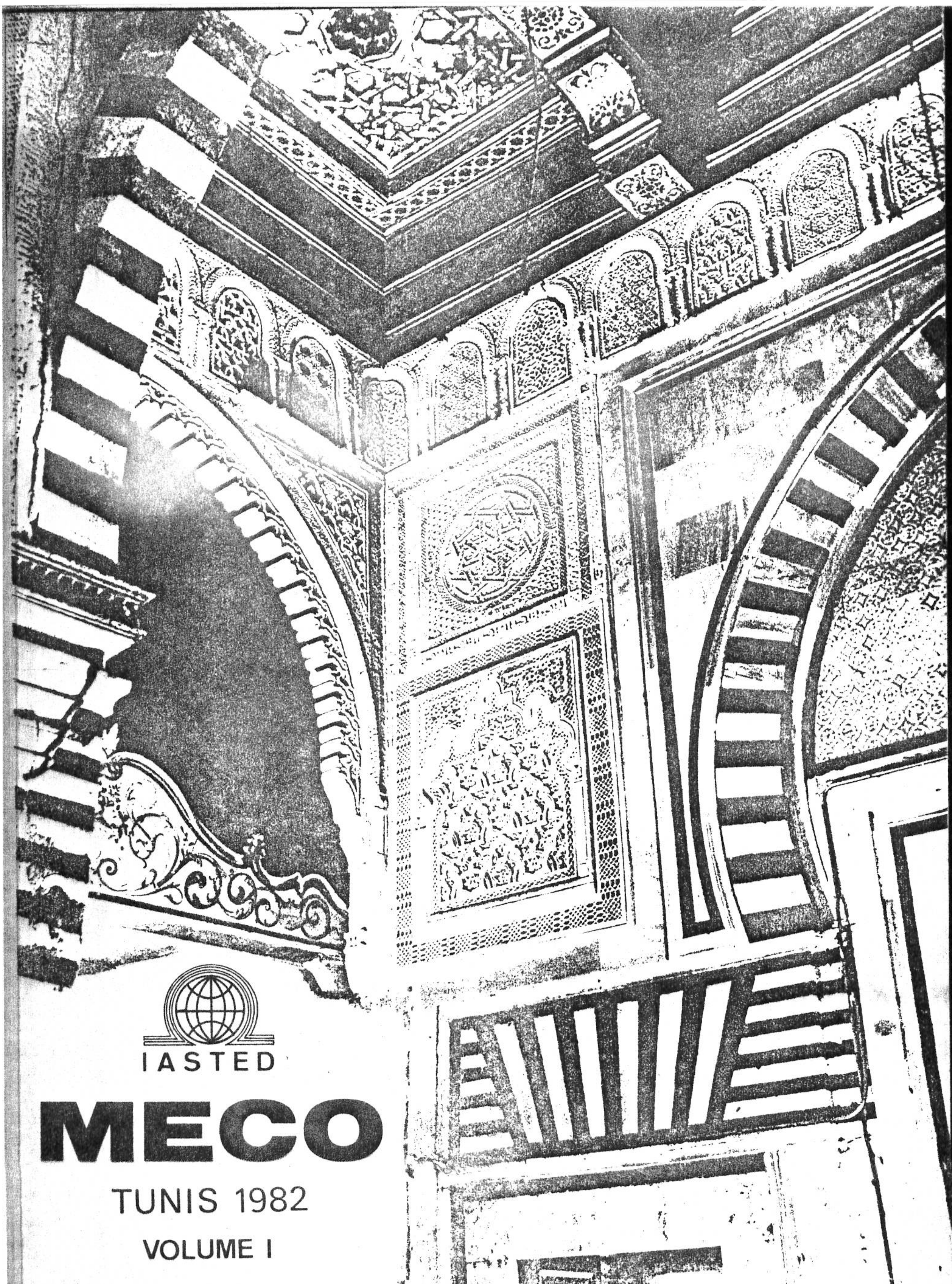

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ON THE RECURSIVE PROCESSING OF TWO DIMENSIONAL DIGITAL DATA BY 2-D SYSTEMS

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ABSTRACT text of first page here.

The analysis of remotely sensed data requires a massive use of digital multidimensional data processing for image filtering. In this paper we shall take into account some formal structure properties of two dimensional filtering, and primarily the recursiveness, with the aim of introducing state space models (2-D systems).

