# On the Problems of Constructing Minimal Realizations for Two-Dimensional Filters

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*Abstract*—The input-output behavior of a two-dimensional linear filter is defined by a formal power series in two variables. If the power series is rational the dynamics of the filter is described by updating equations on finite dimensional local state spaces. The class of realizations considered in this paper is constituted by doubly indexed dynamical systems of reduced structure.

The construction of the class of minimal realizations is based on matrix representation techniques of noncommutative power series.

*Index Terms*—Difference equations, Hankel matrix, noncommutative power series, observability, reachability, realization algorithms, two-dimensional filter.

## I. Introduction

The algebraic realization theory of two-dimensional filters has been formulated by the authors in [1]-[4]. In this contribution we will derive additional results mainly with regard to a reduced structure of the updating equation for the local states.

Reachable and observable realizations of two-dimensional filters are not necessarily minimal. So, reduction algorithms, leading to reachable realizations [2], [4], are not sufficient to obtain minimal realizations.

A further reason why linear systems standard procedures are not sufficient is that the dimension of minimal realizations depends on the ground field.

As we shall show, the updating of local states in doubly indexed dynamical systems, which describe the internal dynamics of two-dimensional filters, is intrinsically a noncommutative phenomenon. Thus, a good deal of theoretical insight can be expected from the representation theory of noncommutative power series.

## II. Realization of Two-Dimensional Filters

This section is devoted to summarize some basic definitions and results which provide the fundamental notions needed in two-dimensional realization.

Consider a two-dimensional linear, time-invariant, causal digital filter with scalar inputs and outputs taken from an arbitrary field $K$. The input-output map $F$ of this filter is completely characterized by the formal power series

$$s = F(1) = \sum_{i,j=1} (F(1), z_1^i z_2^j)\, z_1^i z_2^j \in z_1 z_2 K[[z_1, z_2]] \qquad (1)$$

where $K[[z_1, z_2]]$ denotes the ring of formal power series in two variables.

The coefficient $(F(1), z_1^i z_2^j)$ in $K$ denotes the output value at time $(i, j)$ corresponding to the input $u(h, k) = 1$ for $(h, k) = (0, 0)$ and $u(h, k) = 0$ elsewhere.

Given an input $u$

$$\sum_{-k}^{\infty}{}_{i,j} (u, z_1^i z_2^j)\, z_1^i z_2^j$$

in the space of truncated formal Laurent power series, the corresponding output $y$ is the series

$$y = F(u) = su.$$

*Definition 2.1:* A doubly indexed discrete-time linear, stationary, finite dimensional dynamical system $\Sigma = (A_1, A_2, B, C)$ is defined by a pair of equations of the form

$$x(h+1, k+1) = A_1 x(h+1, k) + A_2 x(h, k+1) + Bu(h, k)$$
$$y(h, k) = Cx(h, k) \qquad (2)$$

where $A_i \in K^{n \times n}$, $i = 1, 2$, $C \in K^{1 \times n}$, $B \in K^{n \times 1}$, and $x$ belongs to the finite dimensional vector space $X = K^n$ (local state space).

The solution of (2) for $h > 0$, $k > 0$, is uniquely determined by $u$ and by the values $x(h, 0)$, $h = 1, 2, \cdots$, and $x(0, k)$ $k = 1, 2, \cdots$ (initial local states).

Associate the monomial $x(h, k)\, z_1^h z_2^k \in K^{n \times 1}[[z_1, z_2]]$ with the local state $x(h, k)$ and assume $x(h, 0) = x(0, k) = 0$, $h, k = 1, 2, \cdots$. Then the input-output relation is given by

$$y = C \sum_{h,k} x(h, k)\, z_1^h z_2^k = C(I - A_1 z_1 - A_2 z_2)^{-1} (z_1 z_2)\, Bu.$$

The series $C(I - A_1 z_1 - A_2 z_2)^{-1}(z_1 z_2)\, B$ is called the *transfer function of* $\Sigma$.

We recall that a formal power series $s \in K[[z_1, z_2]]$ is *rational* if there exist polynomials $p, q \in K[z_1^{-1}, z_2^{-1}]$ with $\deg p \leqslant \deg q$, such that $qs = p$. The polynomial $q$ is called a denominator of $s$.

