Global properties and duality in 2-D systems *

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The paper is concerned with structural properties of the global state of 2-D systems. Global reachability and observability are related to the existence of the inverses over rings of the reachability and observability matrices. This leads to introduce the concept of dual system of a 2-D system in terms of a system defined over a suitable ring of polynomials.

Finally, global reachability and observability conditions are proved to be equivalent to the existence of Bèzout identities in two variables.

Keywords: Linear systems, 2-D systems, Systems over rings, Duality, Reachability and observability.

1. Introduction

A peculiar feature of 2-D system dynamics is that an increasing number of initial local states on a separation set has to be processed as the computation goes on. In fact, while the local state in a given point of the discrete plane depends on a finite number of initial local states, this number increases with the distance of the point from the separation set [1,2,3].

Consequently, the separation property, characteristic of the state variable, is provided by an infinite dimensional "global" state space [4]. Actually the memory function in any dynamical realization of a transfer function in two indeterminates is displayed by an infinite dimensional vector, whatever structure may be given for the local state updating equations.

It is then to be expected that reachability and observability analysis shows both aspects: local and global. So far the local approach has been investigated more extensively in the literature and the global state has been considered mainly in the analysis of internal stability, whose definition is based on the global state dynamics [1,5].

In this paper we deal with global properties of 2-D systems via the introduction of reachability and observability matrices of polynomial type and Bèzout identities in two variables that extend to the 2-D case the well known 1-D coprimeness conditions [6]. Since from a computational point of view only a finite subset of the global state is needed to perform the updating of local states, a natural objection could arise against the use of global states, which are infinite dimensional. Actually, global reachability and observability are equivalent to reachability and observability of finite sequences of local states on a separation set, so that the structure requirements on finite sequences are not weaker than the global ones.

As we shall show, a duality relation between 2-D systems and a class of systems over rings can be established by introducing dynamical systems over the ring of bilateral polynomials $K[\xi, \xi^{-1}]$. Since input, output and state spaces of these systems are the algebraic duals of output, input and global state spaces of 2-D systems, the adjoint transformation theorems lead to consider 2-D global reachability and observability as dual properties of observability and reachability over rings.

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2. Global reachability and observability

The dynamics of a 2-D system is represented by the following updating euqations [2]:

$$\begin{aligned} x(h+1,k+1) &= A_1x(h+1,k)+A_2(h,k+1)+B_1u(h+1,k)+B_2u(h,k+1),\\ y(h,k) &= Cx(h,k), \end{aligned} \tag{1}$$

where the local state x is an n-dimensional vector over a field K, input and output values are scalars and A_1, A_2, B_1, B_2, C are matrices of suitable dimensions with entries in K.

In this paper we are concerned with reachability and observability of global states whose supports are the separation sets

$$\mathcal{C}_i = \{(h,k) \in \mathbb{Z}\times\mathbb{Z}, h+k=i\}, \quad i=0, \pm1, \ldots.$$

Hence global states are the elements of the direct product of the local state spaces $\mathcal{C}_i$ and bilateral Laurent formal power series provide a convenient tool for representing the global state dynamics.

According to this approach, let

$$\mathcal{X}_i = \sum_{j=-\infty}^{+\infty} x(i-j,j)\xi^j \tag{2}$$

represent the global state on $\mathcal{C}_i$ and

$$\mathcal{U}_i = \sum_{j=-\infty}^{+\infty} u(i-j,j)\xi^j, \qquad \mathcal{Y}_i = \sum_{j=-\infty}^{+\infty} y(i-j,j)\xi^j \tag{3}$$

the restrictions to $\mathcal{C}_i$ of input and output functions. With this notation, input and output functions can be written as

$$u = \sum_{i=h}^{+\infty} \mathcal{U}_i\eta^i, \qquad y = \sum_{i=k}^{\infty} \mathcal{Y}_i\eta^i \tag{4}$$

where h and k are integers. The set $K_b^m((\xi))$ of (bilateral) Laurent formal power series with coefficients in K^m can be naturally endowed with the structure of a $K[\xi,\xi^{-1}]$-module, where $K[\xi,\xi^{-1}]$ is the subring of $K(\xi)$ generated by K, ξ and ξ^{-1}. As a consequence of the module structure, the global state updating equations

$$\mathcal{X}_{i+1} = (A_1+A_2\xi)\mathcal{X}_i + (B_1+B_2\xi)\mathcal{U}_i, \qquad \mathcal{Y}_i = C\mathcal{X}_i \tag{5}$$

are easily derived from (1).

