SOME CONNECTIONS BETWEEN ALGEBRAIC PROPERTIES OF PAIRS OF MATRICES AND 2D SYSTEMS REALIZATION

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ABSTRACT

This paper is concerned with some properties of transfer functions in two variables which can be realized by classes of 2D systems characterized by pairs of state updating matrices which generate algebras with special structures. Two situations are mainly considered. The first deals with pairs of matrices which generate a solvable Lie algebra (i.e. are simultaneously triangularizable). The second refers to pairs of matrices which generate abelian Lie algebras (i.e. the matrices commute).

The analysis of the connections between the properties of 2D realizations and transfer functions is based on the representation algorithms of non-commutative rational power series.

1. INTRODUCTION

It is well known [1,2,3] that any proper rational transfer function in two variables can be realized by a finite dimensional 2D system (A_1, A_2, B, C) described by the following state updating and read-out equations:

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h+1,k)+A_2 x(h,k+1)+B_1 u(h+1,k)+B_2 u(h,k+1) \\ y(h,k) &= Cx(h,k) \end{aligned} \tag{1}$$

In general it should be expected that any constraint we assume on the structure of the pairs (A_1,A_2) translates into a restriction of the class of transfer functions which can be realized by (1).

In this communication we shall concentrate our attention on pairs of matrices which can be simultaneously reduced by similarity to upper (lower) triangular form and, in particular, on pairs of commutative matrices.

Commutative matrices have been first considered by Attasi [4], with reference to the special class of systems given by the following equations

$$x(h+1,k+1) = A_1\, x(h+1,k) + A_2\, x(h,k+1) - A_1 A_2\, x(h,k) + B\, u(h,k)$$
$$y(h,k) = C\, x(h,k) \tag{2}$$

with $A_1 A_2 = A_2 A_1$. The transfer functions realizable by this model are (causal) separable functions, that is they can be written in the form $n(z_1,z_2)/p(z_1)q(z_2)$, where n is in $(z_1,z_2)\, K[z_1,z_2]$, p in $K[z_1]$ and q in $K[z_2]$. The converse is also true, in the sense that any (causal) separable transfer function is realizable in the class of Attasi's models.

As we shall see, the main feature of the transfer functions we obtain from (1) when A_1 and A_2 commute, is that their denominators factor completely in the complex field into linear factors [5].

The same is true when the commutativity assumption is weakened and we assume that A_1 and A_2 are simultaneously triangularizable. The difference between the two cases is that the commutativity of A_1 and A_2 imposes some constraints on the numerator of the transfer function while triangularizability does not.

In order to make our analysis simpler, we shall assume that either B_1 or B_2 is the zero vector. So doing the analysis developed in the sequel, applies also to the following models [6,7]:

$$x(h+1,k+1) = A_1 x(h+1,k) + A_2 x(h,k+1) + B\, u(h,k) \tag{3}$$
$$y(h,k) = C\, x(h,k)$$

and:

$$x(h+1,k+1) = A_1 x(h+1,h) + A_2 x(h,k+1) + B\, u(h+1,k+1) \tag{4}$$
$$y(h,k) = C\, x(h,k)$$

If we don't take into account the multiplicative factors z_1, z_2 or $z_1 z_2$, which are unessential to our discussion, the structure of the transfer functions of systems (2) (with B_1 or $B_2 = 0$), (3) and (4) reduces to the following form

$$s = C(I-A_1 z_1 - A_2 z_2)^{-1} B \tag{5}$$

The possibility of representing a proper rational function in the form (5), allows us to associate its realization (A_1,A_2,B,C) with the series

$$\sigma = C(I-A_1\xi_1-A_2\xi_2)^{-1}B \tag{6}$$

in the non-commutative variables ξ_1 and ξ_2 and to exploit known results from the theory of non-commutative power series [8].