Then the transfer function $(z_1 z_2)\, C(I - A_1 z_1 - A_2 z_2)^{-1} B$ belongs to $(z_1 z_2)\, K[(z_1, z_2)] = K_c[(z_1, z_2)]$, where $K[(z_1, z_2)]$ denotes the ring of rational power series in two variables and $K_c[(z_1, z_2)]$ is the ideal of *causal rational power series.*

*Definition 2.2:* A doubly indexed dynamical system $\Sigma = (A_1, A_2, B, C)$ is a zero-state realization of a two-dimensional filter $\mathcal{S}$ represented by a series $s \in K[[z_1, z_2]]$ if

$$s = (z_1 z_2)\, C(I - A_1 z_1 - A_2 z_2)^{-1} B. \qquad (3)$$

The *dimension of a realization* $\Sigma$ is the dimension of the local state space $X$.

The minimality of the realization is naturally related to the dimension of $X$ in the sense that a realization $\Sigma$ is minimal when $\dim \Sigma \leqslant \dim \Sigma'$ for any $\Sigma'$ which realizes $\mathcal{S}$.

*Proposition 2.1 [2]:* Let $s \in K[[z_1, z_2]]$ represent a two-dimensional filter $\mathcal{S}$. Then $\mathcal{S}$ is realizable by a doubly indexed dynamical system if and only if $s \in K_c[(z_1, z_2)]$.

From now on we shall assume that the formal power series $s$ characterizing the input-output map is rational. Hence, there exists a doubly indexed dynamical system $\Sigma = (A_1, A_2, B, C)$ such that

$$s = (z_1 z_2)\, C(I - A_1 z_1 - A_2 z_2)^{-1} B.$$

## III. Examples of Realizations

The structural properties of local reachability and observability of doubly index dynamical systems have been introduced by the authors and we refer to [2], [3] for a linear algorithm leading to locally reachable and observable realizations.

The purpose of Example 1 is to illustrate that local reachability and observability, although necessary, are not sufficient for the minimality of the realization.

The problem of constructing minimal realizations of two-dimensional filters is essentially nonlinear, as proved by the authors in [4]. In fact, the minimal dimension depends on the ground field, as shown in Example 2.

The deep difference with the standard linear theory is also

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emphasized by the failure of Hankel matrix in giving the dimension of minimal realizations (see Example 3).

*Example 1:* Let $K$ be a field of characteristic 0 and consider the filter

$$s = (z_1 z_2) \frac{1 - z_1}{1 - z_1^2 - z_2^2}.$$

The following doubly indexed dynamical systems,

$$\Sigma_1 = (A_1^{(1)}, A_2^{(1)}, B^{(1)}, C^{(1)}):$$

$$A_1^{(1)} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C^{(1)} = [1 \ \ 0]$$

and

$$\Sigma_2 = (A_1^{(2)}, A_2^{(2)}, B^{(2)}, C^{(2)}):$$

$$A_1^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C^{(2)} = [-1 \ \ 1 \ \ 0],$$

are locally reachable and locally observable realizations of $s$ over the field $K$. However, $\Sigma_2$ is not a minimal realization.

*Example 2:* Consider the filter

$$s = (z_1 z_2) \frac{1}{1 - z_1^2 - z_2^2}$$

and assume $K = \mathbb{C}$. It is easy to check that $s$ has a minimal realization of dimension 2 given by $\Sigma_1 = (A^{(1)}, A^{(2)}, B^{(1)}, C^{(1)})$ with

$$A_1^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2^{(1)} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad B^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C^{(1)} = [1 \ \ 0].$$

In [4] it has been proved that the filter $s$ is no longer realizable in dimension 2 over the real field.

*Remark:* The result of Example 2 (i.e., the dependence of minimal dimension on the ground field) was published by the authors in November 1976 [4]. It was "discovered" again in 1978 [5].

*Example 3:* In linear system theory the Hankel matrix associated with a rational series can be obtained multiplying the infinite observability matrix by the infinite reachability matrix and its rank gives the dimension of minimal realizations. These properties do not hold for two-dimensional filters as we shall show below.

Let $s \in K_c[[z_1, z_2]]$, and let $s = (z_1 z_2) s'$. The Hankel matrix $\mathcal{H}(s)$ associated with $s$ is

$$\mathcal{H}(s) = \begin{bmatrix} (s', 1) & (s', z_1) & (s', z_2) & (s', z_1^2) & (s', z_1 z_2) & (s', z_2^2) \cdots \\ (s', z_1) & (s', z_1^2) & (s', z_1 z_2) & (s', z_1^3) & (s', z_1^2 z_2) & (s', z_1 z_2^2) \cdots \\ (s', z_2) & (s', z_1 z_2) & (s', z_2^2) & (s', z_1^2 z_2) & (s', z_1 z_2^2) & (s', z_2^3) \cdots \\ (s', z_1^2) & (s', z_1^3) & (s', z_1^2 z_2) & (s', z_1^4) & (s', z_1^3 z_2) & (s', z_1^2 z_2^2) \cdots \\ & & \cdots & & & \end{bmatrix}.$$

