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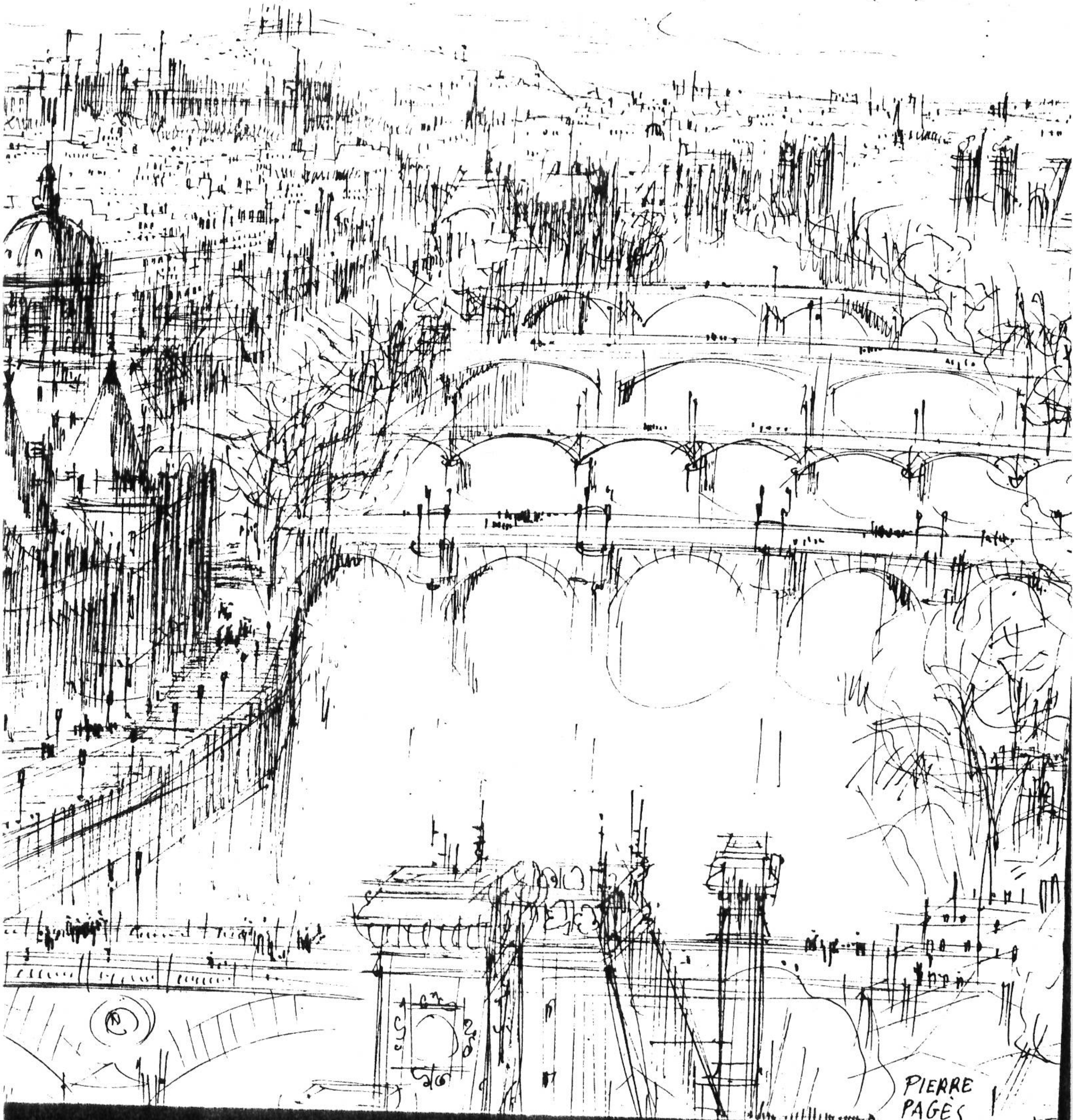

STATE SPACE APPROACH TO TWO DIMENSIONAL FILTERS

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ABSTRACT

In this paper state space models of 2-D filters are introduced as a generalization of 2-D recursive difference equations. The dimension of the recursion and of the state representation are compared and related to structural properties of the state models.