2. REALIZABILITY AND SIMULTANEOUS TRIANGULARIZATION

Two matrices A_1 and A_2 are simultaneously triangularizable if they can be reduced by similarity transformation to upper (lower) triangular form.

Simultaneous triangularizability - also referred in the literature as property P - has been related to other algebraic properties of pairs of matrices. We summarize the principal results in the following theorem [9,10]:

Theorem 1. *Let* A_1 *and* A_2 *belong to* $\mathbb{C}^{nxn}$. *Then the following statements are equivalent:*

(i) *there is an invertible matrix* T *such that* $P^{-1}A_1P$ *and* $P^{-1}A_2P$ *are upper (lower) triangular;*

(ii) *the Lie algebra* $\mathscr{L}$ *defined by matrices* A_1 *and* A_2 *is solvable;*

(iii) *for every scalar polynomial* $\pi(\xi_1,\xi_2)$ *in the non-commutative variables* ξ_1,ξ_2, *each of the matrices* $\pi(A_1,A_2)[A_1,A_2]$ *is nilpotent;*

(iv) *there is an ordering of the eigenvalues* λ_i *of* A_1 *and* μ_i *of* A_2 *such that the eigenvalues of any scalar polynomial* $\pi(A_1,A_2)$ *are* $\pi(\lambda_i,\mu_i)$, $i=1,2,\ldots,n$.

As an obvious consequence of property P, we have that the polynomial $\det(I-A_1z_1-A_2z_2)$ factors completely in the complex field into linear factors:

$$\det(I-A_1z_1-A_2z_2) = \prod_i(1-\lambda_iz_1-\mu_iz_2) \tag{7}$$

The factorization property (7) - also called property L [10] - is weaker than property P, if $n>2$.

The role played by pairs of matrices with property P in the realization of 2D systems is defined by the following theorem.

<u>Theorem 2</u>. *Let* $W(z_1,z_2) = p(z_1,z_2)/d(z_1,z_2)$, $d(0,0) = 1$ *and* p *and* q *coprime polynomials. Then* $W(z_1,z_2)$ *is realizable by a 2D system with* A_1 *and* A_2 *having property* P *if and only if* $d(z_1,z_2)$ *factors completely in the complex field into linear factors.*

Proof. Assume A_1 and A_2 have property P. By (6), since $d(z_1,z_2)$ divides $\det(I-A_1z_1-A_2z_2)$, it factors into linear elements. Conversely, note that starting from 2D systems with A_1 and A_2 having property P, and connecting them in series and parallel, the A_1 and A_2 matrices of the resulting systems still have property P. So, we need only to take into account transfer functions $W_{ij}(z_1,z_2) = z_1^i z_2^j/1-\alpha z_1-\beta z_2$. The following 2D system, with A_1 and A_2 in triangular form,

$$
\begin{array}{cc} & \begin{array}{cc} i+1 & j \end{array} \\
A_1 = & \left[\begin{array}{ccccc|c}
\alpha & -1 & & & & \\
 & 0 & -1 & & & \\
 & & 0 & \ddots & & 0 \\
 & & & \ddots & -1 & \\
 & & & & 0 & \\ \hline
 & & 0 & & & 0
\end{array}\right]
\end{array},
\qquad
\begin{array}{cc} & \begin{array}{cc} i+1 & j \end{array} \\
A_2 = & \left[\begin{array}{cccc|cccc}
\beta & & & & & & & \\
 & 0 & & & & & & \\
 & & \ddots & & & & & \\
 & & & 0 & -1 & & & \\ \hline
 & & & & 0 & -1 & & \\
 & 0 & & & & \ddots & \ddots & \\
 & & & & & & \ddots & -1 \\
 & & & & & & & 0
\end{array}\right]
\end{array},
\quad
B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

provides a 2D realization of the elementary transfer function W_{ij}.

A classical result due to Frobenius [11] states that any pair of commutative matrices satisfies property P. This fact can be viewed as a corollary of Theorem 1, since the commutativity hypothesis $[A_1,A_2] = 0$ implies (iii).