If we restrict global states to belong to $K((\xi))^n$ and input functions to have the form

$$\sum_{i=h}^{+\infty} \mathcal{U}_i\eta^i, \quad \mathcal{U}_i \in K((\xi)),$$

system (5) can be viewed as a linear system over the field $K((\xi))$. Then 1-D linear theory applies and reachability and observability conditions correspond to assume that the matrices

$$\mathcal{R} = \left[(B_1+B_2\xi) \quad (A_1+A_2\xi)(B_1+B_2\xi) \quad \cdots \quad (A_1+A_2\xi)^{n-1}(B_1+B_2\xi)\right],$$

$$\mathcal{O} = \begin{bmatrix} C \\ C(A_1+A_2\xi) \\ \vdots \\ C(A_1+A_2\xi)^{n-1} \end{bmatrix}$$

have full rank over $K((\xi))$.

In the general case, global states and inputs can have infinitely many non-zero elements in both directions of the separation set, so they are really 'bilateral' and cannot be represented on $K((\xi))$ or $K((\xi^{-1}))$. While global reachability of system (1) and reachability of system (5) over the field $K((\xi))$ are equivalent, global observability of (1) is not implied by observability of (5) over $K((\xi))$, so that global states with unilateral support which are distinguishable from each other can be distinguishable from global states with bilateral support.

Theorem 1 (global reachability and observability). *The 2-D system* (1) *is globally reachable if and only if* $\det \mathcal{R} \neq 0$ *and is globally observable if and only if* $\det \mathcal{O}$ *has an inverse in* $K[\xi,\xi]^{-1}$, *that is* $\det \mathcal{O} = k\xi^m$ *for some integer m and some non-zero element k in K.*

Proof. Recalling the Cayley–Hamilton theorem, the system is globally reachable if an only if the equation

$$\mathcal{X}_0 = \mathcal{R}\begin{bmatrix} \mathcal{U}_{-1} \\ \mathcal{U}_{-2} \\ \vdots \\ \mathcal{U}_{-n} \end{bmatrix},$$

where the unknowns $\mathcal{U}_{-1}, \mathcal{U}_{-2}, \ldots, \mathcal{U}_{-n}$ are Laurent formal power series, has a solution for any global state $\mathcal{X}_0$. Depoting by V the unimodular polynomial matrix which reduces $\mathcal{R}$ to its row Hermite form, we have

$$\mathcal{X}_0 = (\mathcal{R}V)V^{-1}\begin{bmatrix} \mathcal{U}_{-1} \\ \mathcal{U}_{-2} \\ \vdots \\ \mathcal{U}_{-n} \end{bmatrix} = \begin{bmatrix} r_{11}(\xi) & 0 & \cdots & 0 \\ r_{21}(\xi) & r_{22}(\xi) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1}(\xi) & r_{n2}(\xi) & \cdots & r_{nn}(\xi) \end{bmatrix}\begin{bmatrix} \mathcal{U}'_{-1} \\ \mathcal{U}'_{-2} \\ \vdots \\ \mathcal{U}'_{-n} \end{bmatrix} \tag{6}$$

where

$$\begin{bmatrix} \mathcal{U}'_{-1} \\ \mathcal{U}'_{-2} \\ \vdots \\ \mathcal{U}'_{-n} \end{bmatrix} = V^{-1}\begin{bmatrix} \mathcal{U}_{-1} \\ \mathcal{U}_{-2} \\ \vdots \\ \mathcal{U}_{-n} \end{bmatrix}$$

If $x(\xi) \in K_b((\xi))$ and $r(\xi) \in K[\xi]$ there exist non-zero Laurent formal power series which satisfy the equation $x(\xi) = r(\xi)u(\xi)$ if and only if $r(\xi) \neq 0$. Hence equation (6) is solvable for any $\mathcal{X}_0$ if and only if the diagonal polynomials $r_{ii}(\xi)$ are all different from 0, that is if and only if $\det \mathcal{U} \neq 0$.

