On the Relevance of Noncommutative Power Series in Spatial Filters Realization

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Abstract—Several properties of noncommutative rational power series are relevant in realizing spatial filters. After a brief survey of previous results, this paper presents an extension of Ho's algorithm, which provides a representation technique for noncommutative rational power series, and outlines the solution to the problem of the partial representation.

Spatial filters, as input/output maps, and doubly indexed dynamical systems are then considered, and a characterization of realizable filters is derived.

Finally an algorithm is constructed, which provides all the minimal realizations of a prescribed filter by exploiting the generalized Ho's algorithm.

Introduction

The CONSTRUCTION of state-space models of linear spatial filters on the basis of input-output data constitutes a typical realization problem in the system-theoretic sense.

This area of research has been developed for the most part in the last few years [1]–[5] and has proved to be a nontrivial generalization of standard realization theory. A number of aspects have already been investigated. These include the concept of Nerode equivalence for input/output maps of spatial filters, the relations between rationality of impulse response and finite dimensional realization, the definition and the properties of reachability and observability in the local state space, and the problem of obtaining explicit realization algorithms.

The deeper complexity that filters exhibit is reflected in some facts which have no counterpart in standard linear systems, i.e., minimality of realizations does not follow from reachability and observability, and minimal dimension for a prescribed impulse response depends on which field we embed the coefficients.

It seems that the first steps to solve these problems by using noncommutative power series were undertaken in [2], mainly with regard to minimization procedures. Here, this kind of approach is further developed. Several previously obtained results are restated in a new context, and a procedure is introduced providing all minimal realizations. Since this algorithm is based on the solution of a finite set of algebraic nonlinear equations, the dependence of minimal dimension on the ground field will be clarified.

This paper can be divided in two parts. The first part (Sections I–IV) is devoted to analyzing the structure of noncommutative rational power series. The main result is an extension of Ho's algorithm to noncommutative power series.

The second part (Sections V and VI) is concerned with the problem of the realization of filters—viewed as input/output maps. Section V is devoted to the relevant definitions and to some consequences, while Section VI contains the realization algorithm.

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I. Noncommutative Rational Power Series

The introduction of noncommutative formal power series in system theory was first done by M. Fliess [6]. He recognized their relevance in the analysis of bilinear systems and successively [7] of larger classes of nonlinear systems. Much of the material in this section is available in the current literature [8]–[10], so that unnecessary proofs will be omitted.

For notational convenience power series in two indeterminates are considered. It is clear how one might generalize to power series in $3,4\cdots$ indeterminates.

Given the set $\Xi=\{\xi_1,\xi_2\}$, called the alphabet, the free monoid Ξ^* with base Ξ is defined as follows. The elements of Ξ^*, called the words, are the n-tuples

$$w=(\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_N}), \qquad N\geqslant 0 \tag{1.1}$$

of elements of Ξ. The integer N is the length of w and is denoted by $|w|$.

If $u=(\xi_{j_1},\xi_{j_2},\cdots,\xi_{j_M})$ is another element of Ξ^*, the product wu is defined by concatenation, i.e.,

$$wu=(\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_N};\xi_{j_1},\xi_{j_2},\cdots,\xi_{j_M}).$$

In this way Ξ^* is a monoid with $1=(\ \ )$ (the only 0-tuple) as unit element.

We agree to write ξ_i instead of the 1-tuple (ξ_i). In this way (1.1) may be written as

$$w=\xi_{i_1}\xi_{i_2}\cdots\xi_{i_N}$$

if $N>0$.

Let K be a field. A noncommutative formal power series with coefficients in K and indeterminates ξ_1 and ξ_2 is an expression such as

$$\sigma=\sum_{w\in\Xi^*}(\sigma,w)w, \qquad (\sigma,w)\in K.$$

Hence σ is just a function $\sigma:\Xi^*\to K:w\mapsto(\sigma,w)$ with codomain the field K and with domain the free monoid Ξ^*. The sum and the product of two series σ, τ are defined by the formulas

$$\sigma+\tau=\sum_{w\in\Xi^*}((\sigma,w)+(\tau,w))w$$

$$\sigma\tau=\sum_{w\in\Xi^*}\sum_{\substack{u,v\\uv=w}}((\sigma,u)(\tau,v))w.$$

Under this sum and product, the set of all noncommutative formal power series with coefficients in K and indeterminates ξ_1 and ξ_2 is a noncommutative ring, and will be written as $K\langle\langle\xi_1,\xi_2\rangle\rangle$. Denote by $K\langle\xi_1,\xi_2\rangle$ the subring of noncommutative polynomials: it consists of all series in $K\langle\langle\xi_1,\xi_2\rangle\rangle$ which have finite support. The integer N is a formal degree of the polynomial $\pi\in K\langle\xi_1,\xi_2\rangle$ if $|w|>N$ implies $(\pi,w)=0$. The smallest such number N is called the degree of π.

A subring $R\subseteq K\langle\langle\xi_1,\xi_2\rangle\rangle$ is said to be rationally closed if $\sigma\in R$ and σ invertible in $K\langle\langle\xi_1,\xi_2\rangle\rangle$ imply $\sigma^{-1}\in R$.

Definition 1.1: The ring $K\langle(\xi_1,\xi_2)\rangle$ of rational noncommutative power series is the minimal rationally closed subring of $K\langle\langle\xi_1,\xi_2\rangle\rangle$ which contains $K\langle\xi_1,\xi_2\rangle$.

Definition 1.1 is very abstract and gives a poor feeling of what rationality means when we deal with noncommutative power series. Actually we would like to get some more concrete information about the structure of this class of series.

As we shall show, every series σ in $K\langle(\xi_1,\xi_2)\rangle$ is identifiable by the assignment of a finite number of parameters—in fact its coefficients result from ring operations (i.e. sums and products) on a finite set of matrices which completely characterizes σ.

In order to describe in a precise and compact way how the coefficients of σ can be generated, we briefly recall some basic definitions.

Let G and H be two (multiplicative) monoids, and let $\rho:G\to H$ be a map of G into H. ρ is called a (monoid) homomorphism [11] if

$$\rho(g_1g_2)=\rho(g_1)\rho(g_2)$$

for all g_1 and g_2 in G, and

$$\rho(1_G)=1_H.$$

Definition 1.2: Let Ξ^* be the free monoid with base $\Xi=\{\xi_1,\xi_2\}$. A representation of Ξ^* into the multiplicative monoid of $K^{N\times N}$ (i.e., into the set of $N\times N$ matrices, equipped with the usual multiplication) is a monoid homomorphism

$$\rho:\Xi^*\to K^{N\times N}:w\mapsto w^\rho.$$

Remark: The image of a word w under the map ρ is denoted by w^ρ. This notation is alternative to the usual one $\rho(w)$. We emphasize that w^ρ is not a power!

Theorem 1.1: Let σ be in $K\langle\langle\xi_1,\xi_2\rangle\rangle$. Then the following facts are equivalent [8]:

i) $\sigma\in K\langle(\xi_1,\xi_2)\rangle$.

ii) There exists a positive integer N, a representation ν of Ξ^* on the multiplicative monoid $K^{N\times N}$

$$\nu:\Xi^*\to K^{N\times N}:w\mapsto w^\nu$$

and a matrix $P\in K^{N\times N}$ such that

$$(\sigma,w)=\text{trace}\,(Pw^\nu),\qquad \forall w\in\Xi^*.$$

iii) There exists a positive integer M, a representation μ of Ξ^* on the multiplicative monoid $K^{M\times M}$, and two matrices $C\in K^{1\times M}$, $B\in K^{M\times 1}$ such that

$$(\sigma,w)=Cw^\mu B,\qquad \forall w\in\Xi^*.$$

The representation is completely known when the matrices $A_1=\xi_1^\mu$ and $A_2=\xi_2^\mu$ are assigned. Assume that ξ_1 and ξ_2 commute with A_1 and A_2; then σ can be written as

$$\sigma=C\sum_{k=0}^{\infty}(A_1\xi_1+A_2\xi_2)^kB. \tag{1.2}$$

With a slight abuse of language, a 4-tuple (A_1,A_2,B,C) is called a representation of σ if (1.2) holds. The dimension of the matrices A_1 and A_2 is the dimension of the representation.