2-D RECURSIVE EQUATIONS AND STATE SPACE MODELS

This paper discusses some aspects of the state space models of 2-D filters (1-3) which are connected with the problem of constructing minimal realizations.

In the 1-D case, reachability and observability are crucial in this framework, and we shall analyze to what extent the corresponding 2-D properties provide an answer to the problem of reducing, and possibly minimizing the dimension of the state space.

Some important aspects of reachability and observability, whose interest is more specific to System Theory (4-6), and other structural properties, as internal stability (7-9), will be omitted in our analysis since their connections with the dimension of the realizations seem to be quite marginal.

As in the 1-D case, the linear processing of two dimensional data can be represented by a convolutional operation or, when the transfer function is rational, by a recursive algorithm.

Let K be any field and $u(h,k) \in K$ and $y(h,k) \in K$ be the input and output signal values in $(h,k) \in Z \times Z$. The convolutional operation is the following

$$y(h,k) = \sum_{i,j=-\infty}^{+\infty} w(h-i,k-j)u(i,j)$$

Here $w(i,j)$ is the unit sample response, i.e. the response of the system to the input whose values are 1 in (0,0) and 0 elsewhere.
When the series

$$w(z_1,z_2) = \sum_{i,j} w(i,j) z_1^i z_2^i$$

is proper rational:

$$w(z_1,z_2) = \sum_{o}^{q_1}{}_i \sum_{o}^{q_2}{}_j b_{ij} z_1^i z_2^j \Big/ \sum_{o}^{p_1}{}_i \sum_{o}^{p_2}{}_j a_{ij} z_1^i z_2^j \,, \quad a_{00}=1 \tag{1}$$

we obtain a 2-D (partial) difference equation relating y and u which provides the output values recursively:

$$y(h,k) = -\sum_{\substack{i=o \\ i+j>0}}^{p_1} \sum_{j=o}^{p_2} a_{ij} y(h-i,k-j) + \sum_{i=o}^{q_1} \sum_{j=o}^{q_2} b_{ij} u(h-i,k-j) \tag{2}$$

We can see some conceptual as well as mathematical key differences that arise in the 2-D case with respect to the 1-D situation:

1. once we have computed $y(h,k)$ the structure of the recursive equation does not give any direction how to select univocally the point in $Z \times Z$ where we have to calculate the "next" output;
2. the values $y(h,k)$, $u(h,k)$ and the input and output data used to calculate the output value in (h,k) are not sufficient to compute the output value in any point of $Z \times Z$ which is not already involved in the recursive equation. For instance, the computation of $y(h+1,k)$ requires $u(h+1,k-j)$ and $y(h+1,k-i)$ $j=0,1\ldots q_2$, $i=0,1\ldots p_2$.

Both of the facts above are intrisically connected with the partial order in $Z \times Z$ that has been implicitly assumed in (2), i.e.

$$(h,k) \le (i,j) \Leftrightarrow h \le i, \quad k \le j$$

This leads to a notion of "future" and "past" which is deeply different from the 1-D case, where the 1-D recursion is provided by the well known difference equation:

$$y(k) = -\sum_{i=1}^{n} a_i y(k-i) + \sum_{i=o}^{m} b_i u(k-i) \tag{3}$$

Here the index k has the interpretation of time.

The recursion structure exhibited by (2) and (3) can be directly exploited in both cases to introduce a state representation.

In fact, starting from (3), let define a state vector at time k as the vector whose elements are the n output values and the m input values preceding k.

Thus the updating equations are given by

$$x(k+1) = A\,x(k) + Bu(k)$$
$$y(k) = C\,x(k) + Du(k) \tag{4}$$

and the state space has dimension n+m.

As it is known, lower dimension state-space models, which realize the i/o map given by (3), exist and n gives their minimal dimension when the polynomials $\Sigma_i a_i z^i$ and $\Sigma_i b_i z^i$ are coprime. Moreover there are linear, finite algorithms based on reachable and observable canonical forms, which give such minimal order models.

