2-D SYSTEMS THEORY - A PROGRESS REPORT

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ABSTRACT. The stability problem of two-dimensional filters is receiving wide attention [1-10].

Here a preliminary account of an "internal stability" theory is given which takes into account the dynamical model point of view.

The 2-D recursive model presented in [6] is considered and a Lyapunov equation is derived.

1. INTRODUCTION AND PRELIMINARY DEFINITIONS

From now on a 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C)$ is identified by the following pair of equations:

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h,k+1) + A_2 x(h+1,k) + B_1 u(h,k+1) + B_2 u(h+1,k) \\ y(h,k) &= C\, x(h,k) \end{aligned} \qquad (1)$$

where

- (h,k) are elements of $\mathbb{Z} \times \mathbb{Z}$ ("time set"), partially ordered by the product of the orderings
- $x(\cdot): \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^n$ is a map whose value at time (h,k) is called the "local state at time (h,k)"
- $u(\cdot): \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ and $y(\cdot): \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ are the input and the output maps respectively, and $u(h,k)$, $y(h,k)$ are the input and the output values at time (h,k)
- $A_1, A_2 \in \mathbb{R}^{n\times n}$, B_1, $B_2 \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ are suitable matrices which completely characterize the 2-D system Σ.

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When an input function $u(\cdot)$ is given, the solution of (1) requires a complete information about a suitable set of local states called "initial global state".

Considered in $\mathbb{Z}\times\mathbb{Z}$ a non empty set $\mathcal{C}$, ("separation set") which satisfies the following characteristic properties:

(i) if $h>i$, $k>j$, (h,k) and (i,j) cannot simultaneously belong to $\mathcal{C}$.

(ii) if (h,k) belongs to $\mathcal{C}$, then $\mathcal{C}$ intersects the sets $\{(h-1,k), (h,k+1), (h-1,k+1)\}$ and $\{(h+1,k), (h+1,k-1)\}$ and does not contain the set $\{(h+1,k), (h,k+1)\}$

(iii) for any (i,j) in $\mathbb{Z}\times\mathbb{Z}$, the relation $(h,k) < (i,j)$ cannot be satisfied by infinitely many elements $(h,\bar{k})$ in $\mathcal{C}$.

The "future of $\mathcal{C}$" is the set

$$\mathcal{F}_{\mathcal{C}} = \{(i,j): (h,k) \leq (i,j) \text{ for some } (h,k) \text{ in } \mathcal{C}\}$$

The "global state" $\mathcal{X}_{\mathcal{C}}$ on the separation set $\mathcal{C}$ is defined as

$$\mathcal{X}_{\mathcal{C}} \triangleq \{x(h,k): (h,k)\in\mathcal{C}\}$$

and the computation of a local state $x(i,j)$, (i,j) in $\mathcal{F}_{\mathcal{C}}$, can be performed starting from $\mathcal{X}_{\mathcal{C}}$, whenever u is known in $\mathcal{F}_{\mathcal{C}}$. In particular if u is identically zero on $\mathcal{F}_{\mathcal{C}}$ the local states in $\mathcal{F}_{\mathcal{C}}$ depend only on $\mathcal{X}_{\mathcal{C}}$.

2. POLYNOMIAL CONDITIONS FOR INTERNAL STABILITY

The notion of internal stability of a 2-D system is related to the behaviour of the free evolution of local states resulting from a bounded global state assignment on a separation set $\mathcal{C}$. Let assume in $\mathbb{Z}\times\mathbb{Z}$ the distance function $d((i,j),(h,k)) \triangleq |i-h| + |j-k|$ and denote by

$$d((i,j),\mathcal{C}) = \min_{(h,k)\in\mathcal{C}} d((i,j),(h,k))$$

the distance between (i,j) and the set $\mathcal{C}$. Introduce the following notation

$$\|\mathcal{X}_{\mathcal{C}}\| = \sup_{x\in\mathcal{X}_{\mathcal{C}}} \|x\| \quad , \quad \|x\| \text{ euclidean norm of } x$$