Clearly, if σ admits a representation, it admits infinitely many, and there exists at least one which has a smaller dimension than the others. Thus it makes sense to look for minimal representations. In Section III an algorithm is presented for obtaining a minimal representation of a given rational series σ.

Note that if (A_1,A_2,B,C) is a representation for σ, so is $(T^{-1}A_1T,T^{-1}A_2T,T^{-1}B,CT)$ for any nonsingular T. Evidently, this constitutes a recipe for constructing several (infinitely many if K is infinite) minimal representations given one minimal representation. The natural question arises as to whether we can construct all minimal representations by this technique. The answer is yes, as in standard linear theory, and the proof rests on the following lemma.

Lemma 1.1: Let σ belong to $K\langle(\xi_1,\xi_2)\rangle$, and let $(\xi_1^\mu,\xi_2^\mu,B,C)$ be a minimal representation of dimension M. Then the matrices

$$\mathcal{R}_M=\begin{bmatrix}B & \xi_1^\mu B & \xi_2^\mu B & \cdots & w^\mu B & \cdots\end{bmatrix}_{|w|<M}$$

$$\mathcal{O}_M=\begin{bmatrix}C\\ C\xi_1^\mu\\ C\xi_2^\mu\\ \cdot\\ \cdot\\ \cdot\\ Cw^\mu\end{bmatrix}_{|w|<M} \tag{1.3}$$

have full rank.

Proof: First we prove that

$$\text{rank}\ \mathcal{R}_M=\text{rank}\ \mathcal{R}_{M+1}=\text{rank}\ \mathcal{R}_{M+2}=\cdots. \tag{1.4}$$

Let $w=\xi_{i_M}\xi_{i_{M-1}}\cdots\xi_{i_1}$ be a word of length M. If the columns

$$B,\ \ \xi_{i_1}^\mu B,\ \ \xi_{i_2}^\mu\xi_{i_1}^\mu B,\cdots,\xi_{i_{M-1}}^\mu\xi_{i_{M-2}}^\mu\cdots\xi_{i_1}^\mu B \tag{1.5}$$

are linearly independent, they give a basis of K^M. Otherwise, in list (1.5), $\xi_{i_T}^\mu\xi_{i_{T-1}}^\mu\cdots\xi_{i_1}^\mu B$ depends on the preceding columns for some $T<M$ and

$$(\xi_{i_T}\xi_{i_{T-1}}\cdots\xi_{i_1})^\mu B=\alpha_0B+\alpha_1\xi_{i_1}^\mu B+\cdots$$
$$+\alpha_{T-1}(\xi_{i_{T-1}}\xi_{i_{T-2}}\cdots\xi_{i_1})^\mu B$$

$\alpha_0, \alpha_1, \cdots, \alpha_{T-1} \in K$. If u denotes the word $\xi_{i_M}\xi_{i_{M-1}}\cdots\xi_{i_{T+1}}$, one gets

$$w^\mu B = \alpha_0 u^\mu B + \alpha_1 (w\xi_1)^\mu B + \cdots + \alpha_{T-1}(w\xi_{i_{T-1}}\cdots\xi_{i_1})^\mu B.$$

In both cases $w^\mu B$ belongs to the space spanned by the columns of $\mathcal{R}_M$. Hence rank $\mathcal{R}_M =$ rank $\mathcal{R}_{M+1}$.

An obvious inductive argument completes the proof of (1.4).

Suppose now rank $\mathcal{R}_M = \overline{M} < M$. Then the columns of $\mathcal{R}_M$ span a proper subspace $R \subset K^M$. By (1.4), R is invariant under the transformation semigroup induced by the matrices w^μ. Let $(e_1, e_2, \cdots, e_M)$ be a basis of K^M, such that $(e_1, e_2, \cdots, e_{\overline{M}})$ is a basis of R, and let $T \in Gl(M, K)$ be the change matrix from the standard basis in K^M to the basis $(e_1, e_2, \cdots, e_M)$.

With respect to the basis $(e_1, e_2, \cdots, e_M)$ a matrix of reduced form

$$Tw^\mu T^{-1} = \overset{\overline{M}}{\begin{bmatrix} * & 0 \\ * & * \end{bmatrix}} \}\overline{M}$$

corresponds to the matrix w^μ, and the last $M - \overline{M}$ elements in TB are zeros.

Thus the representation $(T\xi_1^\mu T^{-1}, T\xi_2^\mu T^{-1}, TB, CT^{-1})$ of dimension M can be reduced. For, define the representation $\nu : \Xi^* \to K^{\overline{M} \times \overline{M}}$:

$$\xi_i^\nu = [\, I_{\overline{M}} \mid 0 \,]\, T\xi_i^\mu T^{-1} \begin{bmatrix} I_{\overline{M}} \\ 0 \end{bmatrix}, \qquad i = 1, 2$$

and construct the matrices

$$\overline{B} = [\, I_{\overline{M}} \mid 0 \,]\, TB$$

$$\overline{C} = CT^{-1} \begin{bmatrix} I_{\overline{M}} \\ 0 \end{bmatrix}.$$

It is easily checked that $(\xi_1^\nu, \xi_2^\nu, \overline{B}, \overline{C})$ is a representation of σ of dimension $\overline{M} < M$. This contradicts the minimality hypothesis. Hence $\mathcal{R}_M$ and (by similar arguments) $\mathcal{O}_M$ have full rank. □

As a consequence of this result, one proves the following theorem.

Theorem 1.2: Let $(A_1^{(1)}, A_2^{(1)}, B^{(1)}, C^{(1)})$ and $(A_1^{(2)}, A_2^{(2)}, B^{(2)}, C^{(2)})$ be two minimal representations of dimension M.

Then there exists a nonsingular T such that

$$A_i^{(2)} = TA_i^{(1)}T^{-1}, \qquad i = 1, 2$$

$$B^{(2)} = TB^{(1)}$$

$$C^{(2)} = C^{(1)}T^{-1}.$$

Proof: Define matrices $\mathcal{O}_M^{(1)}$ and $\mathcal{O}_M^{(2)}$ as in Lemma 1.1. Then the matrix

$$T = (\mathcal{O}_M^{(2)\prime}\mathcal{O}_M^{(2)})^{-1}\mathcal{O}_M^{(2)\prime}\mathcal{O}_M^{(1)}$$

carries one representation into the other (′ denotes transpose).

This can be shown in a completely similar manner, as in standard linear case [12]. Therefore, the rest of the proof will be omitted. □

II. Hankel Matrices

Let $\sigma = \sum_{w \in \Xi^*} (\sigma, w) w$ be in $K\langle\langle \xi_1, \xi_2 \rangle\rangle$. Consider the (infinite) matrix

$$\mathcal{H}(\sigma) = \begin{bmatrix} (\sigma,1) & (\sigma,\xi_1) & (\sigma,\xi_2) & (\sigma,\xi_1^2) & (\sigma,\xi_1\xi_2) & (\sigma,\xi_2\xi_1) & (\sigma,\xi_2^2)\cdots \\ (\sigma,\xi_1) & (\sigma,\xi_1^2) & (\sigma,\xi_1\xi_2) & (\sigma,\xi_1^3) & (\sigma,\xi_1^2\xi_2) & (\sigma,\xi_1\xi_2\xi_1) & \cdots \\ (\sigma,\xi_2) & (\sigma,\xi_2\xi_1) & (\sigma,\xi_2^2) & (\sigma,\xi_2\xi_1^2) & (\sigma,\xi_2\xi_1\xi_2) & \cdots & \\ (\sigma,\xi_1^2) & (\sigma,\xi_1^3) & (\sigma,\xi_1^2\xi_2) & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{2.1}$$

in which the rows and the columns are indexed by the words of Ξ^*. $\mathcal{H}(\sigma)$ is called the Hankel matrix relative to the series σ.

The element in the (v, w) position, that is, in the intersection of the vth row and wth column, is given by (σ, vw).