It has been proved [6] that the rank of $\mathcal{H}(s)$ is finite if and only if $s$ is rational and a denominator $q$ of $s$ can be factorized as $q = q_1 q_2$ with $q_1 \in K[z_1^{-1}]$, $q_2 \in K[z_2^{-1}]$. The series satisfying this property are elements of the ring $K[(z_1)] \otimes K[(z_2)] \triangleq K^{\text{rec}}[(z_1, z_2)]$ called the ring of "*recognizable series.*"

The following remarks are a consequence of the above-mentioned characteristic property of recognizable series and give an account of what the situation is with the Hankel matrix for two-dimensional filters:

1) if $s$ is rational but $s \notin K_c^{\text{rec}}[(z_1, z_2)]$, then rank $\mathcal{H}(s) = \infty$. However, there exist finite dimensional realizations of $s$ (see Proposition 2.1);

2) if $s \in K_c^{\text{rec}}[(z_1, z_2)]$, then rank $\mathcal{H}(s)$ is finite. However, the dimension of minimal realizations of $s$ does not coincide with rank $\mathcal{H}(s)$, as the following example shows.

Assume $s \in K^{\text{rec}}[(z_1, z_2)]$ is given by

$$s = (z_1 z_2) \frac{1 + z_1 + z_2}{1 + z_1 + z_2 + z_1 z_2}.$$

Therefore, the formal power series expansion of $(z_1 z_2)^{-1} s$ is expressed by

$$s' = (1 + z_1 + z_2) \sum_0^{\infty} {}_k (-1)^k (z_1 + z_2 + z_1 z_2)^k$$
$$= 1 - z_1 z_2 + z_1 z_2^2 + z_1^2 z_2 + \cdots$$

and the Hankel matrix is then

$$\mathcal{H}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & \cdots \\ 0 & -1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 1 & \cdots \\ \cdots \end{bmatrix}.$$

Note that rank $\mathcal{H}(s) > 2$. Nevertheless a minimal realization of dimension 2 does exist, i.e., $\Sigma = (A_1, A_2, B, C)$,

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \ \ 1].$$

## IV. Computation of Minimal Realizations

The problem of constructing a realization (not necessarily minimal) of a filter has been solved (see Proposition 2.1) and we can also assume that the realization we have obtained is $L$-reachable and $L$-observable, because of the existence of a reduction algorithm.

Nevertheless, as shown by the examples in Section III, in general a minimal realization cannot be derived by removing unreachable and unobservable parts. We shall now get some insight into the problem of minimal realization by resorting to noncommutative power series techniques.

In particular, in this section we shall present a procedure for obtaining all minimal realizations (modulo similarity transformations) of a filter. This exploits some results on noncommutative power series which are now briefly recalled.

### A. Some Properties of Noncommutative Power Series

Let $\Xi = \{\xi_1, \xi_2\}$, and denote by $\Xi^*$ the free monoid with base $\Xi$. A noncommutative formal power series with coefficients in the field $K$ and indeterminates $\xi_1$ and $\xi_2$ is an expression such as

$$\sigma = \sum_{w \in \Xi^*} (\sigma, w) \, w, \qquad (\sigma, w) \in K.$$

The set of noncommutative formal power series with coefficients in $K$ and indeterminates $\xi_1$ and $\xi_2$ is a noncommutative ring, and will be written as $K\langle\langle \xi_1, \xi_2 \rangle\rangle$. Denote by $K\langle \xi_1, \xi_2 \rangle$

the subring of noncommutative polynomials. The integer $N$ is a formal degree of the polynomial $\pi \in K\langle \xi_1 \xi_2 \rangle$ if length $w \triangleq |w| > N$ implies $(\pi, w) = 0$. The smallest such number $N$ is called the degree of $\pi$.

*Definition 4.1:* A noncommutative formal power series $\sigma \in K\langle\langle \xi_1, \xi_2 \rangle\rangle$ is called rational if there exist a positive integer $m$, and matrices $C \in K^{1 \times m}$, $B \in K^{m \times 1}$, and $A_1, A_2 \in K^{m \times m}$ such that

$$\sigma = C \sum_{k=0}^{\infty} (A_1 \xi_1 + A_2 \xi_2)^k B = C(1 - \xi_1 A_1 - \xi_2 A_2)^{-1} B. \quad (4)$$

The subring of noncommutative rational power series will be denoted by $K\langle(\xi_1, \xi_2)\rangle$.

A 4-tuple $(A_1, A_2, B, C)$ is called a representation of $\sigma$ if (4) holds. The dimension of the matrices $A_1$ and $A_2$ is the dimension of the representation.

