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# ON COMMUTATIVE REALIZATIONS OF 2D TRANSFER FUNCTIONS

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## ABSTRACT

This paper is concerned with structural properties of transfer functions in two variables, which can be realized by the class of 2D systems whose state updating matrices commute.

The analysis of these properties is done by relating commutative realizations with noncommutative exchangeable power series.

## INTRODUCTION

The following communication deals with structural properties of 2D systems whose state updating matrices are assumed to be commutative.

Our aim is to characterize the transfer functions which can be realized by this class at systems and to investigate what kind of problems are raised by their use in the realization problem.

As it is well known, commutative matrices have been first considered in the Attasi's model /1/

$$x(h+1,\ k+1) = A_1\ x(h+1,\ k) + A_2\ x(h,\ k+1) - A_1\ A_2\ x(h,\ k) + B\ u(h,\ k) \quad (1)$$
$$y(h,\ k) = C\ x(h,\ k)$$

where the commutative assumption $A_1\ A_2 = A_2\ A_1$ is made. The transfer functions realizable by model (1) are the so called (causal) separable functions, that is they can be written as $z_1\ z_2\ n(z_1,\ z_2)/p(z_1)\ q(z_2)$, where $n$ is in $K[z_1,\ z_2]$, $p$ in $K[z_1]$ and $q$ in $K[z_2]$. The converse is also true, in the sense that any (causal) separable transfer function is realizable in the class of Attasi's models.

As we can see from (1) if we want to compute the local state at some point $(h+1,\ k+1)$, we need the local states at $(h+1,\ k)$, $(h,\ k)$ and $(h,\ k+1)$, which means that (1) gives a second order state updating.

A wider class of 2D systems includes systems having first order state updating equations, where the state at $(h+1,\ k+1)$ depends only on the local states at $(h+1,\ k)$ and at $(h,\ k+1)$. The general structure of these models is given by the following equations /2/

$$x(h+1,\ k+1) = A_1\ x(h+1,\ k) + A_2\ x(h,\ k+1) + B_1\ u(h+1,\ k) + B_2\ u(h,\ k+1) \quad (2)$$
$$y(h,k) = C\ x(h,\ k)$$

and we shall consider the subclass of these systems characterized by the commutativity assumption $A_1\ A_2 = A_2\ A_1$.

In order to make our analysis simpler, we shall also assume that either $B_1$ or $B_2$ is the zero vector. So doing the analysis applies also to models /3/ with structure

$$x(h+1,\ k+1) = A_1\ x(h+1,\ k) + A_2\ x(h,\ k+1) + B\ u(h,\ k) \quad (3)$$
$$y(h,\ k) = C\ x(h,\ k)$$

and /4/ with structure

$$x(h+1,\ k+1) = A_1\ x(h+1,k) + A_2\ x(h,k+1) + B\ u(h+1,k+1) \quad (4)$$
$$y(h,\ k) = C\ x(h,\ k)$$

If we don't take into account the multiplicative factors $z_1$, $z_2$ or $z_1\ z_2$, which are unessential to our discussion, the structure of the transfer function of systems (2) (with $B_1$ or $B_2 = 0$), (3) and (4) reduce to the following form

$$s = C(I - A_1\ z_1 - A_2\ z_2)^{-1}\ B \quad (5)$$

Since the denominator for the transfer function s is a divisor of the polynomial

$$d(z_1,\ z_2) \triangleq \det\ (I - A_1\ z_1 - A_2\ z_2)$$

the constraints imposed by the commutativity hypothesis $A_1\ A_2 = A_2\ A_1$ on $d(z_1,\ z_2)$ translate into conditions on the singularities of the transfer function s.

An interesting factorization property of $d(z_1,\ z_2)$ is based on the following classical result.

*Theorem 1* /5/ Let $A_1$, $A_2$ be commutative matrices with entries in the complex field. Then there exists a similarity transformation that reduces both matrices simultaneously to upper (or lower) triangular form.