Although unnecessary, it will be convenient to order lexicographically the words in Ξ^*. In this way it makes sense to select in $\mathcal{H}(\sigma)$ the first, the second, the nth row (or column).

We shall partition $\mathcal{H}(\sigma)$ in row blocks and column blocks, indexed by capital letters. They are defined as follows: the Mth row (column) block includes all rows (columns) of $\mathcal{H}(\sigma)$ whose indices are words of length $M-1$. The composition of row and column partitions gives a partition of $\mathcal{H}(\sigma)$ in block matrices of finite size: the block in the (M', M'') position is written as $\mathcal{H}_{M',M''}$ and contains $2^{(M'-1)+(M''-1)}$ elements.

For example, the block in position $(2,3)$ is the matrix

$$\mathcal{H}(\sigma)_{2,3} = \begin{bmatrix} (\sigma,\xi_1^3) & (\sigma,\xi_1^2\xi_2) & (\sigma,\xi_1\xi_2\xi_1) & (\sigma,\xi_1\xi_2^2) \\ (\sigma,\xi_2\xi_1^2) & (\sigma,\xi_2\xi_1\xi_2) & (\sigma,\xi_2^2\xi_1) & (\sigma,\xi_2^3) \end{bmatrix}.$$

We shall denote by $\mathcal{H}_{M' \times M''}(\sigma)$ the $M' \times M''$ block submatrix of $\mathcal{H}(\sigma)$ appearing in the upper left-hand corner of $\mathcal{H}(\sigma)$

$$\mathcal{H}_{M' \times M''}(\sigma) = \begin{bmatrix} \mathcal{H}(\sigma)_{1,1} & \mathcal{H}(\sigma)_{1,2} & \cdots & \mathcal{H}(\sigma)_{1,M''} \\ \mathcal{H}(\sigma)_{2,1} & \mathcal{H}(\sigma)_{2,2} & \cdots & \mathcal{H}(\sigma)_{2,M''} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{H}(\sigma)_{M',1} & \mathcal{H}(\sigma)_{M',2} & \cdots & \mathcal{H}(\sigma)_{M',M''} \end{bmatrix}. \tag{2.2}$$

We will call

$$n_\sigma = \sup_{M', M''} \text{rank } \mathcal{H}_{M' \times M''}(\sigma) \tag{2.3}$$

the rank of the Hankel matrix $\mathcal{H}(\sigma)$ or, briefly, the rank of σ. Suppose now that the series σ belongs to $K\langle(\xi_1, \xi_2)\rangle$

and that $(\xi_1^\mu, \xi_2^\mu, B, C)$ is a representation of σ. Since the matrices $\mathcal{R}_M$ and $\mathcal{O}_M$ in Lemma 1.1 give a factorization of $\mathcal{H}_{M\times M}(\sigma)$

$$\mathcal{H}_{M\times M}(\sigma)=\mathcal{O}_M\mathcal{R}_M \tag{2.4}$$

we get a partial solution to the problem of investigating the relation between the rank of $\mathcal{H}(\sigma)$ and the dimension of the representations of σ. In fact, (2.4) implies that n_σ is a lower bound for the dimension of each representation of σ.

We have therefore established the necessity part of the following

Theorem 2.1 [M. Fliess]: Let σ be in $K\langle\langle\xi_1,\xi_2\rangle\rangle$. Then σ is rational if and only if $\mathcal{H}(\sigma)$ has finite rank. Moreover, rank $\mathcal{H}(\sigma)$, whenever finite, provides the dimension of the minimal representation of σ.

The original proof of the sufficiency part is given by M. Fliess in [8] and is based on some properties of serial modules.

In Section III we shall obtain an alternative proof as a consequence of the generalized Ho's algorithm. In fact Ho's algorithm will provide a representation of σ with dimension n_σ.

Remark: When we know an upper bound $\bar{N}$ for n_σ, Theorem 2.1 allows us to evaluate n_σ as rank $\mathcal{H}_{\bar{N}\times\bar{N}}(\sigma)$. In fact σ has some minimal representation $(\xi_1^\mu,\xi_2^\mu,B,C)$ of dimension $n_\sigma\leq\bar{N}$. The matrices $\mathcal{R}_{\bar{N}}$ and $\mathcal{O}_{\bar{N}}$ associated to this representation have full rank n_σ. Hence $\mathcal{H}_{\bar{N}\times\bar{N}}(\sigma)=\mathcal{O}_{\bar{N}}\mathcal{R}_{\bar{N}}$ has rank n_σ too.

III. Generalized Ho's Algorithm

In linear and bilinear system theory Ho's algorithm enables solution of the problem of passing from a prescribed impulse response to a minimal realization [13]–[16]. The only hypothesis needed for its application is that some upperbound for the dimension of the minimal realization has to be *a priori* known.

In this section we shall generalize Ho's algorithm to noncommutative power series. This leads to the computation of a minimal representation of a given finite rank noncommutative power series σ, starting from its Hankel matrix $\mathcal{H}(\sigma)$. As a consequence of Ho's algorithm, any series σ of finite rank can be written as $\sigma=C(1-\xi_1A_1-\xi_2A_2)^{-1}B$; that is, finite rank implies rationality.

Let $\sigma\in K\langle\langle\xi_1,\xi_2\rangle\rangle$, and let rank $\mathcal{H}(\sigma)=n_\sigma<\infty$. We define the row length of $\mathcal{H}(\sigma)$ as

$$L'=\min\{M':\text{rank }\mathcal{H}_{M'\times\infty}(\sigma)=n_\sigma\}$$

and similarly the column length of $\mathcal{H}(\sigma)$ as

$$L''=\min\{M'':\text{rank }\mathcal{H}_{\infty\times M''}(\sigma)=n_\sigma\}.$$

By the definition of L', each row of the $(L'+1)$th block row of $\mathcal{H}(\sigma)$ is linearly dependent on the rows of the preceding block rows. It follows that for each t in Ξ^*, with $|t|\leq L'$, there exists $d_{t,v}^{(1)}$ and $d_{t,v}^{(2)}$ in K such that

$$(\sigma,t\xi_1w)=\sum_{\substack{v\in\Xi^*\\|v|\leq L'}}d_{t,v}^{(1)}(\sigma,vw),\qquad\forall w\in\Xi^*$$

and

$$(\sigma,t\xi_2w)=\sum_{\substack{v\in\Xi^*\\|v|\leq L'}}d_{t,v}^{(2)}(\sigma,vw),\qquad\forall w\in\Xi^*. \tag{3.1}$$

Obviously, when $|t|<L'$, we can assume $d_{t,v}^{(i)}=1$ if $v=t\xi_i$ and $d_{t,v}^{(i)}=0$ if $v\neq t\xi_i$. The coefficients $d_{t,v}^{(i)}$, $i=1,2$, are arranged in two square matrices D_1 and $D_2\in K^{\rho\times\rho}$, $\rho=2^0+2^1+\cdots+2^{L'}$:

$$D_1=\|d_{t,v}^{(1)}\|\qquad D_2=\|d_{t,v}^{(2)}\|. \tag{3.2}$$

Similarly, for each $w\in\Xi^*$ with $|w|\leq L''$ there exists $f_{u,w}^{(1)}$ and $f_{u,w}^{(2)}$ in K such that

$$(\sigma,t\xi_1w)=\sum_{\substack{u\in\Xi^*\\|u|\leq L''}}f_{u,w}^{(1)}(\sigma,tu),\qquad\forall t\in\Xi^*$$

$$(\sigma,t\xi_2w)=\sum_{\substack{u\in\Xi^*\\|u|\leq L''}}f_{u,w}^{(2)}(\sigma,tu),\qquad\forall t\in\Xi^*. \tag{3.3}$$

For $|w|<L''$, we assume $f_{u,w}^{(i)}=1$ if $\xi_iw=u$ and 0 otherwise. The coefficients $f_{u,w}^{(i)}$, $i=1,2$, are arranged in two square matrices F_1 and $F_2\in K^{\tau\times\tau}$, $\tau=2^0+2^1+\cdots+2^{L''}$:

$$F_1=\|f_{u,w}^{(1)}\|\qquad F_2=\|f_{u,w}^{(2)}\|. \tag{3.4}$$

Let r be in Ξ^*. Denote by $\mathcal{H}^{(r)}(\sigma)$ the infinite matrix whose element in the (u,v) position is given by (σ,urv) for any w and v in Ξ^*.