Clearly, if $\sigma$ admits a representation, it admits infinitely many and there exists at least one which has smaller dimension than the others. Thus it makes sense to look for minimal representations.

Note that if $(A_1, A_2, B, C)$ is a representation for $\sigma$, so is $(T^{-1}A_1 T, T^{-1}A_2 T, T^{-1}B, CT)$ for any nonsingular $T$. Evidently, this constitutes a recipe for constructing several (infinitely many if $K$ is infinite) minimal representations given one minimal representation.

Actually, it has been proved [7] that all minimal representations are related by similarity transformations. Hence, the problem consists in setting up a procedure for constructing a minimal representation.

*Proposition 4.1 [7]:* Let $\sigma = \sum_{w \in \Xi^*} (\sigma, w) w$ be in $K\langle\langle \xi_1, \xi_2 \rangle\rangle$. The rank $n_\sigma$ of the (infinite) Hankel matrix

$$\mathcal{H}(\sigma) = \begin{bmatrix} (\sigma, 1) & (\sigma, \xi_1) & (\sigma, \xi_2) & (\sigma, \xi_1^2) & (\sigma, \xi_1\xi_2) & (\sigma, \xi_2\xi_1) & (\sigma, \xi_2^2) & \cdots \\ (\sigma, \xi_1) & (\sigma, \xi_1^2) & (\sigma, \xi_1\xi_2) & (\sigma, \xi_1^3) & (\sigma, \xi_1^2\xi_2) & (\sigma, \xi_1\xi_2\xi_1) & \cdots & \\ (\sigma, \xi_2) & (\sigma, \xi_2\xi_1) & (\sigma, \xi_2^2) & (\sigma, \xi_2\xi_1^2) & (\sigma, \xi_2\xi_1\xi_2) & \cdots & & \\ (\sigma, \xi_1^2) & (\sigma, \xi_1^3) & (\sigma, \xi_1^2\xi_2) & & \cdots & & & \\ & \cdots & & & & & & \end{bmatrix}$$

is finite if and only if $\sigma$ is rational. $n_\sigma$ is the dimension of the minimal representations of $\sigma$ and is therefore called the rank of $\sigma$.

We shall partition $\mathcal{H}(\sigma)$ in row blocks and column blocks, indexed by capital letters. They are defined as follows: the $M$th row (column) block includes all rows (columns) of $\mathcal{H}(\sigma)$ whose indices are words of length $M - 1$. The composition of row and column partitions gives a partition of $\mathcal{H}(\sigma)$ in block matrices of finite size: the block in the $(M', M'')$ position is written as $\mathcal{H}_{M',M''}$ and contains $2^{(M'-1)+(M''-1)}$ elements.

We shall denote by $\mathcal{H}_{M' \times M''}(\sigma)$ the $M' \times M''$ block submatrix of $\mathcal{H}(\sigma)$ appearing in the upper left-hand corner of $(\sigma)$,

$$\mathcal{H}_{M' \times M''}(\sigma) = \begin{bmatrix} \mathcal{H}(\sigma)_{1,1} & \mathcal{H}(\sigma)_{1,2} & \cdots & \mathcal{H}(\sigma)_{1,M''} \\ \mathcal{H}(\sigma)_{2,1} & \mathcal{H}(\sigma)_{2,2} & \cdots & \mathcal{H}(\sigma)_{2,M''} \\ & & \cdots & \\ \mathcal{H}(\sigma)_{M',1} & \mathcal{H}(\sigma)_{M',2} & \cdots & \mathcal{H}(\sigma)_{M',M''} \end{bmatrix}.$$

In linear system theory Ho's algorithm enables the solution of the problem of passing from a prescribed impulse response to a minimal realization [8]. The only hypothesis needed for its application is that some upper bound for the dimension of the minimal realization has to be *a priori* known.

Let $\sigma \in K\langle(\xi_1, \xi_2)\rangle$ and let rank $\mathcal{H}(\sigma) = n_\sigma < \infty$. We define the row length and the column length of $\mathcal{H}(\sigma)$ as

$$L' = \min \{M': \text{rank } \mathcal{H}_{M' \times \infty}(\sigma) = n_\sigma\}$$

and

$$L'' = \min \{M'': \text{rank } \mathcal{H}_{\infty \times M''}(\sigma) = n_\sigma\}.$$

Denote by $\mathcal{H}^{(\xi_i)}(\sigma)$, $i = 1, 2$, the infinite matrices whose elements in the $(u, v)$ position are given by $(\sigma, u\xi_i v)$ for any $w$ and $v$ in $\Xi^*$.