Similarly, starting from (2), we can obtain a state representation by assuming as a state vector in (h,k) the vector whose elements are the input and output values in the right hand term of (2) excepting u(h,k), u(h-1,k) and u(h,k-1). With this definition of state, the updating equation is given by the following first order vector difference equation:

$$x(h+1,k+1) = A_1 x(h+1,k) + A_2(h,k+1) + B_1 u(h+1,k) + B_2 u(h,k+1) \tag{5}$$
$$y(h,k) = Cx(h,k) + Du(h,k)$$

So doing, the dimension of the state space is in the order of $p_1p_2+q_1q_2$. It is worthwhile to notice that the two state vectors x(h+1,k) and x(h,k+1) are not mutually independent since most of the input and output values which are elements of x(h+1,k) are elements of x(h,k+1) too. These values do not take up the same indexes as vector components in both states but the indexes are mutually related.

A further obvious indication that the dimension of the state vector defined above is exceedingly high is provided by the number of initial data needed to calculate the output values on a segment $\mathcal{S}=\{(-i,i),\ i=1,2,..,N\}$.

Actually, when we solve equation (2) directly we need the output and input values in two strips containing $N(p_1+p_2)+p_1p_2$ and $N(q_1+q_2+1)+q_1q_2$ points respectively. If we use the state model we have to take into account the local states x(-i-1,i), i=0,1,..,N and the input values u(-i-1,i), i=0,1,..,N and u(-i,i), i=1,..,N.

Thus the required storage in terms of local states is in the order of $N(p_1+p_2)+(N+1)p_1p_2+N(q_1+q_2+1)+(N+1)q_1q_2$.

The increase of the storage in the state space model is the price paid when we abandon the parrial order in initial data needed in (2).

Let us remark that in this case even more heavily than before, input and output values in the same points of Z x Z are components of several different state vectors. All this clearly indicates that there exist state models of considerably lower dimension.

In the following we shall consider the connections between dimension and structural properties of the state and discuss how to reduce the dimension of the state model.

STRUCTURAL PROPERTIES AND MINIMALITY

First, consider the 2D transfer function as a quotient of two coprime polynomials in $K[z_1^{-1},z_2^{-1}]$:

$$W(z_1,z_2) = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} z_1^{-i} z_2^{-j} \Big/ \sum_{i=0}^{n}\sum_{j=0}^{m} d_{ij} z_1^{-i} z_2^{-j}, \tag{6}$$

$$c_{n,m}=0, \quad d_{nm}=1$$

In the corresponding formal power series

$$W(z_1,z_2) = \Sigma_{i+j>0}\, w(i,j) z_1^i z_2^j$$

we have w(0,0)=0. This clearly descends from the proper causality assumption, which leads to state space models (5) with D=0.

Let us introduce in (6) the following change of variables

$$\eta = z_1^{-1}, \qquad \xi = z_1^{-1} z_2$$

so we have

$$\sum_{h=0}^{n+m-1} \eta^h \sum_{j=0}^{m} c_{h-j,j}\,\xi^{m-j} \Big/ \sum_{h=0}^{n+m} \eta^h \sum_{j=0}^{m} d_{h-j,j}\,\xi^{m-j} = \sum_{h=0}^{n+m-1} \eta^h c_h(\xi) \Big/ \sum_{h=0}^{n+m} \eta^h d_h(\xi) \tag{7}$$

where the supports of numerator and denominator are in the dashed region of Fig. 1.

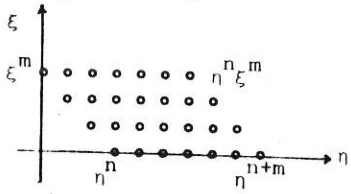

Fig. 1

The coprimeness in $K[z_1^{-1},z_2^{-1}]$ of numerator and denominator of (6) does not imply the coprimeness in $K[\eta,\xi]$ of numerator and denominator of (7). The loss of coprimeness takes place when, for some $\nu>0$, all homogeneous polynomials of degree less than ν vanish in the numerator and denominator of (6). In this case in (7) $c_o(\xi),\dots,c_{\nu-1}(\xi)$ and $d_o(\xi),\dots,d_{\nu-1}(\xi)$ are zero and η^ν is a common factor of the numerator and denominator.