Notice that in general $\mathcal{H}^{(r)}(\sigma)$ does not constitute an Hankel matrix except for $r=1$. In fact, one has

$$\mathcal{H}^{(1)}(\sigma)=\mathcal{H}(\sigma).$$

We partition $\mathcal{H}^{(r)}(\sigma)$ conformably with the partition already introduced in $\mathcal{H}(\sigma)$. In this way $\mathcal{H}^{(r)}_{M'\times M''}(\sigma)$ and $\mathcal{H}^{(r)}(\sigma)_{M',M''}$ should constitute self-explaining notations.

Theorem 3.1 (Generalized Ho's Algorithm): Let σ be in $K\langle\langle\xi_1,\xi_2\rangle\rangle$ and let $n_\sigma=\text{rank }\mathcal{H}(\sigma)<\infty$. Denote by L' and L'' the row length and the column length of $\mathcal{H}(\sigma)$, respectively. The following steps lead to a minimal representation of σ.

1) Find nonsingular matrices P and Q such that

$$P\mathcal{H}_{L'\times L''}(\sigma)Q=\left[\begin{array}{c|c}I_{n_\sigma}&0\\\hline 0&0\end{array}\right]. \tag{3.5}$$

2) Compute

$$A_i=[I_{n_\sigma}|0]P\mathcal{H}^{(\xi_i)}_{L'\times L''}(\sigma)Q\begin{bmatrix}I_n\\\hline 0\end{bmatrix},\qquad i=1,2$$

$$B=[I_{n_\sigma}|0]P\mathcal{H}_{L'\times L''}(\sigma)\begin{bmatrix}1\\0\\\cdot\\\cdot\\0\end{bmatrix}$$

$$C=[1\quad 0\quad\cdots\quad 0]\mathcal{H}_{L'\times L''}(\sigma)Q\begin{bmatrix}I_{n_\sigma}\\\hline 0\end{bmatrix}. \tag{3.6}$$

The crux of the proof is the following lemma.

Lemma 3.1: Let L' and L'' be the row length and the column length of $\mathcal{H}(\sigma)$, respectively. Let $r\in\Xi^*$, $r=\xi_{i_1}\xi_{i_2}\cdots\xi_{i_\nu}$. Then

$$\mathcal{H}^{(r)}_{L'\times L''}(\sigma)=D_{i_1}D_{i_2}\cdots D_{i_\nu}\mathcal{H}_{L'\times L''}(\sigma)$$
$$=\mathcal{H}_{L'\times L''}(\sigma)F_{i_1}F_{i_2}\cdots F_{i_\nu}.$$

Proof: Assume that t and w belong to Ξ^*, $|t|\leqslant L'$, $|w|\leqslant L''$. By repeated applications of (3.1) one gets

$$(\sigma,trw)=\sum_{\substack{v_1\in\Xi^*\\|v_1|\leqslant L'}}h^{(i_1)}_{t,v_1}\sum_{\substack{v_2\in\Xi^*\\|v_2|\leqslant L'}}h^{(i_2)}_{v_1,v_2}\cdots\sum_{\substack{v_\nu\in\Xi^*\\|v_\nu|\leqslant L'}}h^{(i_\nu)}_{v_{\nu-1},v_\nu}(\sigma,v_\nu w).$$

Since (σ,trw) is the element in (t,w) position in $\mathcal{H}^{(r)}(\sigma)$, one obtains

$$\mathcal{H}^{(r)}_{L'\times L''}(\sigma)=D_{i_1}D_{i_2}\cdots D_{i_\nu}\mathcal{H}_{L'\times L''}(\sigma).$$

Similarly, in view of (3.3) one gets

$$(\sigma,trw)=\sum_{\substack{u_\nu\in\Xi^*\\|u_\nu|\leqslant L''}}f^{(i_\nu)}_{u_\nu,w}\sum_{\substack{u_{\nu-1}\in\Xi^*\\|u_{\nu-1}|\leqslant L''}}f^{(i_{\nu-1})}_{u_{\nu-1},u_\nu}\cdots\sum_{\substack{u_1\in\Xi^*\\|u_1|\leqslant L''}}f^{(i_1)}_{u_1,u_2}(\sigma,tu_1).$$

This implies

$$\mathcal{H}^{(r)}_{L'\times L''}(\sigma)=\mathcal{H}_{L'\times L''}(\sigma)F_{i_1}F_{i_2}\cdots F_{i_\nu}. \qquad \square$$

We now turn to the proof of Theorem 3.1.

The dimension of (A_1,A_2,B,C) given by (3.6) is clearly n_σ. Since representations of σ cannot exhibit a dimension smaller than n_σ, (A_1,A_2,B,C) is a minimal representation of σ, provided that it is a representation.

So it suffices to show that the series $\sigma'\in K\langle\langle\xi_1,\xi_2\rangle\rangle$ defined by

$$(\sigma',1)=CB$$
$$(\sigma',\xi_{i_1}\xi_{i_2}\cdots\xi_{i_\nu})=CA_{i_1}A_{i_2}\cdots A_{i_\nu}B$$

coincides with the given series σ. For simplicity, we write $\mathcal{H}$ instead of $\mathcal{H}_{L'\times L''}(\sigma)$, and $\mathcal{H}^{(r)}$ instead of $\mathcal{H}^{(r)}_{L'\times L''}(\sigma)$. The matrix

$$\mathcal{H}^\dagger=Q\left[\begin{array}{c|c}I_{n_\sigma}&0\\\hline 0&0\end{array}\right]P \tag{3.7}$$

is a pseudoinverse of $\mathcal{H}$, that is $\mathcal{H}\mathcal{H}^\dagger\mathcal{H}=\mathcal{H}$. Clearly

$$(\sigma',1)=CB=[1\quad 0\quad\cdots\quad 0]\mathcal{H}\mathcal{H}^\dagger\mathcal{H}\begin{bmatrix}1\\0\\\cdot\\\cdot\\0\end{bmatrix}=(\sigma,1).$$

Next consider the coefficients

$$(\sigma',\xi_{i_1}\xi_{i_2}\cdots\xi_{i_\nu})=CA_{i_1}A_{i_2}\cdots A_{i_\nu}B$$

$$=[1\quad 0\quad\cdots\quad 0]\mathcal{H}\mathcal{H}^\dagger\mathcal{H}^{(\xi_{i_1})}\mathcal{H}^\dagger\mathcal{H}^{(\xi_{i_2})}\mathcal{H}^\dagger\cdots\mathcal{H}^\dagger\mathcal{H}^{(\xi_{i_\nu})}\mathcal{H}\mathcal{H}\begin{bmatrix}1\\0\\\cdot\\\cdot\\0\end{bmatrix}=[1\quad 0\quad\cdots\quad 0]\mathcal{H}\mathcal{H}^\dagger\sum_{j=1}^{\nu}\left(D_{i_j}\mathcal{H}\mathcal{H}^\dagger\right)\mathcal{H}\begin{bmatrix}1\\0\\\cdot\\\cdot\\0\end{bmatrix}.$$

Observe that

$$\left(\prod_{j=1}^{\nu}D_{i_j}\mathcal{H}\mathcal{H}^\dagger\right)\mathcal{H}=\left(\prod_{j=1}^{\nu-1}D_{i_j}\mathcal{H}\mathcal{H}^\dagger\right)D_{i_\nu}\mathcal{H}\mathcal{H}^\dagger\mathcal{H}$$
$$=\left(\prod_{j=1}^{\nu-1}D_{i_j}\mathcal{H}\mathcal{H}^\dagger\right)\mathcal{H}F_{i_\nu}=\prod_{j=1}^{\nu}\mathcal{H}F_{i_1}F_{i_2}\cdots F_{i_\nu}$$

by induction with respect to the integer ν. Hence

$$(\sigma',\xi_{i_1}\xi_{i_2}\cdots\xi_{i_\nu})$$
$$=[1\quad 0\quad\cdots\quad 0]\mathcal{H}\mathcal{H}^\dagger\mathcal{H}F_{i_1}F_{i_2}\cdots F_{i_\nu}\begin{bmatrix}1\\0\\\cdot\\\cdot\\\cdot\\0\end{bmatrix}$$
$$=[1\quad 0\quad\cdots\quad 0]\mathcal{H}^{(\xi_{i_1}\xi_{i_2}\cdots\xi_{i_\nu})}\begin{bmatrix}1\\0\\\cdot\\\cdot\\0\end{bmatrix}. \tag{3.8}$$

Equation (3.8) calls for picking out the top left corner element in $\mathcal{H}^{(\xi_{i_1}\xi_{i_2}\cdots\xi_{i_\nu})}_{L'\times L''}(\sigma)$. This element is $(\sigma,\xi_{i_1}\cdots\xi_{i_\nu})$. $\square$

IV. Partial Representations

It is clear that the main limitation of the generalized Ho's algorithm is that an *a priori* knowledge of an upper bound of rank $\mathcal{H}(\sigma)$ is needed.