Consequently, as 2D systems with commutative matrices A_1 and A_2 are a subclass of 2D systems with triangular matrices, the denominator of their transfer function factors completely into linear elements.

Nevertheless, as we shall see, it is not true that any transfer function with denominator factorizable into linear elements, can be realized by a 2D system with commutative matrices A_1 and A_2. This is due to the fact that when we look for 2D realizations of this type, the numerator of the transfer function cannot be arbitrarily assigned.

The analysis of the constraints to be imposed on the transfer functions for obtaining 2D system realizations with A_1 and A_2 commuting, will be developed in the next section by resorting to non-commutative power series. This will allow us also to give a first insight into the problem of understanding how property P and commutativity affect the structure of minimal realizations.

3. COMMUTATIVITY AND PROPERTY P IN THE REPRESENTATION OF NON-COMMUTATIVE POWER SERIES

Simultaneous triangularization and commutativity assumptions on A_1 and A_2 impose structural constraints on the coefficients of the non-commutative power series (6). The nature of these constraints is relevant for the analysis of 2D systems having A_1 and A_2 matrices with the same properties. In fact we can associate any non-commutative power series σ with its commutative image induced by the algebra morphism $\phi : K \ll \xi_1, \xi_2 \gg \longrightarrow K[[z_1, z_2]]$, assigned by $\phi(k) = k$, $\forall k \in K$, $\phi(\xi_1) = z_1$, $\phi(\xi_2) = z_2$. Then, assuming σ to be represented as in (6), the map ϕ associates the non-commutative series σ with the 2D system (A_1, A_2, B, C) whose transfer function is $\phi(\sigma) = C(I - A_1 z_1 - A_2 z_2)^{-1} B$.

In order to analyze in detail these facts, we need some properties of non-commutative power series that we shall briefly recall in the sequel.

Let K be the ground field. A generic element σ of the algebra $K \ll \xi_1, \xi_2 \gg$ of formal power series in the noncommuting variables ξ_1 and ξ_2 with coefficients in K is written as

$$\sigma = \sum_{w \in \{\xi_1, \xi_2\}^*} (\sigma, w) w$$

where $\{\xi_1, \xi_2\}^*$ is the free monoid generated by ξ_1 and ξ_2 and (σ, w) in K is the <u>coefficient</u> of w in the series σ. The series $\phi(\sigma)$ in $K[[z_1, z_2]]$ is called the <u>commutative image</u> of σ.

A series σ in $K \ll \xi_1, \xi_2 \gg$ is <u>exchangeable</u> if the words which have the same commutative image have the same coefficient in σ.

A series σ in $K \ll \xi_1, \xi_2 \gg$ is <u>rational</u> if there exist a positive integer n and matrices A_1, A_2 in $K^{n\times n}$, B in $K^{n\times 1}$, C in $K^{1\times n}$ such that

$$\sigma = C \sum_{k=0}^{\infty} (A_1\xi_1 + A_2\xi_2)^k B = C(I - A_1\xi_1 - A_2\xi_2)^{-1} B \tag{8}$$

A 4-tuple (A_1, A_2, B_1, C) is called a representation of σ if (8) holds.

The following Theorem [8] shows how the commutativity assumption $[A_1, A_2] = 0$ made on the representation (6) of a non-commutative series σ, can be expressed as a condition on the coefficients of σ itself.

Theorem 3. *Let σ be in $K \ll \xi_1, \xi_2 \gg$. Then the following facts are equivalent:*

i) *σ is rational and exchangeable*

ii) *σ is a linear combination of shuffle products*(*) *of the following form*

$$p(\xi_1)q(\xi_1)^{-1} \sqcup\!\sqcup\; r(\xi_2)t(\xi_2)^{-1} \tag{9}$$

where p, q, r, t are polynomials

iii) *there exists a representation (A_1, A_2, B, C) of σ with $A_1A_2 = A_2A_1$, that is*

$$(\sigma, w) = C\, A_1^{|w|_1} A_2^{|w|_2} B, \ \forall\, w \in \{\xi_1, \xi_2\}^*$$

where $|w|_i$ denotes the number of ξ_i in w, $i = 1,2$.

