SOME ASPECTS OF THE DUALITY THEORY IN 2-D SYSTEMS

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ABSTRACT

The paper presents an application of duality theory to global (zero state) controllability and global reconstructibility of 2-D systems.

The procedure is based on some structural properties of a class of dynamical systems over rings, whose algebraic duals are 2-D systems.

1. INTRODUCTION AND 2-D SYSTEMS REPRESENTATION

The dynamics of a 2-D system is represented by the following updating equations (1):

$$x(h+1,k+1)=A_1 x(h+1,k)+A_2(h,k+1)+B_1 u(h+1,k)+B_2 u(h,k+1),$$

$$y(h,k)=Cx(h,k) \quad . \qquad (1)$$

where the local state x is an n-dimensional vector over a field K, input and output values are scalars and A_1, A_2, B_1, B_2, C are matrices of suitable dimensions with entries in K.

In a previous paper [2] the authors investigated the reachability and observability properties of global states whose supports are the separation sets

$$\mathscr{C}_i = \{(h,k) \in \mathbb{Z}x\mathbb{Z},\ h+k = i\}, \quad i = 0,\pm 1 \ldots$$

In particular, a duality relation between 2-D systems and dynamical systems over the ring of bilateral polynomials $K[\xi,\xi^{-1}]$ led to 2-D global reachability and observability criteria based on observability and reachability criteria of systems over rings.

In this paper the original approach will be further extended to include 2-D global (zero state) controllability and global reconstructibility.

Since the global states are elements of the direct product of the local state spaces on $\mathscr{C}_i$, bilateral Laurent formal power series provide a convenient tool for representing the global state dynamics.

According to this approach, let

$$\mathscr{X}_i = \sum_{j=-\infty}^{+\infty} x(i-j,j)\xi^j \qquad (2)$$

represent the global state on $\mathscr{C}_i$, and

$$\mathscr{U}_i = \sum_{j=-\infty}^{+\infty} u(i-j,j)\xi^j, \quad \mathscr{Y}_i = \sum_{j=-\infty}^{+\infty} y(i-j,j)\xi^j \qquad (3)$$

the restrictions to $\mathscr{C}_i$ of input and output functions. With this notation, input and output functions can be written as

$$u = \sum_{i=h}^{+\infty} \mathscr{U}_i \eta^i, \quad y = \sum_{i=k}^{+\infty} \mathscr{Y}_i \eta^i \qquad (4)$$

where h and k are integers. The set $K_b^n((\xi))$ of (bilateral) Laurent formal power series with coefficients in K^n can be naturally endowed with the structure of a $K[\xi,\xi^{-1}]$-module, where $K[\xi,\xi^{-1}]$ is the subring of $K(\xi)$ generated by K, ξ and ξ^{-1}. As a consequence of the module structure, the global state updating equations

$$\mathscr{X}_{i+1} = (A_1+A_2\xi)\mathscr{X}_i+(B_1+B_2\xi)\mathscr{U}_i, \quad \mathscr{Y}_i = C\mathscr{X}_i \qquad (5)$$

are easily derived from (1).

2. GLOBAL CONTROLLABILITY AND RECONSTRUCTIBILITY

Denote by $\mathscr{R}$ and $\mathscr{O}$ the 2-D global reachability and observability matrices of system (1)

$$\mathscr{R}= \left[(B_1+B_2\xi) \,\vdots\, (A_1+A_2\xi)\ (B_1+B_2\xi) \,\vdots\, \ldots (A_1+A_2\xi)^{n-1}(B_1+B_2\xi)\right],$$

$$\mathscr{O} = \begin{bmatrix} C \\ C(A_1+A_2\xi) \\ \vdots \\ C(A_1+A_2\xi)^{n-1} \end{bmatrix}$$

As it is known [2], global reachability and observability conditions reduce to the invertibility of $\mathscr{R}$ and $\mathscr{O}$ in $K(\xi)^{nxn}$ and in $K[\xi,\xi^{-1}]^{nxn}$ respectively.

<u>Proposition</u> 1. System (1) is globally controllable to zero state if and only if $(A_1+A_2\xi)^n$ factorizes as

$$(A_1+A_2\xi)^n = \mathscr{R}M \, , \qquad (6)$$

for some rational matrix M in $K(\xi)^{nxn}$.