Definition 1: *Let $\mathcal{C}$ be a separation set in $\mathbb{Z}\times\mathbb{Z}$, and assume* $u=0$. *The 2-D system (1) is internally stable with respect to $\mathcal{C}$ if given* $\varepsilon>0$, *for every $\mathcal{X}_{\mathcal{C}}$ with* $\|\mathcal{X}_{\mathcal{C}}\|<\infty$, *there exists a positive integer* m *such that* $\|x(i,j)\|<\varepsilon$ *when* (i,j) *is in the future of $\mathcal{C}$ and* $d((i,j)\ )>m$.

The internal stability depends on the pair (A_1,A_2), and does not depend on the separation set .

Theorem 1: [7] *Internal stability respect to any separation $\mathcal{C}$ implies internal stability w.r. to every separation set.*

At this point the construction of internal stability criteria is perhaps the most natural topic of investigation. Is is well known that internal stability of a discrete linear system

$$x(h+1) = Ax(h) + Bu(h) \qquad y(h) = Cx(h) \tag{3}$$

can be checked (i) by evaluating the roots distribution of the characteristic polynomial of A with respect to the unit circle in the Gauss plane or (ii) by solving the Lyapunov equation.

Up to now the corresponding picture for 2-D systems is incomplete. In fact a 2-D version of Lyapunov equation is not available, despite some steps have already been done [9]. The following theorem provides the 2-D counterpart of the 1-D stability criterion based on the characteristic polynomial.

Theorem 2 [6] *A 2-D system* $\Sigma = (A_1, A_2, B_1, B_2, C)$ *is internally stable if and only if the polynomial* $\det(1 - A_1 z_1 - A_2 z_2)$ *is devoid of zeros in the closed polydisc* $\mathcal{P}_1 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C}: |z_1| \le 1, |z_2| \le 1\}$

Several tests have been proposed in the literature [Shanks, Huang, Jury, Anderson etc.] to check if the unit polydisc $\mathcal{P}_1$ intersects the variety of a polynomial $q \in \mathbb{C}[z_1, z_2]$. The original field of application of these tests has been the external (BIBO) stability analysis, since Shanks theorem [1] states that the external stability of a two dimensional filter with transfer function p/q, p and q coprime, depends only on the zeros distribution of q.

Actually in 1978 Goodman [5] showed that a two dimensional filter can be BIBO stable even when the denominator q vanishes in some points of the torus $T_1 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C}: |z_1| = |z_2| = 1\}$. As a consequence Shanks theorem has to be restated in the following form

Theorem 3 [5] *Let* $G(z_1, z_2) = p(z_1, z_2)/q(z_1, z_2)$ *denote the transfer function of a* BIBO *stable filter. Then* $G(z_1, z_2)$ *has no poles in the closed unit polydisc* $\mathcal{P}_1$ *and no nonessential singularities of the second kind on* $\mathcal{P}_1$ *except possibly on* T_1.

Then the mutual implications between roots distribution of q and BIBO stability are as follows:

$$\text{BIBO stability} \Rightarrow q(z_1, z_2) \neq 0 \ \text{ in } \mathcal{P}_1 - T_1$$
$$\text{BIBO stability} \Leftarrow q(z_1, z_2) \neq 0 \ \text{ in } \mathcal{P}_1$$

This shows that the absence of intersections between $\mathcal{P}_1$ and the variety of q, which can be checked by the above mentioned tests, ensures BIBO stability. Nevertheless we can have BIBO stability even when the intersection is nonempty, that is when the tests would give a negative result.

On the other hand by Theorem 2 these tests fit perfectly to the internal stability analysis. In particular the following Corollary of Theorem 2, derived from Huang's criterion [2], will provide (see Section 4) a frequency dependent Lyapunov equation.

