STATE FEEDBACK STABILIZABILITY OF 2D FILTERS: A DEAD BEAT CONTROLLER APPROACH

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ABSTRACT

In this paper a dead beat controller theory is introduced for 2D filters in state space form. Conditions are derived for the existence of dynamic dead beat controllers and an explicit synthesis technique is given for the single input case. Static feedback controllers are also examined.

INTRODUCTION

The recursive structure of the state equations of 2D systems is naturally related to the causality which, in turns, depends on the partial ordering introduced in the discrete plane.

The fact that in the discrete plane the future and the past sets of any point do not cover the whole plane, plays an important role in the definition of state feedback laws and gives more possibilities than in 1D case. In fact, by allowing the introduction of the so called "weak causality", three types of feedback laws have been presented in the literature [1].

As a consequence, in 2D systems theory we have at our disposal more flexible techniques for solving the stabilization problem. Of course the problems are more involved, essentially because stability criteria rely on the shape of algebraic curves instead of the position of isolated singularities.

In this paper we shall introduce a stabilization technique based on a feedback law which preserves the quarter plane causality.

Consider a 2D filter in state space form (2D system [2]):

$$\begin{aligned} x(h+1,\ k+1) &= A_1 x(h+1,k) + A_2 x(h,k+1) + \\ &\quad + B_1 u(h+1,k) + B_2 u(h,k+1) \\ y(h,k) &= C\, x(h,k) \end{aligned} \tag{1}$$

where the local state x is an n-dimensional vector over the real field $\mathbb{R}$, input and output functions take values in $\mathbb{R}^m$ and $\mathbb{R}^p$, A_1, A_2, B_1, B_2, C are matrices of suitable dimensions with entries in $\mathbb{R}$.

A controller $\bar{\Sigma}$ of Σ is a dynamical system with equations

$$\begin{aligned} \bar{x}(h+1,\ k+1) &= \bar{A}_1 \bar{x}(h+1,k) + \bar{A}_2 \bar{x}(h,k+1) + \\ &\quad + \bar{B}_1 \bar{u}(h+1,k) + \bar{B}_2 u(h,k+1) \end{aligned} \tag{2}$$

$$\bar{y}(h,k) = \bar{C}\ \bar{x}(h,k) + \bar{D}\ \bar{u}(h,k)$$

connected by a state feedback law with Σ, so that

$$\bar{u}(h,k) = x(h,k) \qquad u(h,k) = \bar{y}(h,k) \tag{3}$$

In the sequel we deal with the problem of implementing a dead beat controller, that is a controller which drives to zero the state of the whole system in a finite number of steps.

The theory we shall develop can be extended to more general (non dead beat) controllers [3]. A formal advantage of dealing with dead beat controllers is that only polynomial matrices are needed.

It is interesting to remark that the dynamic feedback control law given by (2) and (3) corresponds to the assumption that the input function u satisfies a 2D recursive equation of the following type

$$u(h,k) = \sum_{\substack{i,j\geq 0\\ i+j>0}} R_{ij} u(h-i,k-j) + \sum_{\substack{i,j\geq 0\\ i+j>0}} S_{ij} x(h-i,k-j) + \bar{D}\ x(h,k)$$

In particular, static feedback control laws correspond to assume $R_{ij} = 0$, $S_{ij} = 0$, so that no state dynamics is involved in the construction of the control signal.

DEAD BEAT CONTROLLERS

The fundamental property needed for studying the existence and the structure of dead beat controllers is local controllability.

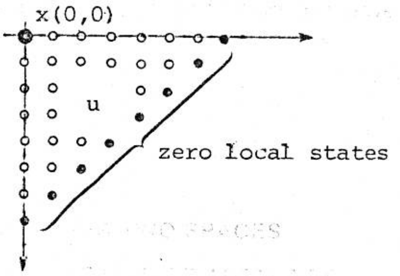
Definition 1. *A system Σ is locally controllable if for any local state* $x(0,0)$ *there exists a poly-*

Author should number the pages in pencil here 1 *nomial vector* $U(z_1,z_2)$ *such that*

$$X(z_1,z_2) = (I-A_1z_1-A_2z_2)^{-1}\left[x(0,0) + (B_1z_1+B_2z_2)U(z_1,z_2)\right]$$

is a polynomial vector.

The above definition of local controllability corresponds to assume that any initial local state can be controlled to zero in a finite number of steps



by using an input function with support in the future of (0,0).