We partition $\mathcal{H}^{(\xi_i)}(\sigma)$ conformably with the partition already introduced in $\mathcal{H}(\sigma)$. In this way $\mathcal{H}^{(\xi_i)}_{M' \times M''}(\sigma)$ and $\mathcal{H}^{(\xi_i)}(\sigma)_{M',M''}$ should constitute self-explaining notations.

*Proposition 4.2 (Generalized Ho Algorithm) [9]:* Let $\sigma$ be in $K\langle(\xi_1, \xi_2)\rangle$ and let $n_\sigma = \text{rank } \mathcal{H}(\sigma) < \infty$. Denote by $L'$ and $L''$ the row length and the column length of $\mathcal{H}(\sigma)$, respectively. The following steps lead to a minimal representation of $\sigma$.

1) Find nonsingular matrices $P$ and $Q$ such that

$$P\mathcal{H}_{L' \times L''}(\sigma)\, Q = \left[\begin{array}{c|c} I_{n_\sigma} & 0 \\ \hline 0 & 0 \end{array}\right]. \quad (5)$$

2) Compute

$$A_i = [I_{n_\sigma} | 0]\, P\mathcal{H}^{(\xi_i)}_{L' \times L''}(\sigma)\, Q \begin{bmatrix} I_{n_\sigma} \\ 0 \end{bmatrix} \qquad i = 1, 2$$

$$B = [I_{n_\sigma} | 0]\, P\mathcal{H}_{L' \times L''}(\sigma) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (6)$$

$$C = [1 0 \cdots 0]\, \mathcal{H}_{L' \times L''}(\sigma)\, Q \begin{bmatrix} I_{n_\sigma} \\ 0 \end{bmatrix}.$$

It is clear that the main limitation of the generalized Ho algorithm is that an *a priori* knowledge of an upper bound of rank $\mathcal{H}(\sigma)$ is needed.

When this kind of information is not available, we cannot find a complete solution to the representation problem of a prescribed series $\sigma$. In this situation we can look for a "partial" representation of $\sigma$, i.e., for a representation $(A_1, A_2, B, C)$ which matches the coefficients of monomials in $\sigma$ up to a given degree.

*Definition 4.2:* Let $B \in K^{m \times 1}$, $C \in K^{1 \times m} A_1, A_2 \in K^{m \times m}$. The 4-tuple $(A_1, A_2, B, C)$ is a partial representation of degree $g$ of the series $\sigma \in K\langle\langle \xi_1, \xi_2 \rangle\rangle$ if

$$(\sigma, \xi_{i_1} \xi_{i_2} \cdots \xi_{i_t}) = C A_{i_1} A_{i_2} \cdots A_{i_t} B$$

holds whenever $t \leq g$.

$(A_1, A_2, B, C)$ will be also called a *partial representation for the polynomial* $\pi_\sigma$ *of formal degree* $g$ which coincides with the initial segment of $\sigma$.

The following theorem extends to noncommutative power series a standard result of realization theory [10].

*Proposition 4.3 [9]:* Let $\sigma \in K\langle\langle \xi_1, \xi_2 \rangle\rangle$ and let $\mathcal{H}(\sigma)$ be the corresponding Hankel matrix. Then the matrices $(A_1, A_2, B, C)$ defined by (5) and (6) constitute a partial representation of $\sigma$ of degree $g = L' + L''$ if and only if

$$\text{rank } \mathcal{H}_{L' \times L''}(\sigma) = \text{rank } \mathcal{H}_{(L'+1) \times L''}(\sigma) = \mathcal{H}_{L' \times (L''+1)}(\sigma). \quad (7)$$

The 4-tuple $(A_1, A_2, B, C)$ we consider in Proposition 4.3 is a minimal partial representation of $\sigma$ of degree $g = L' + L''$. In fact, all partial representations of degree $g$ generate the same truncated Hankel matrix $\mathcal{H}_{L' \times L''}(\sigma)$, so that their dimension is at least rank $\mathcal{H}_{L' \times L''}(\sigma)$.

Furthermore $(A_1, A_2, B, C)$ is the unique minimal partial representation of degree $g$ modulo a similarity transformation.

Hence we have the following.

*Corollary 4.1:* If (7) holds, the rational power series $\sigma'$ satisfying rank $\sigma' \leqslant$ rank $\mathcal{H}_{L' \times L''}(\sigma)$ and $(\sigma', w) = (\sigma, w)$ for $|w| \leqslant L' + L''$ is iniquely determined.

### *B. Minimal Realizations by Means of Noncommutative Power Series*

As we have seen, Ho's algorithm provides an effective technique for obtaining minimal representations of rational noncommutative power series.