The polynomial $d(z_1,\ z_2)$ is invariant under similarity transformations, so we can refer to upper (lower) triangular form of $A_1$ and $A_2$, and we have that $d(z_1,\ z_2)$ and the denominator of s, factor completely in the complex field into linear factors:

$$d(z_1,\ z_2) = \prod_i (1 - a_i^{(1)} z_1 - a_i^{(2)} z_2) \quad (6)$$

The splitting condition (6) is a necessary condition to be satisfied by the denominator of the transfer functions which admit commutative realizations, i.e. realizations in which matrices $A_1$ and $A_2$ commute.

Actually it is not sufficient, since it does not give any constraint on the numerator, which plays an important role in this context.

In order to discuss this fact and the connections between commutative and noncommutative realizations we recall some properties of non commutative power series /6/.

## SOME PROPERTIES OF COMMUTATIVE AND NONCOMMUTATIVE POWER SERIES

Let K be a generic field. The algebra of formal power series in the noncommuting variables $\xi_1$ and $\xi_2$ with coefficients in K is denoted by $K\langle\langle \xi_1,\ \xi_2 \rangle\rangle$, and a generic element $\sigma$ of this algebra is written as

$$\sigma = \sum_{w\ \{\xi_1,\ \xi_2\}^*} (\sigma, w)\, w$$

where $\{\xi_1, \xi_2\}^*$ is the free monoid generated by $\xi_1$ and $\xi_2$ and $(\sigma, w)$ in K is the coefficient of w in the series $\sigma$.

Denote now by $K[[z_1,\ z_2]]$ the algebra of formal power series in the commuting variables $z_1$ and $z_2$ and define the algebra morphism $\emptyset$: $K\langle\langle \xi_1, \xi_2 \rangle\rangle \rightarrow K[[z_1, z_2]]$ by the assignments $\emptyset(\xi_1) = z_1$, $\emptyset(\xi_2) = z_2$ and $\emptyset(k) = k$ for any k in K.

The series $\emptyset(\sigma)$ is the commutative image of $\sigma$.

A series $\sigma$ in $K\langle\langle \xi_1, \xi_2 \rangle\rangle$ is called exchangeable if the words which have the same commutative image have the same coefficient in $\sigma$.

A series $\sigma$ in $K\langle\langle \xi_1, \xi_2 \rangle\rangle$ is called rational if there exist a positive integer n and matrices $A_1$, $A_2$ in $K^{n\times n}$, B in $K^{n\times 1}$, C in $K^{1\times n}$ such that

$$\sigma = C \sum_{k=0}^{\infty} (A_1\xi_1 + A_2\xi_2)^k B = C(I - A_1\xi_1 - A_2\xi_2)^{-1} B \qquad (7)$$

A 4-tuple $(A_1, A_2, B_1, C)$ is called a <u>representation</u> of $\sigma$ if (7) holds.

The following proposition /6/ gives equivalent characterizations of the set of exchangeable series which are also rational.

*Proposition 2.* Let $\sigma$ be in $K \ll \xi_1,\xi_2 \gg$. Then the following facts are equivalent:

i) $\sigma$ is rational and exchangeable
ii) $\sigma$ is a linear combination of series with the following structure

$$p(\xi_1)\ q(\xi_1)^{-1} \sqcup\!\sqcup\ r(\xi_1)\ t(\xi_1)^{-1} \qquad (8)$$

where p, q, r, t are polynomials and $\sqcup\!\sqcup$ denotes the shuffle product.
iii) there exists a representation $(A_1, A_2, B, C)$ of $\sigma$ with $A_1 A_2 = A_2 A_1$, that is

$$(\sigma, w) = C\, A_1^{|w|_1} A_2^{|w|_2} B, \quad \forall\, w \in \{\xi_1,\xi_2\}^*$$

where $|w|_i$ denotes the number of $\xi_i$ in $w, i=1,2$.

A further characterization of exchangeable rational series is given in terms of separable rational functions.