As far as observability is concerned, the global observability condition is equivalent, by definition, to require that $\mathcal{O}\mathcal{X} = 0$ implies $\mathcal{X} = 0$.

Suppose now that W is a unimodular polynomial matrix which reduces $\mathcal{O}$ to its row Hermite form. Then

$$\mathcal{O}WW^{-1}\mathcal{X} = \begin{bmatrix} \omega_{11}(\xi) & 0 & \cdots & 0 \\ \omega_{21}(\xi) & \omega_{22}(\xi) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{n1}(\xi) & \omega_{n2}(\xi) & \cdots & \omega_{nn}(\xi) \end{bmatrix}\mathcal{X}' = 0,$$

with $\mathcal{X}' = W^{-1}\mathcal{X}$, implies $\mathcal{X}' = 0$.

Given a polynomial $\omega(\xi)$ in $K[\xi,\xi^{-1}]$, there exists a unique scalar Laurent formal power series $x(\xi)$ satisfying $\omega(\xi)x(\xi) = 0$ if and only if $\omega(\xi) = k\xi^h$ for some integer h and non-zero k in K. Since the unit elements in $K[\xi,\xi^{-1}]$ have the form $k\xi^h$, $k \neq 0$, the condition above corresponds to assume $\omega(\xi)$ be invertible in $K[\xi,\xi^{-1}]$. Hence a necessary and sufficient condition for global observability is that all polynomials ω_{ii} – and consequently det $\mathcal{O}$ – have an inverse over $K[\xi,\xi^{-1}]$.

Remark 1. If the matrix $\mathcal{O}$ is full rank over $K((\xi))$ but det $\mathcal{O}$ is not invertible in $K[\xi,\xi^{-1}]$, the subspace of global states which are indistinguishable from zero is finite dimensional over K. In fact, let $\alpha_{ij}\xi^j$ be the monomial of minimal degree in $\omega_{ii}(\xi)$ and assume $\nu_i = \deg \omega_{ii}(\xi) - j$. Then $\Sigma_{i=1}^{n} \nu_i$ gives the dimension of the subspace of zero indistinguishable global states. This situation is illustrated in the following example. Assume

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad C = [1 \quad 0].$$

Since the observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1+\xi \end{bmatrix},$$

all states having the structure

$$\mathcal{X} = \begin{bmatrix} 0 \\ k \sum_{i=-\infty}^{+\infty} (-1)^i \xi^i \end{bmatrix}, \qquad k \in K,$$

are indistinguishable from zero.

Notice also that zero indistinguishable global states cannot be represented over $K((\xi))^2$.

Remark 2. The 2-D global reachability conditions det $\mathcal{R} \neq 0$ coincides with the 1-D reachability condition for the $K((\xi))$-linear system (5). However differences arise when considering the uniqueness of the input sequence which produces a given state.

In fact, the $K((\xi))$ valued input sequence $\mathcal{U}_{-n}, \mathcal{U}_{-n+1}, \ldots, \mathcal{U}_{-1}$ of length n which gives $\mathcal{X}_0 \in K((\xi))^n$ in system (5) is always uniquely determined. On the contrary in system (1) admissible $\mathcal{U}_i$'s are Laurent power series, so that the input sequence of length n which produces $\mathcal{X}_0$ is uniquely determined only if det $\mathcal{R}$ is invertible in $K[\xi,\xi^{-1}]$.