Corollary 1 [9] *A 2-D system* $\Sigma = (A_1, A_2, -, -, -)$ *is internally sta-*

ble if and only if the complex matrix $A_1 + e^{j\omega}A_2$ *is stable (i.e. the magnitudes of its eigenvalues are less than 1) for any real* ω

Remark. An obvious necessary 2-D stability condition resulting from Corollary 1 is that $A_1 \pm A_2$ have to be stable.

3. MATRIX CONDITIONS FOR INTERNAL STABILITY

In this section we are concerned with the extension to the two-dimensional case of some properties characterizing internally stable 1-D systems. These properties are recalled in Theorem 4.

Theorem 4. *The following propositions are equivalent:* i) *the system* (3) *is internally stable;* ii) *the series* $\sum_{i=0}^{\infty} \|A^i\|$ *converges;* iii) $\|A^k\| < 1$ *for some integer* $k > 0$; iv) *the series* $\sum_{i=0}^{\infty} (A^i)^T(A^i)$ *converges.*

Note that $P = \sum_{i=0}^{\infty} A^{iT}A^i$ is the solution of the Lyapunov equation

$$X = I + A^T XA$$

As we shall see in Theorem 5, the family of matrices $\{A_1^r \sqcup\!\sqcup^s A_2, r,s \in \mathbb{N}\}$, defined as

$$A_1^r \sqcup\!\sqcup^0 A_2 \triangleq A_1^r \qquad A_1^0 \sqcup\!\sqcup^s A_2 \triangleq A_2^s$$
$$A_1^r \sqcup\!\sqcup^s A_2 \triangleq A_1(A_1^{r-1} \sqcup\!\sqcup^s A_2) + A_2(A_1^r \sqcup\!\sqcup^{s-1} A_2) \quad \text{if} \quad r,s > 0 \tag{4}$$

plays an essential role in extending points i)-iv) of Theorem 4.

Theorem 5 [9] *The following propositions are equivalent:* i) *the 2-D system (1) is internally stable;* ii) *the series* $\sum_{r,s=0}^{\infty} \|A_1^r \sqcup\!\sqcup^s A_2\|$ *converges;* iii) $\sum_{r+s=k} \|A_1^r \sqcup\!\sqcup^s A_2\| < 1$ *for some positive integer* k; iv) *the series* $\sum_{r,s=0}^{\infty} (A_1^r \sqcup\!\sqcup^s A_2)^T(A_1^r \sqcup\!\sqcup^s A_2)$ *converges.*

The problem of relating the series $\sum_{r,s} (A_1^r \sqcup\!\sqcup^s A_2)^T(A_1^r \sqcup\!\sqcup^s A_2)$ to the solution of an algebraic matrix equation is still unsolved, except for n=1 (see section 4).

A way to obtain a Lyapunov equation consists in reducing a two dimensional dynamics to a one dimensional, by assuming a periodic initial global state, and then in applying 1-D theory. As we shall see in the sequel this procedure is not fruitful since the existence of stability criteria under periodic initial conditions does not imply a stable behaviour under generic initial conditions.

From now on, $\mathcal{C}_i$, $i \in \mathbb{Z}$, will denote the following separation sets

$$\mathcal{C}_i = \{(h,k) : h+k = i\}$$

Let call a global state $\mathcal{X}_{\mathcal{C}_i}$ "H periodical" if $x(h,k)\in\mathcal{X}_{\mathcal{C}_i}\Rightarrow x(h,k)=$ $= x(h+H,k-H)$ for any (h,k) in $\mathcal{C}_i$.

Clearly the stability check for a H-periodical global initial state reduces to solve a standard Lyapunov equation in dimension $nH\times nH$.

The 2-D system considered in the following example is unstable. However it shows a stable behaviour corresponding to any periodic global initial state.