When this kind of information is not available, we cannot find a complete solution to the representation problem of a prescribed series σ. In this situation we can look for a "partial" representation of σ, i.e., for a representation (A_1,A_2,B,C) which matches the coefficients of monomials in σ up to a given degree.

Definition 4.1: Let $\mu:\Xi^*\to K^{M\times M}$ be a representation of Ξ^* on the multiplicative monoid of $K^{M\times M}$. Let $B\in K^{M\times 1}$, $C\in K^{1\times M}$. The 4-tuple $(\xi_1^\mu,\xi_2^\mu,B,C)$ is a partial representation of degree G of the series $\sigma\in K\langle\langle\xi_1,\xi_2\rangle\rangle$ if

$$(\sigma,w)=Cw^\mu B \tag{4.1}$$

holds whenever $|w|\leqslant G$.

$(\xi_1^\mu,\xi_2^\mu,B,C)$ will be also called a partial representation for the polynomial π_σ of formal degree G which coincides with the initial segment of σ.

The following theorem extends to noncommutative power series a standard result of realization theory [14], [15].

Theorem 4.1: Let $\sigma\in K\langle\langle\xi_1,\xi_2\rangle\rangle$ and let $\mathcal{H}(\sigma)$ be the corresponding Hankel matrix. Then the matrices

(A_1,A_2,B,C) defined in (3.6) constitute a partial representation of σ of degree $G=L'+L''$ if and only if

$$\begin{aligned}\operatorname{rank} \mathcal{H}_{L'\times L''}(\sigma) &= \operatorname{rank} \mathcal{H}_{(L'+1)\times L''}(\sigma)\\ &= \operatorname{rank} \mathcal{H}_{L'\times(L''+1)}(\sigma). \qquad (4.2)\end{aligned}$$

Proof: Assume that (A_1,A_2,B,C) is a partial representation of σ of degree G. The series $\sigma \in K\langle\langle(\xi_1,\xi_2)\rangle\rangle$ defined by

$$\sigma' = \sum_{k=0}^{\infty} C(\xi_1 A_1 + \xi_2 A_2)^{-1} B$$

satisfies $(\sigma,w)=(\sigma',w)$ when $|w| \leqslant G$. This implies

$$\mathcal{H}_{L'\times L''}(\sigma) = \mathcal{H}_{L'\times L''}(\sigma')$$

$$\mathcal{H}_{(L'+1)\times L''}(\sigma) = \mathcal{H}_{(L'+1)\times L''}(\sigma')$$

$$\mathcal{H}_{L'\times(L''+1)}(\sigma) = \mathcal{H}_{L'\times(L''+1)}(\sigma').$$

On the other hand one has

$$\begin{aligned}\operatorname{rank} \mathcal{H}_{L'\times L''}(\sigma') &= \operatorname{rank} \mathcal{H}_{(L'+1)\times L''}(\sigma')\\ &= \operatorname{rank} \mathcal{H}_{L'\times(L''+1)}(\sigma') = \dim (A_1,A_2,B,C).\end{aligned}$$

This proves the theorem in one direction.

For the other direction, note that by (4.2) the rows of $\mathcal{H}_{(L'+1)\times L''}(\sigma)$ and the columns of $\mathcal{H}_{L'\times(L''+1)}(\sigma)$ linearly depend on the rows and on the columns of $\mathcal{H}_{L'\times L''}(\sigma)$, respectively.

Although (3.1) holds now for $|w| \leqslant L''$, and (3.3) holds for $|t| \leqslant L'$, we can still define the matrices D_1, D_2, F_1, and F_2 as in Section III. In this new situation, Lemma 3.1 is no longer satisfied except for $r=\xi_1$ and $r=\xi_2$:

$$\mathcal{H}^{(\xi_i)}_{L'\times L''}(\sigma) = D_i \mathcal{H}_{L'\times L''}(\sigma) = \mathcal{H}_{L'\times L''}(\sigma) F_i, \qquad i=1,2.$$

Let $T \leqslant G$ and let $T' \leqslant L'$, $T-T' \leqslant L''$.

By the argument used in the proof of Theorem 3.1 we obtain

$$CA_{i_1}A_{i_2}\cdots A_{i_T}B = [\,1 \quad 0 \quad \cdots \quad 0\,] D_{i_1}D_{i_2}\cdots D_{i_{T'-1}} \cdot \mathcal{H}^{(\xi_{i_{T'}})} F_{i_{T'-1}} F_{i_{T'-2}} \cdots F_{i_T} \begin{bmatrix} 1\\ 0\\ \cdot\\ \cdot\\ 0 \end{bmatrix}. \qquad (4.3)$$

When D_i operates on the left on a matrix M, it substitutes t indexed rows of M with $t\xi_1$ indexed if $|t|<L'$, and substitutes t indexed rows of M with t indexed rows of $\mathcal{H}^{(\xi)}$ if $|t|=L'$ and $M=\mathcal{H}$. Similarly, when F_i operates on the right on a matrix M, it substitutes w indexed columns of M with $\xi_i w$ indexed if $|w|<L''$, and substitutes w indexed columns of M with w indexed columns of $\mathcal{H}^{(\xi)}$, if $|w|=L''$ and $M=\mathcal{H}$. Thus

$$[\,1 \quad 0 \quad \cdots \quad 0\,] D_{i_1}D_{i_2}\cdots D_{i_{T'-1}} \mathcal{H}^{(\xi_{i_{T'}})}$$

is both the $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{T'-1}}$ indexed row in $\mathcal{H}^{(\xi_{i_{T'}})}_{L'\times L''}(\sigma)$ and the $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{T'}}$ indexed row in $\mathcal{H}_{L'\times L''}(\sigma)$. It follows that the w indexed element in the row vector

$$[\,1 \quad 0 \quad \cdots \quad 0\,] D_{i_1}D_{i_2}\cdots D_{i_{T'-1}} \mathcal{H}^{(\xi_{i_{T'}})} F_{i_{T'+1}}$$

is $(\sigma, \xi_{i_1}\xi_{i_2}\cdots\xi_{i_{T'+1}}w)$ for any $w \in \Xi^*$ with $|w| \leqslant L''$.

Since $T-(T'+1)$ is smaller than L'', the first element in the row vector $CA_{i_1}A_{i_2}\cdots A_{i_T}$ is $(\sigma, \xi_{i_1}\xi_{i_2}\cdots\xi_{i_T})$. □

The 4-tuple (A_1,A_2,B,C) we considered in Theorem 4.1 is a minimal partial representation of σ of degree $G=L'+L''$. In fact all partial representations of degree G generate the same truncated Hankel matrix $\mathcal{H}_{L'\times L''}(\sigma)$, so that their dimension is at least rank $\mathcal{H}_{L'\times L''}(\sigma)$.