An intermediate problem between global and local reachability and observability consists in reachability nad observability of finite sequences of local states on a separation set. A first approach to this problem considers system (5) over the ring $K[\xi,\xi^{-1}]$: inputs and outputs have compact supports on every separation set and states are element of $K[\xi,\xi^{-1}]^n$. So, by applying the theory of linear systems over rings the reachability condition is

$$\det \mathcal{R} = k\xi^h, \quad k \neq 0, \tag{7}$$

and the observability condition is

$$\det \mathcal{O} \neq 0. \tag{8}$$

A different approach is based on the fact that global reachability and observability conditions of Theorem 1 apply to finite sequences of local states on $\mathcal{C}_0$. Theorem 2 below provides an interpretation in terms of finite sequences of local states that the global reachability condition det $\mathcal{R} \neq 0$ is weaker than (7).

Theorem 2. *System* (1) *is globally reachable if and only if there exists an integer* $N(\geq n^2)$ *such that any set of local states*

$$x(0,0), \quad x(-1,1), \quad \ldots, \quad x(-N+1, N-1) \tag{9}$$

on $\mathcal{S} := \{(0,0), (-1,1), \ldots, (-N+1, N-1)\}$ *is the restriction to* $\mathcal{S}$ *of a global state on* $\mathcal{C}_0$ *produced by some input function with compact support on* $\mathbb{Z} \times \mathbb{Z}$ [1].

[1] Note that this condition does not impose any constraint on the values assumed by the local states on $\mathcal{C}_0\mathcal{S}$.

Proof. Clearly global reachability implies reachability of (9). Conversely, assume that the following sequences of length $N = n^2$,

$$e_i + 0\xi + 0\xi^2 + \cdots + e_i\xi^{N-1}, \quad i = 1, 2, \ldots, n,$$

where e_i denotes the i-th column of the $n \times n$ identity matrix, are reachable. Then there exist scalars a_{ij} and b_{ij} such that the global states

$$\sum_{j=1}^{n} a_{ij}\xi^{-j} + e_i + e_i\xi^{N-1} + \sum_{j=1}^{n} b_{ij}\xi^{j+N}, \quad i = 1, 2, \ldots, n, \tag{10}$$

are reachable. Since the n global states in (10) are linearly independent over $K((\xi))$, det $\mathcal{R} \neq 0$.

Let us now consider the observability case. Here the global condition det $\mathcal{O} = k\xi^h$, $k \neq 0$, is stronger than (8) and corresponds to the following observability condition of finite sequences of local states. Let $\mathcal{S} = \{(0,0), (-1,1), \ldots, (-n,n)\}$ be a subset of $\mathcal{C}_0$. A sequence of local states on $\mathcal{S}$ is observable if the free output values on some finite subset $\mathcal{F} \subseteq \mathbb{Z} \times \mathbb{Z}$ uniquely determine $x(0,0)$, $x(-1,1), \ldots, x(-n,n)$. In fact we have the following theorem:

Theorem 3. *The following facts are equivalent:*

(i) *the 2-D system* (1) *is globally observable;*

(ii) *there exists a finite subset* $\mathcal{F} \subseteq \mathbb{Z} \times \mathbb{Z}$ *such that* $x(0,0)$ *is uniquely determined by the free output values on* $\mathcal{F}$, *whatever local states may be given on* $\mathcal{C}_0 \setminus \{(0,0)\}$;

(iii) *any finite sequence of local states is observable.*

Proof. (i) $\Leftrightarrow$ (ii). Obviously, non-zero local states belonging to an unobservable global state cannot be determined from the output values. Conversely, let the 2-D system (1) be globally observable. Then $\mathcal{O}$ is invertible over $K[\xi, \xi^{-1}]$ and from

$$\mathcal{X} = \sum_{i=+\infty}^{+\infty} x_i\xi^i = \mathcal{O}^{-1}\begin{bmatrix} \mathcal{Y}_0 \\ \mathcal{Y}_1 \\ \vdots \\ \mathcal{Y}_{n-1} \end{bmatrix}$$

we have [2]

$$x(0,0) = \left(\mathcal{O}^{-1}\begin{bmatrix} \mathcal{Y}_0 \\ \mathcal{Y}_1 \\ \vdots \\ \mathcal{Y}_{n-1} \end{bmatrix}, \xi^0 \right). \tag{11}$$

Since the entries of $\mathcal{O}^{-1}$ are in $K[\xi, \xi^{-1}]$, the computation of $x(0,0)$ in (11) involves finitely many coefficients of $\mathcal{Y}_0, \mathcal{Y}_1, \ldots, \mathcal{Y}_{n-1}$, that is the output values on a finite subset $\mathcal{F}$ of $\mathbb{Z} \times \mathbb{Z}$.