A further characterization of exchangeable rational series is given in terms of separable rational functions.

Theorem 4. *Let $\sigma \in K \ll \xi_1, \xi_2 \gg$ be exchangeable and define the map $\bar{\phi}$: by the assignment*

$$\bar{\phi} : \sum_w (\sigma, w) w \mapsto \sum_{i,j=0}^{\infty} (\sigma, \xi_1^i \xi_2^j) z_1^i z_2^j$$

Then σ is rational if and only if $\bar{\phi}(\sigma)$ is (the power series expansion of) a separable rational function.

(*) For any f and g in $\{\xi_1, \xi_2\}^*$ the shuffle product of f and g is defined as $f \sqcup\!\sqcup\, g \stackrel{\Delta}{=} \sum \{f_1g_1 \ldots f_Kg_K \mid f = f_1 \ldots f_K,\ g = g_1 \ldots g_K\}$. By linearity, the definition extends to $K \ll \xi_1, \xi_2 \gg$.

Assume now that the series σ admits a representation with A_1 and A_2 having property P. The following Theorem shows how this assumption reduces to a condition on the coefficients of σ.

Theorem 5. *Let σ be a rational series in $K\langle\langle \xi_1,\xi_2\rangle\rangle$ and admit a representation of dimension* n. *Then σ admits a representation with A_1 and A_2 having property* P *if and only if for any* (n+1)*-tuplet $w_1,\dots,w_{n+1}$ in $\{\xi_1,\xi_2\}^*$ we have*

$$\sum_{i_1,\dots,i_n\in\{1,2\}} (-1)^{i_1+i_2+\dots+i_n}(\sigma, w_1\gamma_{i_1}\dots w_n\gamma_{i_n}w_{n+1}) = 0 \tag{10}$$

where $\gamma_1 \overset{\Delta}{=} \xi_1\xi_2$ *and* $\gamma_2 \overset{\Delta}{=} \xi_2\xi_1$.

Proof. Let (A_1,A_2,B,C), with A_1 and A_2 having property P, be a representation of σ. It is not restrictive to assume that the dimension of this representation is less than or equal to n. In fact any minimal representation $(\bar{A}_1,\bar{A}_2,\bar{B},\bar{C})$ of σ can be obtained (modulo a similarity transformation) from (A_1,A_2,B,C) by standard reducing procedures without destroying property (iii) of Theorem 1 and hence property P. Then, for any (n+1)-tuplet $w_1,\dots,w_{n+1}$, in $\{\xi_1,\xi_2\}^*$, we have

$$w_1(A_1,A_2)\,[A_1,A_2]\,\dots\, w_n(A_1,A_2)\,[A_1,A_2]\,w_{n+1}(A_1,A_2) = 0 \tag{11}$$

as we can check directly by assuming A_1 and A_2 in triangular form.

Let now multiply (11) by C on the left and by B on the right to get (10). Conversely, let (A_1,A_2,B,C) be a minimal representation of σ of dimension $m\le n$. It is known from [8] that there exist m^2 matrices $M_{ij}\in K^{m\times m}$ and two sets of m words, each word with length less than m, $\{d_1,\dots,d_m\}$ and $\{g_1,\dots,g_m\}$ such that for any $w\in\{\xi_1,\xi_2\}^*$, it results

$$w(A_1,A_2) = \Sigma_{h,k}\, M_{h,k}(\sigma, g_h\, w\, d_k)\ .$$

Then, for any n-tuplet $w_1,\dots,w_n$ in $\{\xi_1,\xi_2\}^*$ we have

$$\begin{aligned}\sum_{i_1,\dots,i_n}(-1)^{i_1+\dots+i_n} w_1(A_1,A_2)\gamma_{i_1}(A_1,A_2)\dots w_n(A_1,A_2)\gamma_{i_n}(A_1,A_2) &= \\ = \sum_{h,k} M_{h,k}\sum_{i_1,\dots i_n}(-1)^{i_1+\dots+i_n}(\sigma, g_h w_1\gamma_{i_1}\dots w_n\gamma_{i_n}d_k) &= 0\end{aligned} \tag{12}$$

