 B_2 + A_2 B_1 \vdots A_2 B_2 \vdots \cdots \right.$$
$$\left. \cdots \vdots (A_1^{i-1} \sqcup\!\sqcup^{j} A_2) B_1 + (A_1^{i} \sqcup\!\sqcup^{j-1} A_2) B_2 \cdots \right], \quad i+j < n$$

"reachability matrix"

and

$$O = \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ \vdots \\ C(A_1^{i} \sqcup\!\sqcup^{j} A_2) \\ \vdots \end{bmatrix}, \quad i+j<n, \qquad \text{"observability matrix"}$$

Matrices $A_1{}^{r} \sqcup\!\sqcup^{s} A_2$ are inductively defined as:

$$A_1{}^{r} \sqcup\!\sqcup^{0} A_2 = A_1^{r}, \quad A_1{}^{0} \sqcup\!\sqcup^{s} A_2 = A_2^{s},$$

$$A_1{}^{r} \sqcup\!\sqcup^{s} A_2 = A_1(A_1{}^{r-1} \sqcup\!\sqcup^{s} A_2) + A_2(A_1{}^{r} \sqcup\!\sqcup^{s-1} A_2), r,s \geq 1$$

c) in each class the set of locally reachable and locally observable realizations includes the set of minimal realizations.

Some aspects of equivalence classes analysis have been clarified and some results yet appeared in the literature[5,6], others are still under investigation. The following remarks summarize the present situation:

1. There exist algorithms for computing locally reachable and locally observable realizations which are zero state equivalent to a given one [10].

2. Two locally reachable and locally observable realizations do not necessarily have the same dimension[6]. This gives that in general the set inclusion considered in (c) is proper, i.e. locally reachable and observable realizations are not necessarily minimal.

3. The dimension of minimal realizations of a given filter can be reduced when considering an overfield $K' \supset K$[6,11]. It is still open the problem of finding efficient algorithms for obtaining minimal realizations.

4. Stability and Roundoff Error Propagation

The notion of asymptotic stability of a 2-D system is related to the behaviour of local state free evolutions determined by initial local states on a separation set $\mathcal{C}$, when the distance from $\mathcal{C}$ goes to infinity. From now on $K = \mathbb{R}$ and $\|\cdot\|$ denotes the euclidean norm on $X = \mathbb{R}^n$.

Let refer for simplicity to a separation set as in fig. 1a, denote by $\mathscr{X}_i = \{x(h,k) : h+k = i\}$ and assume $\|\mathscr{X}_i\| = \sup_{n\in\mathbb{Z}} \|x(i-n,n)\|$. We have the following definition:

Definition 1. A 2-D system Σ is asymptotically stable if assuming $u=0$, for every $\mathscr{X}_0$, $\|\mathscr{X}_0\| < \infty$ we have $\lim_{i\to\infty} \|\mathscr{X}_i\| = 0$.

Obviously the internal stability of Σ depends only on the pair (A_1, A_2) and the following Proposition[8,12] gives an algebraic criterion for asymptotic stability.

Proposition 1. A 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C)$ is asymptotically stable if and only if the polynomial $\det(I - z_1A_1 - z_2A_2)$ is devoid of zeros in the closed polydisc:

$$\mathscr{P}_1 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| \le 1,\ |z_2| \le 1\}.$$

External (or input-output) stability is defined as bounded-input bounded-output (BIBO) stability as follows.

Definition 2. A 2-D system is said to be BIBO stable if assuming all local states on $\mathscr{X}_0$ be zero, $\sup_{h+k\ge 0} |y(h,k)|$ is bounded if it is $\sup_{h+k\ge 0} |u(h,k)|$.

It has been proved[13,14,15] that a two dimensional filter is BIBO stable if and only if its transfer function is regular in the closed polydisc $\mathscr{P}_1$. Thus, by Proposition 1, any asymptotically stable 2-D system is BIBO stable. The converse does not hold in general. Some preliminary results on what kind of properties one might add to BIBO stability to get asymptotical stability have been published recently[8,12].

Definition 1 of asymptotic stability and the related criterion have been hystorically introduced having in mind a particular separation set. Actually asymptotic stability definition is easily extended to a generic separation set.

Definition 3. Let $\mathscr{C}$ be a generic separation set and assume $u \equiv 0$. A 2-D system Σ is asymptotically stable with respect to $\mathscr{C}$ if for any set of initial states on $\mathscr{C}$ with $\sup_{(h,k)} \|x(h,k)\| < \infty$ and any $\varepsilon > 0$, there exists a positive integer M such that $\|x(i,j)\| < \varepsilon$ when the distance of (i,j) from $\mathscr{C}$ is greater than M ($d((i,j),\mathscr{C}) > M$).