(ii) $\Leftrightarrow$ (iii) Using (11) and the shifting property, any finite sequence of local states on a separation set can be computed from a finite set of output values.

Remark. When we deal with the local observability condition, all local states on a separation set are assumed to be zero, except the local state we want to determine. At point (ii) of Theorem 3, the assumptions are more general, since no restrictions are imposed on the local states $x(i, -i)$, $i \neq 0$. This explains the known fact that global observability, which has been proved to be equivalent to (ii), is a stronger property than local observability.

[2] As commonly used in formal power series theory, (s, ξ^i) denotes the coefficient of ξ^i in the series s.

3. Duality

In this section global reachability and observability of 2-D systems are related to observability and reachability of systems defined over rings. The basic idea of connecting these properties descends from the remark that for 2-D systems global reachability is equivalent to the nonsingularity of $\mathfrak{R}$ and global observability to the unimodularity of $\mathfrak{O}$, while for systems over rings the conditions on $\mathfrak{R}$ and $\mathfrak{O}$ are in some way exchanged. This suggests the possibility of interpreting 2-D systems as the duals of suitable systems over rings.

Let us first introduce the system

$$w(t+1)=F(\xi)w(t)+G(\xi)v(t), \qquad z(t)=H(\xi)w(t), \tag{12}$$

defined over the ring of polynomials $K[\xi,\xi^{-1}]$. Here the input set is the ring $K[\xi,\xi^{-1}][\eta^{-1}]$, the output set is the ring $K[\xi,\xi^{-1}][[\eta]]$, the states are elements of the free module $K[\xi,\xi^{-1}]^n$ and the matrices $F(\xi)$, $G(\xi)$, $H(\xi)$ have entries in $K[\xi,\xi^{-1}]$.

Since the input and output alphabets and the state set of (10) are K-linear vector spaces, system (12) can be viewed as an infinite dimensional linear system over K.

The construction of the dual system of (12) is based on the following facts:

1. The space $K_b^m((\xi))$ of Laurent formal power series with coefficients in K^m is the algebraic dual of $K^m[\xi,\xi^{-1}]$:

$$\left(K^m[\xi,\xi^{-1}]\right)^*=K_b^m((\xi)).$$

In fact, let $s=\sum_{i=-\infty}^{\infty}s_i\xi^j$ be in $K_b^m((\xi))$ and $r=\sum_j r_j\xi^j$ by any element of $K^m[\xi,\xi^{-1}]$. Then the series s defines a linear functional on $K^m[\xi,\xi^{-1}]$ by assuming

$$\langle s,r\rangle:=\left(\sum_{i=-\infty}^{\infty}s_i^{\mathrm{T}}\xi^{-i}\sum_j r_j\xi^j,0\right) \tag{13}$$

where s_i^{T} is the transpose of s_i. Conversely every linear functional on $K^m[\xi,\xi^{-1}]$ can be represented by a Laurent power series.

Then assuming in (13) $s=\mathfrak{X}$ and $r=w$, it turns out that the global state space of (1) is the algebraic dual of the state space of (12).

Similarly, the input sequences $\mathfrak{U}=\sum_{i=-\infty}^{+\infty}u_i\xi^i$ and the output sequences $\mathfrak{Y}=\sum_{i=-\infty}^{+\infty}y_i\xi^i$ over a separation set are the linear functionals on the input and the output alphabets of (12).