Furthermore, (A_1,A_2,B,C) is the unique minimal partial representation of degree G, modulo a similarity transformation. In fact, let $(\hat{A}_1,\hat{A}_2,\hat{B},\hat{C})$ be another minimal partial representation of degree G. The series $\sigma' = \sum_{k=0}^{\infty}\hat{C}(\xi_1\hat{A}_1+\xi_2\hat{A}_2)^k\hat{B}$ has minimal representation $(\hat{A}_1, \hat{A}_2, \hat{B}, \hat{C})$, since

$$\operatorname{rank} \mathcal{H}_{L'\times L''}(\sigma') = \operatorname{rank} \mathcal{H}_{L'\times L''}(\sigma) = \dim (\hat{A}_1,\hat{A}_2,\hat{B},\hat{C})$$

and minimal representation (A_1,A_2,B,C) via Ho's algorithm, since

$$\mathcal{H}_{L'\times L''}(\sigma) = \mathcal{H}_{L'\times L''}(\sigma')$$

and

$$\mathcal{H}^{(\xi_i)}_{L'\times L''}(\sigma) = \mathcal{H}^{(\xi_i)}_{L'\times L''}(\sigma'), \qquad i=1,2.$$

(A_1,A_2,B,C) and $(\hat{A}_1,\hat{A}_2,\hat{B},\hat{C})$ are minimal representations of the same series σ: thus by Theorem 1.2 they are similar.

We have therefore proved incidentally the following.

Corollary 4.1: If (4.2) holds, the rational power series σ' satisfying rank $\sigma' \leqslant$ rank $\mathcal{H}_{L'\times L''}(\sigma)$ and $(\sigma',w)=(\sigma,w)$ for $|w| \leqslant L'+L''$ is uniquely determined.

Note that Theorem 4.1 does not provide a general solution to the problem of partial representation. However, it is sufficient for our purposes, so we will stop here.

V. Spatial Filters and Doubly Indexed Dynamical Systems

In this section we briefly outline some aspects of the spatial filters realization theory. Several related topics are discussed in [2], [4], [5].

We will consider spatial digital filters with scalar inputs and outputs taken from an arbitrary field K. The input/output representation of such a filter is given by

$$\mathcal{G} \triangleq (T, U, \mathcal{U}, Y, \mathcal{Y}, F) \qquad (5.1)$$

where $T=\mathbf{Z}\times\mathbf{Z}$ (partially ordered by the product of the orderings) is the discrete plane, U and Y are one-dimensional vector spaces over the field K, $\mathcal{U}$ and $\mathcal{Y}$ are the space of truncated formal Laurent series in two commutative variables over K and $F: \mathcal{U} \to \mathcal{Y}$ is the input/output map.

A typical element of $\mathcal{U}$ or $\mathcal{Y}$ will be written

$$r = \sum_{i,j=-k}^{\infty} (r, z_1^i z_2^j) z_1^i z_2^j, \qquad \text{for some integer } k$$

where $(r, z_1^i z_2^j)$ denotes the coefficient of $z_1^i z_2^j$.

The input/output map $F: \mathcal{U} \to \mathcal{Y}$ is assumed to satisfy the following axioms:

i) Linearity.

ii) Two-dimensional shift invariance

$$F(z_1^i z_2^j r) = z_1^i z_2^j F(r), \qquad i,j \in \mathbf{Z}.$$

iii) Two-dimensional strict causality

$$(u_1, z_1^i z_2^j) = (u_2, z_1^i z_2^j), \qquad i < t_1,\ j < t_2$$

implies

$$(Fu_1, z_1^i z_2^j) = (Fu_2, z_1^i z_2^j), \qquad i \leqslant t_1,\ j \leqslant t_2,\ \forall u_1, u_2 \in \mathcal{U}.$$

Under assumption iii) it is easy to verify that the impulse response $F(1)$ is a "strictly causal" power series, i.e.,

$$F(1) = \sum_{i,j=1}^{\infty} (F(1), z_1^i z_2^j) z_1^i z_2^j.$$

More formally we can say that

$$s \triangleq F(1) \in (z_1 z_2) K[[z_1, z_2]] \triangleq K_c[[z_1, z_2]]$$

where $K[[z_1 z_2]]$ denotes the ring of commutative formal power series in two variables and $K_c[[z_1, z_2]]$ is the ideal of *"strictly causal"* power series.

From i) and ii) it follows that

$$F(u) = F(1)u, \qquad \forall u \in \mathcal{U}. \tag{5.2}$$

Thus two-dimensional filters (in their input/output representation) are in one-to-one correspondence with formal power series $K_c[[z_1, z_2]]$.

Definition: A doubly indexed, linear, stationary, finite dimensional dynamical system Σ (DIDS) is defined by a pair of equations of the form

$$x(h+1,k+1) = A_1 x(h+1,k) + A_2 x(h,k+1) + Bu(h,k)$$
$$y(h,k) = Cx(h,k) \tag{5.3}$$

where $A_i \in K^{n \times n}$, $i = 1,2$, $C \in K^{1 \times n}$, $B \in K^{n \times 1}$ and x belongs to some finite dimensional vector space $X = K^n$ (local state space).

The solution of equations (5.3) for $h \geqslant 0$, $k \geqslant 0$, is uniquely determined by u and by the values $x(h,0)$, $h = 1,2,\cdots$, and $x(0,k)$, $k = 0,1,2,\cdots$, (initial local states).

Let $x(h,0) = x(0,k) = 0$, $h,k = 0,1,2,\cdots$, and denote by s_Σ the output of Σ which corresponds to the input $u = 1$. It is easy to check that s_Σ is given by the following commutative power series in $K_c[[z_1, z_2]]$:

$$s_\Sigma = C(I - A_1 z_1 - A_2 z_2)^{-1} z_1 z_2 B. \tag{5.4}$$

As usual, s_Σ will be called the transfer function of Σ. If the system Σ starts from zero initial local states, the output y corresponding to any input u is obtained as

$$y = s_\Sigma u.$$

Clearly the map $u \to s_\Sigma u$ satisfies axioms i)–iii) as stated. Thus the DIDS $\Sigma = (A_1, A_2, B, C)$ determines ("realizes") the input/output map of the spatial filter described by the impulse response $F(1) = s_\Sigma$.

The realization problem is basically concerned with the inverse procedure, i.e., to pass from the input/output description of a filter, assigned as an impulse response $F(1)$, to a DIDS whose transfer function s_Σ satisfies $s_\Sigma = F(1)$.

We therefore have the following definition.

Definition 5.1: A doubly indexed dynamical system $\Sigma = (A_1, A_2, B, C)$ is a zero-state realization of a two-dimensional filter with impulse response $F(1) \in K_c[[z_1, z_2]]$ if

$$F(1) = (z_1 z_2) C(I - A_1 z_1 - A_2 z_2)^{-1} B. \tag{5.5}$$

The *dimension of a realization* Σ is the dimension of the local state space X.

The minimality of the realization is naturally related to the dimension of X in the sense that a realization Σ is minimal when $\dim \Sigma \leqslant \dim \Sigma'$ for any Σ' which realizes $\mathcal{F}$.

When we want to solve the realization problem, we have to answer several questions. For instance, how do we characterize the class of realizable filter? What procedure do we adopt for obtaining all minimal realizations? Is the set of minimal realizations endowed with some algebraic structure? As we shall see, the noncommutative power series approach to these problems is quite fruitful.

We recall that a commutative formal power series $s \in K[[z_1, z_2]]$ is rational if there exist polynomials p, q in $K[z_1, z_2]$, with $q(0,0) \neq 0$, such that $qs = p$. The polynomial q is called a denominator of s. We denote by $K[(z_1, z_2)]$ the ring of rational commutative power series and by $K_c[(z_1, z_2)] \triangleq z_1 z_2 K[(z_1, z_2)]$ the ideal of causal rational power series. Consider the "natural" algebra homomorphism $\phi: K\langle\langle \xi_1, \xi_2 \rangle\rangle \to K[[z_1, z_2]]$ determined by $\phi(\xi_1) = z_1$, $\phi(\xi_2) = z_2$, $\phi(k) = k$, $\forall k \in K$. The ϕ-image of the subalgebra $K\langle(\xi_1, \xi_2)\rangle$ is contained in $K[(z_1, z_2)]$.