<u>proof</u>. By global controllability definition, for any global state $\mathscr{X}_0$ there is an integer ν and a sequence $\mathscr{U}_0, \mathscr{U}_1 \ldots \mathscr{U}_{\nu-1}$ with elements in $K_b((\xi))$ such that

$$(A_1+A_2\xi)^\nu \mathscr{X}_0 + \sum_{i=0}^{\nu-1} (A_1+A_2\xi)^i(B_1+B_2\xi)\mathscr{U}_i = 0 \qquad (7)$$

Cayley Hamilton theorem over commutative rings gives that $\nu = n$ can be assumed in (7), so that

$$\mathrm{Im}(A_1+A_2\xi)^n \subseteq \mathrm{Im}\,\mathscr{R} \qquad (8)$$

is a necessary and sufficient condition for global controllability. Since $(A_1+A_2\xi)^n$ and $\mathscr{R}$ are $K[\xi,\xi^{-1}]$ module-morphisms from $K_b^n((\xi))$ into $K_b^n((\xi))$, the range spaces in (8) are submodules of $K_b^n((\xi))$.

Let U and V be nxn unimodular polynomial matrices which reduce $\mathscr{R}$ to its Smith canonical form, i.e.

$$U\mathscr{R}V = \mathrm{diag}\{\psi_1 \ldots \psi_t\ 0 \ldots 0\} = \varphi$$

By (8), the following equation in the unknowns $\hat{\mathscr{U}}_0 \ldots \hat{\mathscr{U}}_{n-1}$

$$U(A_1+A_2\xi)^n\mathscr{X} = \varphi[\hat{\mathscr{U}}_0\hat{\mathscr{U}}_1 \ldots \hat{\mathscr{U}}_{n-1}]^T$$

can be solved for any global state $\mathscr{X}$. We therefore have

$$U(A_1+A_2\xi)^n = t\left\{\begin{bmatrix} A_{11}(\xi) & A_{12}(\xi) \\ 0 & 0 \end{bmatrix}\right. = \tag{9}$$

$$= \Phi \operatorname{diag}\{\psi_1^{-1}\ldots\psi_t^{-1}\ 0\ldots0\} \begin{bmatrix} A_{11}(\xi) & A_{12}(\xi) \\ 0 & 0 \end{bmatrix}$$

$$= \Phi\bar{M}$$

where $\bar{M} = \operatorname{diag}\{\psi_1^{-1}\ldots\psi_t^{-1}\ 0\ldots0\}$; $U(A_1+A_2\xi)^n$ is a rational matrix. Left multiplication of (9) by U^{-1} gives

$$(A_1+A_2\xi)^n = U^{-1}\Phi V^{-1}V\bar{M} = \mathcal{R}M$$

with $M = V\bar{M}$.

Conversely, let $(A_1+A_2\xi)^n = \mathcal{R}M$, M in $K(\xi)^{n\times n}$, and let d be a common multiple of the denominators of the entries of M. Then $P=dM$ is a polynomial matrix and

$$(A_1+A_2\xi)^n = \frac{1}{d}\mathcal{R}P$$

The $K[\xi,\xi^{-1}]$-module of global states in divisible [5], namely for any p in $K[\xi,\xi^{-1}]$, the map $\mathcal{X}\mapsto p\mathcal{X}$ of the module into itself is surjective [2, Theorem 1].

Therefore, for any global state $\mathcal{X}$, there exists $\mathcal{X}'$ such that $d\mathcal{X}' = \mathcal{X}$ and

$$(A_1+A_2\xi)^n\mathcal{X} = d(A_1+A_2\xi)^n\mathcal{X}' = \mathcal{R}P\mathcal{X}'$$

This proves that the equation

$$(A_1+A_2\xi)^n\mathcal{X} = \mathcal{R}\left[\mathcal{U}_0, \mathcal{U}_1\ldots\mathcal{U}_{n-1}\right]^T$$

can be solved for any $\mathcal{X}$ by assuming

$$\left[\mathcal{U}_0\mathcal{U}_1\ldots\mathcal{U}_{n-1}\right]^T = P\mathcal{X}'$$

□

Proposition 2. System (1) is globally reconstructible if and only if $(A_1+A_2\xi)^n$ factorizes as

$$(A_1+A_2\xi)^n = T\mathcal{O} \tag{10}$$

for some polynomial matrix T in $K[\xi,\xi^{-1}]^{n\times n}$.