Example. The 2-D system $\Sigma = (A_1,A_2,-,-,-)$ with

$$A_1 = 0.5\begin{bmatrix}\cos\sqrt{0.5} & \sin\sqrt{0.5}\\ -\sin\sqrt{0.5} & \cos\sqrt{0.5}\end{bmatrix},\quad A_2 = 0.5\begin{bmatrix}\cos\sqrt{0.3} & \sin\sqrt{0.3}\\ -\sin\sqrt{0.3} & \cos\sqrt{0.3}\end{bmatrix}$$

is internally unstable. In fact the eigenvalues of $A_1+e^{j\omega}A_2$ are $\lambda_{1,2}(\omega) = 0.5(e^{\pm j\sqrt{0.5}}+e^{j(\omega\pm\sqrt{0.3})})$ and the magnitude of one of them is 1 for $\omega=\omega_{1,2}=\pm(\sqrt{0.5}-\sqrt{0.3})$. Any other choice of ω leads to a stable matrix $A_1+e^{j\omega}A_2$.

Let now change the basis in $X=\mathbb{C}^2$ (local state space), and refer to the complex basis

$$v_1 = \begin{bmatrix}-j\\1\end{bmatrix},\quad v_2 = \begin{bmatrix}j\\1\end{bmatrix}$$

which constitutes a spectral basis for A_1, A_2 and $A_1+e^{j\omega}A_2, \forall\omega$. Correspondingly, matrices A_1 and A_2 assume the diagonal form

$$\hat{A}_1 = \frac{e^{j\sqrt{0.5}}}{0.5}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \triangleq \begin{bmatrix}\nu_1 & 0\\ 0 & \nu_2\end{bmatrix};\quad \hat{A}_2 = \frac{e^{j\sqrt{0.3}}}{0.5}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \triangleq \begin{bmatrix}\mu_1 & 0\\ 0 & \mu_2\end{bmatrix}$$

and the free evolution of local states in given by

$$\hat{x}(h+1,k+1) \triangleq \begin{bmatrix}\hat{x}_1(h+1,k+1)\\ \hat{x}_2(h+1,k+1)\end{bmatrix} = \begin{bmatrix}\nu_1\hat{x}_1(h,k+1)+\mu_1\hat{x}_1(h+1,k)\\ \nu_2\hat{x}_2(h,k+1)+\mu_2\hat{x}_2(h+1,k)\end{bmatrix}$$

Assume the local states in $\mathcal{X}_{\mathcal{C}_0}$ have a periodical shape, say $\hat{x}(h,-h)=\hat{x}(h+H,-h-H)$ for some $H>0$ and for every h in $\mathbb{Z}$. Then

$$\hat{A}_1^r \sqcup\!\sqcup\, {}^s\hat{A}_2 = \begin{bmatrix}\binom{r+s}{s}\nu_1^r\mu_1^s & 0\\ 0 & \binom{r+s}{s}\nu_2^r\mu_2^s\end{bmatrix}$$

gives

$$\begin{bmatrix}\hat{x}_1(h,-h+H)\\ \hat{x}_1(h+1,-h-1+H)\\ \vdots\\ \hat{x}_1(h+H-1,-h)\end{bmatrix} = \overbrace{\begin{bmatrix}\binom{H-1}{0}\nu_1^{H-1} & \binom{H-1}{1}\nu^{H-2}\mu_1 \cdots \binom{H-1}{H-1}\mu^{H-1}\\ \binom{H-1}{1}\nu_1^{H-2}\mu_1 & \binom{H-1}{2}\nu^{H-3}\mu_1^2\cdots\binom{H-1}{0}\nu^{H-1}\\ \cdots\cdots & \cdots\cdots\\ \cdots\cdots & \cdots\cdots\\ \binom{H-1}{H-1}\mu_1^{H-1} & \binom{H-1}{0}\nu_1^{H-1}\cdots\binom{H-1}{H-2}\nu_1\mu_1^{H-2}\end{bmatrix}}^{G}\begin{bmatrix}\hat{x}_1(h,-h)\\ \hat{x}_1(h+1,-h-1)\\ \cdots\\ \cdots\\ \hat{x}_1(h+H-1,-h-H+1)\end{bmatrix}$$