The way a 2-D system operates corresponds to the recursion performed by two-dimensional filters in that the one-step filter updating can be derived by a one-step state updating of a 2-D system. The amount of computation at each step strongly depends on the dimension of the state in the dynamical model and this makes worthwhile to look for state space realizations with minimal dimension.

In many cases of interest the procedure presented in this paper provides realizations of minimal dimension.

2-D RECURSIVE EQUATIONS AND STATE SPACE MODELS

This paper discusses some aspects of the state space models of 2-D filters [1-4] which are connected with the problem of constructing minimal realizations.

As in the 1-D case, the linear processing of two dimensional data can be represented by a convolutional operation or, when the transfer function is rational, by a recursive algorithm.

Let K be any field and $u(h,k) \in K$ and $y(h,k) \in K$ be the input and output signal values in $(h,k) \in Z \times Z$. The convolutional operation is the following

$$y(h,k) = \sum_{i,j=-\infty}^{+\infty} w(h-i,k-j)u(i,j)$$

Here $w(i,j)$ is the unit sample response, i.e. the response of the systems to the input whose values are 1 in (0,0) and 0 elsewhere. When the series

$$W(z_1,z_2) = \sum_{i,j} w(i,j) z_1^i z_2^j$$

is proper rational:

$$W(z_1,z_2) = \sum_{0}^{q_1}{}_i \sum_{0}^{q_2}{}_j b_{ij} z_1^i z_2^j \Big/ \sum_{0}^{p_1}{}_i \sum_{0}^{p_2}{}_j a_{ij} z_1^i z_2^j, \qquad a_{00}=1 \tag{1}$$

we obtain a 2-D (partial) difference equation relating y and u which provides the output values recursively:

$$y(h,k) = -\sum_{\substack{0 \\ i+j>0}}^{p_1}{}_i \sum_{0}^{p_2}{}_j a_{ij} y(h-i,k-j) + \sum_{0}^{q_1}{}_i \sum_{0}^{q_2}{}_j b_{ij} u(h-i,k-j) \tag{2}$$

We can see some conceptual as well as mathematical key differences that arise in the 2-D case with respect to the 1-D situation:

1. once we have computed $y(h,k)$ the structure of the recursive equation does not give any direction how to select univocally the point in $Z \times Z$ where we have to calculate the 'next' output;
2. the values $y(h,k)$, $u(h,k)$ and the input and output data used to calculate the output value in (h,k) are not sufficient to compute the output value in any point of $Z \times Z$ which is not already involved in the recursive equation. For instance the computation of $y(h+1,k)$ requires $u(h+1,k-j)$ and $y(h+1,k-j)$, $j = 0,1,\dots q_2$, $i = 0,1,\dots p_2$.

Both of the facts above are intrinsically connected with the partial order in $Z \times Z$ that has been implicitly assumed in (2), i.e.

$$(h,k) \leq (i,j) \qquad h \leq i,\ k \leq j$$

This leads to a notion of 'future' and 'past' which is deeply different from the 1-D case, where the 1-D recursion is provided by the well known difference equation:

$$y(k) = -\sum_{1}^{n}{}_i a_i y(k-i) + \sum_{0}^{m}{}_i b_i u(k-i) \tag{3}$$

Here the index k has the interpretation of time.

The recursion structure exhibited by (2) and (3) can be directly exploited in both cases to introduce a state representation. In fact, starting from (3), let define a state vector at time k as the vector whose elements are the n output values and the m input values preceding k.

Thus the updating equations are given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{4}$$

and the state space has dimension $n+m$.

As it is known, there exist lower dimension state-space models, which realize the i/o map given by (3) and their minimal dimension is n when the polynomials $\Sigma_i a_i z^i$ and $\Sigma_i b_i z^i$ are coprime. Moreover there are linear, finite algorithms based on reachable and observable canonical forms, which give such minimal order models.