One could expect that Definition 3 would lead to different kinds of asymptotical stability depending on the separation set $\mathscr{C}$. On the contrary it has been shown[12] that the changes in the shape of separation set have no effect in the definition of asymptotic stability. This makes unnecessary to bother with the structure of separation sets, and garantees that the stability criterion given by Proposition 1 holds independently of the separation set we deal with and constitutes a test of general validity.

When the filtering process is performed by a digital machine, the finite word length of registers produces modifications of state equations and introduces multiplication roundoff errors.

Whenever the computation is required over a large interval, the question must be considered as to how computational errors introduced at each point in the calculation will propagate.

We shall confine here to analyze roundoff error propagation by means of stability results we have just recalled.

Let make the following assumptions:

- the entries of matrices A_1, A_2, B_1, B_2, C and the input values $u(h,k)$ are initially given as machine numbers, hence not affected by roundoff errors.

- overbarred vectors $\bar{x}(h,k)$ denote machine vectors which correspond to the actual $x(h,k)$ (not affected by any roundoff error).

Since products are involved, the components of the following vector $x(h+1) = A_1\bar{x}(h+1,k) + A_2\bar{x}(h,k+1) + B_1u(h+1,k) + B_2u(h,k+1)$ in general are not machine numbers, so introduce the roundoff error vector e_d:

$$e_d(h+1,k+1) = \bar{x}(h+1,k+1) - x(h+1,k+1)$$

The dynamics of roundoff errors accumulation is governed by updating equations having the structure of the 2-D system which realizes the filter. In fact call $x_\infty(h,k)$ the state vector of the 2-D system cumputed by an ideal machine (not affected by any roundoff error) and denote by $e(h,k)$ the difference $\bar{x}(h,k) - x_\infty(h,k)$. We therefore have:

$$(2) \qquad e(h+1,k+1) = A_1e(h+1,k) + A_2e(h,k+1) + e_d(h+1,k+1)$$

Equation (2) can be viewed as a (modified version) of the 2-D system (1) with input function $e_d(.,.)$. Since e_d is bounded, if we assume zero (or bounded) initial roundoff errors, the error e is also bounded if the system (1) is asymptotically stable.

We can easily obtain an upper bound for the roundoff error e in the following way. Let the polynomial $\det(I - A_1z_1 - A_2z_2)$ be non zero in the closed polydisc:

$$\mathscr{P}_r = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| \le r,\ |z_2| \le r\},\quad r > 1.$$

So in $\mathscr{P}_r$ the series $\Sigma_{ij}(A_1 {}^{i}\sqcup\!\sqcup^{j} A_2)z_1^iz_2^j$ normally converges to $(I - A_1z_1 - A_2z_2)^{-1}$.

By Cauchy's inequalities[16] the (p,q) - entry of the matrix $A_1 {}^{i}\sqcup\!\sqcup^{j} A_2$ is bounded as follows:

$$|(A_1 {}^{i}\sqcup\!\sqcup^{j} A_2)_{p,q}| \le \frac{1}{r^{i+j}} \max_{\substack{0\le\omega_1\le 2\pi \\ 0\le\omega_2\le 2\pi}} |((I - e^{j\omega_1}rA_1 - e^{j\omega_2}rA_2)^{-1})_{p,q}|$$

Consequently

$$\|A_1 {}^{i}\sqcup\!\sqcup^{j} A_2\|^2 = \Sigma_{p,q} |(A_1 {}^{i}\sqcup\!\sqcup^{j} A_2)_{p,q}|^2 \le$$

$$\le \frac{1}{r^{2(i+j)}} \Sigma_{p,q} \max_{\substack{0\le\omega_1\le 2\pi \\ 0\le\omega_2\le 2\pi}} |((I - e^{j\omega_1}rA_1 - e^{j\omega_2}rA_2)^{-1})_{p,q}|^2$$

and calling

$$L = \left[\Sigma_{p,q} \max_{\substack{0\le\omega_1\le 2\pi \\ 0\le\omega_2\le 2\pi}} ((I - e^{j\omega_1}rA_1 - e^{j\omega_2}rA_2)^{-1})_{p,q}|^2\right]^{1/2}$$

we have

$$\|A_1 {}^{i}\sqcup\!\sqcup^{j} A_2\| \le \frac{L}{r^{i+j}}.$$

The accumulated roundoff error in (h,k) is $e(h,k) = \Sigma_{i,j}(A_1 {}^{i}\sqcup\!\sqcup^{j} A_2)\, e_d\,(h-i,k-j)$ when (i,j) runs over some finite sub set of $\mathbb{N} \times \mathbb{N}$ depending on the structure of the initial separation set. Then an error bound is given by:

$$\|e(h,k)\| \le \Sigma_{0\,ij}^{\infty} \|A_1 {}^{i}\sqcup\!\sqcup^{j} A_2\| \sup\|e_d\| \le \sup\|e_d\| L \frac{r^2}{(r-1)^2}.$$

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