Theorem 1 *The following facts are equivalent*

i) Σ *is locally controllable*

ii) *there exist polynomial matrices* $M(z_1,z_2)$ *and* $N(z_1,z_2)$ *such that the Bézout identity*

$$(B_1z_1+B_2z_2)N(z_1,z_2)+(I-A_1z_1-A_2z_2)M(z_1,z_2) = I \quad (4)$$

holds

iii) $$\operatorname{rank}\left[I - A_1z_1-A_2z_2 \vdots B_1z_1 + B_2z_2\right] = n \quad (5)$$

for any (z_1,z_2) *in* $\mathbb{C}\times\mathbb{C}$

iv) *there exist a positive integer* ν *and a polynomial matrix* $T(z_1,z_2)$ *such that*

$$(A_1z_1+A_2z_2)^{\nu} = \left[(B_1z_1+B_2z_2) \vdots (A_1z_1+A_2z_2)(B_1z_1+B_2z_2) \ldots\ldots (A_1z_1+A_2z_2)^{n-1}(B_1z_1+B_2z_2)\right]T(z_1,z_2) . \quad (6)$$

proof

i) $\rightarrow$ ii) Denote by $U_k(z_1,z_2)$ the polynomial input that controls to zero the local state $x(0,0) = e_k$, $k = 1,2,\ldots,n$, and let

$$N(z_1,z_2) \stackrel{\Delta}{=} \left[-U_1(z_1,z_2) \ \ldots -U_n(z_1,z_2)\right] \quad (7)$$

Then, by linearity, for any $x(0,0)$ in R^n, the input $-N(z_1,z_2)x(0,0)$ controls $x(0,0)$ to zero in a finite number of steps, or equivalently

$$X(z_1,z_2) = (I-A_1z_1-A_2z_2)^{-1}\left[I-(B_1z_1+B_2z_2)N(z_1,z_2)\right]x(0,0)$$

is a polynomial vector.

Hence

$$M(z_1,z_2) \stackrel{\Delta}{=} (I-A_1z_1-A_2z_2)^{-1}\left[I-(B_1z_1+B_2z_2)N(z_1,z_2)\right] \quad (8)$$

is a polynomial matrix and the Bézout identity follows by premultiplication of (8) by $I-A_1z_1-A_2z_2$.

ii) $\rightarrow$ i) Immediate, since any initial state $x(0,0)$ is driven to zero in a finite number of steps by the input function

$$U(z_1,z_2) = -N(z_1,z_2)x(0,0)$$

For the proof of the other equivalences, see [3,4]

Now consider any pair of polynomial matrices M,N which satisfies the Bézout identity (4) and let $\bar\Sigma = (\bar A_1,\bar A_2,\bar B_1,\bar B_2,\bar C,\bar D)$ be any realization of the transfer matrix $-NM^{-1}$.

The state feedback connection of Σ and $\bar\Sigma$ has the property that for any set of initial states in Σ

$$\mathscr{X}_o = \Sigma\, x(h,-h)z_1^h z_2^{-h}$$

and for zero initial states in $\bar\Sigma$

$$\bar{\mathscr{X}}_o = \Sigma\, \bar x(h,-h)z_1^h z_2^{-h} = 0 \quad (9)$$

the global state evolution and the input function of Σ are given by

$$X(z_1,z_2) = M(z_1,z_2)\,\mathscr{X}_o$$

$$U(z_1,z_2) = -N(z_1,z_2)\,\mathscr{X}_o$$

Hence inputs and states of Σ vanish in a finite number of steps.

Since this property needs not be satisfied if $\bar{\mathscr{X}}_o \neq 0$, $\bar\Sigma$ will be called a *weak dead beat controller.*

Actually the assumption (9) on the controller initial conditions and the possibility that the state of the controller does not converge to zero in a finite number of steps, are both undesirable properties. So we shall introduce a more restrictive definition of dead beat controller.