*Proposition 3.* Let $\sigma \in K \ll \xi_1,\xi_2 \gg$ be exchangeable and define the map $\breve{\emptyset}$ by the assignment

$$\breve{\emptyset} : \sum_w (\sigma,w)\, w \rightarrow \sum_{i,j=0}^{\infty} (\sigma,\xi_1^i\, \xi_2^j)\, z_1^i\, z_2^j$$

Then $\sigma$ is rational if and only if $\emptyset(\sigma)$ is (the power series expansion of) a separable rational function.

## COMMUTATIVE REALIZATION

Let's now go back to the problem of the existence of commutative realizations. Consider a rational transfer function s and denote by M the set of the 2D systems $\Sigma = (A_1, A_2, B, C)$ which realize s. Denote by N the set of noncommutative rational power series whose commutative image is s.

Then any system $\Sigma = (A_1, A_2, B, C)$ in M is associated with a representation of a noncommutative series $\sigma$ in N, i.e. the series

$$\sigma = C\,(I - A_1\,\xi_1 - A_2\,\xi_2)^{-1}\,B\,.$$

Viceversa, any series $\sigma$ in N admits representations $(A_1, A_2, B, C)$ and, since $\emptyset(\sigma) = s$, the corresponding 2D systems $\Sigma = (A_1, A_2, B, C)$ are realizations of s, that is elements of M /7/.

It is now clear that there exists a commutative realization of s if and only if N contains an exchangeable series, in other terms if and only if the (unique) exchangeable series $\sigma^*$ having s as commutative image is rational. Moreover the full class of the commutative realizations of s is identified with the class of the commutative representations (7) of $\sigma^*$.

A different condition for the existence of a commutative realization of s is given in terms of separability of a commutative power series.

Given $s = \sum_{ij} s_{ij}\, z_1^i\, z_2^j$ introduce the series

$$\bar{s} = \emptyset(\sigma) = \sum_{ij} \bar{s}_{ij}\, z_1^i\, z_2^j \qquad (9)$$

where

$$\bar{s}_{ij} = \binom{i+j}{j}^{-1} s_{ij}\,.$$

Assume s have a commutative realization $\Sigma = (A_1, A_2, B, C)$. Then, from

$$s = C(I - A_1\, z_1 - A_2\, z_2)^{-1}\, B \qquad (10)$$

$$= \sum_{i\,j=0}^{\infty} \binom{i+j}{j}\; C\; A_1^i\, A_2^j\, B\, z_1^i\, z_2^j$$

we have

$$\bar{s} = \sum_{i\,j=0}^{\infty} C\, A_1^i\, A_2^j\, B\, z_1^i\, z_2^j$$

$$= C\,(I - A_1\, z_1)^{-1}\,(I - A_2\, z_2)^{-1}\, B \qquad (11)$$

which shows that $\bar{s}$ is separable.

For the converse, assume $\bar{s}$ be separable. Then $\bar{s}$ can be represented as in (11), with $A_1 A_2 = A_2 A_1$ (see, for instance, /6/), and we go back to (10) following the previous steps in the reverse order.

*Remark.* If s admits a commutative realization, the commutative representations (7) of the associated exchangeable series $\sigma^*$ are in one to one correspondence with the commutative representations (11) of the separable series $\bar{s}$. This shows that, in solving the commutative realization problem, going through $\sigma^*$ or $\bar{s}$ is completely equivalent.

Hankel matrices provide the basic tool for checking the existence of commutative realizations of a transfer function s and for constructing commutative realizations with minimal dimension.

We recall that the Hankel matrix /6/ of a noncommutative (commutative) series $\sigma$ (r) is an infinite matrix whose rows and columns are indexed by the words of the free monoid $\{\xi_1,\xi_2\}^*$ (by the monomials $z_1^i\, z_2^j$).

The matrix element indexed by the pair (u, v) (by the pair $(z_1^i\, z_2^j,\ z_1^h\, z_2^k)$) is the coefficient $(\sigma, u\,v)$ of the word u v (the coefficient $r_{i+h,j+k}$ of the monomial $z_1^{i+h}\, z_2^{j+k}$).