Now take any polynomial π in $K<\xi_1,\xi_2>$ and consider the matrix

$$(\pi(A_1,A_2)\ [A_1,A_2])^n \quad .$$

This turns out to be zero since it is a linear combination of terms of the same type as those in the summation on the left side of (12).

By applying criterion (iii) in Theorem 1 we conclude that A_1 and A_2 satisfy property P.

In view of the applications we shall made, it is worthwhile to state by a separate Theorem the following fact we already used in the proof of Theorem 5.

Theorem 6. *Assume that the rational series* σ *in* $K<<\xi_1,\xi_2>>$ *admits a representation* (A_1,A_2,B,C) *with* A_1 *and* A_2 *simultaneously triangularizable (commutative). Then the matrices* A_1 *and* A_2 *appearing in any minimal representation of* σ *are simultaneously triangularizable (commutative).*

4. COMMUTATIVE REALIZATIONS

Let's now go back to the problem of the existence of commutative realizations. Consider a 2D rational transfer function s and denote by $\mathcal{M}$ the set of the 2D systems $\Sigma=(A_1,A_2,B,C)$ which realize s. Denote by $\mathcal{N}$ the set of noncommutative rational power series whose commutative image is s.

Then any system $\Sigma=(A_1,A_2,B,C)$ in $\mathcal{M}$ is associated with a representation of a noncommutative series σ in $\mathcal{N}$, i.e. the series $\sigma=C(I-A_1\xi_1-A_2\xi_2)^{-1}B$.

Viceversa, any series σ in $\mathcal{N}$ admits representations (A_1,A_2,B,C) and, since $\phi(\sigma)=s$, the corresponding 2D systems $\Sigma=(A_1,A_2,B,C)$ are realizations of s, that is elements of $\mathcal{M}$ [3].

It is now clear that there exists a commutative realization of s if and only if $\mathcal{N}$ contains an exchangeable series, or, in other terms, if and only if the (unique) exchangeable series σ^* having s as commutative image is rational. Moreover the full class of the commutative realizations of s is identified with the class of the commutative representations (8) of σ^*.

Theorem 4 provides another condition for the existence of a commutative realization of s in terms of separability of a commutative power series.

Given $s = \sum_{i,j} s_{ij} z_1^i z_2^j$, introduce the series

$$\bar{s} = \bar{\phi}(\sigma^*) = \sum_{i,j} \bar{s}_{ij} z_1^i z_2^j \quad , \qquad \bar{s}_{ij} = \binom{i+j}{j}^{-1} s_{ij} \tag{13}$$

Assume s have a commutative realization $\Sigma = (A_1, A_2, B, C)$. Then, from

$$s = C(I - A_1 z_1 - A_2 z_2)^{-1} B = \sum_{i,j=0}^{\infty} \binom{i+1}{j} C A_1^i A_2^j B z_1^i z_2^j \tag{14}$$

we have

$$\bar{s} = \sum_{i,j=0}^{\infty} C A_1^i A_2^j B z_1^i z_2^j = C(I - A_1 z_1)^{-1} (I - A_2 z_2)^{-1} B \tag{15}$$

which shows that $\bar{s}$ is separable.

For the converse, assume $\bar{s}$ be separable. Then $\bar{s}$ can be represented as in (15), with $A_1 A_2 = A_2 A_1$ (see, for instance, |8|), and we go back to (14) following the previous steps in the reverse order.