2. The output space $K_b((\xi))[[\eta]]$ of the 2-D system (1), that is the space whose elements are the series

$$\sum_{i=0}^{\infty}\mathfrak{Y}_i\eta^i,$$

where $\mathfrak{Y}_i=\sum_{j=-\infty}^{+\infty}y(i-j,j)\xi^j$ is the algebraic dual of the input space of system (12):

$$\left(\eta^{-1}K[\xi,\xi^{-1}][\eta^{-1}]\right)^*=K_b((\xi))[[\eta]].$$

This follows by assuming

$$\langle\sum_{i=0}^{\infty}\sum_{j=-\infty}^{+\infty}y(i-j,j)\xi^j\eta^i,\ \sum_{q=1}^{Q}\sum_{p=-P}^{P}v(-q-p,p)\xi^p\eta^{-q}\rangle$$

$$=\sum_{q=0}^{Q}\langle\sum_{j=-\infty}^{\infty}y(-q-j,j)\xi^j,\ \sum_p v(-q-1-p,p)\xi^p\rangle.$$

Similarly, the space $\eta^{-1}K_b((\xi))[\eta^{-1}]_n$ of inputs of system (1) with length at most n, i.e. the space whose elements are the series

$$\sum_{i=1}^{n}\mathfrak{U}_{-i}\eta^{-i}$$

is the algebraic dual of the space of the output restrictions to $[0, n-1]$ of system (12)

$$(K[\xi,\xi^{-1}][\eta]_n)^* = \eta^{-1}K_b((\xi))[\eta^{-1}]_n.$$

With these two facts we are now ready to show the global reachability and observability maps in system (1) are the algebraic duals of observability and reachability maps of system (12).

Let ρ and ω be the reachability nad observability maps of (12):

$$\rho:\quad \eta^{-1}K[\xi,\xi^{-1}][\eta^{-1}] \to K[\xi,\xi^{-1}]^n:$$

$$\sum_{q=1}^{Q}\sum_{p} v(-q-p,p)\xi^p\eta^{-q} \mapsto \sum_{q=1}^{Q} F(\xi)^{q-1}G(\xi)\sum_{p} v(-q-p,p)\xi^p,$$

$$\omega:\quad K[\xi,\xi^{-1}]^n \to K[\xi,\xi^{-1}][[\eta]]:\quad \sum_i x_i\xi^i \mapsto \sum_{j=0}^{\infty} F(\xi)^j H(\xi)\sum_i x_i\xi^i\eta^j,$$

and denote by π_n the projection map of $K[\xi,\xi^{-1}][[\eta]]$ onto the subspace $K[\xi,\xi^{-1}][\eta]_n$ of polynomials with degree less than n.

In the diagram

$$\eta^{-1}K[\xi,\xi^{-1}][\eta^{-1}] \xrightarrow{\rho} K[\xi,\xi^{-1}]^n \xrightarrow{\omega_n} K[\xi,\xi^{-1}][\eta]_n,\qquad K[\xi,\xi^{-1}]^n \xrightarrow{\omega} K[\xi,\xi^{-1}][[\eta]] \xrightarrow{\pi_n} K[\xi,\xi^{-1}][\eta]_n$$

ρ is injective if and only if system (12) is reachable, ω and (by the Cayley–Hamilton theorem) $\omega_n := \pi_n \cdot \omega$ are surjective if and only if system (12) is observable.

In the dual sequence

$$K_b((\xi))[[\eta]] \xleftarrow{\rho^*} K_b((\xi))^n \xleftarrow{\omega_n^*} \eta^{-1}K_b((\xi))[\eta^{-1}]_n$$

ρ^* and ω_n^* are the dual maps of ρ and ω_n and provide the observability and the n-steps reachability maps of the dual system of (12) given by

$$\overline{w}(t+1) = F^{\mathrm{T}}(\xi)\overline{w}(t) + H^{\mathrm{T}}(\xi)\overline{z}(t),\qquad \overline{v}(t) = G^{\mathrm{T}}(\xi)\overline{w}(t), \tag{14}$$

where the input and output alphabets and Laurent formal power series and the state space is $K_b((\xi))^n$.

Then, from the theory of dual spaces we have the ρ^* is injective if and only if ρ is surjective and ω_n^* is surjective if and only if ω_n is injective.