Denote by H(r) the Hankel matrix of r.

Then

i) r is separable iff rank H(r) is finite
ii) rank H(r) gives the dimension of minimal (commutative) representations (11) of r
iii) minimal commutative representations (11) are algebraically equivalent. They can be computed from H(r) via Ho's algorithm /1/.

Analogously, let $H(\sigma)$ be the Hankel matrix of $\sigma$. Then

i) is rational iff rank $H(\sigma)$ is finite
ii) rank $H(\sigma)$ gives the dimension of minimal representations (7) of $\sigma$
iii) minimal representations (7) are algebraically equivalent and can be derived from $H(\sigma)$ via Ho's algorithm /7/.

Minimal representations of the exchangeable series $\sigma^*$ are necessarily commutative and coincide with minimal representations (11) of $\bar{s}$. So we have rank $H(\sigma) =$ rank $H(\bar{s})$ (*).

The rank finiteness of $H(\bar{s})$ is equivalent to the existence of commutative realizations of s, and the 4-tuples $(A_1, A_2, B, C)$ which provide minimal (commutative) representations (11) of $\bar{s}$ constitute the minimal commutative realizations of s.

Since minimal representations (11) are algebraically equivalent, minimal commutative realizations are essentially unique, modulo a change of basis in the local state space. This makes a strong difference between commutative and noncommutative realizations, since noncommutative realizations are not necessarily algebraically equivalent /3/.

The realizability condition based on the rank of $H(\bar{s})$ allows us to give a negative answer to the question whether structure conditions on the denominator of the tran-

(*) This can be seen also directly since the rows and columns of $H(\bar{s})$ are obtained by keeping the rows and the columns in $H(\sigma^*)$ which are indexed by $\xi_1^i\, \xi_2^j$ and deleting the others.
Since the columns rows indexed by the words of the polynomial $\xi_1^i \sqcup\!\sqcup\ \xi_2^j$ coincide in $H(\sigma^*)$ with the column row indexed by $\xi_1^i\, \xi_2^j$, the deletion process which leads to $H(\bar{s})$ starting from $H(\sigma^*)$ does not reduce the rank.

sfer function s are sufficient to guarantee the existence of commutative realizations.

This is done by considering the following rational function

$$s = \frac{1}{(1 - z_1)(1 - z_1 - z_2)} \qquad (12)$$

Its power series expansion in a neighbourhood of the origin is

$$s = \sum_{i,j=0}^{\infty} \binom{i+j+1}{j+i} z_1^i z_2^j$$

So, by (9), we have

$$\bar{s} = \sum_{i,j=0}^{\infty} \frac{i+j+1}{j+1} z_1^i z_2^j$$

In the Hankel matrix

$$H(\bar{s}) = \begin{array}{c|c|c|c|} & (1) & (z_1)\ (z_2) & (z_1^2)\ (z_1z_2)\ (z_2^2) \\ \hline (1) & H_{00} & H_{01} & H_{02} \\ \hline (z_1)\ (z_2) & H_{10} & H_{11} & H_{12} \\ \hline (z_1^2)\ (z_1z_2)\ (z_2^2) & H_{20} & H_{21} & H_{22} \\ \hline \end{array}$$

the diagonal block matrices are

$$H_{00} = [\,1\,]$$

$$H_{11} = 3\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$$

$$H_{22} = 5\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

.................................

$$H_{nn} = (2n+1)\begin{bmatrix} 1 & 1/2 & \dots & 1/_{n+1} \\ 1/2 & 1/3 & \dots & 1/_{n+2} \\ \dots & \dots & \dots & \dots \\ 1/_{n+1} & 1/_{n+2} & \dots & 1/2n+1 \end{bmatrix}$$

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Now notice that $H_{nn}/(2n+1)$, $n=0,1,2\dots$ are the $(n+1)\times x(n+1)$ submatrices appearing in the upper left hand corner of the Hankel matrix associated with the nonrational power series $-\log(1-x) = \sum_{n=1}^{\infty} x^n/n$.