Remark. If s admits a commutative realization, the commutative representations (8) of the associated exchangeable series σ^* are in one to one correspondence with the commutative representations (15) of the separable series $\bar{s}$. This shows that the series σ^* and $\bar{s}$ play essentially the same role in the solution of the commutative realization problem.

The existence of commutative realizations of a transfer function s and their construction are essentially based on the properties of Hankel matrices.

The Hankel matrix [8] of a non-commutative series σ (a commutative series r) is an infinite matrix whose rows and columns are indexed by the words of the free monoid $\{\xi_1, \xi_2\}^*$ (by the monomials $z_1^i z_2^j$). The matrix element indexed by the pair (u,v) (by the pair $(z_1^i z_2^j, z_1^h z_2^k)$) is the coefficient (σ, uv) of the word uv (the coefficient $r_{i+h,j+k}$ of the monomial $z_1^{i+h} z_2^{j+k}$).

Denoting by H(r) the Hankel matrix of r, we have that:

i) r is separable if and only if rank H(r) is finite

ii) rank H(r) gives the dimension of minimal, commutative representations (15) of r

iii) minimal, commutative representations (15) are algebraically equivalent. They can be computed from H(r) via Ho's algorithm [4].

Analogously, let $H(\sigma)$ be the Hankel matrix of σ. Then

i) σ is rational if and only if rank $H(\sigma)$ is finite

ii) rank $H(\sigma)$ gives the dimension of minimal representations (8) of σ

iii) minimal representations (8) are algebraically equivalent and can be derived from $H(\sigma)$ via Ho's algorithm [3].

By Theorem 4, minimal representations of the exchangeable series σ^* are necessarily commutative and coincide with minimal representations (15) of $\bar{s}$. So we have rank $H(\sigma)$ = rank $H(\bar{s})$.

The rank finiteness of $H(\bar{s})$ is equivalent to the existence of commutative realizations of s, and the 4-tuples (A_1,A_2,B,C) which provide minimal, commutative representations (15) of $\bar{s}$ constitute the minimal commutative realizations of s. Since minimal representations (15) are algebraically equivalent, minimal commutative realizations are essentially unique, modulo a change of basis in the local state space. This make a strong difference between commutative and non-commutative realizations, since non-commutative realizations are not necessarily algebraically equivalent [6].

The realizability condition based on the rank of $H(\bar{s})$ allows us to give a negative answer to the question whether structure conditions on the denominator of the transfer functions s are sufficient to guarantee the existence of commutative realizations.

This is done by considering the following rational function

$$s = \frac{1}{(1-z_1)(1-z_1-z_2)} = \sum_{i,j=0}^{\infty} \binom{i+j+1}{j+1} z_1^i z_2^j \tag{16}$$

So, by (13), we have

$$\bar{s} = \sum_{i,j=0}^{\infty} \frac{i+j+1}{j+1} z_1^i z_2^j$$

In the Hankel matrix

$$H(\bar{s}) = \begin{bmatrix} H_{00} & H_{01} & H_{02} & \cdots\cdots \\ H_{10} & H_{11} & H_{12} & \cdots\cdots \\ H_{20} & H_{21} & \cdots & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \end{bmatrix}$$

the diagonal block matrices are given by:

$$H_{00} = [1] \ , \quad H_{11} = 3\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \ , \ \ldots . \ H_{nn} = (2n+1)\begin{bmatrix} 1 & 1/2 & \ldots & 1/n+1 \\ 1/2 & 1/3 & \ldots & 1/n+2 \\ \ldots & \ldots & \ldots & \ldots \\ 1/n+1 & 1/n+2 & \ldots & 1/2n+1 \end{bmatrix} \ \ldots$$

Now notice that $H_{nn}/(2n+1)$, $n = 0,1,2\ldots$ are the $(n+1)x(n+1)$ submatrices appearing in the upper left hand corner of the Hankel matrix associated with the nonrational power series $-\log\,(1-x) = \sum_{n=1}^{\infty} x^n/n$.