Proof. By global reconstructibility definition, there is an integer ν such that the output sequence $\mathcal{Y}_0, \mathcal{Y}_1, \ldots \mathcal{Y}_\nu$ of the unforced 2-D system uniquely determines the final state $\mathcal{X}_\nu$.

By Cayley Hamilton theorem, this amounts to require that

$$\ker(A_1+A_2\xi)^n \supseteq \ker\mathcal{O} \tag{11}$$

where $(A_1+A_2\xi)^n$ and $\mathcal{O}$ are $K[\xi,\xi^{-1}]$-module morphisms from $K_b^n((\xi))$ into $K_b^n((\xi))$.

Let U and V be $n\times n$ unimodular polynomial matrices which reduce $\mathcal{O}$ to its Smith canonical form:

$$U\mathcal{O}V = \operatorname{diag}\{\varphi_1\varphi_2\ldots\varphi_s\ 0\ldots0\} = \Phi$$

Note that $\mathcal{O}\mathcal{X} = 0$ is equivalent to $\Phi(V^{-1}\mathcal{X}) = 0$ and $(A_1+A_2\xi)^n\mathcal{X} = 0$ is equivalent to $U(A_1+A_2\xi)^nV(V^{-1}\mathcal{X}) = 0$. Then, by (11),

$$\Phi(V^{-1}\mathcal{X}) = 0 \tag{12}$$

implies

$$U(A_1+A_2\xi)^nV(V^{-1}\mathcal{X}) = 0 \tag{13}$$

Since there are no constraints on the last $n-s$ components of the vectors $V^{-1}\mathcal{X}$ which satisfy (12), by (13) $U(A_1+A_2\xi)^nV$ has the following structure

$$U(A_1+A_2\xi)^nV = s\left\{\begin{bmatrix} A_{11}(\xi) & 0 \\ A_{21}(\xi) & 0 \end{bmatrix}\right. \tag{14}$$

Denote by e_i the i-th column of the $n\times n$ identity matrix, and consider any s_i in $K_b((\xi))$ which satisfies $s_i\varphi_i = 0$. Thus we have

$$\begin{bmatrix} A_{11}(\xi) & 0 \\ A_{21}(\xi) & 0 \end{bmatrix} s_ie_i = 0 \qquad i = 1,2,\ldots s$$

Hence each element of the i-th column in (14) is multiple of φ_i in the ring $K[\xi,\xi^{-1}]$ and there exists $\bar{T}$ in $K[\xi,\xi^{-1}]^{n\times n}$ such that

$$\begin{bmatrix} A_{11}(\xi) & 0 \\ A_{21}(\xi) & 0 \end{bmatrix} = \bar{T}\Phi$$

and

$$(A_1+A_2\xi)^n = U^{-1}\bar{T}\Phi V^{-1} = U^{-1}\bar{T}U\mathcal{O}VV^{-1}$$

$$= U^{-1}\bar{T}U\mathcal{O} = T\mathcal{O}$$

where $T = U^{-1}\bar{T}U$ is in $K[\xi,\xi^{-1}]^{n\times n}$.

Conversely, (11) is an immediate consequence of (10). □

In discrete time 1-D system theory zero state controllability and reconstructibility are weaker properties than reachability and observability.

The situation for 2-D systems is very similar. In fact if (1) is globally reachable, $\mathcal{R}^{-1}$ exists in $K(\xi)^{n\times n}$ and condition (6) can be fulfilled by $M = \mathcal{R}^{-1}(A_1+A_2\xi)^n$. Hence global reachability implies global controllability.

Also, if (1) is globally observable, $\mathcal{O}^{-1}$ exists in $K[\xi,\xi^{-1}]^{n\times n}$ and condition (10) is satisfied by $T=(A_1+A_2\xi)^n\mathcal{O}^{-1}$.

Thus global observability implies global reconstructibility.

3. SYSTEMS OVER POLYNOMIAL RINGS AND DUALITY

In this section, 2-D global controllability and reconstructibility will be shown to be dual of reconstructibility and controllability of systems defined over the ring $K[\xi,\xi^{-1}]$. This provides different proofs of propositions 1 and 2, based on algebraic duality arguments.