Similarly, starting from (2), we can obtain a state representation by assuming as a state vector in (h,k) the vector whose elements are the input and output values in the right hand term of (2) excepting $u(h,k)$, $u(h-1,k)$, $u(h,k-1)$. With this definition of state, the updating equation is given by the following first order vector difference equation

$$x(h+1,k+1)=A_1x(h+1,k)+A_2x(h,k+1)+B_1u(h+1,k)+B_2u(h,k+1)$$
$$y(h,k) = Cx(h,k)+Du(h,k) \tag{5}$$

So doing, the dimension of the state space is in the order of $p_1p_2+q_1q_2$. Also in this case it is possible to reduce the dimension of the state model by using linear algorithms [5]. Nevertheless they do not provide minimal dimension models and this makes a deep difference from the 1-D case.

Actually, the analysis of structural properties of 2-D models and of the connections between these properties and those of transfer functions is more involved than in the 1-D case, and 2-D system theory did not yet reach an equivalent degree of completeness.

In the following we shall account for some partial results and open problems in 2-D realization.

2D TRANSFER FUNCTIONS REALIZATION

Consider a 2-D transfer function as a quotient of two coprime polynomials in $K[z_1^{-1},z_2^{-1}]$:

$$W(z_1,z_2) = \sum_{0}^{n}{}_i \sum_{0}^{m}{}_j c_{ij} z_1^{-i} z_2^{-j} \Big/ \sum_{0}^{n}{}_i \sum_{0}^{m}{}_j d_{ij} z_1^{-i} z_2^{-j}, \tag{6}$$

$$c_{nm} = 0\ , \quad d_{nm} = 1$$

In the corresponding formal power series

$$W(z_1,z_2) = \sum_{i+j>0} w(i,j) z_1^i z_2^j$$

we have $w(0,0) = 0$. This is equivalent to a proper causality assumption and leads to state space models (5) with $D=0$.

Let us introduce in (6) the following change of variables

$$\eta = z_1^{-1}\ , \quad \xi = z_1^{-1} z_2$$

so we have

$$\sum_{0}^{n+m-1}{}_h \eta^h \sum_{0}^{m}{}_j c_{h-j,j}\xi^{m-j} \Big/ \sum_{0}^{n+m}{}_h \eta^h \sum_{0}^{m}{}_j d_{h-j,j}\xi^{m-j} = \tag{7}$$
$$= \sum_{0}^{n+m-1}{}_h \eta^h c_h(\xi) \Big/ \sum_{0}^{n+m}{}_h \eta^h d_h(\xi)$$

where the supports of numerator and denominator are in the region of fig. 1.

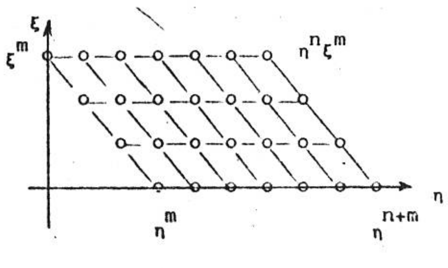

Fig. 1

The coprimeness in $K[z_1^{-1},z_2^{-1}]$ of numerator and denominator of (6) does not imply the coprimeness in $K[\eta,\xi]$ of numerator and denominator of (7). The loss of coprimeness takes place when, for some $\nu>0$, all homogeneous polynomials of degree less than ν vanish in the numerator and denominator of (6). In this case in (7) $c_0(\xi),\ldots,c_{\nu-1}(\xi)$ and $d_0(\xi),\ldots,d_{\nu-1}(\xi)$ are zero and η^ν is a common factor of the numerator and denominator.

A) If in (7) there are no cancellations which reduce the degree in η of the denominator, $n+m$ is a lower bound for the dimension of 2-D systems which realize (7). In fact, since the transfer function of (5) is given by

$$C(\eta I-A_1-A_2\xi)^{-1}\ (B_1+B_2\xi),$$

a denominator of degree $n+m$ in η cannot be obtained with dimension lower than $n+m$.

As far as the existence of a $n+m$ dimensional realization is concerned, there are several techniques for constructing matrix pairs A_1,A_2 of dimension $(n+m)\times(n+m)$ which satisfy

$$\det(\eta I-A_1-A_2\xi)^{-1} = \sum_{0}^{n+m}{}_h \eta^h d_h(\xi) \tag{8}$$

However it is not known if (8) and

$$C\ \mathrm{adj}(\eta I-A_1-A_2\xi)(B_1+B_2\xi) = \sum_{h=0}^{n+m-1} \eta^h c_h(\xi) \tag{9}$$

can be simultaneously satisfied by a suitable choice of A_1, A_2, B_1,B_2, C.