Intuitively, the procedure we shall give is based on the following idea: first, associate the commutative power series $s$ with the set of noncommutative power series having $s$ as commutative image, then construct minimal realizations of $s$ using minimal representations of the corresponding noncommutative rational power series.

Let $s \in K_c[(z_1, z_2)]$ be the impulse response of a given filter $\mathcal{S}$.

Define the algebra morphism $\phi$: $K\langle(\xi_1, \xi_2)\rangle \to K[[z_1, z_2]]$ by $\phi(k) = k$, $\forall k \in K$, $\phi(\xi_1) = z_1$, $\phi(\xi_2) = z_2$. Since

$$\phi: \; C(I - A_1\xi_1 - A_2\xi_2)^{-1}B \to C(I - A_1 z_1 - z_2 z_2)^{-1}B$$

all rational series in the commutative variables $z_1$ and $z_2$ are obtained by varying $A_1, A_2, B, C$, and the map $\phi$ is onto $K[(z_1, z_2)]$.

Pick any $\sigma \in K\langle(\xi_1, \xi_2)\rangle$ satisfying $\phi(\sigma) = z_1^{-1}z_2^{-1}s$. Then each representation $(A_1, A_2, B, C)$ of $\sigma$ is a realization of $\mathcal{S}$. Conversely, if $\Sigma = (A_1, A_2, B, C)$ is a realization of $\mathcal{S}$, $\Sigma$ is a representation of the noncommutative rational power series $\sigma_\Sigma = \sum_{i=0}^{\infty} C(\xi_1 A_1 + \xi_2 A_2)^i B$, which satisfies $\phi(\sigma_\Sigma) = z_1^{-1}z_2^{-1}s$. Hence, the class of realizations of $\mathcal{S}$ coincides with the class of representations of noncommutative rational power series $\sigma$ satisfying

$$\phi(\sigma) = z_1^{-1}z_2^{-1}s.$$

Obviously, there are infinitely many noncommutative rational power series which satisfy the last equality.

As an interesting consequence, we point out that two minimal realizations of a given filter $\mathcal{S}$ are not necessarily similar: that happens whenever these realizations come from minimal representations of two different noncommutative power series which have both $z_1^{-1}z_2^{-1}s$ as image.

In order to obtain all minimal realizations (modulo similarity transformations) we shall therefore go through the following steps.

*Step 1:* Determine the set $\mathcal{M} \subseteq K\langle(\xi_1, \xi_2)\rangle$ of series having minimal rank and $z_1^{-1}z_2^{-1}s$ as $\phi$-image.

*Step 2:* For each series $\sigma \in \mathcal{M}$ obtain, via Ho's algorithm, a minimal representation.

Let

$$s z_1^{-1} z_2^{-1} = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a'_{ij} z_1^{-i} z_2^{-j} \Big/ \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} b_{ij} z_1^{-i} z_2^{-j} \tag{8}$$

and denote by $m$ an upper bound for the dimension of minimal realizations of $\mathcal{S}$ provided, e.g., by the realization introduced in [2], [4].

Consider the set $\mathcal{P}$ of noncommutative polynomials which fulfill the following conditions:

i) $\deg \pi \leqslant 2\overline{P} \triangleq 2\left(m + \left[\dfrac{n_1 + n_2}{2}\right]\right)$ (9)[1]

ii) $\phi(\pi) = \sum_{i+j \leqslant 2\overline{P}} (z_1^{-1}z_2^{-1}s, z_1^i z_2^j)\, z_1^i z_2^j$ (10)

iii) $\text{rank}\, \mathcal{H}_{\overline{P} \times \overline{P}}(\pi) = \text{rank}\, \mathcal{H}_{(\overline{P}+1) \times \overline{P}}(\pi) = \text{rank}\, \mathcal{H}_{\overline{P} \times (\overline{P}+1)}(\pi) \leqslant m.$ (11)

The set $\mathcal{P}$ is not empty. In fact, let $\Sigma = (A_1, A_2, B, C)$ be a realization of $\mathcal{S}$, and let $P \geqslant \dim \Sigma$.