Moreover ϕ carries $K\langle(\xi_1, \xi_2)\rangle$ onto $K[(z_1, z_2)]$ as a consequence of the following theorem.

Theorem 5.1: Let $F(1) \in K_c[[z_1, z_2]]$ be the impulse response of a filter $\mathcal{F}$. Then $\mathcal{F}$ is realizable (by a DIDS) if and only if $F(1) \in K_c[(z_1, z_2)]$.

Proof: Necessity is trivial. To prove sufficiency, let $F(1)$ be in $K_c[(z_1, z_2)]$, $F(1) = z_1 z_2 p(z_1, z_2)/q(z_1, z_2)$, $p, q \in K[z_1, z_2]$, $q(0,0) \neq 0$. Choose π and η in $K\langle \xi_1, \xi_2 \rangle$ such that

$$\phi(\pi) = p$$
$$\phi(\eta) = q.$$

Since the coefficient $(\eta, 1)$ is different from zero, the polynomial η is an invertible element in $K\langle\langle \xi_1, \xi_2 \rangle\rangle$. The coefficients of the series η^{-1} can be obtained by recursion

$$(\eta^{-1}, w) = (\eta, w)^{-1}, \qquad \text{if } |w| = 0$$

$$(\eta^{-1}, w) = -(\eta, 1)^{-1} \sum_{\substack{ut = w \\ |w| > 0}} (\eta, u)(\eta^{-1}, t), \qquad \text{if } |w| > 0.$$

Hence, by Definition 1.1, η^{-1} belongs to $K\langle(\xi_1, \xi_2)\rangle$. Plainly the series $\sigma = \pi\eta^{-1}$ belongs to $K\langle(\xi_1, \xi_2)\rangle$ and satisfies

$$\phi(\sigma)=\phi(\pi)\phi(\eta^{-1})=p/q$$

because ϕ is an homomorphism.

If (A_1,A_2,B,C) is a representation of σ then the DIDS $\Sigma=(A_1,A_2,B,C)$ is a realization of $\mathcal{G}$. In fact

$$F(1)=z_1z_2p(z_1,z_2)/q(z_1,z_2)=\phi(\xi_1\xi_2\pi\eta^{-1})$$
$$=\phi\left(\xi_1\xi_2\sum_{i=0}^{\infty}C(\xi_1A_1+\xi_2A_2)^iB\right)$$
$$=z_1z_2\sum_{i=0}^{\infty}C(A_1z_1+A_2z_2)^iB=z_1z_2C(I-A_1z_1-A_2z_2)^{-1}B.$$

□

VI. Algorithm for Minimal Realizations

An interesting application of noncommutative power series to filters theory is the development of algorithms for generating minimal realizations.

We have already succeeded in obtaining nonminimal realization algorithms and some reduction procedures [5], [6]. It is a remarkable fact that the problem of getting a minimal realization from a generic one cannot be solved in general by removing incontrollable and unobservable parts of a DIDS as we pointed out in [6].

The procedure outlined in this section is fitted to the problem of avoiding minimization techniques and obtaining directly minimal realizations.

Let $F(1)\in K_c[(z_1,z_2)]$ be the impulse response of a given filter $\mathcal{G}$.

Pick any $\sigma\in K\langle(\xi_1,\xi_2)\rangle$ satisfying $\phi(\sigma)=z_1^{-1}z_2^{-1}F(1)$. Then each representation (A_1,A_2,B,C) of σ is a realization of $\mathcal{G}$. Conversely if $\Sigma=(A_1,A_2,B,C)$ is a realization of $\mathcal{G}$, Σ is a representation of the rational series $\sigma_\Sigma=\Sigma_{i=0}^{\infty}C(\xi_1A_1+\xi_2A_2)^iB$, which satisfies $\phi(\sigma_\Sigma)=z_1^{-1}z_2^{-1}F(1)$. Hence the class of realizations of $\mathcal{G}$ coincides with the class of representations of rational series σ satisfying $\phi(\sigma)=z_1^{-1}z_2^{-1}F(1)$.

In order to obtain all minimal realizations (modulo similarity transformations) we shall therefore go through the following steps.

Step 1: Determine the set $\mathfrak{M}\subseteq K\langle(\xi_1,\xi_2)\rangle$ of series having minimal rank and $z_1^{-1}z_2^{-1}F(1)$ as ϕ-image.

Step 2: For each series $\sigma\in\mathfrak{M}$ obtain, via Ho's algorithm, a minimal representation.

Let $\Sigma=(\xi_1^{\mu},\xi_2^{\mu},B,C)$ be a realization of $\mathcal{G}$, and let M and P integers satisfying $\dim\Sigma\leqslant M\leqslant P$. Introduce the following noncommutative polynomial

$$\pi_\Sigma=\sum_{\substack{w\in\Xi^*\\|w|<2P}}Cw^{\mu}B.$$

It is easy to see that

$$\phi(\pi_\Sigma)=\sum_{i+j\leqslant 2P}(z_1^{-1}z_2^{-1}F(1),z_1^iz_2^j)z_1^iz_2^j \qquad (6.1)$$

and

$$\operatorname{rank}\mathcal{H}_{P\times P}(\pi_\Sigma)=\operatorname{rank}\mathcal{H}_{(P+1)\times P}(\pi_\Sigma)$$
$$=\operatorname{rank}\mathcal{H}_{P\times(P+1)}(\pi_\Sigma)\leqslant M.$$

As a consequence of corollary 4.1, partial representation of π_Σ of dimension $\leqslant M$ are (global) representations of the series σ_Σ, hence realizations of $\mathcal{G}$.

Assume now $\pi\in K\langle\xi_1,\xi_2\rangle$ have formal degree $2P$ and verify relations (6.1). Let (A_1,A_2,B,C) be a partial representation of π of dimension $\leqslant M$. What conditions on the integers P and M do guarantee that (A_1,A_2,B,C) realizes $\mathcal{G}$?

To answer this question, we use the following.

Lemma 6.1: Let r and r' belong to $K[(z_1,z_2)]$ and

$$r=\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}a_{ij}z_1^{-i}z_2^{-j}\Big/\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}b_{ij}z_1^{-i}z_2^{-j},\qquad b_{N_1N_2}=1$$
$$r'=\sum_{i=0}^{N_1'}\sum_{j=0}^{N_2'}a'_{ij}z_1^{-i}z_2^{-j}\Big/\sum_{i=0}^{N_1'}\sum_{j=0}^{N_2'}b'_{ij}z_1^{-i}z_2^{-j},\qquad b'_{N_1N_2}=1. \qquad (6.2)$$

Let $h_\nu^{(r)}$ and $h_\nu^{(r')}$ denote the homogeneous polynomials of degree ν in r and r', respectively. Then

$$h_\nu^{(r)}=h_\nu^{(r')},\qquad \nu=0,1,2,\cdots,N_1+N_2+N_1'+N_2'$$

imply $r=r'$.