Since the case $m = 0$ corresponds to a 1-D realization problem and can be solved by standard linear techniques, we do not consider it and we assume $m\geq 1$. In the sequel we shall introduce a constructive procedure which gives matrices A_1,A_2 which satisfy (8) and such that the number of free parameters in A_1,A_2, B_1,B_2, C exceeds the number of coefficients in $\Sigma_h\eta^h c_h(\xi)$. However condition (9) produces a system of nonlinear equations in these parameters whose solvability over the real field has not been proved for the generic case.

First, let us observe that the determinant of the following matrix

$$\left[\begin{array}{ccccc|cccc} \eta & & & & & & & & \\ -1 & \eta & & & & & & & \\ & -1 & \eta & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & -1 & \eta & & & & \\ \hline & & & & -1 & \eta & & & \\ & & & & & -\xi & \eta & & \\ & & O & & & & \ddots & \ddots & \\ & & & & & & & -\xi & \eta \end{array}\right] + \left[\begin{array}{c|c} O & \begin{matrix} r_0\\ r_1\\ r_2\\ \vdots\\ r_n \end{matrix} \\ \hline O & O \end{array}\right] \begin{matrix} \left.\vphantom{\begin{matrix} r_0\\ r_1\\ r_2\\ \vdots\\ r_n \end{matrix}}\right\} n+1 \\ \left.\vphantom{O}\right\} m-1 \end{matrix}$$
$$\hphantom{xxxxxxxxxxxxxxxxxxxx}\underbrace{\hphantom{xxxxxx}}_{n}\ \underbrace{\hphantom{xxx}}_{m}$$

$r_i\in\mathbb{R}^{1\times m}$, can be expressed as

$$\eta^{n+m}+\eta^n\det\begin{bmatrix} r_n & & \\ \hline -\xi & \eta & \\ & \ddots & \ddots \\ & -\xi & \eta \end{bmatrix} -\eta^{n-1}\det\begin{bmatrix} r_{n-1} & & \\ \hline -\xi & \eta & \\ & \ddots & \ddots \\ & -\xi & \eta \end{bmatrix} \ldots +\det\begin{bmatrix} r_0 & & \\ \hline -\xi & \eta & \\ & \ddots & \ddots \\ & -\xi & \eta \end{bmatrix} \tag{10}$$

Each determinant in (10) is an homogeneous polynomial of degree $m-1$ in η, ξ.

Now, rewrite the denominator of the transfer function as the sum of forms $\eta^p\psi_p(\eta,\xi)$ of degrees $m,m+1,\ldots m+n$

$$\sum_{p=0}^{n} \eta^p \sum_{j=0}^{m} d_{p,j}\eta^j\xi^{m-j} \triangleq \sum_{p=0}^{n} \eta^p\psi_p(\eta,\xi)$$

Note that for $p < n$ one has

$$\psi_p(\eta,\xi) = \eta \det \begin{bmatrix} d_{pm} & (d_{p,m-1}-f_{p,m-1}) & \cdots & (d_{p1}-f_{p1}) \\ -\xi & \eta & & \\ & \ddots & \ddots & \\ & & -\xi & \eta \end{bmatrix} +$$

$$+ \det \begin{bmatrix} f_{p,m-1} & f_{p,m-2} & \cdots & f_{p1} & d_{po} \\ -\xi & \eta & & & \\ & \ddots & \ddots & & \\ & & & -\xi & \eta \end{bmatrix} =$$

$$\triangleq \eta \det \begin{bmatrix} & r_{p+1} & \\ \hline -\xi & \eta & \\ & \ddots & \ddots \\ & -\xi & \eta \end{bmatrix} + \xi \det \begin{bmatrix} & s_p & \\ \hline -\xi & \eta & \\ & \ddots & \ddots \\ & -\xi & \eta \end{bmatrix}$$

and for $p = n$,

$$\psi_n(\eta,\xi) = \eta^m + \xi \det \begin{bmatrix} d_{n,m-1} & d_{n,m-2} & \cdots & d_{n1} & d_{n0} \\ -\xi & \eta & & & \\ & \ddots & \ddots & & \\ & & & -\xi & \eta \end{bmatrix}$$

$$\triangleq \eta^m + \xi \det \begin{bmatrix} & s_n & \\ \hline -\xi & \eta & \\ & \ddots & \ddots \\ & -\xi & \eta \end{bmatrix}$$

independently on the values of the real parameters f_{ij}.