Definition 2. *A system* $\bar\Sigma$ *is a dead beat controller of* Σ *if the state feedback connection of* Σ *and* $\bar\Sigma$ *is a finite memory system, i.e. for any set of ini* *tial conditions $\mathcal{X}_o$ and $\bar{\mathcal{X}}_o$ the state of the composite system goes to zero in a finite number of steps.*

Since the system matrices of the composite systems given by (1), (2) and (3) are the following

$$F_1 = \left[\begin{array}{c|c} A_1+B_1\bar{D} & B_1\bar{C} \\ \hline \bar{B}_1 & \bar{A}_1 \end{array}\right], \quad F_2 = \left[\begin{array}{c|c} A_2+B_2\bar{D} & B_2\bar{C} \\ \hline \bar{B}_2 & \bar{A}_2 \end{array}\right] \quad (10)$$

the finite memory condition of Definition 2 corresponds to assume

$$\det(I-F_1z_1-F_2z_2) = 1 \quad (11)$$

To express the conditions which guarantee that a weak dead beat controller is a dead beat controller, we need the dual property of local controllability, i.e. causal reconstructibility [4]. We recall that a system Σ is causally reconstructible if and only if there exist polynomial matrices $P(z_1,z_2)$ and $Q(z_1,z_2)$ which satisfy the following Bézout identity

$$P(z_1,z_2)C + Q(z_1,z_2)(I-A_1z_1-A_2z_2) = I \quad (12)$$

Theorem 2. *A weak dead beat controller $\bar{\Sigma}$ of Σ is a dead beat controller if and only if $\bar{\Sigma}$ is locally controllable and causally reconstructible.*

proof. Assume $\bar{\Sigma}$ to be locally controllable and causally reconstructible and partition the matrix $(I-F_1z_1-F_2z_2)^{-1}$ conformably with the partition given by (10).

$$(I-F_1z_1-F_2z_2)^{-1} = \left[\begin{array}{c|c} G_{11}(z_1,z_2) & G_{12}(z_1,z_2) \\ \hline G_{21}(z_1,z_2) & G_{22}(z_1,z_2) \end{array}\right] \quad (13)$$

Condition (11) will be proved by showing that (13) is a polynomial matrix.

Let $\bar{\mathcal{X}}_o = 0$ and $\mathcal{X}_o = \Sigma\, x(h,-h)z_1^h z_2^{-h} = x(0,0)$. Then since $\bar{\Sigma}$ is a weak dead beat controller, the state evolution of Σ is expressed by a polynomial vector,

$$X(z_1,z_2) = [I \mid 0] \begin{bmatrix} X(z_1,z_2) \\ \bar{X}(z_1,z_2) \end{bmatrix} = [I \mid 0] \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} x(0,0) \\ 0 \end{bmatrix}$$

$$= G_{11}x(0,0) \quad (14)$$

Under the same assumptions the input function of Σ is given by the polynomial vector

$$U(z_1,z_2) = \bar{Y}(z_1,z_2) = [\bar{D}\ \bar{C}] \begin{bmatrix} X(z_1,z_2) \\ \bar{X}(z_1,z_2) \end{bmatrix}$$

$$= [\bar{D}\ \bar{C}] \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} x(0,0) \\ 0 \end{bmatrix} = \quad (15)$$

$$= (\bar{D}\,G_{11} + \bar{C}\,G_{21})x(0,0)$$

Hence G_{11} and $\bar{C}\,G_{21}$ are both polynomial matrices.

Since the input function can be written also in the form

$$U(z_1,z_2) = \bar{W}(z_1,z_2)\bar{U}(z_1,z_2) = \bar{W}\,G_{11}x(0,0) \quad (16)$$

where

$$\bar{W}(z_1,z_2) = \bar{D} + \bar{C}(I-\bar{A}_1z_1-\bar{A}_2z_2)^{-1}(\bar{B}_1z_1+\bar{B}_2z_2)$$

denotes the transfer matrix of $\bar{\Sigma}$, comparing (15) and (16) we get

$$\bar{W}\,G_{11} = \bar{D}\,G_{11} + \bar{C}\,G_{21} \quad (17)$$

By causal reconstructibility of $\bar{\Sigma}$, the Bézout identity

$$P(z_1,z_2)\bar{C} + Q(z_1,z_2)(I-\bar{A}_1z_1-\bar{A}_2z_2) = I \quad (18)$$

holds for some polynomial matrices P and Q. Then equation (17) and postmultiplication of (18) by $(I-\bar{A}_1z_1-\bar{A}_2z_2)^{-1}(\bar{B}_1z_1+\bar{B}_2z_2)G_{11}$ give

$$\begin{aligned} & P\,\bar{C}\,G_{21} + Q(\bar{B}_1z_1+\bar{B}_2z_2)G_{11} = \\ & = (I-\bar{A}_1z_1-\bar{A}_2z_2)^{-1}(\bar{B}_1z_1+\bar{B}_2z_2)G_{11} \end{aligned} \quad (19)$$

The matrix

$$(I-\bar{A}_1z_1-\bar{A}_2z_2)G_{21} - (\bar{B}_1z_1+\bar{B}_2z_2)G_{11} \quad (20)$$

is zero, since it is the block submatrix in position (2,1) in $(I-F_1z_1-F_2z_2)(I-F_1z_1-F_2z_2)^{-1}$.