Proof: First rewrite (6.2) as

$$r\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}b_{ij}z_1^{-i}z_2^{-j}=\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}a_{ij}z_1^{-i}z_2^{-j}.$$

This shows that $h_0^{(r)}=h_1^{(r)}=\cdots=h_{N_1+N_2}^{(r)}=0$ represent a necessary and sufficient condition for $r=0$. Next observe that a denominator of $r-r'$ has degree $N_1+N_2+N_1'+N_2'$, and finally apply the condition above to $r-r'$. □

We are now in the position to answer the previous question. In fact, let

$$F(1)z_1^{-1}z_2^{-1}=\sum_{i=0}^{N_1}\sum_{j=0}^{N_1}a'_{ij}z_1^{-i}z_2^{-j}\Big/\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}b_{ij}z_1^{-i}z_2^{-j} \qquad (6.3)$$

and suppose we know an upper bound M for the dimension of minimal realizations of $\mathcal{G}$. Denote by $\mathcal{P}$ the set of all polynomials π which fulfill the following conditions:

i)[1] $$\deg\pi\leqslant 2\bar{P}\triangleq 2\left(M+\left[\frac{N_1+N_2}{2}\right]\right) \qquad (6.4)$$

ii) $$\phi(\pi)=\sum_{i+j\leqslant 2\bar{P}}(z_1^{-1}z_2^{-1}F(1),z_1^iz_2^j)z_1^iz_2^j \qquad (6.5)$$

iii) $$\operatorname{rank}\mathcal{H}_{\bar{P}\times\bar{P}}(\pi)=\operatorname{rank}\mathcal{H}_{(\bar{P}+1)\times\bar{P}}(\pi)$$
$$=\operatorname{rank}\mathcal{H}_{\bar{P}\times(\bar{P}+1)}(\pi)\leqslant M. \qquad (6.6)$$

Proposition 6.1: Let $A_i\in K^{N\times N}$, $i=1,2$, $B\in K^{N\times 1}$, $C\in K^{1\times N}$, $N\leqslant M$. Then (A_1,A_2,B,C) is a partial representa-

[1] $[x]$ denotes the smallest integer n such that $n\geqslant x$.

tion of some polynomial in $\mathcal{P}$ if and only if (A_1, A_2, B, C) is a realization of $\mathcal{S}$.

Proof: Let (A_1, A_2, B, C) be a partial representation of $\pi \in \mathcal{P}$. Then $s_\Sigma = C(I - A_1 z_1 - A_2 z_2)^{-1}B$ agrees with $z_1^{-1}z_2^{-1}F(1)$ up to the degree $2\bar{P}$. σ_Σ has a denominator in $K[z_1^{-1}, z_2^{-1}]$ of degree $\leqslant 2M$ and $z_1^{-1}z_2^{-1}F(1)$ has a denominator given by (6.3). Hence $\sigma_\Sigma = z_1^{-1}z_2^{-1}F(1)$ by Lemma 6.1 and by the definition of $\bar{P}$.

This proves the only if part of the proposition. The converse is obvious. □

Noncommutative polynomials which satisfy degree condition (6.4) constitute a finite dimensional K linear space $V^{(i)}$. Since the restriction of ϕ to $V^{(i)}$, denoted $\phi | V^{(i)}$, is linear, the set $V^{(ii)}$ of polynomials satisfying (6.4) and (6.5) is a coset relative to the subspace ker $(\phi | V^{(i)})$.

Thus we are interested in characterizing $\mathcal{P}$ in $V^{(ii)}$. For that purpose we shall give a parametric representation of polynomials in $V^{(ii)}$, and then we shall find the parameters corresponding to the elements in $\mathcal{P}$ having minimal partial representations.

The procedure is summarized as follows.

1) Let $\Xi^{(h,k)} \subset \Xi^*$ denote the set of words which contain ξ_1 and ξ_2, respectively, h and k times, let $\mathcal{J} = \{w : w \in \Xi^*, |w| \leqslant 2\bar{P}, w \neq \xi_1^h \xi_2^k, \forall h,k\}$, and let $K^{\mathcal{J}}$ be the space of $\mathcal{J}$ indexed sequences of elements of K. Each polynomial π in $V^{(ii)}$ is biuniquely represented onto $K^{\mathcal{J}}$ as follows.

$$\pi_{(t_w)} = \sum_{w \in \mathcal{J}} t_w w + \sum_{i+j \leqslant 2\bar{P}} \left(\left(z_1^{-1}z_2^{-1}F(1), z_1^i z_2^j \right) - \sum_{w \in \mathcal{J} \cap \Xi^{(i,j)}} t_w \right) \xi_1^i \xi_2^j. \tag{6.7}$$

2) Consider the following sets in $K^{\mathcal{J}}$

$$\begin{aligned} \mathcal{V}_n' &= \left\{ (t_w) : (t_w) \in K^{\mathcal{J}}, \text{rank } \mathcal{H}_{\bar{P} \times \bar{P}}(\pi_{(t_w)}) \leqslant n \right\} \quad n \leqslant M \\ \mathcal{V}_n'' &= \left\{ (t_w) : (t_w) \in K^{\mathcal{J}}, \text{rank } \mathcal{H}_{\bar{P} \times (\bar{P}+1)}(\pi_{(t_w)}) \leqslant n \right\} \quad n \leqslant M \\ \mathcal{V}_n''' &= \left\{ (t_w) : (t_w) \in K^{\mathcal{J}}, \text{rank } \mathcal{H}_{(\bar{P}+1) \times \bar{P}}(\pi_{(t_w)}) \leqslant n \right\} \quad n \leqslant M. \end{aligned} \tag{6.8}$$

Each condition on the rank of matrices in (6.8) is equivalent to a number of conditions on the minors of order $n+1$ and is expressed by a system of algebraic equations in the parameters t_w, $w \in \mathcal{J}$. Thus $\mathcal{V}_n'$, $\mathcal{V}_n''$ and $\mathcal{V}_n'''$ are algebraic varieties in $K^{\mathcal{J}}$.

3) Evaluate the smallest value of the index n such that

$$\mathcal{W}_n \triangleq (\mathcal{V}_n - \mathcal{V}_{n-1}') \cap (\mathcal{V}_n'' - \mathcal{V}_{n-1}'') \cap (\mathcal{V}_n''' - \mathcal{V}_{n-1}''') \neq \varnothing.$$

Let $\bar{M}$ denote this value. Then $(t_w) \in \mathcal{W}_{\bar{M}} = (\mathcal{V}_{\bar{M}}'' \cap \mathcal{V}_{\bar{M}}''') \cap (\mathcal{V}_{\bar{M}}' - \mathcal{V}_{\bar{M}-1}')$ if and only if $\pi_{(t_w)}$ belongs to $\mathcal{P}$ and has minimal partial representation. In fact $\bar{M}$ is the smallest value of n for which the equations chain

$$\begin{aligned} \text{rank } \mathcal{H}_{\bar{P} \times \bar{P}}(\pi_{(t_w)}) &= \text{rank } \mathcal{H}_{\bar{P} \times (\bar{P}+1)}(\pi_{(t_w)}) \\ &= \text{rank } \mathcal{H}_{(\bar{P}+1) \times \bar{P}}(\pi_{(t_w)}) = n \end{aligned}$$

admits a (t_w) solution.

As obvious consequence minimal partial representations have dimension $\bar{M}$.

4) Use (6.7) for constructing polynomials in $\mathcal{P}$ which have minimal partial representation.

Clearly, $\mathcal{M}$ is constituted by all rational power series of rank $\bar{M}$ which extend the polynomials obtained by the above steps. It is interesting to remark that two minimal realizations of a given filter are not necessarily similar—in fact whenever $\mathcal{M}$ contains more than one element, minimal representations of two different series in $\mathcal{M}$ cannot be similar.

In conclusion, the key points of the realization algorithm are the following.

a) Evaluate an upper bound M for the dimension of minimal realizations. This can be done in several ways (see, for instance, [4], [5]). Obviously, the smaller will be the integer M, the easier will be the subsequent computation.

b) Evaluate the integer $\bar{M}$ and construct the set of polynomials in $\mathcal{P}$ which exhibit minimal partial representations of dimension $\bar{M}$. This point has no counterpart in standard realization algorithms and is the most difficult to be implemented because it involves the solution of several nonlinear algebraic equations. On the other hand the necessity of introducing nonlinear algorithms is intrinsic to the problem, as the dimension of minimal realizations depends on the ground field.

c) Use the generalized Ho's algorithm for getting minimal partial representations of polynomials obtained in the previous point. The set of these representations (modulo similarity transformations) gives all minimal realizations of $\mathcal{S}$.

Conclusions

We have attempted in this paper to show how noncommutative power series represent a fruitful tool in the area of filter realization.

Various aspects of the theory of rational noncommutative power series have been discussed and an extension of Ho's algorithm has been derived.

The partial representation problem, in which one is trying to obtain a recursive model for the coefficients of a series on the basis of incomplete data is subsequently presented.

The problem of generating all minimal realizations of a given filter is finally considered, and it is shown how to construct all minimal state space models of the filter by solving algebraic equations and applying the generalized Ho's algorithm.

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