Then the matrices A_1, A_2 defined as

$$A_1 = \left[\begin{array}{ccccc|c} 0 & 0 & \cdots & & 0 & 0 \\ -1 & & & & & r_1 \\ & -1 & & & & r_2 \\ & & -1 & & & r_3 \\ & & & \ddots & & \vdots \\ & & & & -1 & r_n \\ \hline & & 0 & & & 0 \end{array}\right]$$

$$A_2 = \left[\begin{array}{c|ccc} & s_0 & & \\ & s_1 & & \\ 0 & s_2 & & \\ & s_3 & & \\ & \vdots & & \\ & s_n & & \\ \hline & -1 & & 0 \\ 0 & & \ddots & \vdots \\ & & -1 & 0 \end{array}\right]$$

satisfy (8) and have $n(m+1)$ free real parameters. These parameters and the $3(n+m)$ entries of B_1, B_2, C may be used to fit the right hand term in (9) (i.e. the numerator of the transfer function).

B) If in (7) cancellations occur which reduce the degree in η of the denominator, realizations of dimension lower than $n+m$ are possible and their minimal dimension may depend on the field where we consider the entries of A_1, A_2, B_1, B_2, C.

Example. Consider the transfer function

$$\frac{z_1^{-2} z_2^{-1} + z_1^{-1} z_2^{-2}}{z_1^{-2} z_2^{-2} - z_1^{-2} - z_2^{-2}} = \frac{\eta(\xi+1)}{\eta^2 - (\xi^2+1)}$$

It admits realizations of dimension 2 over C but the lowest dimension over R is 3 [6,7].

A further aspect which differentiates the 2-D case from the 1-D is the following. The 1-D minimal realizations of the same transfer function are algebraically equivalent, i.e. unique modulo a change of basis in the state space, while there exist 2-D minimal realizations of the same filter which are not algebraically equivalent.

Example

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$\bar{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

are minimal realizations of the same transfer function, but they are not algebraically equivalent.

REMARKS ON COMMUTATIVE REALIZATIONS

In general, matrices A_1 and A_2 which appear in the realization of a 2-D transfer function $W(z_1,z_2)$ do not commute, i.e. $A_1A_2 \neq A_2A_1$.

The possibility of realizing a transfer function by using pairs of commutative matrices imposes strong conditions on the structure of the denominator of $W(z_1,z_2)$. In fact, assume that A_1 and A_2 satisfy $A_1A_2 = A_2A_1$. Then there exists a similarity transformation over C that reduces A_1 and A_2 simultaneously to lower (upper) triangular form [8] . Hence the denominator of $W(z_1,z_2)$, which is a factor of

$$\det(I - A_1 z_1 - A_2 z_2)$$

splits into linear factors as

$$\prod_i (1 - a_i^{(1)} z_1 - a_i^{(2)} z_2) \qquad (11)$$

Clearly we have the same factorization of the denominator if we refer to the state model introduced in [9]:

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h+1,k) + A_2 x(h,k+1) + Bu(h+1,k+1) \\ y(h,k) &= C\, x(h,k) \end{aligned} \qquad (12)$$

where $A_1A_2 = A_2A_1$. In general, the commutative relation $A_1A_2 = A_2A_1$ does not imply that the transfer function of (5) and (12) is a recognizable power series (contrary to what is stated in [9]), unless all factors in (11) are polynomials in one indeterminate.

The separability of the denominator of the transfer function is a characteristic property of the following model [10]:

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h-1,k) + A_2 x(h_1 k+1) - A_1 A_2 x(h,k) + Bu(h,k) \\ y(h,k) &= C\, x(h,k) \end{aligned}$$

where $A_1A_2 = A_2A_1$. In fact the denominator of the transfer function

$$W(z_1,z_2) = C(I-A_1z_1)^{-1}(I-A_2z_2)^{-1}Bz_1z_2$$

is the product $\det(I-A_1z_1)\det(I-A_2z_2)$. In this case the transfer function is a recognizable power series.

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