Let us first introduce the system

$$w(t+1) = F(\xi)w(t)+G(\xi)v(t), \quad z(t) = H(\xi)w(t), \tag{15}$$

defined over the ring of polynomials $K[\xi,\xi^{-1}]$. Here the input set is the ring $K[\xi,\xi^{-1}][\eta^{-1}]$, the output set is the ring $K[\xi,\xi^{-1}][[\eta]]$, the states are elements of the free module $K[\xi,\xi^{-1}]^n$ and the matrices $F(\xi), G(\xi), H(\xi)$ have entries in $K[\xi,\xi^{-1}]$.

Let $\mathcal{R}_p$ and $\mathcal{O}_p$ denote the reachability and observability matrices of (14)

$$\mathcal{R}_p = \left[G(\xi)\quad F(\xi)G(\xi)\ \ldots F^{n-1}(\xi)G(\xi)\right]$$

$$\mathcal{O}_p = \begin{bmatrix} H(\xi) \\ H(\xi)F(\xi) \\ \vdots \\ H(\xi)F^{n-1}(\xi) \end{bmatrix}$$

Proposition 3. System (15) is (zero state) controllable if and only if $F^n(\xi)$ factorizes as

$$F^n(\xi) = \mathcal{R}_p(\xi)P(\xi) \tag{16}$$

with $P(\xi)$ in $K[\xi,\xi^{-1}]^{n\times n}$ and is reconstructible if and only if $F^n(\xi)$ factorizes as

$$F^n(\xi) = L(\xi)\mathcal{O}_p(\xi) \tag{17}$$

with $L(\xi)$ in $K(\xi)^{n\times n}$.

Proof. Zero state controllability is equivalent to the $K[\xi,\xi^{-1}]$-module inclusion

$$\operatorname{Im} F^n(\xi) \subseteq \operatorname{Im} \mathcal{R}_p(\xi) \tag{18}$$

which corresponds [3] to the existence of the factorization (16).

State reconstructibility of system (15) is expressed by the $K[\xi,\xi^{-1}]$-module inclusion

$$\ker F^n(\xi) \supseteq \ker \mathcal{O}_p(\xi) \tag{18'}$$

This is equivalent to the corresponding inclusion of $K(\xi)$ spaces and to factorization (16) [4]. □

Proposition 3 shows that reconstructibility and controllability of systems over the ring $K[\xi,\xi^{-1}]$ are expressed by factorization properties which correspond to 2-D global controllability and reconstructibility conditions given by (6) and (7). This fact is formally explained by viewing 2-D systems as dual of systems over the ring $K[\xi,\xi^{-1}]$. Let us briefly recall from [2] the main steps in the construction of the dual system of (14).

1. The global state space of 2-D system (1), namely the space $K_b^n((\xi))$, is the algebraic dual of $K^n[\xi,\xi^{-1}]$, which is the state space of system (14):

$$(K^n[\xi,\xi^{-1}])^* = K_b^n((\xi))$$

2. The output space $K_b((\xi))[[\eta]]$ of (1) is the algebraic dual of the input space $\eta^{-1}K_b[\xi,\xi^{-1}][\eta^{-1}]$ of system (14).

Similarly the space of 2-D inputs whose support is in $\ldots\cup\mathcal{L}_{-1}$, i.e. whose elements are represented by series $\eta^{-1}K_b((\xi))[\eta^{-1}]_n$, is the algebraic dual of the space $K[\xi,\xi^{-1}][\eta]_n$ of output restrictions to $[0,n-1]$ of system (14)

$$(K[\xi,\xi^{-1}][\eta]_n)^* = \eta^{-1}K_b((\xi))\,[\eta^{-1}]_n$$

If we consider (14) and assume

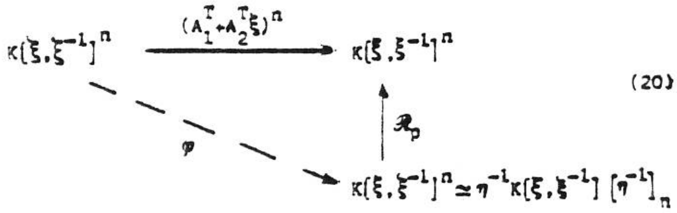
$$F(\xi) = A_1^T + A_2^T\xi\,, \quad G(\xi) = C^T, \quad H(\xi) = B_1^T + B_2^T\xi \tag{19}$$

the global reachability and observability maps of 2-D system (1), given by the polynomial matrices $\mathcal{R}$ and $\mathcal{O}$, are the algebraic duals of the observability and reachability maps of system (19), given by the polynomial matrices $\mathcal{O}_p$ and $\mathcal{R}_p$. In fact $\mathcal{R}_p = \mathcal{O}^T$ and $\mathcal{O}_p = \mathcal{R}^T$.