Since the reachability map of system (14)

$$\omega' : \eta^{-1}K_b((\xi))[\eta^{-1}] \to K_b((\xi))^n$$

is surjective if and only if ω_n^* is surjective, reachability (observability) of system (14) is equivalent to observability (reachability) of system (12). In particular, assuming

$$F(\xi) = A_1^{\mathrm{T}}A_2^{\mathrm{T}}\xi,\qquad G(\xi) = C^{\mathrm{T}},\qquad H(\xi) = B_1^{\mathrm{T}} + B_2^{\mathrm{T}}\xi,$$

we have proved the following theorem:

Theorem 4 (duality). *The 2-D system* (1) *is globally reachable* (*observable*) *if and only if the system*

$$w(t+1) = (A_1^{\mathrm{T}} + A_2^{\mathrm{T}}\xi)w(t) + C^{\mathrm{T}}v(t),\qquad z(t) = (B_1^{\mathrm{T}} + B_2^{\mathrm{T}}\xi)w(t), \tag{15}$$

defined over the ring $K[\xi,\xi^{-1}]$ *is observable* (*reachable*).

This result provides also an alternative approach to global reachability and observability criteria, based on the theory of systems over rings. In fact, system (15) is reachable if and only if its reachability matrix

$$\mathcal{R}' = \left[C^{\mathrm{T}}\quad (A_1^{\mathrm{T}} + A_2^{\mathrm{T}}\xi)C^{\mathrm{T}}\quad \cdots\quad (A_1^{\mathrm{T}} + A_2^{\mathrm{T}}\xi)^{n-1}C^{\mathrm{T}}\right]$$

has an inverse in the ring $K[\xi, \xi^{-1}]$, i.e. if and only if det $\mathcal{R}' = k\xi^h$, for some non-zero k in K and some integer h. Then, since $\mathcal{O} = \mathcal{R}'^{\mathrm{T}}$, the duality theorem provides the global observability condition given in Theorem 1.

Analogously, the 2-D global reachability condition of Theorem 1 follows by duality from the non-singularity of the observability matrix of system (15)

$$\mathcal{O}' = \begin{bmatrix} B_1^{\mathrm{T}} + B_2^{\mathrm{T}}\xi \\ (B_1^{\mathrm{T}} + B_2^{\mathrm{T}}\xi)(A_1^{\mathrm{T}} + A_2^{\mathrm{T}}\xi) \\ \vdots \\ (B_1^{\mathrm{T}} + B_2^{\mathrm{T}}\xi)(A_1^{\mathrm{T}} + A_2^{\mathrm{T}}\xi)^{n-1} \end{bmatrix}.$$

4. Global properties and Bézout identities

As we have seen in the previous sections, global reachability and observability are equivalent to the invertibility of $\mathcal{R}$ over $K(\xi)$ and of $\mathcal{O}$ over $K[\xi, \xi^{-1}]$ respectively. In this section we prove that these conditions on $\mathcal{R}$ and $\mathcal{O}$ are equivalent to the existence of two Bézout identities. More precisely, we have the following results:

Theorem 5. *The 2-D system (1) is globally reachable if and only if there exist matrices X_1 and X_2, with entries in $K(\xi)[\eta]$, which satisfy*

$$(\eta I - A_1 - A_2\xi)X_1 + (B_1 + B_2\xi)X_2 = I_n. \tag{16}$$

Theorem 6. *The 2-D system (1) is globally observable if and only if there exist matrices Y_1 and Y_2 with entries in $K[\xi, \xi^{-1}][\eta]$ which satisfy*

$$Y_1(\eta I - A_1 - A_2\xi) + Y_2C = I_n. \tag{17}$$

Since $K(\xi)$ is a field, the proof of Theorem 5 follows from standard results in the theory of polynomial matrices.

Proof of Theorem 6 [3]. Assume first that system (1) is not globally observable. If det $\mathcal{O} = 0$, identity (17) cannot be satisfied by matrices Y_1 and Y_2 with entries in $K(\xi)[\eta]$ and a fortiori in $K[\xi, \xi^{-1}][\eta]$. So, let det $\mathcal{O} = r(\xi) \neq k\xi^h$ and by contradiction suppose that there exist Y_1 and Y_2 satisfying (17).