Since G is a circulant matrix [11], its eigenvalues are

$$\chi_\ell = (\nu_1+\mu_1 r_\ell)^{H-1} = 0.5(e^{j\sqrt{0.5}}\pm r_\ell e^{j\sqrt{0.3}})^{H-1}$$

where r_ℓ, $\ell = 1,2,\ldots H$ denote the H-th roots of 1. Due to the fact that $|\nu_1+\mu_1 e^{j\omega}|$ is less than 1 for any real ω, except $\omega = \sqrt{0.5}-\sqrt{0.3}$ (mod. 2π), and that $r_\ell = e^{j2\pi\ell/H}$, $\ell = 1,2,\ldots,H$, $|\nu_1+r_\ell\mu_1|$ is always less than 1. Consequently G is a stable matrix and the components $\hat{x}_1$ of local states in $\mathcal{X}_{\mathcal{C}_i}$ eventually decay to 0 as $i \to +\infty$.

A similar argument holds for $\hat{x}_2$ components. This shows that any periodic global state $\mathcal{X}_{\mathcal{C}_o}$ determines a stable free evolution.

4. STABILITY CONDITIONS BY FOURIER ANALYSIS

By Corollary 1, $\Sigma = (A_1,A_2,-,-,-)$ is internally stable if and only if the Lyapunov equation

$$P(\omega) = I + (A_1 + e^{-j\omega}A_2)^T P(\omega)(A_1 + e^{j\omega}A_2) \tag{5}$$

admits a positive definite Hermitian solution $P(\omega)$ for every real ω.

The entries of $P(\omega)$ are rational functions of $e^{j\omega}$, periodic in ω with period 2π. The positive definite character of $P(\omega)$ can be checked by applying Sturm's test to the principal minors of $P(\omega)$.

The Fourier coefficients P_k of the expansion $P(\omega)=\sum_{k=-\infty}^{+\infty} P_k e^{j\omega k}$ satisfy the following properties:

i) since the entries of $P(\omega)$ are in $L^2[-\pi,\pi]$, the sequence $\{P_k\}$ belongs to $\ell^2(\mathbb{R}^{n\times n})$

ii) for any integer k, $P_k = P_{-k}^T$, and the following set of equalities holds:

$$\begin{aligned} P_o &= I + A_1^T P_o A_1 + A_2^T P_o A_2 + A_1^T P_1^T A_2 + A_2^T P_1 A_1 \\ P_k &= A_1^T P_k A_1 + A_2^T P_k A_2 + A_1^T P_{k-1} A_2 + A_2^T P_{k+1} A_1, \quad k = 1,2,\ldots \end{aligned} \tag{6}$$

iii) the doubly infinite block Toeplitz matrix

$$\mathcal{P} = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & P_{-1} & P_o & P_1 & \\ & \ddots & P_{-1} & P_o & P_1 & \ddots \\ & & \ddots & P_{-1} & P_o & P_1 \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \tag{7}$$

induces a positive definite scalar product in the space $\ell^2(\mathbb{C}^n)$.

In fact for every non zero vector function $v(\cdot):[-\pi,\pi] \to \mathbb{C}^n$, with elements in $L^2[-\pi,\pi]$ we have

$$\int_{-\pi}^{\pi} \check{v}^T(\omega)P(\omega)v(\omega)d\omega > 0$$

Then $\sum_{h,k} \check{v}_h^T P_{h-k} v_k > 0$ where v_k are the Fourier coefficients of $v(\cdot)$: $v(\omega) = \sum_{k=-\infty}^{+\infty} v_k e^{j\omega k}$.

By Riesz-Fischer theorem, there is a bijection between $\ell^2(\mathbb{C}^n)$ and the space of n-tuples with elements in $L^2[-\pi,\pi]$. Therefore the relation

$$(u,w) = \sum_{h,k} \check{u}_h^T P_{h-k} w_k$$

defines a positive definite scalar product in $\ell^2(\mathbb{C}^n)$.