Then the noncommutative polynomial

$$\pi_\Sigma = \sum_{t=0}^{2P} \sum_{(i_1, \cdots, i_t)} CA_{i_1}A_{i_2}, \cdots, A_{i_t}B\xi_{i_1}\xi_{i_2}, \cdots, \xi_{i_t}$$

satisfies

$$\phi(\pi_\Sigma) = \sum_{i+j \leqslant 2\overline{P}} (z_1^{-1}z_2^{-1}s, z_1^i z_2^j)\, z_1^i z_2^j$$

and

$$\text{rank}\, \mathcal{H}_{P \times P}(\pi_\Sigma) = \text{rank}\, \mathcal{H}_{(P+1) \times P}(\pi_\Sigma) = \text{rank}\, \mathcal{H}_{P \times (P \times 1)}(\pi_\Sigma) \leqslant \dim \Sigma.$$

*Proposition 4.4:* Let $A_i \in K^{n \times n}$, $i = 1, 2$, $B \in K^{n \times 1}$, $C \in K^{1 \times n}$, $n \leqslant m$. Then $(A_1, A_2, B, C)$ is a partial representation of some polynomial in $\mathcal{P}$ if and only if $\Sigma = (A_1, A_2, B, C)$ is a realization of $\mathcal{S}$.

The proof of Proposition 4.4 depends on the following lemma.

*Lemma 4.1 [9]:* Let $r$ and $r'$ belong to $K[(z_1, z_2)]$ and

$$r = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_{ij} z_1^{-i} z_2^{-j} \Big/ \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} b_{ij} z_1^{-i} z_2^{-j} \qquad b_{n_1 n_2} = 1$$

$$r' = \sum_{i=0}^{n'_1}\sum_{j=0}^{n'_2} a'_{ij} z_1^{-i} z_2^{-j} \Big/ \sum_{i=0}^{n'_1}\sum_{j=0}^{n'_2} b'_{ij} z_1^{-i} z_2^{-j} \qquad b_{n'_1 n'_2} = 1.$$

Let $h_\nu^{(r)}$ and $h_\nu^{(r')}$ denote the homogeneous polynomials of degree $\nu$ in $r$ and $r'$, respectively. Then

$$h_\nu^{(r)} = h_\nu^{(r')} \qquad \nu = 0, 1, 2, \cdots, n_1 + n_2 + n'_1 + n'_2$$

imply $r = r'$.

*Proof:* Let $(A_1, A_2, B, C)$ be a partial representation of $\pi \in \mathcal{P}$. Then $s_\Sigma = C(I - A_1 z_1 - A_2 z_2)^{-1}B$ agrees with $z_1^{-1}z_2^{-1}s$ up to the degree $2P$ by condition ii). $s_\Sigma$ has a denominator in $K[z_1^{-1}, z_2^{-1}]$ of degree $\leqslant 2m$ and $z_1^{-1}z_2^{-1}s$ has a denominator given by (8). Hence $s_\Sigma = z_1^{-1}z_2^{-1}s$ by Lemma 4.1 and by the definition of $\overline{P}$. This proves the only if part of the proposition. The converse is obvious.

Noncommutative polynomials which satisfy degree condition (9) constitute a finite dimensional $K$-linear space $V^{(i)}$. Since the restriction of $\phi$ to $V^{(i)}$, denoted $\phi | V^{(i)}$, is linear, the set $V^{(ii)}$ of polynomials satisfying (9) and (10) is a coset relative to the subspace ker $(\phi | V^{(i)})$.

Thus, we are interested in characterizing $\mathcal{P}$ in $V^{(ii)}$. For that purpose we shall give a parametric representation of polynomials in $V^{(ii)}$, and then we shall find the parameters corresponding to the elements in $\mathcal{P}$ having minimal partial representations.

The procedure is summarized as follows.

1) Let $\Xi^{(h,k)} \subset \Xi^*$ denote the set of words which contain $\xi_1$ and $\xi_2$ respectively $h$ and $k$ times, let

$$\mathcal{I} = \{w\colon\; w \in \Xi^*, |w| \leqslant 2\overline{P}, w \neq \xi_1^h \xi_2^k, \forall h, k\}$$

and let $K^{\mathcal{I}}$ be the space of $\mathcal{I}$ indexed sequences of elements of $K$. To every sequence $(t_w) \in K^{\mathcal{I}}$ biuniquely corresponds in $V^{(ii)}$ the polynomial $\pi_{(t_w)}$:

[1] $[x]$ denotes the smallest integer $n$ such that $n > x$.

$$\pi(t_w) = \sum_{w \in \mathcal{I}} t_w w + \sum_{i+j \leq 2\overline{P}} (z_1^{-1} z_2^{-1} s, z^i z^j) - \sum_{w \in \mathcal{I} \cap \Xi} (i,j) t_w \xi_1^i \xi_2^j. \quad (12)$$