Then (19) and (20) imply

$$G_{21} = P\,\bar{C}\,G_{21} + Q(\bar{B}_1z_1+\bar{B}_2z_2)G_{11}$$

which is obviously polynomial.

Finally, by local controllability of $\bar{\Sigma}$, the Bézout identity

$$(I-\bar{A}_1z_1-\bar{A}_2z_2)\bar{M}(z_1,z_2) + (\bar{B}_1z_1+\bar{B}_2z_2)\bar{N}(z_1,z_2) = I \quad (21)$$

holds for some polynomial matrices $\bar{M}$ and $\bar{N}$.

Premultiplication of (21) by G_{12} and G_{22} respectively gives

$$\begin{aligned} G_{12} &= G_{12}(I-\bar{A}_1 z_1-\bar{A}_2 z_2)\bar{M} + G_{12}(\bar{B}_1 z_1+\bar{B}_2 z_2)\bar{N} \\ G_{22} &= G_{22}(I-\bar{A}_1 z_1-\bar{A}_2 z_2)\bar{M} + G_{22}(\bar{B}_1 z_1+\bar{B}_2 z_2)\bar{N} \end{aligned} \tag{22}$$

Explicit computation of the four blocks in the partitioned matrix $(I-F_1z_1-F_2z_2)^{-1}(I-F_1z_1-F_2z_2)$ and their substitution in (29) gives

$$G_{12} = -G_{11}(B_1 z_1+B_2 z_2)\bar{C}\, M + G_{11}(I-A_1 z_1-A_2 z_2 - B_1\bar{D}z_1-B_2\bar{D}z_2)N$$

$$G_{22} = \left[I + G_{21}(B_1 z_1+B_2 z_2)\bar{C}\right]M + G_{21}(I-A_1 z_1-A_2 z_2 - B_1\bar{D}z_1-B_2\bar{D}z_2)N$$

which show that G_{12} and G_{22} are polynomial matrices.

Viceversa, assume that $\bar{\Sigma}$ is a dead beat controller. Then G_{11}, G_{12}, G_{21}, G_{22} are polynomial matrices and the explicit evaluation of the blocks in $(I-F_1z_1-F_2z_2)^{-1}(I-F_1z_1-F_2z_2)$ and in $(I-F_1z_1-F_2z_2)(I-F_1z_1-F_2z_2)^{-1}$ provides the Bézout identities (18) and (21).

Remark The last step of the proof above provides four Bézout identities. Two of them correspond to local controllability and causal reconstructibility of $\bar{\Sigma}$, while the remaining identities correspond to local controllability and to causal reconstructibility of Σ (provided that state and output coincide). This could be expected since local controllability of Σ is a necessary condition for the existence of a dead beat controller.

SINGLE INPUT CASE

In this section an explicit synthesis procedure will be given for obtaining a dead beat controller when Σ is a single input locally controllable system.

The problem can be restated as a realization problem of the row transfer matrix $-NM^{-1}$, where N and M satisfy the Bézout identity (4), under the constraint that the realization has to be locally controllable and causally reconstructible.

Rewrite the transfer matrix $-NM^{-1}$ as a left matrix fraction

$$(\det M)^{-1}(-N\,\mathrm{Adj}\,M)$$

$$= (\Sigma\, d_{ij} z_1^i z_2^j)^{-1}\left[\Sigma\, n_{ij}^{(1)} z_1^i z_2^j \;\;\ldots\;\; \Sigma\, n_{ij}^{(n)} z_1^i z_2^j\right] \tag{23}$$

As a consequence of the Bézout identity (4), the matrix

$$\begin{bmatrix} N(z_1,z_2) \\ M(z_1,z_2) \end{bmatrix} \tag{24}$$

has full rank at any point in $\mathbb{C}\times\mathbb{C}$. Since the polynomials in (23) are the n-th order minors of (24), the variety $\mathcal{V}(\det M, n^{(1)}, n^{(2)},\ldots,n^{(n)})$ is empty and there exist polynomials p_i, $i=1,2,\ldots,n$, and q which satisfy the identity

$$(\det M)q + \left[n^{(1)}\, n^{(2)} \ldots n^{(n)}\right]\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = 1 \tag{25}$$