If in (7) there are no cancellations which reduce the degree in η of the denominators, n+m is a lower bound for the dimension of 2-D systems which realize (7). Infact, since the transfer function of (5) is given by

$$C(\eta I - A_1 - A_2\xi)^{-1}\,(B_1 + B_2\xi),$$

a denominator of degree n+m in η cannot be obtained with dimension lower than $n+m$.

In general it is not known if the lowest bound $n+m$ is always obtainable and if it can be reached using matrices with entries in K or in some finite extension of K.

Whatever solution we may look for, we must face with the following facts:

(i) the transfer functions with unitary numerator are always realizable with dimension $n+m$;
(ii) all transfer functions (7) are realizable with dimension $n+2m$ (or $m+2n$) (10,4);
(iii) if cancellations reducing the denominator degree in η occur in (7), realizations of dimension lower than $n+m$ are possible and their minimal dimension may depend on the field where we consider the entries of A_1, A_2, B_1, B_2, C.

Example. Consider the transfer function

$$\frac{z_1^{-2} z_2^{-1} + z_1^{-1} z_2^{-2}}{z_1^{-2} z_2^{-2} - z_1^{-2} - z_2^{-2}} = \frac{\eta(\xi+1)}{\eta^2 - (\xi^2+1)} \tag{8}$$

It admits realizations of dimension 2 over C but the lowest dimension over R is 3 (11,12).

A further aspect which differentiates the 2-D case from the 1-D is the following. The 1-D minimal realizations of the same transfer function are algebraically equivalent, i.e. unique modulo a change of basis in the state space, while there exist 2-D minimal realizations of the same filter which are not algebraically equivalent.

Example:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$\bar{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

are minimal realizations of the same transfer function, but they are not algebraically equivalent.

A well known technique for obtaining minimal realizations in the 1-D case consists in eliminating the unreachable and unobservable state subspaces. The generalization of this method to the 2-D case encountres from the beginning the difficulty of defining the reachability and observability properties. In fact there exist two types of structural properties which are related to the local state and to the global state, the latter being the set of local states on a separation set of the discrete plane. If we refer to local reachability and observability, the 1-D reduction procedures (13) can be extended to the 2-D case but in general the lower order realizations we obtain are not minimal. On the other side, globally reachable and observable realizations are minimal but some examples exist of transfer functions (see for instance (8)) which do not admit globally reachable and observable realizations (at least over R).

Moreover, as far as we know, there are no systematic procedures which give globally reachable and observable realizations whenever they exist.

Let us associate with the 2-D system (5) the following 1-D system

$$\begin{aligned} w(t+1) &= (A_1+A_2\xi)w(t) + (B_1+B_2\xi)v(t) \\ r(t) &= C\,w(t) \end{aligned} \tag{9}$$

defined over the field $K(\xi)$.

Is is easily verified (6) that global reachability and observability of the 2-D system (5) imply reachability and observability of (9). Nevertheless it is important to notice that if $(A(\xi),B(\xi),C(\xi))$ is a minimal realization over $K(\xi)$ of (7), in general $A(\xi)$, $B(\xi)$ and $C(\xi)$ do not exhibit the structure

$$\begin{aligned} A(\xi) &= A_1 + A_2\xi \\ B(\xi) &= B_1 + B_2\xi \\ C(\xi) &= C \end{aligned} \tag{10}$$

where A_1, A_2, B_1, B_2 and C have entries in K, and it is not always possible to reduce them to the structure (10) by a change of basis induced by an invertible matrix on $K(\xi)$.

Example. Consider again the transfer function (8). The following is a 1-D minimal realization of dimension 2:

$$A(\xi) = \begin{bmatrix} 0 & 1 \\ \xi^2+1 & 0 \end{bmatrix}, \quad B(\xi) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C(\xi) = \begin{bmatrix} 0 & \xi+1 \end{bmatrix}$$

It is strightforward to see that there is no change of basis in $R^2(\xi)$ which reduces the matrices $A(\xi)$, $B(\xi)$, $C(\xi)$ to have structure (10). This agrees with the fact that there are no 2-D systems of dimension 2 which realize (8) over R.

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