Taking on both sides of rank $H(\bar{s}) >$ rank $H_{nn}$ the limit as n goes to infinity, we obtain rank $H(\bar{s}) = \infty$.

This implies that (12) cannot be realized using commutative matrices $A_1$ and $A_2$, despite the denominator of s factorizes as a product of linear factors.

## FURTHER REMARKS

We shall now give a first insight into the problem of a complete characterization of transfer functions which are realizable by commutative matrices.

For this, we use jointly the following facts:

i) a rational function s admits a commutative realization iff it is the commutative image of a rational exchangeable noncommutative series $\sigma$

ii) a noncommutative series can be represented as a linear combination of series with structure (8).

So it appears essential to study the structures of series (8) and of their commutative images. By exploiting the partial fraction expansion on the complex field, structure (8) reduces to a linear combination of the noncommutative series $\xi_1^m \sqcup\!\sqcup\, \xi_2^n$, $\xi_1^m \sqcup\!\sqcup\, (1 - b\,\xi_2)^{-n}$, $(1 - a\,\xi_1)^{-m} \sqcup\!\sqcup\, \xi_2^n$, $(1-a\xi_1)^{-m} \sqcup\!\sqcup\, (1 - b\xi_2)^{-n}$, $m,n \in \mathbb{N}$.

Thus the commutative image of a rational exchangeable series is a linear combination of

i) $z_1^m\ z_2^n$

ii) $\dfrac{\partial^m}{\partial z_2^m} \dfrac{(z_1\ z_2)^m}{(1 - b\ z_2)^n}$

iii) $\dfrac{\partial^n}{\partial z_1^n} \dfrac{(z_1\ z_2)^n}{(1 - a\ z_1)^m}$

iv) $\dfrac{\partial^{m+n}}{\partial z_1^m\,\partial z_2^n} \dfrac{z_1^m\ z_2^n}{1 - a\ z_1 - b\ z_2}$

Viceversa, any linear combination of series having structure i) - iv) is the commutative image of an exchangeable rational series, hence it admits a commutative realization.

Commutative realizations of a transfer function s are always a proper subset of the set M of all realizations (commutative and noncommutative). So, in general, minimal commutative realizations are not minimal in M.

In some cases, as shown by the following exàmple, the increase in the dimension of the local state seems to be very heavy.

*Example.* Consider the polynomial transfer function $s=z_1^m z_2^m$. The minimal dimension of its commutative realizations is given by rank $H(\bar{s})$, where $\bar{s} = \binom{2m}{m} z_1^m z_2^m$.

The non zero rows in $H(\bar{s})$ are indexed by the following monomials

| | |
|---|---|
| $1$ | (1 row) |
| $z_1,\ z_2$ | (2 rows) |
| $z_1^2,\ z_1z_2,\ z_2^2$ | (3 rows) |
| - - - - - - - - | - - - - |
| $z_1^m,\ z_1^{m-1} z_2,\ \dots,\ z_1 z_2^{m-1},\ z_2^m$ | (m+1 rows) |
| $z_1^m z_2,\ z_1^{m-1} z_2^2,\ \dots z_1 z_2^m$ | (m rows) |
| $z_1^{m-1} z_2^2,\ z_1^{m-2} z_2^3 \dots z_1^2 z_2^{m-1}$ | (m-1 rows) |
| - - - - - - - - | - - - - |
| $z_1^m\ z_2^m$ | (1 row) |

These rows are linearly independent, so we have rank $H(\bar{s}) = (m+1)^2$. On the other side, a minimal noncommutative realization of s is the following

$$A_1 = \left[\begin{array}{c|c} \begin{matrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right] \begin{matrix} \}m+1 \\ \}m \end{matrix} \qquad \text{(columns: } m+1,\ m)$$

$$A_2 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{matrix} \end{array}\right] \begin{matrix} \}m \\ \}m+1 \end{matrix} \qquad \text{(columns: } m,\ m+1)$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

whose dimension is 2m+1.

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