Conversely, assume that there exists a sequence $\{P_k\}$ of matrices in $\mathbb{R}^{n\times n}$ satisfying conditions i), ii), iii). Then the series $\sum_{k=-\infty}^{+\infty} P_k e^{j\omega k}$ defines a.e. a matrix function $P(\omega)$ with elements in $L^2[-\pi,\pi]$ which solves equation (5). In fact, using (6), we check directly that

$$I+A_1^TP(\omega)A_1+A_2^TP(\omega)A_2+A_1^Te^{j\omega}P(\omega)A_2+A_2^Te^{-j\omega}P(\omega)A_1 = \sum_k P_k e^{j\omega k} = P(\omega)$$

Hence the entries of $P(\omega)$ are a.e. real rational functions of $e^{j\omega}$. Since

$$P_o = \int_{-\pi}^{\pi} P(\omega)d\omega$$

is finite, and $P(\omega)$ is a.e. positive definite, the analytic extension $Q(z)$ of $Q(e^{j\omega}) \triangleq P(\omega)$ has no poles on the unit circle.

Since by continuity $P(\omega)$ is at least positive semidefinite and satisfies equation (5) for all real ω, then it is positive definite for all real ω.

Assume $\Sigma = (A_1,A_2,-,-,-)$ to be internally stable and let $P(\omega) = \sum_{k=-\infty}^{+\infty} P_k e^{j\omega k}$ be the solution of equation (5). Then equations (6) can be used to investigate the structure of matrices P_k in terms of the family of matrices (4).

First notice that the Toeplitz matrix $\mathcal{P}$ satisfies the equation

$$\mathcal{P} = \mathcal{I} + \mathcal{A}^*\mathcal{P}\mathcal{A} \tag{8}$$

where $\mathcal{I}$ is the (infinite) identity matrix and $\mathcal{A}$ and $\mathcal{A}^*$ are the doubly infinite block Toeplitz matrices:

$$\mathcal{A} = \begin{bmatrix} \ddots & 0 & A_1 & A_2 & 0 & & & \\ & & 0 & A_1 & A_2 & 0 & & \\ & & & 0 & A_1 & A_2 & 0 & \ddots \end{bmatrix} \qquad \mathcal{A}^* = \begin{bmatrix} \ddots & 0 & A_2^T & A_1^T & 0 & & & \\ & & 0 & A_2^T & A_1^T & 0 & & \\ & & & 0 & A_2^T & A_1^T & 0 & \ddots \end{bmatrix}$$

Whe therefore have the following Theorem.

Theorem 6. *Let $\mathcal{P}$ as in (7). Then $\mathcal{P}$ is the sum of the series*

$$\mathcal{I} + \sum_{i=1}^{\infty} \mathcal{A}^{*i}\mathcal{A}^i$$

of Toeplitz matrices and the blocks P_k are expressed as

$$P_k = \sum_{r,s=0}^{\infty} (A_1^{r+k} \sqcup\!\sqcup^{s} A_2)^T (A_1^{r} \sqcup\!\sqcup^{s+k} A_2) \qquad k = 0,1,2,\dots \tag{9}$$

Proof. Since the P's are the Fourier coefficients of the continuous bounded function $P(\omega)$, the operator $\mathcal{P}: \ell^2(\mathbb{C}^n) \to \ell^2(\mathbb{C}^n)$ is continuous [12]. Hence there exists a positive integer M such that $\check{v}^T \mathcal{P} v < M \check{v}^T v$ for every v in $\ell^2(\mathbb{C}^n)$. Then by (8) we have

$$\check{v}^T v + \check{v}^T \mathcal{A}^* \mathcal{A} v + \ldots + \check{v}^T \mathcal{A}^{*M} \mathcal{A}^M v < M \check{v}^T v$$

$$\sigma \triangleq \sup_{\|v\|=1} \min\{\check{v}^T \mathcal{A}^* \mathcal{A} v, \ldots, \check{v}^T \mathcal{A}^{*M} \mathcal{A}^M v\} < 1$$

Let t be a positive integer such that $\sigma^{\frac{t}{M}-1} < \frac{1}{2M}$.