Letting n go to infinity in rank $H(\bar{s}) > \text{rank } H_{nn}$, we obtain rank $H(\bar{s}) = \infty$. This implies that (16) cannot be realized using commutative matrices A_1 and A_2, despite the denominator of s factorizes as a product of linear factors.

An existence condition for commutative realizations may be obtained by using jointly the following facts:

i) a rational function s admits a commutative realization if and only if it is the commutative image of a rational exchangeable noncommutative series σ

ii) a noncommutative series can be represented as a linear combination of series with structure (9).

By exploiting partial fraction expansion of rational functions in one variable, the series having structure (9) reduce to linear combinations of the noncommutative series $\xi_1^m \sqcup\!\sqcup\, \xi_2^n$, $\xi_1^m \sqcup\!\sqcup\, (1-b\xi_2)^{-n}$, $(1-a\xi_1)^{-m} \sqcup\!\sqcup\, \xi_2^n$, $(1-a\xi_1)^{-m} \sqcup\!\sqcup\, (1-b\xi_2)^{-n}$, $m, n \in N$.

Thus the commutative image of a rational exchangeable series is the power series expansion of a linear combination of the following functions:

$$z_1^m z_2^n \ , \quad \frac{\partial^m}{\partial z_2^m} \frac{(z_1 z_2)^m}{(1-bz_2)^n} \ , \quad \frac{\partial^n}{\partial z_1^n} \frac{(z_1 z_2)^n}{(1-az_1)^m} \ , \quad \frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} \frac{z_1^m z_2^n}{1-a z_1 -b z_2} \ . \qquad (17)$$

Viceversa, any linear combination of rational functions (17) is the commutative image of an exchangeable rational series, hence it admits a commutative realization.

5. FURTHER REMARKS

In general, given a rational transfer function, the class of its realizations with matrices A_1 and A_2 having property P, does not share all properties with the class of commutative realizations.

For instance, minimal realizations with A_1 and A_2 having property P, need not be algebraically equivalent.

Example. The following 2D systems

$$\Sigma_1 : \left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1\\ 1\end{bmatrix}, \begin{bmatrix}1 & 0\end{bmatrix}\right)$$

$$\Sigma_2 : \left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}-1\\ 1\end{bmatrix}, \begin{bmatrix}1, 2\end{bmatrix}\right)$$

are minimal realizations of (16) with A_1 and A_2 triangular matrices. Yet, Σ_1 and Σ_2 are not algebraically equivalent. This follows checking that the non-commutative power series associated with Σ_1 and Σ_2 are different.

Moreover, Σ_1 and Σ_2 represent (modulo similarity transformations) the whole class of minimal realizations of (16) which is then wholly constituted by 2D systems with A_1 and A_2 triangularizable.

This is not surprising. In fact, minimal realizations of any rational transfer function whose denominator factors into linear elements, have matrices A_1 and A_2 with property P, if their dimension is 2. If the dimension is greater than 2 the following example shows that matrices A_1 and A_2 of minimal realizations need not simultaneously triangularize.

Example. The following 2D systems

$$\Sigma_1 : \left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}-1 & 0 & 0\end{bmatrix}\right)$$

$$\Sigma_2 : \left(\begin{bmatrix}0 & -1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\end{bmatrix}\right)$$

are minimal realizations of the polynomial $1-z_1z_2$. It is easy to check that A_1 and A_2 from Σ_1 do not have property P. Actually $[A_1, A_2]$ is not nilpotent.

Finally, we observe that minimal commutative realizations of a transfer fu. have higher dimension than minimal realizations with property P and, a fortiori, than minimal unconstrained realizations of the same transfer function. As an example, $z_1^m z_2^m$ has minimal commutative realizations of dimension $(m+1)^2$, while the dimension of minimal realizations with property P is 2m+1 [5].

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