2) Consider the following sets in $K$:

$$T_q = \{(t_w): (t_w) \in K^{\mathcal{I}}, \operatorname{rank} \mathcal{H}_{\overline{P} \times \overline{P}} (\pi_{(t_w)}) \leq q\} \quad q \leq m$$
$$V_q = \{(t_w): (t_w) \in K^{\mathcal{I}}, \operatorname{rank} \mathcal{H}_{\overline{P} \times (\overline{P}+1)} (\pi_{(t_w)}) \leq q\} \quad q \leq m$$
$$U_q = \{(t_w): (t_w) \in K^{\mathcal{I}}, \operatorname{rank} \mathcal{H}_{(\overline{P}+1) \times \overline{P}} (\pi_{(t_w)}) \leq q\} \quad q \leq m. \quad (13)$$

Each condition on the rank of matrices in (13) in equivalent to a number of conditions on the minors of order $q + 1$ and is expressed by a system of algebraic equations in the parameters $t_w$, $w \in \mathcal{I}$. Thus, $T_q$, $V_q$, and $U_q$ are algebraic varieties in $K^{\mathcal{I}}$.

3) Evaluate the smallest value of the index $q$ such that

$$\mathfrak{W}_q \triangleq (T_q - T_{q-1}) \cap (V_q - V_{q-1}) \cap (U_q - U_{q-1}) \neq \phi. \quad (14)$$

Let $\overline{m}$ denote this value. Then $(t_w) \in \mathfrak{W}_{\overline{m}} = (V_{\overline{m}} \cap U_{\overline{m}}) \cap (T_{\overline{m}} - T_{\overline{m}-1})$ if and only if $\pi(t_w)$ belongs to $\mathcal{P}$ and has minimal partial representation. In fact, $\overline{m}$ is the smallest value of $q$ for which the equations chain

$$\operatorname{rank} \mathcal{H}_{\overline{P} \times \overline{P}} (\pi_{(t_w)}) = \operatorname{rank} \mathcal{H}_{\overline{P} \times (\overline{P}+1)} (\pi_{(t_w)}) = \operatorname{rank} \mathcal{H}_{(\overline{P}+1) \times \overline{P}} (\pi_{(t_w)}) = q$$

admits a $(t_w)$ solution.

As obvious consequence, minimal partial representation have dimension $\overline{m}$.

4) Use (12) for constructing polynomials in $\mathcal{P}$ which have minimal partial representations.

Clearly, $\mathfrak{M}$ is constituted by all noncommutative rational power series of rank $\overline{m}$ which extend the polynomials obtained by the above steps.

Once we obtained the set $\mathfrak{M}$, we can use the generalized Ho algorithm for getting minimal partial representations. The set of these representations gives, modulo similarity transformations, all minimal realizations of $\mathcal{S}$.

Clearly, points 2) and 3) are the most difficult to be implemented because they involve the solution of several nonlinear algebraic equations. On the other hand, the necessity of introducing nonlinear algorithms is intrinsic to the problem as the dimension of minimal realizations depends on the ground field.

## V. Conclusions

In this paper by pursuing the idea of introducing a state space model of two-dimensional filters, the realization problem has been further investigated along the directions outlined in previous works [1], [2].

The class of realizations introduced in this paper is characterized by a local state updating equation of the following form:

$$x(h+1, k+1) = A_1 x(h+1, k) + A_2 x(h, k+1) + Bu(h, k).$$

The minimality of the realizations is not guaranteed by reachability and observability. In general, the dimension of minimal realizations depends on the field $K$ and does not coincide with the rank of $\mathcal{H}(s)$. Nevertheless, we can associate the commutative series $s$ with noncommutative recognizable power series whose representations provide all the realizations of $s$. Hence, the problem of determining minimal realizations of $s$ can be approached looking for minimal representations of noncommutative power series.

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# The Approximation of Image Blur Restoration Filters by Finite Impulse Responses

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*Abstract*—Image blur can often be modeled by a linear spatially invariant, symmetric point spread function. For this class of functions, several restoration filters are known in the literature.

The approximation of their frequency transfer functions (ftf's) by the ftf's of small finite impulse response (FIR) filters has been studied. Accurate approximations will be possible by 9 × 9 FIR's with 8-bit elements if the approximation is done in a weighted MMSE sense, and if the truncation of the element values will be carried out such that the errors are small. A heuristic truncation algorithm MINIM will be described. An example of restoration by a 9 × 9 FIR will be shown.


*Index Terms*—Approximation, finite impulse response, finite register length, image blur, restoration filter.

## I. Introduction

In the literature on image processing [1]–[3] a number of restoration filters are known, which can be applied in the case of spatially invariant image blurring. In general, these filters are implemented by means of the discrete Fourier transform (DFT), with help of the FFT algorithm. These filters, however, are operational only for static images as a consequence of the computational complexity.

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