For any $\nu > t$ and v in $\ell^2(\mathbb{C}^n)$, $\|v\| = 1$, there exists a partition of $\nu = \nu_1 + \nu_2 + \ldots + \nu_{r+1}$, $\nu_i \leq M$, $i = 1, 2 \ldots r+1$ such that

$$\check{v}^T \mathcal{A}^{*\nu_1} \mathcal{A}^{\nu_1} v \leq \sigma$$

$$\check{v}^T \mathcal{A}^{*\nu_1} \mathcal{A}^{*\nu_2} \mathcal{A}^{\nu_1} \mathcal{A}^{\nu_2} v \leq \sigma^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\check{v}^T \mathcal{A}^{*\nu_1} \ldots \mathcal{A}^{*\nu_r} \mathcal{A}^{\nu_r} \ldots \mathcal{A}^{\nu_1} v \leq \sigma^r$$

and

$$\check{v}^T \mathcal{A}^{*\nu} \mathcal{A}^{\nu} v \leq \sigma^r M \leq \sigma^{\frac{\nu}{M}-1} M \leq \sigma^{\frac{t}{M}-1} M < \frac{1}{2}$$

Thus the sequence of operators $\mathcal{P}_0 = \mathcal{I}, \mathcal{P}_i = \mathcal{P}_{i-1} + \mathcal{A}^{*i} \mathcal{A}^i$, $i > 0$, is a Cauchy sequence, as the following inequalities

$$\|(\mathcal{P}_n - \mathcal{P}_m) v\| \leq \sum_{i=m+1}^{n} \|\mathcal{A}^{*i} \mathcal{A}^i v\| < \sum_{i=m+1}^{n} \frac{2 M^2}{2^{i/t}}$$

hold for $n > m > t$ and $\|v\| = 1$. Since $\ell^2(\mathbb{C}^n)$ is complete, the operator $\sum_{i=1}^{\infty} \mathcal{A}^{*i} \mathcal{A}^i + \mathcal{I}$ is well defined and solves equation (8).

Finally $\mathcal{P}$ coincides with the series $\mathcal{I} + \sum_{i=1}^{\infty} \mathcal{A}^{*i} \mathcal{A}^i$.

In fact $\Delta = \mathcal{P} - (\mathcal{I} + \Sigma \mathcal{A}^{*i} \mathcal{A}^i)$ satisfies the equations chain

$$\Delta = \mathcal{A}^* \Delta \mathcal{A} = \ldots = \mathcal{A}^{*i} \Delta \mathcal{A}^i \ldots$$

and $\Delta = 0$, as $\lim_{i \to \infty} \mathcal{A}^i = 0$.

Example (scalar case). If the local state space is one dimensional, the matrices A_1 and A_2 become scalars a_1 and a_2 respectively, and a closed form computation of the Fourier coefficients P_k is possible when the 2-D system is internally stable.

The solution of equation (5), given by

$$P(\omega) = (1 - a_1^2 - a_2^2 - 2a_1 a_2 \cos\omega)^{-1}$$

is positive for every real ω if and only if

$$1 - a_1^2 - a_2^2 > |2a_1 a_2| \qquad (10)$$

Incidentally note that condition (10) can be restated in the following equivalent forms

i) $|a_1| + |a_2| < 1$

ii) $q(z_1,z_2) = 1 - a_1 z_1 - a_2 z_2 \neq 0$ in $\mathcal{P}_1$

iii) $\begin{bmatrix} 1-a_1^2-a_2^2 & 2a_1a_2 \\ 2a_1a_2 & 1-a_1^2-a_2^2 \end{bmatrix} > 0$

When condition (10) is satisfied, we obtain

$$P_o = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-a_1^2-a_2^2-2a_1a_2\cos\omega)^{-1}d\omega = \left[(1-a_1^2-a_2^2)^2-4a_1^2a_2^2\right]^{-1/2}$$

When once P_o is computed, the coefficients P_k, $k=1,2\ldots$ are obtained recursively from eqns. (6).

Observe that in general the sum of the series (9) is not a rational function of the matrices A_1 and A_2.

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