The realization of (23) is a strightforward extension of the following realization of the scalar transfer function

$$n(z_1,z_2)/d(z_1,z_2) = n_{oo} + \sum_{i+j\geq 1} n_{ij} z_1^i z_2^j / \Sigma\, d_{ij} z_1^i z_2^j$$

with $d_{oo}=1$. It is easy to check that the following 2D system

$$\bar{A}_1 = \begin{bmatrix} \ddots & & & & & & & & & \\ & 0&0&0&0 & & & & & 0 \\ & 0&0&0&0 & & & & & 0 \\ & 0&0&0&1 & & & & & -d_{30} \\ & & & & & 0&0&0 & & 0 \\ & & & & & 0&0&1 & & -d_{20} \\ & & & & & & & & 0\;\;1 & -d_{10} \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ n_{30} \\ 0 \\ n_{20} \\ n_{10} \end{bmatrix}$$

$$\bar{A}_2 = \begin{bmatrix} \ddots & & & & & & & & & \\ & 1&0&0&0 & & & & & -d_{03} \\ & 0&1&0&0 & & & & & -d_{12} \\ & 0&0&1&0 & & & & & -d_{21} \\ & & & & & 1&0&0 & & -d_{02} \\ & & & & & 0&1&0 & & -d_{11} \\ & & & & & & & & 1\;\;0 & -d_{01} \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} \vdots \\ n_{03} \\ n_{12} \\ n_{21} \\ n_{02} \\ n_{11} \\ n_{01} \end{bmatrix}$$

$$\bar{C} = \left[\;\ldots\; 0\;0\;0\;0\;0\;\;1\;\right] \qquad \bar{D} = n_{oo}$$

realizes n/d. In order to get a realization of (23), denote by $\bar{\Sigma}_k = (\bar{A}_1,\bar{A}_2,\bar{B}_1^{(k)},\bar{B}_2^{(k)},\bar{C},\bar{D}^{(k)})$ the realizations of the transfer functions $n^{(k)}(z_1,z_2)/\det M$,

$k = 1,2,\ldots,n$, given by the previous formulas, and consider the system $\bar{\Sigma} = (\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2, \bar{C}, \bar{D})$, with

$$\bar{B}_1 = \begin{bmatrix} \bar{B}_1^{(1)} & \bar{B}_1^{(2)} & \cdots\cdots & \bar{B}_1^{(n)} \end{bmatrix}$$
$$\bar{B}_2 = \begin{bmatrix} \bar{B}_2^{(1)} & \bar{B}_2^{(2)} & \cdots\cdots & \bar{B}_2^{(n)} \end{bmatrix}$$
$$\bar{D} = \begin{bmatrix} \bar{D}^{(1)} & \bar{D}^{(2)} & \cdots\cdots & \bar{D}^{(n)} \end{bmatrix}$$

Clearly $\bar{\Sigma}$ is a realization of (23). Moreover such a realization is locally controllable and causally reconstructible. Causal reconstructibility follows directly from the structure of matrices $\bar{A}_1$, $\bar{A}_2$ and $\bar{C}$, while local controllability depends on the Bézout identity (25).

CONCLUDING REMARKS

The rank condition (5) of Theorem 1 is analogous to the condition:

$$\text{rank}\begin{bmatrix} I-Az & B \end{bmatrix} = n, \quad \forall z \in \mathbb{C}$$

which guarantees stabilizability of a 1D system (A,B,C) by means of a *static* dead beat controller. The 2D situation looks different, essentially because the existence of static dead beat controllers, in general, is not guaranteed by (5). In other words, the 1D state static feedback gives essentially the same performances as the dynamic one, while dynamical compensation exhibits greater potentialities in 2D controller theory.

As an example, the following 2D system

$$A_1 = \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is locally controllable, since $\begin{bmatrix} I-A_1z_1-A_2z_2 & B_1z_1+B_2z_2 \end{bmatrix}$ is full rank in $\mathbb{C}\times\mathbb{C}$. Hence the system admits a dynamical dead beat controller.

Nevertheless a static feedback law

$$u(h,k) = \begin{bmatrix} \alpha & \beta & \gamma & \delta \end{bmatrix} x(h,k) = K\,x(h,k)$$

gives

$$\det(I-A_1z_1-A_2z_2-(B_1z_1+B_2z_2)K) = 1-\delta z_1-z_1z_2-\gamma z_2^2$$

which is different from 1 for any choice of K. So a static dead beat controller does not exist.

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