By projectivity of the module $K[\xi,\xi^{-1}]^n$, the controllability condition of (19),

$$\operatorname{Im}(A_1^T + A_2^T\xi)^n \subseteq \operatorname{Im}\mathcal{R}_p$$

is equivalent to the existence of a $K[\xi,\xi^{-1}]$-module morphism φ which makes the following diagram



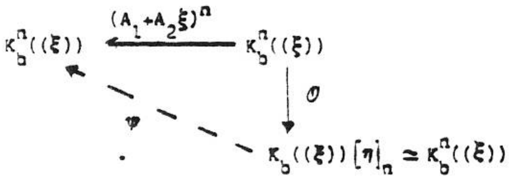
(20)

commutative.

On the other side by the injectivity of the $K[\xi,\xi^{-1}]$ module $K_b^n((\xi))$ the reconstructibility condition of (1),

$$\ker(A_1 + A_2\xi)^n \supseteq \ker\mathcal{O} \tag{21}$$

is equivalent to the existence of a $K[\xi,\xi^{-1}]$-module morphism ψ which makes the following diagram



(22)

commutative.

We therefore have the following result

Proposition 4. Global reconstructibility of the 2-D system (1) is equivalent to controllability of the system (19) defined over $K[\xi,\xi^{-1}]$.

Proof. Assume first commutativity of the diagram (20). Since each of its maps admits a dual map, we have

$$(A_1^T + A_2^T\xi)^{n*} = \varphi^*\mathcal{R}_p^*$$

$$(A_1 + A_2\xi)^n = \varphi^*\mathcal{O}$$

which guarantees the commutativity of (22) with $\psi = \varphi^*$.

Conversely, assume reconstructibility of (1), i.e. the existence of ψ which makes (22) commutative. Then, by taking the orthogonal complements of (21)

$$\ker(A_1 + A_2\xi)^{n\perp} \subseteq (\ker\mathcal{O})^\perp$$

and

$$(\ker(A_1^T + A_2^T\xi)^{n*})^\perp \subseteq (\ker\mathcal{R}_p^*)^\perp$$

Hence by the properties of the spaces of linear functionals we have

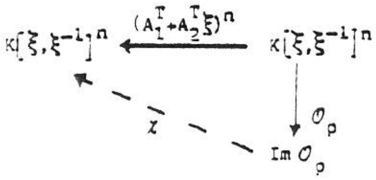
$$\operatorname{Im}(A_1^T + A_2^T\xi)^n \subseteq \operatorname{Im}\mathcal{R}_p$$

Then there exists φ which makes (20) commutative, and in (22) ψ can be assumed as φ^*. □

The reconstructibility condition of system (19)

$$\ker(A_1^T + A_2^T\xi)^n \supseteq \ker\mathcal{O}_p \tag{23}$$

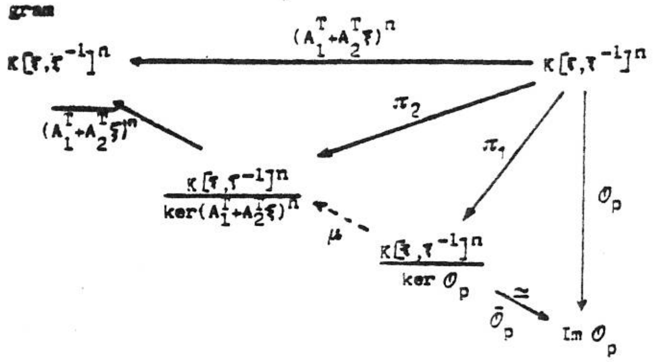
is equivalent to the existence of a $K[\xi,\xi^{-1}]$-module morphism χ which makes the following diagram



(24)

commutative.