Let $g(\xi)$ be an irreducible factor of $r(\xi)$ and consider the field $K[\xi, \xi^{-1}]/(g)$, where (g) denotes the ideal of $K[\xi, \xi^{-1}]$ generated by $g(\xi)$. Let $\overline{p(\xi)} := p(\xi) + (g)$ and, given a matrix $H(\xi) = \| h_{ij}(\xi) \|$, denote by $\overline{H(\xi)}$ the matrix $\| \overline{h_{ij}(\xi)} \|$. Then, the natural homomorphism of $K[\xi, \xi^{-1}]$ onto $K[\xi, \xi^{-1}]/(g)$ applies to (17) and we have

$$\overline{Y}_1\left(\eta I - \left(\overline{A_1 + A_2\xi}\right)\right) + \overline{Y}_2 C = I_n.$$

This implies right coprimeness of $\eta I - (A_1 + A_2\bar{\xi})$ and C. Since

$$\det \overline{\mathcal{O}} = \det \begin{bmatrix} C \\ C(A_1 + A_2\bar{\xi}) \\ \vdots \\ C(A_1 + A_2\bar{\xi})^{n-1} \end{bmatrix} = \bar{0}$$

we get a contradiction.

[3] The referee informed the authors that a proof of this Theorem will be included in: Kargonegar, Sontag "On the relation between stable matrix fraction factorizations and regulable realizations of linear systems over rings", to appear in *IEEE Trans. Automat. Control.*

Conversely, let system (1) be globally observable, that is

$$\det \mathcal{O} = k\xi^h \neq 0.$$

Then there exists a matrix $T(\xi) \in K[\xi, \xi^{-1}]^{n \times n}$, whose inverse is also in $K[\xi, \xi^{-1}]^{n \times n}$, such that

$$A_0(\xi) = T^{-1}(\xi)(A_1 + A_2\xi)T(\xi) = \begin{bmatrix} 0 & & & -a_0(\xi) \\ 1 & & & -a_1(\xi) \\ & \ddots & & \vdots \\ 0 & \dots & 1 & -a_{n-1}(\xi) \end{bmatrix},$$

$$C_0 = CT(\xi) = [0 \quad 0 \quad \cdots \quad 1], \tag{18}$$

where the polynomial $a_i(\xi)$ is the coefficient of η^i in $\det(\eta I - A_1 - A_2\xi)$.

To obtain Y_1 and Y_2 in (17), we can proceed as follows. First compute matrices $\hat{Y}_1$ and $\hat{Y}_2$ with entries in $K[\xi, \xi^{-1}][\eta]$ which satisfy

$$\hat{Y}_1(\eta I - A_0(\xi)) + \hat{Y}_2 C_0 = I_n. \tag{19}$$

This can be easily done by assuming

$$\hat{Y}_1 = \begin{bmatrix} 0 & -1 & -\eta & -\eta^2 & \cdots & -\eta^{n-2} \\ 0 & 0 & -1 & -\eta & \cdots & -\eta^{n-3} \\ & & & \ddots & & \\ \vdots & & & & \ddots & -1 \\ 0 & & \cdots & & & 0 \end{bmatrix},$$

so that the first $n-1$ columns of $\hat{Y}_1(\eta I - A_0(\xi))$ are $e_1, e_2, \dots, e_{n-1}$, and then by properly selecting the row matrix $\hat{Y}_2$.

From (18) and (19) we obtain

$$\hat{Y}_1\left(\eta I - T^{-1}(\xi)(A_1 + A_2\xi)T(\xi)\right) + \hat{Y}_2 CT(\xi) = I_n.$$

Right multiplication by $T^{-1}(\xi)$ and left multiplication by $T(\xi)$ give identity (17), with

$$Y_1 = T(\xi)\hat{Y}_1 T^{-1}(\xi) \quad \text{and} \quad Y_2 = T(\xi)\hat{Y}_2.$$

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