In fact, (23) is an obvious consequence of the existence of χ. Conversely (23) implies the existence of a $K[\xi,\xi^{-1}]$-module morphism μ which makes the following diagram

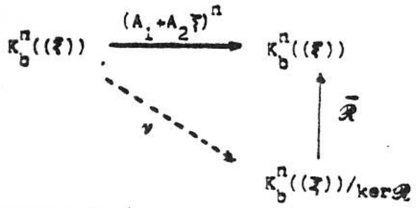


commutative, and we can assume $\chi = (A_1^T+A_2^T\xi)^n \circ \mu \circ \bar{\mathcal{O}}_p$.

The global controllability condition of system (1)

$$\mathrm{Im}(A_1+A_2\xi)^n \subseteq \mathrm{Im}\,\mathcal{R} \qquad (25)$$

is equivalent to the existence of a $K[\xi,\xi^{-1}]$-module morphism ν which makes the following diagram



(26)

commutative.

In fact (25) is an easy consequence of the existence of ν. Viceversa, assuming $\bar{\mathcal{R}}^{-1}$ as the inverse of $\bar{\mathcal{R}}$ on $\mathrm{Im}\,\bar{\mathcal{R}}$, the inclusion (25) allows to define $\nu = \bar{\mathcal{R}}^{-1} \circ (A_1+A_2\xi)^n$ which makes the diagram (26) commutative.

To prove that diagrams (24) and (26) are dual it is first necessary to show that $K_b^n((\xi))/\ker\mathcal{R}$ can be viewed as the algebraic dual of $\mathrm{Im}\,\mathcal{O}_p$. Let s be any element in $K_b^n((\xi))$ and denote by $[s]$ its equivalence class modulo $\ker \mathcal{R}$. Then for any q in $K[\xi,\xi^{-1}]^n$, the relation (*)

$$<\mathcal{O} q,[s]> = (q^T\mathcal{O}_p^T\, s, \xi^0)$$

defines a linear functional on $\mathcal{O}_p K[\xi,\xi^{-1}]^n$. Viceversa, given a linear functional $f: \mathcal{O}_p K[\xi,\xi^{-1}]^n \to K$, there exists a bilateral power series s in $K_b^n((\xi))$ such that

$$f(\mathcal{O}_p q) = <\mathcal{O}_p q, [s]>$$

for any q in $K[\xi,\xi^{-1}]^n$, and $[s]$ is uniquely determined.

Assume that the map χ in (24) exists, and consider an irreducible matrix fraction representation of it given by NQ^{-1}. Then $NQ^{-1}\mathcal{O}_p$ is a polynomial matrix and $\mathcal{O}_p$ factorizes as

$$\mathcal{O}_p = Q\, H \qquad (27)$$

for some H in $K[\xi,\xi^{-1}]^{n\times n}$. (27) follows from the Bézout identity $AN+BQ = I_n$ by premultiplication by Q and postmultiplication by $Q^{-1}\mathcal{O}_p$.

For any s in $K_b^n((\xi))$, the bilateral series g which solve the equation

$$N^T s = Q^T g \qquad (28)$$

are equivalent modulo $\ker\mathcal{R}$, and the map

$$\nu : K_n^b((\xi)) \to K_n^b((\xi))/\ker\mathcal{R} : s \mapsto g$$

is a well defined $K[\xi,\xi^{-1}]$-module morphism.

ν is the dual map of χ. In fact

$$<\chi\mathcal{O}_p q,s> = (q^T\mathcal{O}_p^T(Q^{-1})^T N^T s, \xi^0)$$
$$= (q^T H^T Q^T (Q^{-1})^T N^T s, \xi^0) = (q^T H^T N^T s, \xi^0)$$

is equal to

$$<\mathcal{O}_p q, \nu[s]> = <\mathcal{O}_p q, [g]> = (q^T\mathcal{O}_p^T g, \xi^0)$$
$$= (q^T H^T Q^T g, \xi^0) = (q^T H^T N^T s, \xi^0)$$

for any s in $K_b^n((\xi))$ and q in $K[\xi,\xi^{-1}]^n$.

Proposition 5. Global controllability of the 2-D system (1) is equivalent to reconstructibility of the system (19).

The proof is similar to that we gave for Proposition 4.

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(*) As commonly used in formal power series theory, (s,ξ^i) denotes the coefficient of ξ^i in the series s.